KÄHLER-EINSTEIN TORIC SUBMANIFOLDS OF THE PROJECTIVE SPACE

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Abstract. We show that the Kähler-Einstein metrics on the four families of symmetric toric Fano manifolds due to Batyrev and Selivanova cannot be induced by immersions into projective spaces equipped with the Fubini-Study metric. We also show a similar conclusion for the non-symmetric examples discovered by Nill and Paffenholz. As an application we obtain that a centrally symmetric toric Fano manifold admits a Kähler-Einstein metric induced by a projective immersion if and only if it is a product of projective lines. These results confirm for each of these examples a broader conjecture characterizing the Kähler-Einstein metrics that can be induced by projective immersions.

1. Introduction

It is well-known that the Fubini-Study metric g_{FS} on a complex projective space $\mathbb{P}^s$ is Einstein. Less known are the compact complex manifolds X having an immersion $\varphi\colon X \to \mathbb{P}^s$ such that the pullback Kähler metric $g = \varphi^* g_{\mathrm{FS}}$ is Einstein. A result of Hulin [Hul00] shows that in this case the Ricci curvature is a positive multiple of the metric, which implies that X is a Fano manifold.

After the proof of the Yau-Tian-Donaldson conjecture, the existence of a Kähler-Einstein metric g on a Fano manifold X can be characterized by the algebro-geometric notion of K-polystability [CDS15, Tia15]. When this condition is verified, it is natural to ask if the metric g is *projectively induced* in the sense that there exists an immersion $\varphi\colon X \to \mathbb{P}^s$ such that

$$g = \varphi^* g_{\mathrm{FS}}. \tag{1.1}$$

In [Cal53] Calabi showed that for an arbitrary Kähler manifold (X, g) there are strong restrictions for the existence of a projective immersion inducing the metric, and so one expects (1.1) to occur only in special situations. This contrasts sharply with the analogue in Riemannian geometry, since by Nash's theorem every Riemannian manifold can be isometrically embedded into a real Euclidean space [Nas56].

The problem is therefore to classify the Kähler-Einstein compact manifolds that satisfy the condition (1.1). In this direction Calabi showed that a Kähler-Einstein metric g on a curve X satisfies this condition if and only if X is the projective line and g is a positive integer multiple of the Fubini-Study metric [Cal53]. Continuing with the work of Smyth [Smy67] and Chern [Che67] on hypersurfaces, all known examples of Kähler-Einstein submanifolds of a projective space turned out to be homogeneous [Han75, Nak76, Mat85, Tsu86]. This has motivated the following folklore conjecture.

Conjecture 1.1. *Let g be a projectively induced Kähler-Einstein metric on a compact manifold X. Then (X, g) is a homogeneous Kähler manifold.*

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Ogiue's problem about irreducible or Einstein submanifolds of the projective space with parallel second fundamental form [Ogi74, Problem 7] can be regarded as a predecessor of this statement. In its current form, the conjecture appears for instance in Tosatti's Zentralblatt review of [HWZ15] and in [LZ18, Conjecture 4.2]. The analogous problem of classifying the Kähler-Einstein submanifolds of the complex Euclidean and hyperbolic spaces was also proposed by Ogiue [Ogi74, Problem 8] and solved by Umehara [Ume87]. Indeed the homogeneous Kähler-Einstein submanifolds of a projective space have been determined by Takeuchi [Tak78], and so a positive answer to Conjecture 1.1 would complete the classification of the Kähler-Einstein submanifolds of the complex space forms.

The study of Kähler-Einstein metrics on compact toric manifolds has experienced an important development in recent years, see for instance [BB13] and the references therein. Their classification in dimensions up to 4 was obtained by Batyrev and Selivanova [BS99] and then in dimensions up to 7 combining the classification of unimodular Fano polytopes by Obro [Obr07] with the combinatorial characterization for the existence of a Kähler-Einstein metric on the associated Fano manifolds by Wang and Zhu [WZ04].

Using these results Arezzo, Loi and Zuddas [ALZ12] and Manno and Salis [MS26] proved that a Kähler-Einstein toric manifold of dimension at most 6 is not projectively induced unless it is a product of projective spaces, under the additional hypothesis that the immersion is given by a complete linear series. Unfortunately a complete classification of Kähler-Einstein compact toric manifolds is not available yet, and so this approach to the toric case of the conjecture cannot be extended to the higher dimensional situation.

In this paper we develop algebraic tools to determine handle such higher dimensional situations with the aim of. To explain our results, for an integer $n \geq 1$ let X be an n-dimensional toric Fano manifold and Σ its associated fan on a vector space $N_{\mathbb{R}} \simeq \mathbb{R}^n$. Following [VK85, BS99] this toric Fano manifold is said to be *centrally symmetric* if the reflection on $N_{\mathbb{R}}$ with respect to the origin is an automorphism of Σ, and *symmetric* if the group of automorphisms of Σ fixes only the origin of $N_{\mathbb{R}}$. Clearly a centrally symmetric toric Fano manifold is symmetric, but the converse does not always hold [JRZ25].

In [BS99] Batyrev and Selivanova showed that every symmetric toric Fano manifold admits a Kähler-Einstein metric, and they presented four families of such toric manifolds including many of the previously known examples. These are the del Pezzo toric manifolds V_k of Voskresenskij and Klyachko [VK85], and the symmetric toric Fano manifolds $S_{m,k}$, $X_{m,k}$ and W_m respectively introduced by Sakane [Sak86], Nakagawa [Nak94] and themselves, see [BS99, Section 4] or Section 4.2 for details.

Here is our main result, which is a particular case of the more technical Theorem 4.9.

Theorem 1.2. *Let X be a symmetric toric Fano manifold of type V_k, $S_{m,k}$, $X_{m,k}$ or W_m, and g a Kähler-Einstein metric on it. Then g is not projectively induced.*

By a result of Voskresenskij and Klyachko, every centrally symmetric toric Fano manifold is a product of projective lines and del Pezzo toric manifolds [VK85] . Applying this together with Theorem 4.9 we obtain the next consequence.

Corollary 1.3. *Let X be a centrally symmetric toric Fano manifold. Then X admits a projectively induced Kähler-Einstein metric if and only if it is a product of projective lines.*

In [NP11] Nill and Paffenholz proved that there exist Kähler-Einstein toric Fano manifolds that are not symmetric by exhibiting two examples in dimensions 7 and 8. As another application of our techniques we show that these metrics are neither projectively induced (Theorem 4.10), thus providing further evidence for Conjecture 1.1.

We next sketch our strategy. We work with the equivalent point of view of forms instead of that of metrics, and so we consider a pair (X, ω) consisting of an n-dimensional toric Fano manifold and a Kähler-Einstein form induced by an immersion $\varphi\colon X \to \mathbb{P}^s$ from the Fubini-Study form of this projective space.

Let $M \simeq \mathbb{Z}^n$ be the group of characters of the complex torus acting on X. Applying Calabi's rigidity theorem and the basic properties of critical metrics we reduce to the situation where φ is equivariant with respect to the usual toric structure on $\mathbb{P}^s$ (Lemmas 2.7 and 2.8). In this situation the immersion is represented by a family of monomials that we can use to construct a Laurent polynomial $p_\varphi \in \mathbb{R}_{>0}[M]$.

We then define an algebraic differential operator $\mu\colon \mathbb{C}[M] \to \mathbb{C}[M]$ allowing to translate the Einstein condition for ω into the equation (Proposition 3.16)

$$\mu(p_\varphi) = p_\varphi^n,$$

similar to that considered in [ALZ12, MS26]. To study it we consider the more flexible requirement that $\mu(p_\varphi)$ divides p_φ^κ for an integer $\kappa > 0$, which we call the *generalized Einstein condition (GEC)*. This condition is useful because it is inherited by initial parts of polynomials. Hence the existence of a single obstructing invariant subvariety of X rules out a projectively induced Kähler-Einstein metric on X.

Our key technical results are an explicit description of the Newton polytope of $\mu(p_\varphi)$ and a factorization formula for the initial parts of this Monge-Ampère polynomial (Theorems 3.10 and 3.14). The latter can be interpreted as an algebraic version the adjunction formula on X for an invariant hypersurface, and implies that GEC is hereditary on the invariant submanifolds of X (Proposition 3.19). We then study some specific toric surfaces to show that GEC cannot be satisfied on them (Corollaries 4.4 and 4.7). Since all the considered higher dimensional toric Fano manifolds contain at least one of these toric surfaces as an invariant submanifold, we deduce that GEC can be neither satisfied on them. In particular, none of them admits a projectively induced Kähler-Einstein metric (Theorems 4.9 and 4.10).

In view of this approach, it is natural to ask if for every toric Fano manifold X that admits a Kähler-Einstein metric and is not a product of projective spaces we have that X contains one of these specific toric surfaces as an invariant submanifold. Whereas we do not have a conceptual argument to support this claim, we could verify it for every example we considered, thus confirming Conjecture 1.1 for each of them.

The paper is organized as follows. Section 2 contains the preliminary notions and facts from differential and toric geometries. In Section 3 we give a combinatorial formula for the polynomial Monge-Ampére operator and study its properties. Section 4 contains the proof of our main results.

2. Preliminaries

In this section we recall the basic constructions and properties of complex toric manifolds and start to study the toric Kähler forms that are induced by projective immersions. We show that each of these Kähler forms can be encoded by a Laurent polynomial, which allows to translate the corresponding Einstein condition into an algebraic equation.

2.1. **Toric manifolds.**
For an integer $n \geq 1$ we denote by $\mathbb{T}$ an n-dimensional (complex) torus, that is a complex group isomorphic to $(\mathbb{C}^\times)^n$, and by $\mathbb{S} \simeq (S^1)^n$ its maximal compact subtorus.

Definition 2.1 ([Don08, Section 2])**.** A (complex) manifold X is *toric* if it contains $\mathbb{T}$ as a dense open subset and is equipped with an action of $\mathbb{T}$ extending the action of this torus on itself by translations. A Kähler form ω on X is *toric* if it is invariant under the action of $\mathbb{S}$, in which case the pair (X, ω) is called a *toric* Kähler manifold.

Recall that a Kähler manifold (X, ω) is *Einstein* if its Ricci form ρ verifies

$$\rho = \lambda\, \omega \quad \text{ with } \lambda \in \mathbb{R}.$$

Example 2.2. The n-dimensional (complex) projective space $\mathbb{P}^n$ is a toric complex manifold for the torus $(\mathbb{C}^\times)^n$ with the action defined for $t = (t_1, \dots, t_n) \in (\mathbb{C}^\times)^n$ and $z = (z_0 : z_1 : \dots : z_n) \in \mathbb{P}^n$ as

$$t \cdot z = (z_0 : t_1 z_1 : \dots : t_n z_n).$$

The *Fubini-Study form* of $\mathbb{P}^n$ is the Kähler form ω_{FS} defined at each point as the pullback of the 2-form on $\mathbb{C}^{n+1} \setminus \{0\}$

$$i\, \partial\overline{\partial} \log\Big(\sum_{j=0}^{n} |z_j|^2\Big)$$

with respect to any local section of the projection $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$. This Kähler form is invariant under the action of the projective unitary group $\mathrm{PU}(n+1)$ and is Einstein with constant $\lambda = n+1$ [Mor07, Chapter 13]. In particular $(\mathbb{P}^n, \omega_{\mathrm{FS}})$ is a toric Kähler-Einstein manifold.

In the sequel we focus on the case when X is a compact smooth toric variety, that is a compact toric manifold which is also an algebraic variety. We present the necessary notions and facts, referring to [Ful93, CLS11] for the proofs and more details.

Set

$$M = \mathrm{Hom}(\mathbb{T}, \mathbb{C}^\times) \quad \text{and} \quad N = \mathrm{Hom}(\mathbb{C}^\times, \mathbb{T})$$

for the lattices of characters and co-characters of the torus $\mathbb{T}$. Both are isomorphic to $\mathbb{Z}^n$ and dual of each other, that is $N = M^\vee$ and $M = N^\vee$. Set then $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ and $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$. These vector spaces are also dual of each other, and for $u \in N_\mathbb{R}$ and $x \in M_\mathbb{R}$ we denote their pairing by $\langle u, x \rangle$. Let $\mathbb{C}[M]$ be the group algebra of M over $\mathbb{C}$, and for each $m \in M$ we set $\chi^m \in \mathbb{C}[M]$ for the corresponding monomial.

