CLOSING THE GAP AROUND THE ESSENTIAL MINIMUM OF HEIGHT FUNCTIONS WITH LINEAR PROGRAMMING

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Abstract. For many common height functions, it is notoriously hard to compute the essential minimum. Nevertheless there are two classical methods, one giving lower bounds and the other giving upper bounds. In this paper, we show that the two methods are actually dual to each other in the sense of linear programming. The main theorem is that they satisfy strong duality, which closes the gap around the essential minimum from both ends. As applications we prove that this essential minimum can be realized by a generic sequence of algebraic integers, and that if the associated Green function is computable then this essential minimum is a computable real number.

Contents

1. Introduction

1.1. The essential minimum of height functions. The height of an algebraic point in a quasi-projective variety is a measure of its arithmetic complexity, which makes it an important tool in the study of Diophantine equations. In particular it is crucial in the proof of results like the Mordell–Weil theorem and the Mordell–Faltings theorem, and appears in far-reaching conjectural statements such as the effective Mordell conjecture and Vojta's conjecture, see for instance [BG06].

Date: March 20, 2026.

2020 *Mathematics Subject Classification.* Primary: 11G50; Secondary: 14G40, 30C85, 49N15, 11Y16.

Key words and phrases. Height function, essential minimum, potential of a measure, strong duality, computable real number.

Burgos Gil was partially supported by grants PID2022-142024NB-I00 and CEX2023-001347-S funded by MICIU/AEI/10.13039/501100011033. Menares was partially supported by ANID FONDECYT Regular grant 1250734. Qu was partially supported by grant PID2022-142024NB-I00 funded by MICIU/AEI/10.13039/501100011033. Sombra was partially supported by grants PID2023-147642NB-I00 and CEX2020-001084-M funded by MICIU/AEI/10.13039/501100011033.

Among the different properties of heights, the equidistribution of small points is both of intrinsic interest and helpful in problems about unlikely intersections such as the Manin–Mumford and the Bogomolov conjectures. To formulate it, let $X/\mathbb{Q}$ be a quasi-projective variety and $h\colon X(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}$ a height function on its set of algebraic points. The *essential minimum* of h is the quantity defined as

$$\mathrm{ess}(h) \coloneqq \inf \left\{ \liminf_{n\to\infty} h(\alpha_n) \,\middle|\, (\alpha_n) \text{ is a generic sequence in } X(\overline{\mathbb{Q}}) \right\} \in \mathbb{R} \cup \{-\infty\},$$

where a sequence of algebraic points of X is called *generic* if it eventually escapes every proper Zariski closed subset. A generic sequence (x_n) in $X(\overline{\mathbb{Q}})$ is called *small* if $\lim_{n\to\infty} h(x_n) = \mathrm{ess}(h)$, namely if the height of these points converges to the smallest possible value. Then the equidistribution of small points is the property that the Galois orbit of the points in any small generic sequence converge towards a prescribed measure.

The central result in this direction is Yuan's equidistribution theorem [Yua08], which most notably applies to the canonical heights associated to dynamical systems, and in particular to the canonical heights on toric varieties and the Néron–Tate heights on abelian varieties. Beyond Yuan's theorem, other cases that are understood include toric heights on toric varieties [BPRS19] and canonical heights on semiabelian varieties [Küh22].

Recently Ballaÿ and the fourth author obtained a more general equidistribution theorem unifying all these previous results [BS25]. However its implementation in any other setting is challenging since in the first place it assumes the knowledge of the essential minimum, which is a very difficult problem.

For example consider the deceptively simple *Zhang–Zagier height* $h_{\mathrm{ZZ}}\colon \mathbb{A}^1(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}} \longrightarrow \mathbb{R}$ defined by

$$h_{\mathrm{ZZ}}(\alpha) = h_{\mathrm{W}}(\alpha) + h_{\mathrm{W}}(1-\alpha) \quad \text{for } \alpha \in \overline{\mathbb{Q}},$$

where h_{W} denotes the Weil height on $\overline{\mathbb{Q}}$. Both lower and upper bounds are known for its essential minimum [Zag93, Doc01a, Doc01b] but the problem of computing this quantity or even approximating it up to three significant digits remains open.

1.2. **Height functions on $\overline{\mathbb{Q}}$.** In this paper we will study height functions on $\mathbb{A}^1(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}}$ whose archimedean part is governed by an arbitrary Green function. Precisely, a *Green function* is a continuous function $\mathrm{g}\colon \mathbb{C} \longrightarrow \mathbb{R}$ that is invariant under complex conjugation and obeys the asymptotics

$$\mathrm{g}(z) = \log|z| + o(\log|z|) \quad \text{as } |z| \longrightarrow \infty. \tag{1.1}$$

The associated height function $h_{\mathrm{g}}\colon \overline{\mathbb{Q}} \longrightarrow \mathbb{R}$ is then defined for $\alpha \in \overline{\mathbb{Q}}$ as

$$h_{\mathrm{g}}(\alpha) = \frac{1}{\deg(\alpha)} \left(\log|c_\alpha| + \sum_{\beta\in O(\alpha)} \mathrm{g}(\beta) \right), \tag{1.2}$$

where $\deg(\alpha)$ denotes the degree of this algebraic number, c_α the leading coefficient of its minimal polynomial $P_\alpha \in \mathbb{Z}[x]$ and $O(\alpha) \subseteq \mathbb{C}$ its *Galois orbit*, that in this situation might be defined as the set of complex zeros of P_α. For instance, the choice $\mathrm{g}(z) = \log^+|z| = \log\max(1,|z|)$ gives the Weil height on $\overline{\mathbb{Q}}$.

In the language of Arakelov geometry, these are the height functions corresponding to adelic metrics on the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^1$ that for the archimedean place have mild singularities at the point at infinity whereas for the non-archimedean places are canonical, see Section 6.1 for details.

There are two classical methods for bounding the essential minimum of h_{g}, one giving lower bounds and the other giving upper bounds. In this paper we show that both methods are actually dual to each other in the sense of linear programming. Our main result (Theorem D below) shows that the strong duality property holds in this setting, closing the gap around the essential minimum from both ends. A surprising consequence is that for every such height function, the essential minimum can always be attained by a generic sequence of algebraic integers.

Theorem A. *Let* $\mathrm{g}\colon \mathbb{C} \longrightarrow \mathbb{R}$ *be a Green function. Then there exists a sequence* (α_n) *of distinct algebraic integers such that* $h_{\mathrm{g}}(\alpha_n)$ *is monotonically decreasing and* $\lim_n h_{\mathrm{g}}(\alpha_n) = \mathrm{ess}(h_{\mathrm{g}})$.

In fact we obtain that the set of height values $h_{\mathrm{g}}(\overline{\mathbb{Z}})$ is dense in the interval $[\mathrm{ess}(\mathrm{h_g}), \infty)$ (Corollary 6.7), which in particular gives Theorem A.

The Faltings height $h_{\mathrm{F}}\colon \overline{\mathbb{Q}} \longrightarrow \mathbb{R}$ is defined as the stable Faltings height of the semistable elliptic curve with j-invariant equal to α, for $\alpha \in \overline{\mathbb{Q}}$. As a specialization of the previous result, we obtain that the essential minimum of Faltings height can be attained by a generic sequence of j-invariants of different elliptic curves having good reduction everywhere (Theorem 6.16).

1.3. **Lower and upper bounds.** Let $P_1, \ldots, P_k \in \mathbb{Z}[x]$ be nonzero polynomials with integer coefficients and $a_1, \ldots, a_k \in \mathbb{R}_{\geq 0}$ non negative real numbers. For every $\alpha \in \overline{\mathbb{Q}}$ such that $P_i(\alpha) \neq 0$ for all i we have

$$h_{\mathrm{g}}(\alpha) \geq \inf_{z\in\mathbb{C}} \left(\mathrm{g}(z) - \sum_{i=1}^{k} a_i \log \left| P_i(z) \right| \right).$$

This inequality follows from the definition of the height and the product formula on number fields and is at the basis of Smyth's method [Smy81]. It readily implies

$$\mathrm{ess}(h_{\mathrm{g}}) \geq \inf_{z\in\mathbb{C}} \left(\mathrm{g}(z) - \sum_{i=1}^{k} a_i \log \left| P_i(z) \right| \right)$$

and so the best lower bound that one can obtain in this way is

$$\mathrm{ess}(h_{\mathrm{g}}) \geq \sup \left\{ \inf_{z\in\mathbb{C}} \left(\mathrm{g}(z) - \sum_{i=1}^{k} a_i \log \left| P_i(z) \right| \right) \;\middle|\; k \in \mathbb{Z}_{\geq 0}, a_i \in \mathbb{R}_{\geq 0}, P_i \in \mathbb{Z}[x] \setminus \{0\} \right\}.$$

To describe the upper bounds, we denote by $\mathscr{P}_{\log}(\mathbb{C})$ the space of probability measures μ on $\mathbb{C}$ such that

$$\int \log^{+} |x| \, \mathrm{d}\mu < \infty.$$

A function $f\colon \mathbb{C} \longrightarrow \mathbb{R}$ has *logarithmic growth* if there exist constants $A, B > 0$ such that $|f(z)| \leq A + B \log^{+} |z|$ for all $z \in \mathbb{C}$. We say that a sequence (μ_n) in $\mathscr{P}_{\log}(\mathbb{C})$ converges log-*weakly* to $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ if for every continuous function f with logarithmic growth we have

$$\lim_{n\to\infty} \int f \, \mathrm{d}\mu_n = \int f \, \mathrm{d}\mu.$$

Then we denote by $\mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C})$ the subset of $\mathscr{P}_{\log}(\mathbb{C})$ consisting of the limit points of $\{\delta_{O(\alpha)} \mid \alpha \in \overline{\mathbb{Z}}\}$ in the log-weak topology, where $\delta_{O(\alpha)}$ denotes the the uniform probability measure on the Galois orbit of the algebraic integer α.

The basic observation is that for $\mu \in \mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C})$ and a sequence of distinct algebraic integers (α_n) such that $\delta_{O(\alpha_n)}$ converges log-weakly to μ we have

$$\lim_{n\to\infty} h_{\mathrm{g}}(\alpha_n) = \int \mathrm{g}\,\mathrm{d}\mu, \tag{1.3}$$

because the height of an algebraic integer reduces to the average of the Green function over its Galois orbit and the fact that Green functions have logarithmic growth. Hence for every $\mu \in \mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C})$ we have

$$\mathrm{ess}(h_{\mathrm{g}}) \leq \int \mathrm{g}\,\mathrm{d}\mu.$$

So, the best upper bound we can obtain this way is

$$\mathrm{ess}(h_{\mathrm{g}}) \leq \inf\left\{ \int \mathrm{g}\,\mathrm{d}\mu \,\middle|\, \mu \in \mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C}) \right\}.$$

This is the principle applied by the first and second authors together with Rivera-Letelier in their study of the Faltings height [BMR18]. To take advantage of this upper bound for the essential minimum it is important to understand the probability measures that can be arbitrarily approached by Galois orbits of algebraic integers.

1.4. Approximation of measures by algebraic integers.

We now describe different ways to approximate measures by integers. To this end we first introduce some definitions and recall the previous results in this direction.

We denote by $\mathscr{P}_c(\mathbb{C})$ the space of probability measures on $\mathbb{C}$ with compact support, and say that a sequence (μ_n) in $\mathscr{P}_c(\mathbb{C})$ *converges properly* to $\mu \in \mathscr{P}_c(\mathbb{C})$ if it converges weakly and there exists a compact subset $K \subseteq \mathbb{C}$ containing the support of μ_n for all n. Then we denote by $\mathscr{P}_c^{\overline{\mathbb{Z}}}(\mathbb{C}) \subseteq \mathscr{P}_c(\mathbb{C})$ the subset consisting of the limit points of $\{\delta_{O(\alpha)} \mid \alpha \in \overline{\mathbb{Z}}\}$ with respect to the topology of the proper convergence.

Many measures in $\mathscr{P}_c^{\overline{\mathbb{Z}}}(\mathbb{C})$ can be constructed using potential theory. Let μ_K be the equilibrium measure of a compact subset $K \subseteq \mathbb{C}$ that is invariant under the complex conjugation and has capacity $\mathrm{cap}(K) = 1$. Combining the Fekete–Szegő theorem [FS55] with a result of Rumely [Rum99] it can be shown that $\mu_K \in \mathscr{P}_c^{\overline{\mathbb{Z}}}(\mathbb{C})$ (Remark 2.7).

A recent groundbreaking result by Smith [Smi24, Theorem 1.5] and by Orloski and Sardari [OS24, Theorem 1.2] gives the next elegant characterization: for $\mu \in \mathscr{P}_c(\mathbb{C})$ we have that $\mu \in \mathscr{P}_c^{\overline{\mathbb{Z}}}(\mathbb{C})$ if and only if μ is invariant under the complex conjugation and

$$\int \log|Q|\,\mathrm{d}\mu \geq 0 \quad \text{for all } Q \in \mathbb{Z}[x] \setminus \{0\}. \tag{1.4}$$

In particular $\mathscr{P}_c^{\overline{\mathbb{Z}}}(\mathbb{C})$ is a convex subset of $\mathscr{P}_c(\mathbb{C})$.

Actually both the Fekete–Szegő–Rumely result and the Smith–Orloski–Sardari theorem are stronger (Remark 2.10) but the current versions suffice for our purposes.

The use of log-weak convergence allows us to extend the Smith–Orloski–Sardari theorem to probability measures whose support is not necessarily compact.