To a compact smooth toric variety X corresponds a fan $\Sigma = \Sigma_X$ on $N_\mathbb{R}$ that is *complete* and *unimodular*, that is it covers the whole of this vector space and each of its n-dimensional cones is generated by a basis of the lattice N. For each integer $0 \leq r \leq n$ we denote by Σ^r the collection of cones of Σ of dimension r.

There is an inclusion-reversing bijection $\sigma \mapsto X_\sigma$ between the cones of Σ and the affine toric varieties that glue up to build X. The origin $0 \in \Sigma$ corresponds to the *principal open subset* $X_0 \subset X$ and is canonically identified with the torus. The fact that X is smooth implies that for each $\sigma \in \Sigma^r$ we have

$$X_\sigma \simeq \mathbb{C}^r \times (\mathbb{C}^\times)^{n-r}.$$

There is also a dimension-reversing bijection $\sigma \mapsto V(\sigma)$ between the cones of Σ and the invariant subvarieties of X. In particular to each 1-dimensional cone (or *ray*) it corresponds an invariant hypersurface.

Every line bundle L on X can be realized as the line bundle associated to a toric divisor, that is

$$L = \mathcal{O}_X\Big(\sum_\tau a_\tau V(\tau)\Big),$$

the sum being over the rays τ of the fan and where $a_\tau \in \mathbb{Z}$ for each τ. This representation is unique up to a principal divisor of the form $\operatorname{div}(\chi^m)$ with $m \in M$. Then we associate to L the lattice polytope in the vector space $M_\mathbb{R}$ defined as

$$\Delta_L = \{x \in M_\mathbb{R} \mid \langle u_\tau, x\rangle \geq -a_\tau \text{ for all } \tau \in \Sigma^1\},$$

where u_τ denotes the smallest nonzero lattice vector in the ray τ. It is well-defined up to a translation by an element of M.

When L is ample we have that the polytope Δ_L is n-dimensional and its normal fan coincides with Σ. In particular there are dimension-reversing bijections

$$F \longmapsto \sigma_F \quad \text{and} \quad \sigma \longmapsto F_\sigma$$

between the faces of Δ_L and the cones of Σ.

The anticanonical line bundle of X can be represented as $-K_X = \mathcal{O}_X(\sum_\tau V(\tau))$ and so its associated lattice polytope can be fixed to

$$\Delta_{-K_X} = \{x \in M_\mathbb{R} \mid \langle u_\tau, x\rangle \geq -1 \text{ for all } \tau \in \Sigma^1\}. \tag{2.1}$$

If X is Fano then $-K_X$ is ample and so Δ_{-K_X} is n-dimensional. Hence this is a reflexive polytope, and in particular the origin is its unique interior lattice point.

2.2. **Projectively induced forms.** Here we establish some general facts about the Kähler-Einstein forms on toric manifolds that are induced by projective immersions.

Definition 2.3. A Kähler form ω on a compact manifold X is *projectively induced* if there exists an immersion $\varphi\colon X \to \mathbb{P}^s$ such that

$$\omega = \varphi^*\omega_{\mathrm{FS}}.$$

Up to reducing to a linear subspace of $\mathbb{P}^s$ we can assume that the immersion is *full*, that is its image is not contained in any proper linear subspace.

Remark 2.4. If (X, ω) is a projectively induced compact Kähler manifold then the line bundle $L = \varphi^*\mathcal{O}(1)$ is positive because ω is a curvature form on it. Hence by Kodaira's embedding theorem L is ample, and in particular X is an algebraic variety.

Furthermore if (X, ω) is Einstein then its constant is positive by a theorem of Hulin [Hul00]. Since the Ricci form ρ is a curvature form on $-K_X$, this line bundle is positive and so ample. Hence in this case X is a Fano variety.

Let X be a compact toric manifold with torus $\mathbb{T}$. A map $\varphi\colon X \to \mathbb{P}^s$ is *toric* if there exists a homomorphism of tori $\zeta\colon \mathbb{T} \to (\mathbb{C}^\times)^s$ such that

$$\varphi(t \cdot_X x) = \zeta(t) \cdot_{\mathbb{P}^s} \varphi(x) \quad \text{for all } t \in \mathbb{T} \text{ and } x \in X.$$

The restriction of this map to the principal open subset is described in terms of monomials: there are $\alpha_j \in \mathbb{C}$, $j = 0, \dots, s$, not all zero and $m_j \in M$, $j = 0, \dots, s$, such that

$$\varphi(x) = (\alpha_0 \chi^{m_0}(x) : \cdots : \alpha_s \chi^{m_s}(x)) \quad \text{for all } x \in X_0.$$

These scalars and vectors form a set of *coefficients* and *exponents* of φ.

The pullback $L = \varphi^* \mathcal{O}(1)$ is a line bundle on X whose polytope coincides with the convex envelope of the set of exponents of the map, that is

$$\Delta_L = \mathrm{conv}(m_0, \dots, m_s) \subset M_{\mathbb{R}}. \tag{2.2}$$

We next characterize the toric maps that are full immersions.

Definition 2.5. A finite subset $S \subset M$ is *unimodular* if for every vertex of its convex hull $v \in \mathrm{conv}(S)$ and every edge $E \preceq \mathrm{conv}(S)$ containing v there exists $w_E \in E \cap S$ such that the vectors $w_E - v$ form a basis of the lattice M.

Lemma 2.6. *Let $\varphi \colon X \to \mathbb{P}^s$ be a toric map with coefficients $\alpha_j \in \mathbb{C}$, $j = 0, \dots, s$, and exponents $m_j \in M$, $j = 0, \dots, s$. The following conditions are equivalent:*

(1) φ is a full immersion,

(2) $\alpha_j \neq 0$ for all j, $m_j \neq m_k$ for all $j \neq k$, and $\{m_0, \dots, m_s\}$ is unimodular.

Proof. Assume that (1) holds. Since φ is full the first two conditions in (2) are necessary because otherwise the image of this map would be contained in a hyperplane. For the third condition, let $v = m_{j_0}$ be a vertex of the polytope $\mathrm{conv}(m_0, \dots, m_s)$. For convenience we rescale and reorder the coefficients and exponents so that $\alpha_{j_0} = 1$ and then $j_0 = 0$.

Since φ is an immersion we have that X is a compact smooth toric variety and $L = \varphi^* \mathcal{O}(1)$ is an ample line bundle on it. By (2.2) we have that m_0 is a vertex of the polytope Δ_L, and so it corresponds to an n-dimensional cone σ of the fan associated to X. Hence φ restricts to a monomial map between the charts $X_\sigma \simeq \mathbb{C}^n$ and $(z_0 \neq 0) \simeq \mathbb{C}^s$ that writes down in these coordinates as

$$\mathbb{C}^n \longrightarrow \mathbb{C}^s, \quad z \longmapsto (\alpha_1 z^{a_1}, \dots, \alpha_s z^{a_s}), \tag{2.3}$$

where $a_j \in \mathbb{Z}^n$ corresponds to the difference $m_j - m_0$ through the isomorphism $M \simeq \mathbb{Z}^n$ given by the identification $X_\sigma \simeq \mathbb{C}^n$, and where $z^{a_j} = z_1^{a_{j,1}} \cdots z_n^{a_{j,n}}$ for $z = (z_1, \dots, z_n) \in \mathbb{C}^n$.

The Jacobian matrix of this map writes down as

$$J_\varphi(z) = (\alpha_j a_{j,k} z^{a_j - e_k})_{j,k} \in \mathbb{C}^{s \times n} \quad \text{for } z \in \mathbb{C}^s,$$

where e_k denotes the k-th vector in the standard basis of $\mathbb{Z}^n$. Evaluating at the origin we get $J_\varphi(0)_{j,k} \neq 0$ if and only if $a_j = e_k$. Since φ is an immersion we have

$$\mathrm{rank}(J_\varphi(0)) = n$$

and so there are $j_1, \dots, j_n \in \{0, \dots, s\}$ such that $a_{j_k} = e_k$ for each k. From here we get that $\mathrm{conv}(m_0, \dots, m_s)$ has exactly n edges E_k, $i = 1, \dots, n$, containing v, and for each k there is $m_{j_k} \in E_k$ such that $m_{j_k} - m_0$ corresponds to e_k under the above isomorphism. Hence

$$m_{j_1} - m_0, \dots, m_{j_n} - m_0$$

is a basis of M, and since this holds for every vertex of the polytope we conclude that the set $\{m_0, \dots, m_s\}$ is unimodular.

Conversely if (2) holds then the toric map φ is full: otherwise there would be a linear relation between the monomials $\alpha_j\chi^{m_j}$, $j = 0, \dots, s$, which is not possible because they are different and nonzero. Finally the assumption that the set of exponents is unimodular implies that the local map in (2.3) is a graph, and in particular an immersion. □

We now show that every projectively induced toric Kähler form on X is induced by a toric full immersion. In particular this is also the case for a projectively induced Kähler-Einstein form on this compact toric manifold.

Lemma 2.7. *Let ω be a toric Kähler form on X induced by a full immersion $\varphi\colon X \to \mathbb{P}^s$. Then there is $A \in \mathrm{PU}(s+1)$ such that the full immersion $A\,\varphi\colon X \to \mathbb{P}^s$ is toric.*

Proof. Since ω is invariant under the action of the compact subtorus $\mathbb{S}$, by Calabi's rigidity theorem [Cal53, Theorem 9] (see also [LZ18, Theorem 2.2]) there is a group homomorphism $\theta\colon \mathbb{S} \to \mathrm{PU}(s+1)$ such that

$$\varphi(t \cdot_X x) = \theta(t)\,\varphi(x) \quad \text{for all } t \in \mathbb{S} \text{ and } x \in X. \tag{2.4}$$

Since the representations of $\mathbb{S}$ are diagonalizable there exists $A \in \mathrm{PU}(s+1)$ and a homomorphism $\zeta\colon \mathbb{S} \to (S^1)^s \hookrightarrow \mathrm{PU}(s+1)$ such that

$$\theta(t) = A^{-1}\zeta(t)A \quad \text{for all } t \in \mathbb{S}. \tag{2.5}$$

Recall that X_0 and $\mathbb{T}$ are canonically isomorphic and denote by $x_0 \in X_0$ the point of the principal open subset corresponding to the unit of the torus. Setting $\alpha = A\,\varphi(x_0) \in \mathbb{P}^s$ we deduce from (2.4) and (2.5) that

$$A\,\varphi(t \cdot_X x_0) = \zeta(t) \cdot_{\mathbb{P}^s} \alpha \quad \text{for all } t \in \mathbb{S}.$$

The homomorphism ζ is defined by a sequence of characters, and so it extends to a homomorphism $\zeta\colon \mathbb{T} \to (\mathbb{C}^\times)^s$. Hence both $t \mapsto A\,\varphi(t \cdot_X x_0)$ and $t \mapsto \zeta(t) \cdot_{\mathbb{P}^s} \alpha$ are holomorphic maps that coincide on $\mathbb{S}$, and so they are equal.

Now let $t, t' \in \mathbb{T}$ and $x = t' \cdot_X x_0 \in X_0$. Hence $t \cdot_X x = (t \cdot_{\mathbb{T}} t') \cdot_X x_0$ and so

$$A\,\varphi(t \cdot_X x) = \zeta(t \cdot_{\mathbb{T}} t') \cdot_{\mathbb{P}^s} \alpha = (\zeta(t) \cdot_{(\mathbb{C}^\times)^s} \zeta(t')) \cdot_{\mathbb{P}^s} \alpha = \zeta(t) \cdot_{\mathbb{P}^s} (\zeta(t') \cdot_{\mathbb{P}^s} \alpha) = \zeta(t) \cdot_{\mathbb{P}^s} A\,\varphi(x),$$

showing that the map $A\,\varphi\colon X \to \mathbb{P}^s$ is toric. □

Lemma 2.8. *Let ω be a projectively induced Kähler-Einstein form on X. Then ω is induced by a toric full immersion into a projective space.*

Proof. Since ω is Kähler-Einstein it is critical in the sense of [Cal85]. By Theorem 3 in *loc. cit.* we have that ω is toric, and so the statement follows from Lemma 2.7. □

2.3. **Toric potentials.** Let $\varphi\colon X \to \mathbb{P}^s$ be a toric full immersion of a compact toric manifold X with torus $\mathbb{T}$, and let ω and ρ be the associated Kähler and Ricci forms.