Theorem B (Theorem 4.1)**.** *Let $\mu \in \mathscr{P}_{\log}(\mathbb{C})$. Then $\mu \in \mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C})$ if and only if μ is invariant under the complex conjugation and*

$$\int \log|Q|\,\mathrm{d}\mu \geq 0 \quad \textit{for all } Q \in \mathbb{Z}[x] \setminus \{0\}.$$

Even though this characterization may seem daunting because it involves infinitely many inequalities, it allows us to obtain structural information on $\mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C})$. For instance, Theorem B readily implies that this is a convex set. Also, we can use it to construct a countable dense subset that allows a complementary approach to $\mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C})$. To this end, for each pair $P, Q \in \mathbb{Z}[x]$ of different monic irreducible polynomials we denote by $\mu_{P,Q}$ the normalized pullback of the Haar measure on the unit circle under the rational map

$$\frac{P^{\deg(Q)+1}}{Q^{\deg(P)}} \colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1.$$

Its support is the lemniscate $\{z \in \mathbb{C} \mid |P(z)|^{\deg(Q)+1} = |Q(z)|^{\deg(P)}\}$, which is compact because the above rational function has a pole at infinity. We show that $\mu_{P,Q}$ lies in $\mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C})$ and that the set of all such measures is dense in $\mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C})$ with respect to the log-weak topology (Theorem 4.5).

1.5. **Potential theory.** Potential theory plays an important role in the previous results. Recall that the *potential* of a probability measure $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ is the function $U^{\mu} \colon \mathbb{C} \longrightarrow \mathbb{R} \cup \{\infty\}$ defined as

$$U^{\mu}(z) = \int \log \frac{1}{|z-w|} \, \mathrm{d}\mu(w).$$

As a matter of fact, $\mathscr{P}_{\log}(\mathbb{C})$ is the largest space of probability measures with a well-defined potential [BLW15]. Furthermore a sequence (μ_n) in $\mathscr{P}_{\log}(\mathbb{C})$ converges to μ log-weakly if and only if the corresponding sequence of potential functions (U^{μ_n}) converges to U^{μ} as distributions (Lemma 2.1, Lemma 2.5 and Proposition 2.17). Hence arguably the log-weak convergence is the right topology on $\mathscr{P}_{\log}(\mathbb{C})$ to do potential theory.

We can characterize the log-weak closure of the set of equilibrium measures of compact sets with capacity one as the set of measures with negative potential.

Theorem C (Theorem 4.7)**.** *Let $\mu \in \mathscr{P}_{\log}(\mathbb{C})$. Then μ is the* log*-weak limit of a sequence of equilibrium measures (μ_{K_n}) of compact subsets $K_n \subseteq \mathbb{C}$ with* $\operatorname{cap}(K_n) = 1$ *if and only if $U^{\mu}(z) \leq 0$ for all $z \in \mathbb{C}$. In particular, if μ is invariant under the complex conjugation and $U^{\mu}(z) \leq 0$ for all $z \in \mathbb{C}$ then $\mu \in \mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C})$.*

Using this result it is easy to exhibit measures in $\mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C})$ that are not compactly supported. A first example is the probability measure μ_{FS} induced by the Fubini–Study form

$$\omega_{\mathrm{FS}} = \frac{i \, \mathrm{d}z \wedge \mathrm{d}\bar{z}}{2\pi \, (|z|^2+1)^2},$$

whose potential is the function $U^{\mu_{\mathrm{FS}}}(z) = (-1/2) \log(1+|z|^2) \leq 0$.

As a byproduct, this also allows to check that the set of equilibrium measures of compact sets of capacity one are not dense in $\mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C})$. Indeed let $P_0 = x-1$ and $Q_0 = x^2 - x + 1$. Applying Proposition 4.4 we find

$$U^{\mu_{P_0,Q_0}}\left(\frac{1}{2}\right) = \frac{1}{3} \log\left(\frac{4}{3}\right) > 0, \tag{1.5}$$

and so by Theorem C this measure cannot be log-weakly approximated by equilibrium measures of compact subsets of capacity one.

1.6. **The essential minimum as a linear programming problem.** Let $\mathrm{g}\colon \mathbb{C} \longrightarrow \mathbb{R}$ be a Green function and set

$$\mathcal{D}(\mathrm{g}) = \sup \left\{ \inf_{z\in\mathbb{C}} \left(\mathrm{g}(z) - \sum_{i=1}^{k} a_i \log \left|P_i(z)\right| \right) \middle| k \in \mathbb{Z}_{\geq 0}, a_i \in \mathbb{R}_{\geq 0}, P_i \in \mathbb{Z}[x] \setminus \{0\} \right\},$$

$$\mathcal{P}(\mathrm{g}) = \inf \left\{ \int \mathrm{g}\, \mathrm{d}\mu \middle| \mu \in \mathscr{P}_{\log}(\mathbb{C}) \text{ conjugation invariant and } \int \log|Q|\, \mathrm{d}\mu \geq 0 \text{ for all } Q \in \mathbb{Z}[x] \setminus \{0\} \right\}.$$

Combining the lower and upper bounds in Section 1.3 and the characterization of $\mathscr{P}^{\mathbb{Z}}_{\log}(\mathbb{C})$ in Theorem B we obtain

$$\mathcal{D}(\mathrm{g}) \leq \mathrm{ess}(h_{\mathrm{g}}) \leq \mathcal{P}(\mathrm{g}). \tag{1.6}$$

A key observation is that:

1. $\mathcal{P}(\mathrm{g})$ is a linear programming problem, in the sense that the objective function and the constraints are all linear functionals on μ,
2. $\mathcal{D}(\mathrm{g})$ is the dual problem of $\mathcal{P}(\mathrm{g})$ in the sense of linear programming.

These problems are akin to those considered by Smyth [Smy84], Smith [Smi24] and Orloski, Sardari and Smith [OSS24] in their works on totally real algebraic integers of small trace, which were indeed a major source of inspiration for us.

The previous observation can be made rigorous (Appendix A), which gives the weak duality property $\mathcal{D}(\mathrm{g}) \leq \mathcal{P}(\mathrm{g})$ in accordance to (1.6). Our main result is the following strong duality theorem showing the equality between these terms.

Theorem D (Theorem 6.5)**.** *We have* $\mathcal{D}(\mathrm{g}) = \mathrm{ess}(h_{\mathrm{g}}) = \mathcal{P}(\mathrm{g})$.

Assuming g is subharmonic and $\mathrm{g}(z) = \log|z| + a + o(1)$ as $z \longrightarrow \infty$, the equality $\mathcal{D}(\mathrm{g}) = \mathrm{ess}(\mathrm{g})$ is a particular case of a theorem of Ballaÿ [Bal21, Theorem 1.2], see Remark 6.11.

1.7. **The computability of the essential minimum.** Our closing point concerns the computability of the essential minimum. To place this problem into context, we first recall the known approximations for the specific heights we mentioned before.

In [Zag93] Zagier obtained a lower bound for the essential minimum of the Zhang–Zagier height which was later improved by Doche [Doc01a, Doc01b]. Doche also obtained an upper bound, giving

$$0.248247 \leq \mathrm{ess}(h_{\mathrm{ZZ}}) \leq 0.254437.$$

In [Löb17] Löbrich obtained a lower bound for the essential minimum of the Faltings height, and both lower and upper bounds were obtained in [BMR18]. This gave

$$-0.748629 \leq \mathrm{ess}(h_{\mathrm{F}}) \leq -0.748622.$$

The above are the best known bounds for these essential minima, and actually no practical algorithm is known to approximate them up to any arbitrary precision. Indeed, after a few iterations the methods that produce these bounds reach a point where it is unclear how to continue, due to the enormous size of the search space and the lack of an efficient criterion to find the optimal direction, see for instance the discussions in [Zag93, Section 3] and [Doc01a, Section 5] for the Zhang–Zagier height and [BMR18, Section 8] for the Faltings height.

As an application of our results we show that these methods do converge towards these essential minima. Indeed, for each of these heights we can reduce both $\mathcal{D}(\mathrm{g})$ and $\mathcal{P}(\mathrm{g})$ to countable subsets (Propositions 6.9 and Theorem 4.5) allowing to set up a theoretical algorithm that produces sequences of lower and upper bounds converging to the same limit. When these bounds are closer than the given precision, it has found the required approximation and stops.

Loosely speaking, a real number λ is *computable* if there exists an algorithm that is able to compute a rational approximation of λ up to any given precision. We also say that a Green function $\mathrm{g}\colon \mathbb{C} \longrightarrow \mathbb{R}$ is *computable* if the asymptotics of g can be effectively controlled and there exists an algorithm that can compute rational approximations of g of arbitrary precision within any bounded rectangle, see Section 7 for the formal definitions.

The theoretical algorithm described before can be extended to this more general situation and leads to the following result.

Theorem E (Theorem 7.6)**.** *Let* $\mathrm{g}\colon \mathbb{C} \longrightarrow \mathbb{R}$ *be a computable Green function. Then* $\operatorname{ess}(h_{\mathrm{g}})$ *is a computable real number.*

Ultimately we would like to have an practical algorithm to compute the essential minimum of specific height functions like those of Zhang–Zagier and Faltings. The fact that it can be reached from both sides through linear programming might suggest strategies allowing to produce numerical approximations with arbitrary precision.

Acknowledgments. We thank François Ballaÿ, Norman Levenberg, Mayuresh Londhe, Joaquim Ortega Cerdà, Cristóbal Rojas and Michał Szachniewicz for illustrating discussions, and specially Nuno Hultberg for sharing with us the example in Section 6.4. We also thank the organizers of the conference "Arithmetic and algebraic geometry week" held at Universitatea Alexandru Ioan Cuza in Iaşi on September 2025, where part of this work was done.

2. POTENTIAL THEORY AND LIMIT DISTRIBUTION OF ALGEBRAIC INTEGERS

2.1. **Potential theory on the complex plane and the Fekete–Szegő theorem.** Let $\mathscr{P}(\mathbb{C})$ be the set of all probability Borel measures on $\mathbb{C}$ and let $\mathscr{P}_{\log}(\mathbb{C})$ be the subspace of those that can integrate $\log^+|z|$, i.e.

$$\mathscr{P}_{\log}(\mathbb{C}) \coloneqq \left\{\mu \in \mathscr{P}(\mathbb{C}) \,\middle|\, \int \log^+|z|\,\mathrm{d}\mu < \infty\right\}.$$

Let $\mathscr{P}_c(\mathbb{C})$ be the set of compactly supported measures. Then $\mathscr{P}_c(\mathbb{C}) \subseteq \mathscr{P}_{\log}(\mathbb{C})$.

Potential theory of measures with non-necessarily compact support has been developed for some time, see for example [BLW15],[OSW19]. We recall the basic facts. For any $\mu \in \mathscr{P}_{\log}(\mathbb{C})$, we define its *potential function* as

$$U^{\mu}\colon \mathbb{C} \longrightarrow \mathbb{R} \cup \{+\infty\}, \quad U^{\mu}(z) \coloneqq \int \log \frac{1}{|w-z|}\,\mathrm{d}\mu(w).$$

The value $U^{\mu}(z)$ is well-defined in $\mathbb{R} \cup \{+\infty\}$ because the negative part

$$\int_{|w-z|\geq 1} -\log|w-z|\,\mathrm{d}\mu(w)$$

is finite.

Lemma 2.1. *Let* $\mu \in \mathscr{P}_{\log}(\mathbb{C})$*. Then* $-U^{\mu}$ *is subharmonic and hence locally integrable.*

Proof. The first statement is [BLW15, Lemma 3.2] and the second is a classical result on subharmonic functions. □

To be on the safe side, we next check that Fubini–Tonelli theorem can be applied to potentials of measures in $\mathscr{P}_{\log}(\mathbb{C})$.

Lemma 2.2. *Let* $\mu_1, \mu_2 \in \mathscr{P}_{\log}(\mathbb{C})$*. Then*

$$\iint \log|w-z|^{-1}\,\mathrm{d}\mu_1(w)\mathrm{d}\mu_2(z) = \iint \log|w-z|^{-1}\,\mathrm{d}\mu_2(z)\mathrm{d}\mu_1(w).$$

In other words, if U_1 *and* U_2 *are the potentials of* μ_1 *and* μ_2*, then*

$$\int U_1 \mathrm{d}\mu_2 = \int U_2 \mathrm{d}\mu_1.$$

Proof. We write $\log^-|z| = -\min\{0, \log|z|\}$ so that $\log|z| = \log^+|z| - \log^-|z|$. We first check that

$$\iint \log^+|w-z|\,\mathrm{d}\mu_1(w)\mathrm{d}\mu_2(z) = \iint \log^+|w-z|\,\mathrm{d}\mu_2(z)\mathrm{d}\mu_1(w) < \infty. \tag{2.7}$$

Indeed, using that $\mu_1 \in \mathscr{P}_{\log}(\mathbb{C})$ we have that

$$\int \log^+|w-z|\,\mathrm{d}\mu_1(w) \le \int \log(2) + \log^+|w| + \log^+|z|\,\mathrm{d}\mu_1(w) \le A + \log^+|z|$$

for some real number $A > 0$. Since $\mu_2 \in \mathscr{P}_{\log}(\mathbb{C})$, we deduce

$$\iint \log^+|w-z|\,\mathrm{d}\mu_1(w)\mathrm{d}\mu_2(z) < \infty$$

and equation (2.7) follows from the classical Fubini–Tonelli theorem. We turn to $\log^-|w-z|$. If one of the integrals

$$\iint \log^-|w-z|\,\mathrm{d}\mu_1(w)\mathrm{d}\mu_2(z), \ \iint \log^-|w-z|\,\mathrm{d}\mu_2(z)\mathrm{d}\mu_1(w)$$

is finite then by the Fubini–Tonelli theorem both integrals are finite and agree. If both are ∞ then of course they also agree. Since the positive part is finite, we conclude. ☐