To choose coordinates we identify the principal open subset X_0 with $(\mathbb{C}^\times)^n$. The exponential

$$\exp\colon \mathbb{C}^n/2\pi i\mathbb{Z}^n \to (\mathbb{C}^\times)^n$$

is an isomorphism of complex spaces, and we will study the Einstein condition for these forms on the chart of X given by this map.

By Lemma 2.6 we have

$$\varphi(x) = (\alpha_0\chi^{m_0}(x) : \cdots : \alpha_s\chi^{m_s}(x)) \quad \text{for } x \in X_0 = (\mathbb{C}^\times)^n$$

with $\alpha_j \in \mathbb{C}^\times$ and $m_j \in M = \mathbb{Z}^n$ such that $m_j \neq m_k$ for all $j \neq k$ and the set $\{m_0, \dots, m_s\} \subset \mathbb{Z}^n$ is unimodular. Then we associate to the toric full immersion φ the Laurent polynomial

$$p_\varphi = \sum_{j=0}^{s} |\alpha_j|^2 \chi^{m_j} \in \mathbb{R}_{>0}[\mathbb{Z}^n]. \tag{2.6}$$

By (2.2) its Newton polytope coincides with the polytope Δ_L associated to the line bundle $L = \varphi^*\mathcal{O}(1)$.

Now let $\mathbb{C}[\mathbb{Z}^n] = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be the algebra of Laurent polynomials over the complex numbers and $\mathbb{C}(\mathbb{Z}^n) = \mathbb{C}(x_1, \dots, x_n)$ its field of rational functions. Let $\mathbb{L}$ be any differential extension of $\mathbb{C}[\mathbb{Z}^n]$ containing the logarithms of all nonzero Laurent polynomials and consider the differential operators on $\mathbb{L}$ defined as

$$D_i = x_i \frac{\partial}{\partial x_i}, \quad i = 1, \dots, n. \tag{2.7}$$

Definition 2.9 (δ-operator)**.** For $p \in \mathbb{C}[\mathbb{Z}^n] \setminus \{0\}$ we set

$$\delta_{\mathbb{Z}^n}(p) = \det(D^2 \log(p)) \in \mathbb{C}(\mathbb{Z}^n),$$

where $D^2 \log(p) = (D_i D_j \log(p))_{i,j} \in \mathbb{C}(\mathbb{Z}^n)^{n \times n}$ denotes the D-Hessian matrix of the logarithm of this Laurent polynomial. When the lattice is clear from the context we denote this rational function simply by $\delta(p)$.

Lemma 2.10. *Let $f, h \colon \mathbb{C}^n/2\pi i\mathbb{Z}^n \to \mathbb{R}$ be the functions respectively defined as*

$$f(u+iv) = \log(p_\varphi(e^{2u})) \quad \text{and} \quad h(u+iv) = \log(\delta(p_\varphi)(e^{2u})).$$

Then $\omega = i\partial\overline{\partial} f$ and $\rho = -i\partial\overline{\partial} h$.

Proof. We have

$$f(u+iv) = \log\Big(\sum_{j=0}^{s} |\alpha_j e^{\langle m_j, u+iv\rangle}|^2\Big) = (\Phi \circ \exp)^* \log \|\cdot\|_2 \quad \text{for all } u+iv \in \mathbb{C}^n/2\pi i, \mathbb{Z}^n$$

where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{C}^s$ and $\Phi\colon (\mathbb{C}^\times)^n \to \mathbb{C}^s \setminus \{0\}$ the monomial map $x \mapsto (\alpha_0 \chi^{m_0}(x), \dots, \alpha_s \chi^{m_s}(x))$. By the definition of the Fubini-Study form (Example 2.2) this function is a potential for ω on $\mathbb{C}^n/2\pi i\mathbb{Z}^n$.

In this chart we have

$$\omega = i \sum_{k,l=1}^{n} \frac{\partial^2 f}{\partial z_k \partial \overline{z}_l} \, dz_k \wedge d\overline{z}_l$$

with $z_k = u_k + iv_k$ for each k. Setting $H = (\frac{\partial^2 f}{\partial z_k \partial \overline{z}_l})_{k,l}$ we have that $H(u+iv)$ is a positive Hermitian matrix for all $u+iv$ and so its determinant gives a function

$$\det(H)\colon \mathbb{C}^n/2\pi i\mathbb{Z}^n \longrightarrow \mathbb{R}_{>0}.$$

By [Mor07, Formula (12.6)] the Ricci form can be defined as

$$\rho = -i\partial\overline{\partial} \log(\det(H)).$$

Our potential depends only on the real part and so $\frac{\partial^2 f}{\partial z_k \partial \overline{z}_l} = \frac{1}{4} \frac{\partial^2 f}{\partial u_k \partial u_l}$ for each k, l. We have

$$\frac{\partial f}{\partial u_k}(u+iv) = 2\, e^{2u_k} \frac{\partial \log(p_\varphi)}{\partial x_k}(e^{2u}) = 2 D_k \log(p_\varphi)(e^{2u})$$

and then $\frac{\partial^2 f}{\partial u_k \partial u_l}(u+iv) = 4D_kD_l\log(p_\varphi)(e^{2u})$, which implies

$$\frac{\partial^2 f}{\partial z_k \partial \overline{z}_l}(u+iv) = D_kD_l\log(p_\varphi)(e^{2u}).$$

Thus $H(u+iv) = \det(D^2\log(p_\varphi))(e^{2u})$. □

We give an algebraic characterization of the Einstein condition.

Proposition 2.11. *The Kähler form ω verifies the Einstein condition with constant $\lambda > 0$ if and only if there exist $c \in \mathbb{R}_{>0}$ and $m \in \mathbb{Z}^n$ such that*

$$\delta(p_\varphi) = c\chi^m p_\varphi^{-\lambda}. \tag{2.8}$$

Proof. In the notation of in Lemma 2.10 the Einstein condition for ω reduces to $h \equiv -\lambda f \pmod{\operatorname{Ker}(\partial\overline{\partial})}$. Hence

$$h(u+iv) = -\lambda f(u+iv) + F(u+iv) + \overline{F(u+iv)} \quad \text{for all } u+iv \in \mathbb{C}^n/2\pi i\mathbb{Z}^n$$

with F holomorphic, and so entire.

Since f and h do not depend on the variable v this is also the case for $F+\overline{F}$. Considering the representation of F as a power series in $u+iv$ we see that this happens exactly when F is affine, that is when $F(u+iv) = \alpha + \langle\beta, u+iv\rangle$ with $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{R}^n$. We get

$$\delta(p_\varphi)(e^{2u}) = e^{\alpha+\overline{\alpha}+\langle\beta,2u\rangle} p_\varphi(e^{2u})^{-\lambda} \quad \text{for all } u \in \mathbb{R}^n,$$

which readily gives (2.8) with $c = e^{\alpha+\overline{\alpha}} \in \mathbb{R}_{>0}$ and $m = \beta$. This exponent lies in M because this equation implies that $\chi^m \in \mathbb{R}(x_1,\dots,x_n)$. □

Example 2.12. The Laurent polynomial associated to the identity map on $\mathbb{P}^n$ is

$$p_{\mathrm{FS}} = 1 + \sum_{i=1}^{n} x_i.$$

We have $\delta(p_{\mathrm{FS}}) = \left(\prod_{i=1}^n x_i\right) p_{\mathrm{FS}}^{-(n+1)}$ as it can be checked by a direct computation or applying Lemma 3.1 below. Using Proposition 2.11 this confirms that the Fubini-Study form ω_{FS} is Kähler-Einstein with constant $\lambda = n+1$, in agreement with Example 2.2.

We deduce an algebraic characterization for the existence of a Kähler-Einstein form on X that is induced by a projective immersion.

Corollary 2.13. *The following conditions are equivalent:*

(1) X admits a projectively induced Kähler-Einstein form with constant $\lambda > 0$,

(2) there exists $p \in \mathbb{R}_{>0}[\mathbb{Z}^n]$ such that $\operatorname{supp}(p)$ is unimodular, $\operatorname{NP}(p) = \lambda^{-1}\Delta_{-K_X}$ and $\delta(p) = c\chi^m p^{-\lambda}$ for some $c \in \mathbb{R}_{>0}$ and $m \in \mathbb{Z}^n$.

Proof. If (1) holds then by Lemmas 2.6 and 2.8 the considered Kähler-Einstein form is induced by a toric full immersion, which combined with Proposition 2.11 gives (2).

Conversely assume that (2) holds. Write

$$p = \sum_{j=0}^{s} |\alpha_j|^2 \chi^{m_j} \in \mathbb{R}_{>0}[\mathbb{Z}^n]$$

with $\alpha_j \in \mathbb{C}^\times$ and $m_j \in \mathbb{Z}^n$, and consider the monomial map $X_0 \to \mathbb{P}^s$ defined as

$$x \longmapsto (\alpha_0\chi^{m_0} : \cdots : \alpha_s\chi^{m_s}) \tag{2.9}$$

The condition $\mathrm{NP}(p) = \lambda^{-1}\Delta_{-K_X}$ implies that this lattice polytope is compatible with the fan of X because so is the case for this anticanonical polytope. Hence the monomials in (2.9) extend to global sections of a line bundle on X, and so this map extends to a toric map $\varphi\colon X \to \mathbb{P}^s$. Since $\mathrm{supp}(p)$ is unimodular, by Lemma 2.6 it is a toric full immersion with $p_\varphi = p$. We conclude again with Proposition 2.11. □

3. Algebraic potentials

Motivated by our previous results we now place ourselves in a purely algebraic setting and study systematically the corresponding Monge-Ampère operator. Throughout this section we denote by $\mathbb{T}$ a torus of dimension $n \geq 1$, by M and N its lattices of characters and co-characters, and by $M_\mathbb{R}$ and $N_\mathbb{R}$ their vector spaces.

3.1. **The algebraic Monge-Ampère operator.** Let

$$p = \sum_{j=0}^{s} c_j \chi^{m_j} \in \mathbb{C}[\mathbb{Z}^n] \setminus \{0\}$$

with $c_j \neq 0$ for all j and $m_j \neq m_k$ for all $j \neq k$. With notation as in (2.7) we have

$$D_l p = \sum_{j=0}^{s} m_{j,l} c_j \chi^{m_j} \quad \text{and} \quad D_k D_l p = \sum_{j=0}^{s} m_{j,k} m_{j,l} c_j \chi^{m_j} \quad \text{for each } k, l.$$

Moreover $D_l \log(p) = D_l p/p$ and $D_k D_l \log(p) = (((D_k D_l p)\, p - (D_k p)(D_l p))/p^2$ and so

$$\delta(p) = \frac{\det((D_k D_l p)\, p - (D_k p)(D_l p))_{k,l}}{p^{2n}} \in \mathbb{C}(\mathbb{Z}^n).$$

Hence p^{2n} is a denominator for the rational function $\delta(p)$. The next lemma gives a smaller one and a formula for the corresponding numerator.

Lemma 3.1. *We have*

$$p^{n+1}\delta(p) = \sum_J \big(n! \operatorname{vol}_{\mathbb{Z}^n}(\operatorname{conv}(\{m_j\}_{j\in J}))\big)^2 \prod_{j\in J} c_j \chi^{m_j} \in \mathbb{C}[\mathbb{Z}^n],$$

where the sum is over the subsets $J \subset \{0, \dots, s\}$ of cardinality $n+1$ and $\operatorname{vol}_{\mathbb{Z}^n}$ denotes the Lebesgue measure of $\mathbb{R}^n$.

Proof. For each k, l we have that $(D_k D_l p)\, p - (D_k p)(D_l p)$ equals

$$\Big(\sum_{i=0}^{s} m_{i,k} m_{i,l} c_i \chi^{m_i}\Big)\Big(\sum_{j=0}^{s} c_j \chi^{m_j}\Big) - \Big(\sum_{i=0}^{s} m_{i,k} c_i \chi^{m_i}\Big)\Big(\sum_{j=0}^{s} m_{j,l} c_j \chi^{m_j}\Big).$$

Set

$$E = (m_0 \cdots m_s), \quad K = \operatorname{diag}(c_0\chi^{m_0}, \dots, c_s\chi^{m_s}), \quad w = \sum_{j=0}^{s} m_j c_j \chi^{m_j}.$$

Then E is an $n \times (s+1)$ matrix, K a diagonal $(s+1) \times (s+1)$ matrix and w an n vector that can be used to write the D-Hessian matrix of $\log(p)$ as

$$p^2 D^2 \log(p) = p\, EKE^T - w\, w^T.$$

Write $EKE^T = (a_1 \cdots a_n)$ and $w = (w_1 \cdots w_n)^T$ where each a_j is an n vector and each w_j a scalar. Then

$$p\, EKE^T - w\, w^T = (pa_1 - w_1 w, \dots, pa_n - w_n w).$$

By the multilinearity of the determinant function we get

$$p^{2n}\det(D^2\log(p)) = \det(pEKE^T) - \sum_{k=1}^n w_k \det(p\,a_1,\dots,p\,a_{k-1},w,p\,a_{k+1},\dots,p\,a_n),$$

which readily implies

$$p^{n+1}\delta(p) = p\det(EKE^T) - \sum_{k=1}^n w_k\det(a_1,\dots,a_{k-1},w,a_{k+1},\dots,a_n). \tag{3.1}$$

Now consider the $(n+1)\times(s+1)$-matrix $\widetilde{E} = \begin{pmatrix} E \\ \mathbf{1}^T\end{pmatrix} = \begin{pmatrix} m_0 & \cdots & m_s \\ 1 & \cdots & 1\end{pmatrix}$. We have

$$\widetilde{E}K\widetilde{E}^T = \begin{pmatrix} E \\ \mathbf{1}^T\end{pmatrix} K\,(E^T\,\mathbf{1}) = \begin{pmatrix} EKE^T & EK\,\mathbf{1} \\ \mathbf{1}^TKE^T & \mathbf{1}^TK\,\mathbf{1}\end{pmatrix} = \begin{pmatrix} EKE^T & w \\ w^T & p\end{pmatrix}.$$

Developing the determinant of this matrix by the last row we deduce that it coincides with the right hand side of (3.1). By the Cauchy-Binet formula for the determinant of a product of rectangular matrices we have

$$p^{n+1}\delta(p) = \det(\widetilde{E}K\widetilde{E}^T) = \sum_J \det((\widetilde{E}K)_J)\det(\widetilde{E}_J^T) = \sum_J \det(\widetilde{E}_J)^2\det(K_J),$$

the sum being over the subsets $J\subset\{0,\dots,s\}$ of cardinality $n+1$. Finally for each J

$$\det(\widetilde{E}_J) = \pm n!\operatorname{vol}_{\mathbb{Z}^n}(\operatorname{conv}(\{m_j\}_{j\in J})) \quad\text{and}\quad \det(K_J) = \prod_{j\in J}^n c_j\chi^{m_j},$$

which gives the formula. □

Remark 3.2. Write $p_J = \sum_{j\in J} c_j\chi^{m_j}$ for each $J\subset\{0,\dots,s\}$ with $\#J = n+1$. Then Lemma 3.1 gives the decomposition

$$p^{n+1}\delta(p) = \sum_J p_J^{n+1}\delta(p_J).$$

This lemma also shows that the δ-operator is invariant with respect to unimodular changes of coordinates of $\mathbb{Z}^n$.

In the 1-dimensional case we can compute the δ-operator of a Laurent polynomial from a complete factorization of it.

Lemma 3.3. *Let $p = c\,\chi^m\prod_{k=1}^t(\chi^1+\xi_k)^{e_k}\in\mathbb{C}[\mathbb{Z}]\setminus\{0\}$ with $c\in\mathbb{C}^\times$, $e_k\in\mathbb{Z}_{>0}$ and $\xi_k\in\mathbb{C}^\times$ such that $\xi_k\neq\xi_l$ for all $k\neq l$. Then*

$$\delta(p) = \sum_{k=1}^t \frac{e_k\xi_k\chi^1}{(\chi^1+\xi_k)^2}.$$

Proof. We have

$$\delta(p) = D^2\log(p) = D^2\log(c\chi^m) + \sum_{k=1}^t e_kD^2\log(\chi^1+\xi_k) = \delta(c\chi^m) + \sum_{k=1}^t e_k\,\delta(\chi^1+\xi_k).$$

Then we get the stated formula applying Lemma 3.1 to each of these terms. □

The δ-operator also satisfies the following properties.

Lemma 3.4. *Let $p\in\mathbb{C}[\mathbb{Z}^n]\setminus\{0\}$. Then*

(1) for all $c \in \mathbb{C}^\times$ *and* $m \in \mathbb{Z}^n$ *we have* $\delta(c\chi^m p) = \delta(p)$*,*

(2) for all $\lambda \in \mathbb{Q}_{>0}$ *such that* $p^\lambda \in \mathbb{C}[\mathbb{Z}^n]$ *we have* $\delta(p^\lambda) = \lambda^n \delta(p)$*,*

(3) for all $q \in \mathbb{C}[\mathbb{Z}^l] \setminus \{0\}$ *we have* $\delta_{\mathbb{Z}^{n+l}}(p\,q) = \delta_{\mathbb{Z}^n}(p)\,\delta_{\mathbb{Z}^l}(q)$*.*

Proof. We have

$$D^2\log(c\chi^m p) = D^2\log(c\chi^m) + D^2\log(p) = D^2\log(p) \quad \text{and} \quad D^2\log(p^\lambda) = \lambda\, D^2\log(p).$$

Taking the determinants of these matrices gives the formulae in (1) and (2). For (3) it holds

$$D^2\log(p\,q) = D^2\log(p) + D^2\log(q) = \begin{pmatrix} D^2\log(p) & 0 \\ 0 & D^2\log(q) \end{pmatrix}$$

for the D-Hessian matrices on $\mathbb{C}[\mathbb{Z}^{n+l}]$, $\mathbb{C}[\mathbb{Z}^n]$ and $\mathbb{C}[\mathbb{Z}^l]$, and the formula follows again taking determinants. □

3.2. **A coordinate-free setting.** It will be convenient to avoid a specific choice of coordinates. Let then

$$p = \sum_{j=0}^{s} c_j \chi^{m_j} \in \mathbb{C}[M] \setminus \{0\}$$

with $c_j \neq 0$ for all j and $m_j \neq m_k$ for all $j \neq k$. Set

$$\mathrm{supp}(p) = \{m_0, \dots, m_s\} \subset M \quad \text{and} \quad \mathrm{NP}(p) = \mathrm{conv}(\mathrm{supp}(p)) \subset M_\mathbb{R}$$

for the *support* and the *Newton polytope* of this Laurent polynomial, and consider also the associated vector space and lattice, respectively defined as

$$M_{p,\mathbb{R}} = \sum_{j,k} \mathbb{R}(m_j - m_k) \subset M_\mathbb{R} \quad \text{and} \quad M_p = M_{p,\mathbb{R}} \cap M \subset M.$$

We have that M_p is a subgroup of $M_{p,\mathbb{R}}$ with compact quotient, and we denote by vol_{M_p} the Haar measure on this vector space normalized so that

$$\mathrm{vol}_{M_p}(M_{p,\mathbb{R}}/M_p) = 1.$$

Note that $\mathrm{supp}(\chi^{-m_0}p) \subset M_p$ and $\mathrm{rank}(M_p) = \dim(M_{p,\mathbb{R}} = \dim(\mathrm{NP}(p))$.

We introduce a polynomial variant of the δ-operator that contains its nontrivial information and gives a meaningful extension of this operator to the rank-deficient case.

Definition 3.5 (μ-operator)**.** For $p \in \mathbb{C}[M] \setminus \{0\}$ we set

$$\mu(p) = p^{\mathrm{rank}(M_p)+1} \delta_{M_p}(\chi^{-m_0}p) \in \mathbb{C}[M_p] \subset \mathbb{C}[M],$$

where δ_{M_p} denotes the δ-operator acting on the algebra $\mathbb{C}[M_p]$.

By Remark 3.2 the δ-operator does not depend on the choice of a basis of the lattice, and by Lemma 3.4(1) it is invariant by translations of exponents. Hence the formula in Definition 3.5 produces a well-defined rational function, which by Lemma 3.1 is a Laurent polynomial.

We restate in terms of μ-operator the formula given by this latter result.

Lemma 3.6. *Set* $r = \mathrm{rank}(M_p)$*. Then*

$$\mu(p) = \sum_J (r!\,\mathrm{vol}_{M_p}(\mathrm{conv}(\{m_j\}_{j\in J})))^2 \Big(\prod_{j\in J} c_j \chi^{m_j}\Big),$$

the sum being over the subsets $J \subset \{0, \dots, s\}$ *of cardinality* $r+1$*.*

We also restate in our current setting both the formula for the 1-dimensional case and the basic properties of the δ-operator.

Lemma 3.7. *Let $p = c\,\chi^m \prod_{k=1}^t (\chi^1+\xi_k)^{e_k} \in \mathbb{C}[\mathbb{Z}]\setminus\mathbb{C}$ with $e_k \in \mathbb{Z}_{>0}$ and $\xi_k \in \mathbb{C}^\times$ such that $\xi_k \neq \xi_l$ for all $k \neq l$. Then*

$$\mu(p) = c^2\chi^{2m+1}\sum_{k=1}^t e_k\xi_k(\chi^1+\xi_k)^{2e_k-2}\prod_{l\neq k}(\chi^1+\xi_l)^{2e_l}.$$

Proof. This follows directly from Lemma 3.3 noting that here $\mu(p) = p^2\delta(p)$. □

Lemma 3.8. *Let $p \in \mathbb{C}[M]$ and set $r = \operatorname{rank}(M_p)$. Then*
(1) for all $c \in \mathbb{C}^\times$ and $m \in M$ we have $\mu(c\chi^m p) = c^{r+1}\chi^{(r+1)m}\mu(p)$,
(2) for all $\lambda \in \mathbb{Q}_{>0}$ such that $p^\lambda \in \mathbb{C}[M]$ we have $\mu(p^\lambda) = \lambda^r p^{(r+1)(\lambda-1)}\mu(p)$,
(3) for another lattice M' of rank r' and $q \in \mathbb{C}[M']$ we have $\mu(p\,q) = p^{r'}q^r\mu(p)\,\mu(q)$.

Proof. It is enough to prove these properties when $M_p = \mathbb{Z}^n$, in which case they are a direct consequence of Lemma 3.4 and the fact that $\mu(p) = p^{n+1}\delta(p)$. For (1) we have

$$\mu(c\chi^m p) = (c\chi^m p)^{n+1}\delta(c\chi^m p) = (c\chi^m p)^{n+1}\delta(p) = c^{n+1}\chi^{(n+1)m}\mu(p).$$

For (2) we have

$$\mu(p^\lambda) = (p^\lambda)^{n+1}\delta(p^\lambda) = \lambda^n p^{(n+1)\lambda}\delta(p) = \lambda^n p^{(n+1)(\lambda-1)}\mu(p).$$

Finally for (3) let $q \in \mathbb{C}[\mathbb{Z}^l]$. Then

$$\mu(p\,q) = (p\,q)^{n+l+1}\delta_{\mathbb{Z}^{n+l}}(p\,q) = (p\,q)^{n+l+1}\delta_{\mathbb{Z}^n}(p)\,\delta_{\mathbb{Z}^l}(q) = p^l q^n\mu(p)\,\mu(q),$$

completing these verifications. □

3.3. Newton polytopes.

Now fix

$$p = \sum_{j=0}^s c_j\chi^{m_j} \in \mathbb{C}[M]$$

such that $\operatorname{supp}(p)$ is a unimodular subset of M (Definition 2.5).

Definition 3.9. For each face $F \preceq \mathrm{NP}(p)$, the *angle* of $\mathrm{NP}(p)$ at F is the cone in $N_\mathbb{R}$ defined as

$$\sigma_F = \{u \in N_\mathbb{R} \mid \langle u, x-y\rangle \geq 0 \text{ for all } x \in \mathrm{NP}(p) \text{ and } y \in F\}.$$

The collection $\Sigma_{\mathrm{NP}(p)} = \{\sigma_F \mid F \preceq \mathrm{NP}(p)\}$ is a complete and unimodular fan on $N_\mathbb{R}$, called the *normal fan* of $\mathrm{NP}(p)$.

The next result gives the behavior of the Newton polytope with respect to the μ-operator. Set $\Sigma = \Sigma_{\mathrm{NP}(p)}$ for short. Recall that Σ^1 denotes the set of rays of this fan and $u_\tau \in N$ the smallest nonzero lattice vector in each ray τ.