Lemma 2.3. *Let* $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ *and* $\varphi \in C_c^\infty(\mathbb{C})$*. Then*

$$\iint \varphi(w)\log|z-w|^{-1}\,\mathrm{d}\mu(z)\mathrm{d}\lambda(w) = \iint \varphi(w)\log|z-w|^{-1}\,\mathrm{d}\lambda(w)\mathrm{d}\mu(z),$$

where λ *is the Lebesgue measure on* $\mathbb{C}$*.*

Proof. It follows from the fact that

$$z \longmapsto \int \varphi(w)\log|z-w|^{-1}\mathrm{d}\lambda(w)$$

is a continuous function of logarithmic growth, hence it is absolutely integrable by $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ and the Fubini–Tonelli theorem applies. ☐

As in the case of compactly supported measures, we recover the measure from its potential. We denote by Δ the distributional Laplacian.

Lemma 2.4. *Let* $\mu \in \mathscr{P}_{\log}(\mathbb{C})$*. Then* $-\Delta U^\mu = 2\pi\mu$ *as distributions.*

Proof. Let $\varphi \in C_c^\infty(\mathbb{C})$ be a smooth function with compact support. For smooth functions we abuse notation and write $\Delta\varphi = \Delta(\varphi)\lambda$, where $\Delta(\varphi)$ is a the Laplacian of φ as a function and λ is the Lebesgue measure in $\mathbb{C}$. Then

$$\begin{aligned}(-\Delta U^\mu)(\varphi) &= \int -U^\mu(z)\Delta(\varphi)(z)\,\mathrm{d}\lambda(z)\\ &= \iint \log|z-w|\Delta(\varphi)(z)\,\mathrm{d}\mu(w)\mathrm{d}\lambda(z)\\ &= \iint \log|z-w|\Delta(\varphi)(z)\,\mathrm{d}\lambda(z)\mathrm{d}\mu(w)\\ &= \int 2\pi\varphi(w)\mathrm{d}\mu(w)\\ &= 2\pi\mu(\varphi).\end{aligned}$$

Here the third equality is Lemma 2.3, and the fourth equality is a classical application of Green's formula (e.g. as in the proof of [Ran95, Theorem 3.7.4])

$$\int \log|z-w|\,\mathrm{d}(\Delta\varphi)(z) = 2\pi\varphi(w).$$

□

We quote a result that will be useful later.

Lemma 2.5. *Let $f, f_n \colon \mathbb{C} \longrightarrow \mathbb{R}\cup\{-\infty\}$, $n \geq 1$, be subharmonic functions. Then f_n converges to f in L^1_{loc} if and only if f_n converges to f as distributions.*

Proof. [Hör07, Theorem 3.2.13]. □

We next recall the definition of capacity and the Fekete–Szegő theorem. The *energy* of a measure $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ as is defined as

$$I(\mu) = \int U^\mu\,\mathrm{d}\mu = \iint -\log|w-z|\,\mathrm{d}\mu(w)\mathrm{d}\mu(z) \in \mathbb{R}\cup\{+\infty\}.$$

Let $K \subseteq \mathbb{C}$ be a compact subset, and let $\mathscr{P}(K)$ be the space of all probability measures supported on K. We say K is *polar* if $I(\mu) = +\infty$ for all $\mu \in \mathscr{P}(K)$.

Now assume K is non-polar, then there exists a unique measure $\mu_K \in \mathscr{P}(K)$ such that $I(\mu_K) = \inf_{\mu\in\mathscr{P}(K)} I(\mu)$ [Ran95, Theorem 3.7.6]. This μ_K is called the *equilibrium measure* of K, and the *capacity* of K is defined as $\operatorname{cap}(K) \coloneqq e^{-I(\mu_K)}$.

Let α be an algebraic number and $O(\alpha) \coloneqq \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\cdot\alpha$ be the Galois orbit of α, that is the complete set of conjugates of α. Let $S \subseteq \mathbb{C}$ be a subset, we say α is *totally in* S if $O(\alpha) \subseteq S$. We denote by $\delta_{O(\alpha)}$ the associated discrete measure

$$\delta_{O(\alpha)} = \frac{1}{\deg(\alpha)}\sum_{\beta\in O(\alpha)}\delta_\beta.$$

Theorem 2.6 (Fekete–Szegő [Fek23, FS55])**.** *Let $K \subseteq \mathbb{C}$ be a compact subset. Then*

(1) *if $\operatorname{cap}(K) < 1$, then there exists an open neighborhood U of K, such that there are only finitely many algebraic integers totally in U.*
(2) *if $\operatorname{cap}(K) \geq 1$ and K is invariant under complex conjugation, then for any open neighborhood U of K, there are infinitely many algebraic integers totally in U.*

Remark 2.7. In the critical case $\operatorname{cap}(K) = 1$, let (α_n) be a sequence of algebraic integers such that $O(\alpha_n) \subseteq K_{1/n} \coloneqq \{z \mid |z - w| < 1/n \text{ for some } w \in K\}$. Then the sequence of measures $\delta_{O(\alpha_n)}$ converges to the equilibrium measure μ_K [Rum99, Theorem 1].

2.2. **Smith and Orloski–Sardari theorems.** When $\operatorname{cap}(K) \geq 1$, by Fekete–Szegő there will be infinitely many algebraic integers totally contained in any open neighborhood of K and we would like to know what are the possible accumulation measures describing their asymptotic distribution. For $\operatorname{cap}(K) = 1$ this is answered by a result of Rumely as in Remark 2.7, while for $\operatorname{cap}(K) > 1$, this is answered in the recent breakthrough by Smith and Orloski–Sardari.

We say a sequence (μ_n) in $\mathscr{P}(\mathbb{C})$ *converges weakly* to μ, if we have for any continuous and compactly supported function $f \colon \mathbb{C} \longrightarrow \mathbb{R}$,

$$\lim_{n\to\infty} \int f \, \mathrm{d}\mu_n = \int f \, \mathrm{d}\mu.$$

We can replace "compactly supported" by "bounded" [Bil97, Lemma 2.2].