Theorem 3.10. *Let $p \in \mathbb{C}[M]$ with $\operatorname{supp}(p)$ unimodular and write*

$$\mathrm{NP}(p) = \{x \in M_\mathbb{R} \mid \langle u_\tau, x\rangle \geq -a_\tau \textit{ for all } \tau \in \Sigma^1\}$$

with $a_\tau \in \mathbb{Z}_{>0}$. Then

$$\mathrm{NP}(\mu(p)) = \{x \in M_\mathbb{R} \mid \langle u_\tau, x\rangle \geq -(n+1)a_\tau + 1 \textit{ for all } \tau \in \Sigma^1\}.$$

This will be derived from a characterization of the face structure and vertex set of the Newton polytope of the Monge-Ampère polynomial of p. To state it let

$$\mathrm{NP}(p)^0 \quad \text{and} \quad \mathrm{NP}(\mu(p))^0$$

be the vertex sets of these Newton polytopes, and for each $v \in \mathrm{NP}(p)^0$ denote by B_v the (unique) basis of the lattice M such that for every edge $E \preceq \mathrm{NP}(p)$ containing v there is $b \in B_v$ with $v + b \in E$.

Furthermore, given a cone $\sigma \subset N_{\mathbb{R}}$ and a compact subset $C \subset M_{\mathbb{R}}$ put

$$C_\sigma = \{y \in C \mid \langle u, x - y\rangle \geq 0 \text{ for all } u \in \sigma \text{ and } x \in C\} \tag{3.2}$$

for the subset of elements of C of minimal weight in the direction of σ.

Proposition 3.11. *The following properties hold:*

(1) $\Sigma_{\mathrm{NP}(\mu(p))} = \Sigma_{\mathrm{NP}(p)}$*,*

(2) for each $v \in \mathrm{NP}(p)^0$ *we have* $\mathrm{NP}(\mu(p))_{\sigma_v} = \{(n+1)v + \sum_{b\in B_v} b\}$*,*

(3) $\mathrm{NP}(\mu(p))^0 = \{(n+1)v + \sum_{b\in B_v} b \mid v \in \mathrm{NP}(p)^0\}$.

Proof. Let $v \in \mathrm{NP}(p)^0$. For each subset $J \subset \{0, \dots, s\}$ of cardinality $n+1$ whose associated exponents are affinely independent set $m_J = \sum_{j\in J} m_j \in M$. For each $j \in J$ write $m_j = v + \sum_{b\in B_v} \gamma_{j,b}\, b$ with $\gamma_{j,b} \in \mathbb{Z}_{\geq 0}$ and then for each $b \in B_v$ set $\gamma_{J,b} = \sum_{j\in J} \gamma_{j,b}$. Thus

$$m_J = (n+1)v + \sum_{b\in B_v} \gamma_{J,b}\, b.$$

Since the exponents m_j, $j \in J$, are affinely independent we have $\gamma_{J,b} \geq 1$ for all b, and $\gamma_{J,b} = 1$ for all b exactly when $J = J_0$ for the only index subset such that $\{m_j\}_{j\in J_0} = \{v + b\}_{b\in B_v}$.

Let $u \in \sigma_v^\circ$ be an interior point of the n-dimensional cone of Σ corresponding to the vertex v. Then $\langle u, b\rangle > 0$ for all b and so

$$\langle u, m_J\rangle = (n+1)\langle u, v\rangle + \sum_{b\in B_v} \gamma_{J,b}\, \langle u, b\rangle \geq (n+1)\langle u, v\rangle + \sum_{b\in B_v} \langle u, b\rangle,$$

with the equality occurring exactly when $J = J_0$. Combining this with Lemma 3.6 we get that the minimal weight with respect to u of the exponents of $\mu(p)$ is realized at the lattice point

$$w_v \coloneqq m_{J_0} = (n+1)v + \sum_{b\in B_v} b.$$

Hence $\mathrm{NP}(\mu(p))_{\sigma_v} = \{w_v\}$, thus proving (2). This also implies that the angle σ_{w_v} of $\mathrm{NP}(\mu(p))$ at the vertex w_v contains the cone σ_v. Since the collection σ_v, $v \in \mathrm{NP}(p)^0$, covers $N_{\mathbb{R}}$ we deduce that $\sigma_{w_v} = \sigma_v$ for all v, giving both (1) and (3). □

Proof of Theorem 3.10. By Proposition 3.11(1) we have that $\mathrm{NP}(\mu(p))$ has the same face structure of $\mathrm{NP}(p)$. In particular their rays coincide and so by Proposition 3.11(2)

$$\mathrm{NP}(\mu(p)) = \{x \in M_{\mathbb{R}} \mid \langle u_\tau, x\rangle \geq b_\tau \text{ for all } \tau \in \Sigma^1\}$$

with $b_\tau = \langle u_\tau, (n+1)v + \sum_{b\in B_v} b\rangle$ for each ray $\tau \in \Sigma^1$ and any vertex $v \in \mathrm{NP}(p)^0$ lying in the facet $\mathrm{NP}(p)_\tau$. Then

$$b_\tau = (n+1)\langle u_\tau, v\rangle + \sum_{b\in B_v} \langle u_\tau, b\rangle = -(n+1)a_\tau + 1$$

because $\langle u_\tau, v\rangle = -a_\tau$ and $\langle u_\tau, b\rangle = 0$ for all but one $b \in B_v$, for which this weight equals 1. □

Corollary 3.12. *If* $\mathrm{NP}(p)$ *is a reflexive polytope then* $\mathrm{NP}(\mu(p)) = n\,\mathrm{NP}(p)$.

Proof. This follows directly from Theorem 3.10 and the fact that when $\mathrm{NP}(p)$ is reflexive we have $a_\tau = 1$ for all $\tau \in \Sigma^1$. □

3.4. **Initial parts.** Consider again a Laurent polynomial

$$p = \sum_{j=0}^{s} c_j \chi^{m_j} \in \mathbb{C}[M]$$

with unimodular support. The *restriction* of p to a subset $S \subset M_\mathbb{R}$, denoted by $p|_S$, is the sum of the terms of p whose exponents lie in S, that is

$$p|_S = \sum_{m_j \in S} c_j \chi^{m_j} \in \mathbb{C}[M]. \tag{3.3}$$

Given a cone $\sigma \subset N_\mathbb{R}$, the *initial part* of p in the direction of σ is the restriction of p to the face of its Newton polytope defined by σ as in (3.2), that is

$$\mathrm{init}_\sigma(p) = p|_{\mathrm{NP}(p)_\sigma} \in \mathbb{C}[M].$$

Definition 3.13. Let $\Delta \subset M_\mathbb{R}$ be an n-dimensional lattice polytope and $F \preceq \Delta$ a facet of it. The *adjacent polytope* of F is the subset of Δ defined as

$$F' = \{x \in \Delta \mid \langle u_F, x - y\rangle = 1 \text{ for all } y \in F\}$$

with $u_F \in N$ the primitive inner normal vector of F. It coincides with the intersection of Δ with the inner parallel lattice hyperplane that is closest to F.

We now turn to the study of the initial parts of the Monge-Ampère polynomial of p. Recall that by Proposition 3.11(1) the Newton polytopes of p and $\mu(p)$ have the same face structure, that is

$$\Sigma_{\mathrm{NP}(p)} = \Sigma_{\mathrm{NP}(\mu(p))}.$$

Hence the faces of these polytopes are in dimension-reversing bijection with the cones of this fan, that we denote again by Σ for short.

Theorem 3.14. *Let* $p \in \mathbb{C}[M]$ *with* $\mathrm{supp}(p)$ *unimodular and* $\tau \in \Sigma^1$. *Then*

$$\mathrm{init}_\tau(\mu(p)) = \mu(\mathrm{init}_\tau(p))\; p|_{F'_\tau}$$

with F'_τ *the adjacent polytope of the facet* $F_\tau \preceq \mathrm{NP}(p)$.

Proof. Since $\mathrm{supp}(p)$ is unimodular, up to a translation and an isomorphism $M \simeq \mathbb{Z}^n$ we can assume

$$\mathrm{NP}(p) \subset (x_n \geq 0) \quad \text{and} \quad F_\tau = \mathrm{NP}(p) \cap (x_n = 0).$$

Denote by $H_\tau \preceq \mathrm{NP}(\mu(p))$ the facet defined by the ray τ. By Theorem 3.10 we have

$$\mathrm{NP}(\mu(p)) \subset (x_n \geq 1) \quad \text{and} \quad H_\tau = \mathrm{NP}(\mu(p)) \cap (x_n = 1).$$

Then by Lemma 3.6

$$\mathrm{init}_\tau(\mu(p)) = \mu(p)|_{H_\tau} = \sum_J (n!\,\mathrm{vol}_{\mathbb{Z}^n}(\mathrm{conv}(\{m_j\}_{j\in J}))^2 \prod_{j\in J} c_j \chi^{m_j},$$

the sum being over the subsets $J \subset \{0, \dots, s\}$ of cardinality $n+1$ whose corresponding exponents are affinely independent and

$$\sum_{j \in J} m_{j,n} = 1.$$

These are the index subsets of the form $J = I \cup \{k\}$ for any $I \subset \{0, \dots, s\}$ of cardinality n whose exponents are affinely independent and verify $m_{i,n} = 0$ for all $i \in I$, and any $0 \le k \le s$ with $m_{k,n} = 1$. For each decomposition $J = I \cup \{k\}$ we have

$$n! \operatorname{vol}_{\mathbb{Z}^n}(\operatorname{conv}(\{m_j\}_{j \in J}) = (n-1)! \operatorname{vol}_{\mathbb{Z}^{n-1}}(\operatorname{conv}(\{m_i\}_{i \in I}))$$

and $\prod_{j \in J} c_j \chi^{m_j} = c_k \chi^{m_k} \prod_{i \in I} c_i \chi^{m_i}$. Hence

$$\begin{aligned} \operatorname{init}_\tau(\mu(p)) &= \sum_{I,k} ((n-1)! \operatorname{vol}_{\mathbb{Z}^{n-1}}(\operatorname{conv}(\{m_i\}_{i \in I}))^2 \, c_k \chi^{m_k} \prod_{i \in I} c_i \chi^{m_i} \\ &= \Big(\sum_I ((n-1)! \operatorname{vol}_{\mathbb{Z}^{n-1}}(\operatorname{conv}(\{m_i\}_{i \in I}))^2 \prod_{i \in I} c_i \chi^{m_i} \Big) \Big(\sum_k c_k \chi^{m_k} \Big) \\ &= \mu(\operatorname{init}_\tau(p)) \; p_{F'_\tau}, \end{aligned}$$

which is the stated formula. □

By a descent argument we obtain a formula for an arbitrary initial part of $\mu(p)$.

Corollary 3.15. *Let $p \in \mathbb{C}[M]$ with $\operatorname{supp}(p)$ unimodular and $\sigma \in \Sigma^r$ with $1 \le r \le n$. Let F_σ be the $(n-r)$-dimensional face of $\operatorname{NP}(p)$ corresponding to σ and G_i, $i = 1, \dots, r$, the faces of $\operatorname{NP}(p)$ of dimension $n - r + 1$ containing F_σ. Then*

$$\operatorname{init}_\sigma(\mu(p)) = \mu(\operatorname{init}_\sigma(p)) \prod_{i=1}^{r} p|_{F_\sigma^{(i)}}$$

with $F_\sigma^{(i)} \subset G_i$ the adjacent polytope of F_σ as a facet of G_i.

Proof. Choose a decomposition $\sigma = \varsigma + \tau$ with $\varsigma \in \Sigma^{r-1}$ and $\tau \in \Sigma^1$, which is possible by the unimodularity of Σ. By induction

$$\operatorname{init}_\varsigma(\mu(p)) = \mu(\operatorname{init}_\varsigma(p)) \prod_{i=1}^{r-1} p|_{F_\varsigma^{(i)}}, \tag{3.4}$$

where $F_\varsigma^{(i)}$, $i = 1, \dots, r-1$, are the adjacent polytopes of the $(n-r+1)$-dimensional face F_ς. We also have

$$\operatorname{init}_\tau(\operatorname{init}_\varsigma(\mu(p))) = \operatorname{init}_\sigma(\mu(p)), \quad \operatorname{init}_\tau(\operatorname{init}_\varsigma(p)) = \operatorname{init}_\sigma(p)$$

and $\operatorname{init}_\tau(p|_{F_\varsigma^{(i)}}) = p_{F_\sigma^{(i)}}$, $i = 1, \dots, r-1$, for the adjacent polytopes $F_\sigma^{(i)}$ of the face F_σ that are not contained in F_ς.