For compactly supported measures there is a stronger notion of convergence. We say a sequence (μ_n) in $\mathscr{P}_c(\mathbb{C})$ *converges properly* to $\mu \in \mathscr{P}_c(\mathbb{C})$, if it converges weakly and there exists a compact subset $K \subseteq \mathbb{C}$ such that $\operatorname{supp}(\mu_n) \subseteq K$ for all n. Note that this is equivalent to

$$\lim_{n\to\infty} \int f \, \mathrm{d}\mu_n = \int f \, \mathrm{d}\mu \tag{2.8}$$

for any continuous function $f \colon \mathbb{C} \longrightarrow \mathbb{R}$.

A reformulation of the result by Smith [Smi24, Theorem 1.5] and Orloski–Sardari [OS24, Theorem 1.2] is

Theorem 2.8 (Smith, Orloski–Sardari). *Assume $\mu \in \mathscr{P}_c(\mathbb{C})$ is invariant under complex conjugation. Then the following are equivalent*

1. *there exists a sequence of distinct algebraic integers (α_n) such that $\delta_{O(\alpha_n)}$ converges properly to μ,*
2. $\displaystyle\int \log|Q| \, \mathrm{d}\mu \geq 0$ *for any $Q \in \mathbb{Z}[x]$.*

Remark 2.9. Let $h \colon \overline{\mathbb{Q}} \longrightarrow \mathbb{R}$ be the standard Weil height function. Suppose (α_n) is a sequence of distinct algebraic integers such that $\delta_{O(\alpha_n)}$ converges properly to $\mu \in \mathscr{P}_c(\mathbb{C})$, then

$$\lim_{n\to\infty} h(\alpha_n) = \lim_{n\to\infty} \int \log^+|z| \, \mathrm{d}\delta_{O(\alpha_n)} = \int \log^+|z| \, \mathrm{d}\mu < \infty.$$

This forces $\deg(\alpha_n)$ to converge to ∞ because otherwise it would contradict the Northcott property.

Remark 2.10. The original results of Smith and Orloski–Sardari are in fact stronger and are more of Fekete–Szegő style, in the sense that they prove that for any $\varepsilon > 0$ there exists a sequence of algebraic integers (α_n) that are totally contained in

$$\operatorname{supp}(\mu)_\varepsilon \coloneqq \{z \in \mathbb{C} \mid |z - w| < \varepsilon \text{ for some } w \in \operatorname{supp}(\mu)\}.$$

and such that $\delta_{O(\alpha_n)}$ converges weakly to μ. This is much stronger than just proper convergence. Besides, when μ is supported on $\mathbb{R}$, they prove that the approximating algebraic integers α_n can be chosen to be totally real.

2.3. $\log$**-weak convergence of measures.** We say a function $f\colon \mathbb{C} \longrightarrow \mathbb{R}$ has *logarithmic growth*, if there exist constants $A, B \geq 0$ such that for all $z \in \mathbb{C}$, we have $|f(z)| \leq A + B\log^+|z|$.

Definition 2.11. Let $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ and (μ_n) be a sequence in $\mathscr{P}_{\log}(\mathbb{C})$. We say μ_n *converges* $\log$*-weakly* to μ, if for any continuous function f with logarithmic growth, we have

$$\lim_{n\to\infty} \int f \,\mathrm{d}\mu_n = \int f \,\mathrm{d}\mu.$$

Definition 2.12. Let $\varphi\colon \mathbb{C} \longrightarrow \mathbb{R}_{\geq 0}$ be a continuous function. We say that a sequence (μ_n) is φ*-tight*, if

$$\lim_{R\to\infty} \sup_{n\in\mathbb{N}} \int_{|z|\geq R} \varphi \,\mathrm{d}\mu_n = 0.$$

In particular, we say that a sequence (μ_n) is *tight* if it is 1-tight, and we say that it is $\log$*-tight*, if it is $\log^+$-tight.

The next two lemmas are elementary and follow easily from the definition.

Lemma 2.13. *Let $\varphi, \psi\colon \mathbb{C} \longrightarrow \mathbb{R}_{\geq 0}$ be continuous functions with $\psi = o(\varphi)$ as $|z| \longrightarrow \infty$. Suppose we have a sequence (μ_n) such that*

$$\sup_{n\in\mathbb{N}} \int \varphi \,\mathrm{d}\mu_n < \infty.$$

Then (μ_n) is ψ-tight.

Lemma 2.14. *Let $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ and (μ_n) be a sequence in $\mathscr{P}_{\log}(\mathbb{C})$ that converges weakly to μ. Then, the following assertions are equivalent:*

1. *the sequence (μ_n) converges* $\log$*-weakly to μ;*
2. *the sequence (μ_n) is log-tight;*
3. *we have that*

$$\lim_{n\to\infty} \int \log^+|z| \,\mathrm{d}\mu_n(z) = \int \log^+|z| \,\mathrm{d}\mu(z).$$

We record here the following fact for future use.

Lemma 2.15. *Let (μ_n) be a sequence in $\mathscr{P}(\mathbb{C})$ that converges* $\log$*-weakly to μ. Then, for all polynomials $Q(x) \in \mathbb{C}[x]$, we have that*

$$\int \log|Q| \,\mathrm{d}\mu \geq \liminf_{n\to\infty} \int \log|Q| \,\mathrm{d}\mu_n.$$

Proof. Using the monotone convergence theorem and (2.8), we have that

$$\begin{aligned} \int \log|Q| \,\mathrm{d}\mu &= \lim_{M\to\infty} \int \max\{-M, \log|Q|\} \,\mathrm{d}\mu \\ &= \lim_{M\to\infty} \lim_{n\to\infty} \int \max\{-M, \log|Q|\} \,\mathrm{d}\mu_n \\ &\geq \lim_{M\to\infty} \liminf_{n\to\infty} \int \log|Q| \,\mathrm{d}\mu_n, \\ &= \liminf_{n\to\infty} \int \log|Q| \,\mathrm{d}\mu_n \end{aligned}$$

the second equality follows from the fact that $H^M(z) = \max\big\{-M, \log|Q(z)|\big\}$ is continuous of logarithmic growth. ☐

It will be useful in what follows to be able to do diagonal arguments when dealing with weak convergence and log-weak convergence.

Lemma 2.16. *Consider a measure* $\mu \in \mathscr{P}(\mathbb{C})$, *and a sequence* (μ_n) *and a double sequence* $(\mu_{n,m})$ *in* $\mathscr{P}(\mathbb{C})$. *Assume that*

- μ_n *converges weakly to* μ *as* $n \longrightarrow \infty$,
- *for each* n, $\mu_{n,m}$ *converges weakly to* μ_n *as* $m \longrightarrow \infty$.

Then there is a diagonal subsequence $(\mu_{n_i,m_i})_{i\in\mathbb{N}}$ *that converges weakly to* μ *as* $i \longrightarrow \infty$.