Hence taking initial parts in the direction of the ray τ in (3.4) and applying the multiplicativity of initial parts and Theorem 3.14 we get

$$\operatorname{init}_\sigma(\mu(p)) = \operatorname{init}_\tau(\mu(\operatorname{init}_\varsigma(p))) \prod_{i=1}^{r-1} \operatorname{init}_\tau(p|_{F_\varsigma^{(i)}}) = \mu(\operatorname{init}_\sigma(p)) \; p|_{F_\sigma^{(r)}} \prod_{i=1}^{r-1} p|_{F_\sigma^{(i)}},$$

which gives the formula. □

3.5. **Algebraic variants of the Einstein condition.** Now let X be a toric Fano manifold with torus $\mathbb{T}$, and recall that X admits a projectively induced Kähler-Einstein form with constant $\lambda > 0$ if and only if there exists $p \in \mathbb{R}_{>0}[M]$ such that $\operatorname{supp}(p)$ is unimodular, $\operatorname{NP}(p) = \lambda^{-1}\Delta_{-K_X}$ and

$$\mu(p) = c\chi^m p^{n+1-\lambda} \tag{3.5}$$

for some $c \in \mathbb{R}_{>0}$ and $m \in M$ (Corollary 2.13).

In the study of the Einstein condition it is standard to normalize the Kähler form to reduce from an arbitrary constant $\lambda > 0$ to the case $\lambda = 1$. For projectively induced metrics this normalization is more delicate because one cannot ensure that it can be done through a projective immersion, contrary to what might be inferred from [ALZ12, page 485] and [MS26, Section 2.5.7], see Remark 3.17 below. Taking this issue into account we introduce the subset of Laurent polynomials

$$\mathbb{R}[M]^+ = \{p \in \mathbb{R}[M] \mid p^\nu \in \mathbb{R}_{>0}[M] \text{ for some } \nu \in \mathbb{Z}_{>0}\}.$$

We obtain the next algebraic characterization of the Einstein condition for a Kähler form that is induced by a full toric immersion, similar to that considered in [ALZ12, Lemma 4.1] and [MS26, Remark 2.4].

Proposition 3.16. *The following conditions are equivalent:*

(1) X admits a projectively induced Kähler-Einstein form,

(2) there is $p \in \mathbb{R}[M]^+$ with $\operatorname{supp}(p)$ unimodular, $\operatorname{NP}(p) = \Delta_{-K_X}$ and $\mu(p) = p^n$.

Proof. Assume (2) and choose an integer $\nu > 0$ such that $q \coloneqq p^\nu \in \mathbb{R}_{>0}[M]$. We have that $\operatorname{supp}(q)$ is unimodular and $\operatorname{NP}(q) = \nu \operatorname{NP}(p) = \nu\, \Delta_{-K_X}$, and by Lemma 3.8(2) we also have

$$\mu(q) = \mu(p^\nu) = \nu^n p^{(n+1)(\nu-1)} \mu(p) = \nu^n p^{(n+1)(\nu-1)+n} = \nu^n q^{n+1-1/\nu}.$$

By (3.5) X admits a projectively induced Kähler-Einstein form with constant ν^{-1}.

Conversely assume that X admits a projectively induced Kähler-Einstein form with constant $\lambda > 0$, so that the conditions in (3.5) holds. With notation as therein set

$$q = \gamma\, p^\lambda \quad \text{with } \gamma = (c\lambda^n)^{-1/(n+1)}. \tag{3.6}$$

Hence $q \in \mathbb{R}[M]^+$ because by (3.5) it is a rational function that has a positive integral power lying in $\mathbb{R}_{>0}[M]$. Furthermore $\operatorname{supp}(q)$ is unimodular and $\operatorname{NP}(q) = \lambda \operatorname{NP}(p) = \Delta_{-K_X}$ and by Lemma 3.8(1,2)

$$\mu(q) = \mu(\gamma p^\lambda) = \gamma^{n+1}\lambda^n p^{(n+1)(\lambda-1)}\mu(p) = \gamma^{n+1}\lambda^n p^{(n+1)(\lambda-1)} c\chi^m p^{n+1-\lambda} = \chi^m q^n.$$

Considering the associated Newton polytopes and applying Corollary 3.12 we get $m = 0$, thus proving (2). □

Remark 3.17. There are Laurent polynomials with positive coefficients that are powers of Laurent polynomials with some negative coefficients. As an example take $p = 2 + 2x - x^2 + 2x^3 + 2x^4 \in \mathbb{R}[x^{\pm 1}]$, for which

$$p^2 = 4 + 8x + 4x^3 + 17x^4 + 4x^5 + 8x^7 + 4x^8 \in \mathbb{R}_{>0}[x^{\pm 1}].$$

Hence in (3.6) we cannot ensure that q has positive coefficients, and so this Laurent polynomial is not necessarily given by a toric full immersion as in (2.6).

We introduce a more flexible version of the Einstein condition in our algebraic setting that is better suited for a recursive analysis.

Definition 3.18. Let $p \in \mathbb{C}[M]$ with $\mathrm{supp}(p)$ unimodular. We say that p satisfies the *generalized (algebraic) Einstein condition (GEC)* if $\mu(p) \mid p^\kappa$ for some $\kappa \in \mathbb{Z}_{>0}$.

As a consequence of our previous results we can show that this condition is hereditary. To properly state this property, for a Laurent polynomial $p \in \mathbb{C}[M] \setminus \{0\}$ and a cone $\sigma \in \Sigma_{\mathrm{NP}(p)}$ we define the *translated initial part* of p in the direction of σ as

$$p_\sigma = \chi^{-m} \,\mathrm{init}_\sigma(p) \in \mathbb{C}[M \cap \sigma^\perp]$$

for any $m \in \mathrm{supp}(\mathrm{init}_\sigma(p))$, where $\sigma^\perp \subset M_\mathbb{R}$ denotes the orthogonal subspace of σ.

Proposition 3.19. *Let $p \in \mathbb{C}[M]$ with $\mathrm{supp}(p)$ unimodular and satisfying GEC, and let $\sigma \in \Sigma_{\mathrm{NP}(p)}$. Then the translated initial part $p_\sigma \in \mathbb{C}[M \cap \sigma^\perp]$ also verifies that $\mathrm{supp}(p_\sigma)$ is unimodular as a subset of the lattice $M \cap \sigma^\perp$ and satisfies GEC.*

Proof. It is clear that $\mathrm{supp}(p_\sigma)$ is unimodular as a subset of $M \cap \sigma^\perp$ because so is the case for $\mathrm{supp}(p)$ as a subset of M.

On the other hand there exists an integer $\kappa > 0$ such that $\mu(p)$ divides p^κ because p satisfies GEC. By the multiplicativity of initial parts we get that $\mathrm{init}_\sigma(\mu(p))$ divides $\mathrm{init}_\sigma(p)^\kappa$, and by Corollary 3.15 we have that $\mu(\mathrm{init}_\sigma(p))$ divides $\mathrm{init}_\sigma(\mu(p))$. Hence $\mu(p_\sigma)$ divides p_σ^κ and so p_σ satisfies GEC, as stated. □

Finally we state the application of this property to the study of projectively induced Kähler-Einstein forms on X. We denote by Σ the fan of this toric Fano manifold.

Corollary 3.20. *Assume that X admits a projectively induced Kähler-Einstein form, and for a cone $\sigma \in \Sigma$ denote by $F_\sigma \preceq \Delta_{-K_X}$ the associated face and $v \in F_\sigma$ a lattice point in it. Then there exists $q \in \mathbb{C}[M \cap \sigma^\perp]$ with $\mathrm{supp}(q) \subset M \cap \sigma^\perp$ unimodular and $\mathrm{NP}(q) = F_\sigma - v$ satisfying GEC.*

Proof. This follows readily from Propositions 3.16 and 3.19. □

4. Applications

In this section we apply our constructions and results to the study of the algebraic Einstein conditions in several concrete situations.

For simplicity here we assume that $\mathbb{T}$ is the standard torus $(\mathbb{C}^\times)^n$, so that $M = N = \mathbb{Z}^n$, $M_\mathbb{R} = N_\mathbb{R} = \mathbb{R}^n$ and $\mathbb{C}[M] = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. To ease the exposition we switch to the usual notation for monomials.

4.1. Small dimensions.

We first consider the 1-dimensional case.

Proposition 4.1. *Let $p \in \mathbb{C}[x^{\pm 1}]$ with $\mathrm{supp}(p)$ unimodular. Then p satisfies GEC if and only if*

$$p = c\, x^m (x + \xi)^\nu$$

with $c, \xi \in \mathbb{C}^\times$, $m \in \mathbb{Z}$ and $\nu \in \mathbb{Z}_{>0}$.

Proof. Set $\mathrm{NP}(p) = [m, m']$ with $m \le m'$. Write

$$p = \sum_{j=0}^{s} c_j x^{m_j} = c\, x^m \prod_{k=1}^{t} (x + \xi_k)^{\nu_k}$$

with $m_j \in \mathbb{Z}$, $\nu_k \in \mathbb{Z}_{>0}$ and $\xi_k \in \mathbb{C}^\times$ such that $m = m_0 < \cdots < m_s = m'$ and $\xi_k \neq \xi_l$ for all $k \neq l$. By Lemma 3.7 we have

$$\mu(p) = c^2\, x^{2m+1} q \prod_{k=1}^{t} (x+\xi_k)^{2\nu_k - 2} \quad \text{ with } q = \sum_{k=1}^{t} \nu_k \xi_k \prod_{l\neq k} (x+\xi_l)^2.$$

The leading and tail coefficients of q are respectively equal to

$$\sum_{k=1}^{t} \nu_k \xi_k = c^{-1} c_{s-1} \quad \text{and} \quad \sum_{k=1}^{t} \nu_k \xi_k \prod_{l\neq k} \xi_l^2 = c^{-1} c_1 \prod_{k=1}^{t} \xi_k^{2-\nu_k}.$$

These quantities are nonzero because $\mathrm{supp}(p)$ is unimodular, and so $\mathrm{NP}(q) = [0, 2t-2]$. Since this polynomial is coprime with p, we have that $\mu(p)$ divides p^κ for a positive integer κ if and only if q is a monomial. This is equivalent to the fact that $t = 1$, which gives the statement. □

Corollary 4.2. *Let ω be a projectively induced toric Kähler form on $\mathbb{P}^1$ induced by a toric full immersion $\varphi\colon \mathbb{P}^1 \to \mathbb{P}^s$ whose associated Laurent polynomial $p_\varphi \in \mathbb{R}_{>0}[x^{\pm 1}]$ satisfies GEC. Then $(\mathbb{P}^1, \omega) \simeq (\mathbb{P}^1, \nu\, \omega_{\mathrm{FS}})$ with $\nu \in \mathbb{Z}_{>0}$.*

Proof. By Lemma 2.6 we have that $\mathrm{supp}(p_\varphi)$ is unimodular, and so by Proposition 4.1 we have that $p_\varphi = cx^m(x+\xi)^\nu$ for an integer $\nu > 0$ and real numbers $c, \xi > 0$. Applying Lemma 2.10 we get

$$\omega = i\,\partial\overline{\partial} \log(c\, e^{2mu}(e^{2u}+\xi)^\nu) = \nu\, i\,\partial\overline{\partial}\log(e^{2u}+\xi) = \nu A^* \omega_{\mathrm{FS}}$$

for any linear map $A\colon (z_0 : z_1) \mapsto (\alpha z_0 : z_1)$ with $\alpha \in \mathbb{C}$ such that $|\alpha| = \xi^{1/2}$. □

We now focus on the 2-dimensional case to prove a result that severely restricts the possible Newton polygons of the bivariate Laurent polynomials that satisfy GEC. It is a generalization of [MS26, Lemma 2.20].

Recall that for an edge E of a lattice polygon $\Delta \subset \mathbb{R}^2$ we denote by E' its *adjacent segment*, defined as the intersection of Δ with the inner parallel lattice line that is closest to E (Definition 3.13). For a lattice segment $F \subset \mathbb{R}^2$ we denote by $\ell(F)$ its *lattice length*, defined as the number of lattice points in F minus 1.

Recall also that $p|_S$ denotes the restriction of a Laurent polynomial $p \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ to a subset $S \subset \mathbb{R}^2$ as in (3.3).

Proposition 4.3. *Let $p \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ with $\mathrm{supp}(p)$ unimodular and satisfying GEC. Let E be an edge of $\mathrm{NP}(p)$ and E' its adjacent segment.*

(1) Let v be a vertex of $\mathrm{NP}(p)$ and a, b be the basis of $\mathbb{Z}^2$ at v induced by $\mathrm{supp}(p)$ with $v + a \in E$. Then there exist $c, c' \in \mathbb{C}^\times$ such that

$$p|_E = c\chi^v(\chi^a + \xi)^{\ell(E)} \quad \text{and} \quad p|_{E'} = c'\chi^{v+b}(\chi^a+\xi)^{\ell(E')}.$$

(2) Let F be another edge of $\mathrm{NP}(p)$ and F' its adjacent segment. Then

$$\frac{\ell(F')}{\ell(F)} = \frac{\ell(E')}{\ell(E)}.$$

The figure below illustrates the objects in this proposition.

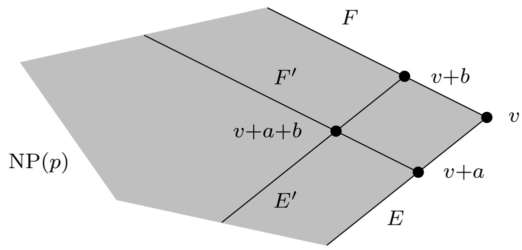

FIGURE 1. Edges and adjacent segments

Proof. For (1) let τ be the inner normal ray of E so that this edge realizes as $E = \mathrm{NP}(p)_\tau$, and set $H = \mathrm{NP}(\mu(p))_\tau$ for the corresponding edge of the Newton polygon of $\mu(p)$. Then Theorem 3.14 gives the factorization

$$\mu(p)|_H = \mu(p|_E)\, p|_{E'}. \tag{4.1}$$

By GEC we have that $\mu(p)$ divides p^κ for some integer $\kappa > 0$. By the multiplicativity of initial parts $\mu(p)|_H$ divides $(p|_E)^\kappa$, and so the factorization (4.1) implies that $\mu(p|_E)\, p|_{E'}$ divides $(p|_E)^\kappa$. Hence the restriction $p|_E$ is a univariate Laurent polynomial that satisfies GEC and whose Newton polytope coincides with E. With Proposition 4.1 we get

$$p|_E = c\chi^v(\chi^a + \xi)^{\ell(E)}.$$

The corresponding expression for $p|_{E'}$ follows from the fact that this is univariate Laurent polynomial is a factor of $(\chi^a + \xi)^\kappa$ whose Newton polytope coincides with E'.

For (2) it is enough to consider the case when E and F share a vertex, as in Figure 1. By (1) we have

$$\begin{aligned} p|_E &= c_1\, \chi^v(\chi^a + \xi_1)^{\ell(E)}, & p|_F &= c_2\, \chi^v(\chi^b + \xi_2)^{\ell(F)}, \\ p|_{E'} &= c_1'\, \chi^{v+b}(\chi^a + \xi_1)^{\ell(E')}, & p|_{F'} &= c_2'\, \chi^{v+a}(\chi^b + \xi_2)^{\ell(F')}, \end{aligned}$$

with $c_i, c_i', \xi_i \in \mathbb{C}^\times$. Examining the overlaps of these Laurent polynomials at the lattice points v, $v+a$, $v+b$ and $v+a+b$ we obtain the equations

$$\begin{aligned} c_1\xi_1^{\ell(E)} &= c_2\xi_2^{\ell(F)}, & \ell(E)\, c_1\xi_1^{\ell(E)-1} &= c_2'\xi_2^{\ell(F')}, \\ c_1'\xi_1^{\ell(E')} &= \ell(F)\, c_2\xi_2^{\ell(F)-1}, & \ell(E')\, c_1'\xi_1^{\ell(E')-1} &= \ell(F')\, c_2'\xi_2^{\ell(F')-1}. \end{aligned}$$

The statement follows by considering the ratio between the product of the first equation by the fourth and that of the second by the third. □

Passing to the geometric setting we obtain the next consequence.

Corollary 4.4. *Let X be a toric surface and $\varphi\colon X \to \mathbb{P}^s$ a toric full immersion such that $p_\varphi \in \mathbb{R}_{>0}[x^{\pm1}, y^{\pm1}]$ satisfies GEC. Then the ratio $\ell(E')/\ell(E)$ for an edge $E \preceq \mathrm{NP}(p_\varphi)$ and its adjacent segment E' does not depend on the choice of the edge.*

In particular, if X admits a projectively induced Kähler-Einstein form then the ratio $\ell(E')/\ell(E)$ for an edge $E \preceq \Delta_{-K_X}$ and its adjacent segment E' does not depend on the choice of the edge.

Proof. The first statement follows directly from Proposition 4.3(2). For the second, let ω be a projectively induced Kähler-Einstein form on X with constant $\lambda > 0$. By Lemma 2.8 it is induced by a toric full immersion $\varphi\colon X \to \mathbb{P}^s$, and by the Einstein condition $\Delta_{-K_X} = \lambda\, \mathrm{NP}(p)$. This statement follows then from the previous one. □

Example 4.5. Let

$$p = \alpha_0 x^{-1}y^{-1} + \alpha_1 y^{-1} + \alpha_2 xy^{-1} + \alpha_3 x^2 y^{-1} + \alpha_4 x^{-1} + \alpha_5 + \alpha_6 x + \alpha_7 x^{-1}y + \alpha_7 y \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$$

with $\alpha_j \neq 0$ for all $j \neq 5$. Its Newton polytope is the trapezoid in Figure 2.

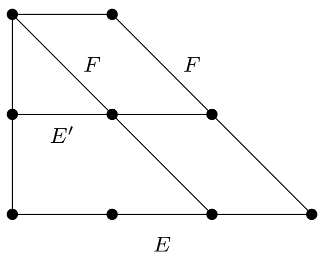

FIGURE 2. A reflexive trapezoid

The edges meeting at the lattice point $(2, -1)$ verify

$$\frac{\ell(E')}{\ell(E)} = \frac{2}{3} \neq 1 = \frac{\ell(F')}{\ell(F)},$$

and so p cannot satisfy GEC by Proposition 4.3(2).

On the other hand, for the hexagon in Figure 3 we have $\ell(E')/\ell(E) = 2$ for every edge E and its adjacent segment E'. In spite of this, no Laurent polynomial with

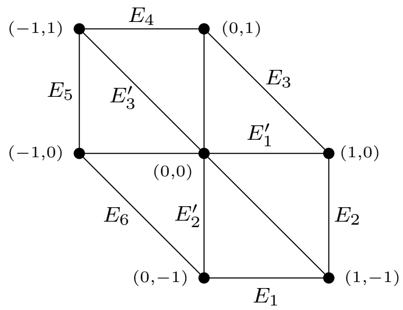

FIGURE 3. A reflexive hexagon

unimodular support and having this Newton polytope can satisfy GEC.

Proposition 4.6. *Let*

$$p = \alpha_0 y^{-1} + \alpha_1 xy^{-1} + \alpha_2 x^{-1} + \alpha_3 + \alpha_4 x + \alpha_5 x^{-1}y + \alpha_6 y \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$$

with $\alpha_j \neq 0$ *for all* j *except possibly* $j = 3$. *Then* p *does not satisfy GEC.*

Proof. Let E_i, $i = 1, \dots, 6$, and E'_j, $j = 1, 2, 3$, be the edges and adjacent segments of the hexagon, and p_i, $i = 1, \dots, 6$, and p'_j, $j = 1, 2, 3$, the corresponding univariate

Laurent polynomials. Note that E_1' is adjacent to both E_1 and E_4, and a similar remark holds for E_2' and E_3'. By Proposition 4.3(1)

$$\begin{array}{lll} p_1 = c_1\, y^{-1}(x+\xi_1), & p_2 = c_2\, xy^{-1}(y+\xi_2), & p_3 = c_3\, x(x^{-1}y+\xi_3), \\ p_4 = c_4\, x^{-1}y(x+\xi_1), & p_5 = c_5\, x^{-1}(y+\xi_2), & p_6 = c_6\, y^{-1}(x^{-1}y+\xi_3), \\ p_1' = c_1'\, x^{-1}(x+\xi_1)^2, & p_2' = c_2'\, y^{-1}(y+\xi_2)^2, & p_3' = c_3'\, xy^{-1}(x^{-1}y+\xi_3)^2, \end{array}$$

for some nonzero parameters c_i, c_j' and ξ_k. Hence these parts of p are encoded by 12 parameters, and their overlapping at each lattice point gives the following system of 14 equations:

$$\begin{array}{llll} (0,-1): & c_1\xi_1 = c_2'\xi_2^2 = c_6\,\xi_3, & (1,-1): & c_1 = c_2\xi_2 = c_3'\xi_3^2, \\ (-1,0): & c_1'\xi_1^2 = c_5\xi_2 = c_6, & (0,0): & 2c_1'\xi_1 = 2c_2'\xi_2 = 2c_3'\xi_3, \\ (1,0): & c_1' = c_2 = c_3\xi_3, & (-1,1): & c_4\xi_1 = c_5 = c_3', \\ (0,1): & c_4 = c_2' = c_3. & & \end{array}$$

Taking logarithms this transforms into a system of 14 linear equations in 12 variables. Its solution is

$$\begin{array}{lll} \log(c_1) = -r_1+2r_2+2r_3, & \log(c_2) = -r_1+r_2+2r_3, & \log(c_3) = r_3, \\ \log(c_4) = r_3, & \log(c_5) = r_1, & \log(c_6) = r_1+r_2, \\ \log(c_1') = -r_1+r_2+2r_3, & \log(c_2') = r_3, & \log(c_3') = r_1, \\ \log(\xi_1) = r_1-r_3, & \log(\xi_2) = r_2, & \log(\xi_3) = -r_1+r_2+r_3, \end{array}$$

for $r_1, r_2, r_3 \in \mathbb{C}/2\pi i\mathbb{Z}$. Setting $\rho_i = e^{r_i}$ this implies

$$\begin{aligned} p &= p_1 + p_1' + p_4 \\ &= \rho_1\rho_3^2 y^{-1} + \rho_1\rho_2\rho_3 xy^{-1} + \rho_1\rho_2^{-1}\rho_3^2 x^{-1} + 2\rho_1\rho_3 + \rho_1\rho_2 x + \rho_1\rho_2^{-1}\rho_3 x^{-1}y + \rho_1 y. \end{aligned}$$

Setting furthermore $x = \rho_2^{-1}\rho_3 u$, $y = \rho_3 v$ and dividing by $\rho_1\rho_3$, this Laurent polynomial reduces to

$$q = v^{-1} + uv^{-1} + u^{-1} + 2 + u + u^{-1}v + v = u^{-1}v^{-1}(u+v)(u+1)(v+1) \in \mathbb{C}[u^{\pm 1}, v^{\pm 1}].$$

The corresponding Monge-Ampère polynomial is

$$\begin{aligned} \mu(q) =& u^2 + 2uv + v^2 + 10u + 2u^2v^{-1} + 10v + 2u^{-1}v^2 + u^2v^{-2} + 10uv^{-1} + 10u^{-1}v \\ &+ u^{-2}v^2 + 10u^{-1} + 2uv^{-2} + 10v^{-1} + 2u^{-2}v + u^{-2} + v^{-2} + 2u^{-1}v^{-1} + 18 \\ =& u^{-2}v^{-2}(u^2v + uv^2 + u^2 + 6uv + v^2 + u + v)(u+v)(u+1)(v+1), \end{aligned}$$

Its first nontrivial factor is quadratic and so if it divides a power of p then it is necessarily equal to $(u+v)^2$, $(u+v)(u+1)$, $(u+v)(v+1)$ or $(u+1)(v+1)$, which is is clearly not the case. Hence p does not satisfy GEC, as stated. □

This lattice polygon is the anticanonical polytope of the blow up of $\mathbb{P}^2$ at its three fixed points [CLS11, Exercise 8.3.8(a)].

Corollary 4.7. *Let X be the blowup of $\mathbb{P}^2$ at its three fixed points. Then there is no $p \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ with* $\operatorname{supp}(p)$ *unimodular and* $\operatorname{NP}(p) = \Delta_{-K_X}$ *that satisfies GEC. In particular X does not admit a projectively induced Kähler-Einstein form.*

4.2. **Large dimensions.** Finally we apply our results to the four families of symmetric toric Fano manifolds discussed by Batyrev and Selivanova in [BS99] and the non-symmetric examples of Nill and Paffenholz [NP11].