Moreover, the same holds if we consider μ, μ_n *and* $\mu_{n,m}$ *in* $\mathscr{P}_{\log}(\mathbb{C})$ *and replace weak convergence with* log*-weak convergence.*

Proof. Weak convergence on $\mathscr{P}(\mathbb{C})$ is metrizable (e.g. see [Bil99, Theorem 6.8], [Pol84, Example IV.22-23]), so the first assertion is just the diagonal argument in a metric space.

Let d_{weak} be a distance in $\mathscr{P}(\mathbb{C})$ whose associated topology is weak convergence. For $\mu, \mu' \in \mathscr{P}_{\log}(\mathbb{C})$, define

$$d_{\log^+}(\mu, \mu') = \left| \int \log^+ |z| \, \mathrm{d}\mu - \int \log^+ |z| \, \mathrm{d}\mu' \right|$$

and $d = \max\{d_{\text{weak}}, d_{\log^+}\}$. Then d is a distance in $\mathscr{P}_{\log}(\mathbb{C})$ and, by Lemma 2.14, its associated topology is log-weak convergence. Thus the second statement is again the diagonal argument in a metric space. □

2.4. **Subharmonic functions and L^1_{loc} convergence.** Let $f, f_n \colon \mathbb{C} \longrightarrow \mathbb{R}$, $n \geq 0$, be functions that are locally integrable. We say f_n converges to f *in* L^1_{loc}, if for any $z \in \mathbb{C}$, there exists an open neighborhood U of x such that

$$\lim_{n\to\infty} \int_U |f_n - f| \, \mathrm{d}\lambda = 0,$$

where λ is the usual Lebesgue measure on the complex plane. In this section we show that the log-weak convergence of measures corresponds exactly to L^1_{loc} convergence of potentials.

Proposition 2.17. *Let* $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ *and* (μ_n) *be a sequence in* $\mathscr{P}_{\log}(\mathbb{C})$. *Then* μ_n *converges* log*-weakly to* μ *if and only if* U^{μ_n} *converges to* U^{μ} *in* L^1_{loc}.

Proof. We start by the direct implication. By Lemmas 2.1 and 2.5 we need only to show that U^{μ_n} converges to U^{μ} as distributions. Taking $\varphi \in C_c^{\infty}(\mathbb{C})$, we have that

$$\begin{aligned}
\lim_{n\to\infty} \int \varphi(z) U^{\mu_n}(z) \, \mathrm{d}\lambda(z) &= \lim_{n\to\infty} \iint \varphi(z) \log|z-w|^{-1} \mathrm{d}\mu_n(w) \, \mathrm{d}\lambda(z) \\
&= \lim_{n\to\infty} \iint \varphi(z) \log|z-w|^{-1} \, \mathrm{d}\lambda(z) \mathrm{d}\mu_n(w) \\
&= \iint \varphi(z) \log|z-w|^{-1} \, \mathrm{d}\lambda(z) \mathrm{d}\mu(w) \\
&= \iint \varphi(z) \log|z-w|^{-1} \, \mathrm{d}\mu(w) \mathrm{d}\lambda(z) \\
&= \int \varphi(z) U^{\mu}(z) \, \mathrm{d}\lambda(z).
\end{aligned}$$

The second and fourth equalities are Lemma 2.3. The third equality follows from the fact that the function

$$w \longmapsto \int \varphi(z)\log|z-w|^{-1}\mathrm{d}\lambda(z)$$

is continuous of logarithmic growth and the hypothesis of log-weak convergence.

Now we prove the reverse implication. Assume that U^{μ_n} converges to U^{μ} in L^1_{loc}. It follows that ΔU^{μ_n} converges to ΔU^{μ} as distributions. By Lemma 2.4, we conclude that μ_n converges to μ as distributions which implies that μ_n converges to μ weakly. By Lemma 2.14 it only remains to be proven that

$$\lim_{n\to\infty}\int \log^+|z|\,\mathrm{d}\mu_n(z) = \int \log^+|z|\,\mathrm{d}\mu(z). \tag{2.9}$$

Let $\nu = \frac{1}{\pi}\lambda_{B_1}$ be the Lebesgue measure on the unit ball normalized so that it has total mass one. Let U^{ν} be its potential. Then $-U^{\nu}(z) - \log^+|z|$ is a continuous and bounded function on $\mathbb{C}$. Since μ_n converges to μ weakly, equation (2.9) is equivalent to

$$\lim_{n\to\infty}\int U^{\nu}(z)\,\mathrm{d}\mu_n(z) = \int U^{\nu}(z)\,\mathrm{d}\mu(z). \tag{2.10}$$

Using Lemma 2.2 and the convergence in L^1_{loc}, we have that

$$\lim_{n\to\infty}\int U^{\nu}\mathrm{d}\mu_n = \lim_{n\to\infty}\int U^{\mu_n}\,\mathrm{d}\nu = \int U^{\mu}\,\mathrm{d}\nu = \int U^{\nu}\mathrm{d}\mu,$$

obtaining (2.10). □

3. The sweetened truncation

The goal of this section is to define the sweetened truncation of a measure and prove related technical lemmas and propositions. They will be used in the next sections.

3.1. **Sweetened truncation.** Let $\mu \in \mathscr{P}_{\log}(\mathbb{C})$. For any $R > 1$, we want to produce a new measure μ_R^{sw} supported on the ball $B_R \coloneqq \{|z| \leq R\}$ such that μ_R^{sw} converges to μ log-weakly as R goes to ∞. A first candidate for this purpose is the *naïve truncation*

$$\mu'_R \coloneqq \mu|_{B_R} + (1-m_R)\cdot\lambda_{S_R},$$

where $m_R \coloneqq \mu(B_R)$ and λ_{S_R} is the equilibrium measure on $S_R \coloneqq \{|z| = R\}$.

As we will see in Section 4, we are interested in measures that satisfy

$$\int \log|Q|\,\mathrm{d}\mu \geq 0 \quad \text{for all } Q \in \mathbb{Z}[x]\setminus\{0\}. \tag{3.11}$$

We want to refine the naïve truncation in such a way that, if μ satisfies condition (3.11), then the measure μ_R^{sw} still satisfies the same condition. The following definition is inspired by Smith's concept of *sweetened measure*, see [Smi24, Definition 5.5].