Definition 4.8. Let X be a toric manifold with torus $\mathbb{T}$ and denote by Σ its fan on the vector space $N_{\mathbb{R}}$. Then X is *symmetric* if the the group of automorphisms of Σ fixes only the origin. This toric manifold is *centrally symmetric* if the map $N_{\mathbb{R}} \to N_{\mathbb{R}}$ defined as $u \mapsto -u$ is an automorphism of Σ.

Following [BS99, Section 4], for each integer $k \geq 1$ we denote by V_k the k-th del Pezzo toric manifold introduced by Voskresenskij and Klyachko [VK85], see [BS99, Example 4.1]. It is the centrally symmetric toric Fano manifold of dimension $n = 2k$ with a fan whose cones are generated by the vectors

$$\pm e_1, \dots, \pm e_n, \pm(e_1 + \dots, e_n), \tag{4.2}$$

where $e_1, \dots, e_n$ is the standard basis of $N = \mathbb{Z}^n$. Note that V_1 is the blowup of $\mathbb{P}^2$ at its three fixed points in Corollary 4.7.

Next for integers $1 \leq k \leq m$ we denote by $S_{m,k}$ the symmetric toric Fano manifold of dimension $n = 2m + 1$ introduced by Sakane [Sak86], see [BS99, Example 4.2]. It is the projectivization of the vector bundle $\mathcal{O} \oplus \mathcal{O}(k, -k)$ over $\mathbb{P}^m \times \mathbb{P}^m$, and its fan is consisting of cones generated by the vectors

$$\begin{aligned} &e_1, \dots, e_{2m}, \pm e_{2m+1}, -(e_1 + e_2 + \dots + e_m + k e_{2m+1}), \\ &\qquad - (e_{m+1} + e_{m+2} + \dots + e_{2m} - k e_{2m+1}). \end{aligned} \tag{4.3}$$

Then for integers $0 \leq k \leq m$ we denote by $X_{m,k}$ the symmetric toric Fano manifold of dimension $n = 2m + 2$ introduced by Nakagawa [Nak94], see [BS99, Example 4.3]. It is defined by a fan whose cones are generated by the vectors

$$\begin{aligned} &e_1, \dots, e_{2m}, \pm e_{2m+1}, \pm e_{2m+2}, \pm(e_{2m+1} + e_{2m+2}) \\ &\quad - (e_1 + e_2 + \dots + e_m - k e_{2m+1}), -(e_{m+1} + e_{m+2} + \dots + e_{2m} + k e_{2m+1}). \end{aligned} \tag{4.4}$$

Finally for an integer $m \geq 1$ we denote by W_m the symmetric toric Fano manifold of dimension $n = 2m$ introduced in [BS99, Example 4.4]. It is the blowup of $\mathbb{P}^m \times \mathbb{P}^m$ along certain $m + 1$ invariant subvarieties of codimension 2, and it is defined by a fan whose cones are generated by the vectors

$$\begin{aligned} &e_1, \dots, e_{2m}, e_1 + e_{m+1}, \dots, e_m + e_{2m}, \\ &\qquad - (e_1 + \dots + e_m), -(e_{m+1} + \dots + e_{2m}), -(e_1 + \dots + e_{2m}). \end{aligned} \tag{4.5}$$

By [BS99, Theorem 1.1], all these toric Fano manifolds admit a Kähler-Einstein form.

The next result contains Theorem 1.2 from the introduction.

Theorem 4.9. *Let X be a symmetric toric Fano manifold of type V_k, $S_{m,k}$, $X_{m,k}$ or W_m. Then for every $p \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ with* $\operatorname{supp}(p)$ *unimodular and* $\operatorname{NP}(p) = \Delta_{-K_X}$ *we have that p does not satisfy GEC. In particular X does not admit a projectively induced Kähler-Einstein form.*

Proof. Let $k \geq 1$ be an integer and denote by Δ_k the anticanonical polytope of the del Pezzo V_k. By the description of this polytope in (2.1) and of the fan of this toric manifold in (4.2) we have

$$\Delta_k = \{x \in \mathbb{R}^n \mid -1 \leq x_1, \dots, x_n, x_1 + \dots + x_n \leq 1\}.$$

The subset

$$F=\{x\in\Delta_k \mid x_i=(-1)^i \text{ for } i=3,\dots,n\}$$

is a face of Δ_k because it is given by the equality in some of the inequalities defining this polytope. This face is an hexagon in the $(1,2)$-plane of $\mathbb{R}^n$ identifying with that in Proposition 4.6, namely

$$F\simeq\{(x_1,x_2)\in\mathbb{R}^2 \mid -1\le x_1,x_2,x_1+x_2\le 1\}.$$

By Proposition 3.19, if the Laurent polynomial p satisfies GEC then this is also the case for its restriction $p|_F$, which is excluded by Proposition 4.6.

Now for integers $1\le k\le m$ we denote by $\Delta_{m,k}$ the anticanonical polytope of the Sakane toric manifold $S_{m,k}$. By (4.3) we have

$$\begin{aligned}\Delta_{m,k}=\{(x_1,\dots,x_m,y_1,\dots,y_m,z)\in\mathbb{R}^n \mid x_i,y_j\ge -1 \text{ for } i,j=1,\dots,m,\\ -1\le z\le 1,\ x_1+x_2+\dots+x_m\le 1-kz,\ y_1+y_2+\dots+y_m\le 1+kz\}\end{aligned}$$

The subset of this polytope defined by the equations

$$x_1=\dots=x_m=y_1=\dots=y_{m-1}=-1$$

is a face F that identifies with a trapezoid in the $(2m,2m+1)$-plane of $\mathbb{R}^n$, namely

$$F\simeq\{(y_m,z)\in\mathbb{R}^2 \mid -1\le y_m\le m+kz,-1\le z\le 1\}.$$

As in the previous case, if p satisfies GEC then this is also the case for $p|_F$. Now the ratios of the lattice length of the two edges of this trapezoid meeting at the lattice point $(-1,1)$ and its respective adjacent segments are $(m+k+1)/(m+1)$ and 1, and so by Proposition 4.3(2) the condition GEC for $p|_F$ is not possible.

Next for integers $0\le k\le m$ we denote by $\Delta_{m,k}$ the anticanonical polytope of the Nakagawa toric manifold $X_{m,k}$. By (4.4) we have

$$\begin{aligned}\Delta_{m,k}=\{(x_1,\dots,x_m,y_1,\dots,y_m,z,w)\in\mathbb{R}^n\\ \mid x_i,y_j\ge -1 \text{ for } i,j=1,\dots,m,\ -1\le z,w,z+w\le 1,\\ x_1+x_2+\dots+x_m\le 1+kz,\ y_1+y_2+\dots+y_m\le 1-kz\}.\end{aligned}$$

The subset defined by the equations

$$x_1=x_2=\dots,x_m=y_1=y_2=\dots=y_m=-1$$

is a face F that identifies with an hexagon in the $(2m+1,2m+2)$-plane of $\mathbb{R}^n$:

$$F\simeq\{(z,w)\in\mathbb{R}^2 \mid -1\le z,w,z+w\le 1\}.$$

It is the same that appears for the del Pezzo V_k's, and as therein we deduce that p cannot satisfy GEC.

Finally for an integer $m\ge 1$ we let Δ_m be the anticanonical polytope of the Batyrev-Selivanova toric manifold W_m. By (4.5) we have

$$\begin{aligned}\Delta_m=\{(x_1,\dots,x_m,y_1,\dots,y_m)\in\mathbb{R}^n \mid x_i,y_i,x_i+y_i\ge -1 \text{ for } i=1,\dots,m,\\ x_1+\dots+x_m,\ y_1+\dots+y_m,\ x_1+\dots+x_m+y_1+\dots+y_m\le 1\}.\end{aligned}$$

The face of this polytope defined by the equations

$$x_1=\dots=x_{m-1}=-1,\quad x_1+y_1=\dots=x_{m-1}+y_{m-1}=-1$$

is an hexagon in the $(m,2m)$-plane:

$$F\simeq\{(x_m,y_m)\in\mathbb{R}^2 \mid x_m,y_m,x_m+y_m\ge -1,\ x_m,x_m+y_m\le m,\ y_m\le 1\},$$

When $m = 1$ this is the hexagon considered in previous cases. When $m \geq 2$ the ratios of the lattice length of the two edges meeting at the lattice point $(-1, 1)$ and their adjacent segments are

$$\frac{m+1}{m} \neq 2.$$

Applying again Proposition 4.3(2) we get that p can neither satisfy GEC in this case.

The last statement is a direct consequence of Corollary 3.20. □

Proof of Corollary 1.3. Let X be a centrally symmetric toric Fano manifold. By [VK85, Theorem 6] it isomorphic to a product of projective lines and del Pezzo toric manifolds.

If X is a product of projective lines then its Segre embedding gives a projectively induced Kähler-Einstein form on it. On the other hand, if X has a del Pezzo factor V_k then the anticanonical polytope of V_k is a face of the anticanonical polytope of X. Theorem 4.9 and Corollary 3.20 then imply that X does not admit a projectively induced Kähler-Einstein form. □

In [NP11] Nill and Paffenholz presented two non-symmetric toric Fano manifolds in dimensions 7 and 8 admitting a Kähler-Einstein form. The first is the toric manifold X_1 associated to a fan whose cones are generated by the vectors

$$e_1, e_2, e_3, e_4, e_5, e_6, \pm e_7, -e_1 - e_7, -e_2 - e_7, -e_3 - e_7, -e_4 - e_5 - e_6 + 2e_7 \in \mathbb{R}^7, \quad (4.6)$$

whereas the second is the toric manifold X_2 associated to a fan whose cones are generated by the vectors

$$\begin{aligned} &e_1, e_2, e_3, e_4, e_5, e_6, \pm e_7, \pm e_8, \pm(e_7 - e_8), \\ &\qquad - e_1 - e_8, -e_2 - e_8, -e_3 - e_8, -e_4 - e_5 - e_6 + 2e_8 \in \mathbb{R}^8. \quad (4.7) \end{aligned}$$

Theorem 4.10. *The toric Fano manifolds X_1 and X_2 do not admit a projectively induced Kähler-Einstein form.*

Proof. Denote by Δ_i the anticanonical polytope of X_i, $i = 1, 2$. By (4.6) we have

$$\begin{aligned} \Delta_1 = \{(x_1, \ldots, x_7) \in \mathbb{R}^7 \mid -1 \leq x_1, \ldots, x_6,\ -1 \leq x_7 \leq 1, \\ x_1 + x_7, x_2 + x_7, x_3 + x_7, x_4 + x_5 + x_6 - 2x_7 \leq 1\}. \end{aligned}$$

The subset of this polytope defined by the equations $x_2 = \cdots = x_6 = -1$ is a face in the $(1, 7)$-plane identifying with a trapezoid:

$$F \simeq \{(x_1, x_7) \in \mathbb{R}^2 \mid -1 \leq x_1,\ -1 \leq x_7 \leq 1,\ x_1 + x_7 \leq 1\}.$$

Let $p \in \mathbb{C}[\mathbb{Z}^7]$ with $\mathrm{supp}(p)$ unimodular. By Proposition 4.3 we have that $p|_F$ does not satisfy GEC, and so p does not satisfy GEC by the hereditary character of this condition (Proposition 3.19). Hence by Proposition 3.16 we have that X_1 does not admit a projectively induced Kähler-Einstein form.

Similarly by (4.7) we have

$$\begin{aligned} \Delta_2 = \{(x_1, \ldots, x_8) \in \mathbb{R}^8 \mid -1 \leq x_1, \ldots, x_6,\ -1 \leq x_7, x_8, x_7 - x_8 \leq 1, \\ x_1 + x_8, x_2 + x_8, x_3 + x_8, x_4 + x_5 + x_6 - 2x_8 \leq 1\}. \end{aligned}$$

The subset defined by the equations $x_1 = \cdots = x_6 = -1$ is a face in the $(7, 8)$-plane identifying with an hexagon:

$$F \simeq \{(x_7, x_8) \in \mathbb{R}^2 \mid -1 \leq x_7, x_8, x_7 - x_8 \leq 1\}.$$

As before, we deduce that X_2 does not admit a projectively induced Kähler-Einstein form. □

Remark 4.11. As is clear from its proof, it is possible to state a GEC version of this result similar to that in Theorem 4.9.

These examples were generalized by Nakagawa [Nak15], and it would be interesting to see if the previous analysis extends to this series of non-symmetric toric Fano manifolds.

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