Definition 3.1. Let $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ be a probability measure. Let $R > 1$ and set

$$T_R \coloneqq \int_{|z|>R}\log^+|z|\mathrm{d}\mu,\ L_R \coloneqq (1-m_R)\log 2 + T_R,\ \eta_R \coloneqq \frac{\log R}{\log R + L_R}.$$

We define the *sweetened truncation* μ_R^{sw} as

$$\mu_R^{\mathrm{sw}} \coloneqq \eta_R\cdot\mu|_{B_R} + (1-m_R\eta_R)\cdot\lambda_{S_R}. \tag{3.12}$$

We note that Jensen's formula implies that the potential of λ_{S_R} is given by

$$U^{\lambda_{S_R}}(z) = -\log\max\big\{|z|, R\big\}. \tag{3.13}$$

Proposition 3.2. *Let* $\mu \in \mathscr{P}_{\log}(\mathbb{C})$*. Then*

1. μ_R^{sw} *converges to* μ log*-weakly as* R *goes to* ∞*,*
2. *for all* $R > 1$*, if* μ *is conjugation invariant, then* μ_R^{sw} *is conjugation invariant as well,*
3. *for all* $R > 1$*, the inequality* $U^{\mu_R^{\mathrm{sw}}} \le \eta_R U^{\mu}$ *holds. In particular, if* $\int \log|Q|\,\mathrm{d}\mu \ge 0$ *for some* $Q \in \mathbb{Z}[x]\setminus\{0\}$*, then* $\int \log|Q|\,\mathrm{d}\mu_R^{\mathrm{sw}} \ge 0$ *as well.*

We will prove this statement after the following lemma.

Lemma 3.3. *The naïve truncation* μ'_R *satisfies* $U^{\mu'_R} \le U^{\mu} + L_R$*.*

Proof. Assume $R > 1$. By equation (3.13),

$$U^{\mu'_R}(z) - U^{\mu}(z) = \int_{|w|>R} \log|z-w|\,\mathrm{d}\mu(w) - (1-m_R)\log\max\big\{|z|, R\big\}.$$

Assume $|z| < R$. Then for all w with $|w| > R$, we have $|z-w| \le |z| + |w| \le 2|w|$ and hence

$$\begin{aligned}
U^{\mu'_R}(z) - U^{\mu}(z) &= \int_{|w|>R} \log|z-w|\,\mathrm{d}\mu(w) - (1-m_R)\log R\\
&\le \int_{|w|>R} \log|2w|\,\mathrm{d}\mu(w)\\
&\le L_R.
\end{aligned}$$

Assume now $|z| \ge R$. Then,

$$\begin{aligned}
U^{\mu'_R}(z) - U^{\mu}(z) &= \int_{|w|>R} \log|z-w|\,\mathrm{d}\mu(w) - (1-m_R)\log|z|\\
&= \int_{|w|>R} \log\left|1-\frac{w}{z}\right|\,\mathrm{d}\mu(w)\\
&= \int_{|w|>|z|\ge R} \log\left|1-\frac{w}{z}\right|\,\mathrm{d}\mu(w) + \int_{|z|\ge|w|>R} \log\left|1-\frac{w}{z}\right|\,\mathrm{d}\mu(w)\\
&\le \int_{|w|>|z|\ge R} \log\left|\frac{2w}{R}\right|\,\mathrm{d}\mu(w) + \int_{|z|\ge|w|>R} \log 2\,\mathrm{d}\mu(w)\\
&= \int_{|w|>|z|\ge R} \log\left|\frac{w}{R}\right|\,\mathrm{d}\mu(w) + \int_{|w|>R} \log 2\,\mathrm{d}\mu(w)\\
&\le \int_{|w|>R} \log|w|\,\mathrm{d}\mu(w) + \int_{|w|>R} \log 2\,\mathrm{d}\mu(w)\\
&\le L_R.
\end{aligned}$$

□

Proof of Proposition 3.2. Item (2) follows readily from the definitions. Now we show (1). It is easy to see from (3.12) that μ_R^{sw} converges to μ weakly as R goes to ∞. So by Lemma 2.14, it suffices to test against $\log^+|z|$:

$$\begin{aligned}\int \log^+|z|\,\mathrm{d}\mu_R^{\mathrm{sw}} - \int \log^+|z|\,\mathrm{d}\mu \\ = (1-m_R\eta_R)\log R + (\eta_R-1)\int_{|z|\le R}\log^+|z|\,\mathrm{d}\mu - \int_{|z|>R}\log^+|z|\,\mathrm{d}\mu.\end{aligned}$$

Clearly the last two terms vanish when R goes to ∞. With regards to the first term, we have that

$$(1-m_R\eta_R)\log R = \big((1-m_R)\log R + L_R\big)\frac{\log R}{\log R + L_R}. \tag{3.14}$$

Then, the estimate $0 \le (1-m_R)\log R \le T_R$ and the facts that T_R and L_R are positive and converge to zero when R goes to ∞ imply that $(1-m_R\eta_R)\log R$ also converges to zero.

Finally we prove (3). For the first assertion, we use Lemma 3.3

$$\begin{aligned}U^{\mu_R^{\mathrm{sw}}} &= \eta_R U^{\mu'_R} + (1-\eta_R)U^{\lambda_{S_R}} \\ &\le \eta_R U^{\mu} + \eta_R L_R - (1-\eta_R)\log R \\ &\le \eta_R U^{\mu}.\end{aligned}$$

For the second one, let $a\in\mathbb{Z}$ be the leading coefficient of Q and $\alpha_1,\dots,\alpha_n$ be the roots of Q. Then

$$\begin{aligned}\int \log|Q|\,\mathrm{d}\mu_R^{\mathrm{sw}} &= \log|a| - \sum U^{\mu_R^{\mathrm{sw}}}(\alpha_i) \\ &\ge \eta_R\log|a| - \eta_R\sum U^{\mu}(\alpha_i) \\ &\ge \eta_R\int \log|Q|\,\mathrm{d}\mu \\ &\ge 0.\end{aligned}$$

□

3.2. Sweetened truncation and asymptotically logarithmic functions.

Definition 3.4. A function $\mathrm{g}\colon \mathbb{C}\longrightarrow\mathbb{R}$ is called *asymptotically logarithmic at infinity*, if it obeys the asymptotic

$$\mathrm{g}(z) = \log|z| + o(\log|z|), \quad \text{as } |z|\longrightarrow\infty.$$

A *Green function* is a function $\mathrm{g}\colon\mathbb{C}\longrightarrow\mathbb{R}$ which is continuous, invariant under complex conjugation and asymptotically logarithmic at infinity. Such Green functions arise when considering singular metrics on $\mathcal{O}(1)$, see Section 6.1 for a discussion.

The purpose of this section is to prove the following statement that will play an essential role in the proof of strong duality in Section 5.

Proposition 3.5. *Let* $\mathrm{g}\colon\mathbb{C}\longrightarrow\mathbb{R}$ *be a continuous function that is asymptotically logarithmic at infinity. Let* (μ_n) *be a sequence in* $\mathscr{P}(\mathbb{C})$ *such that*

$$\limsup_{n\to\infty}\int \mathrm{g}\,\mathrm{d}\mu_n < \infty.$$

Then, for all $\varepsilon > 0$ *there exists* $R_0 > 1$*, such that for all* $R \geq R_0$

$$\limsup_{n\to\infty} \int \mathrm{g}\, \mathrm{d}\mu_{n,R}^{\mathrm{sw}} \leq \limsup_{n\to\infty} \int \mathrm{g}\, \mathrm{d}\mu_n + \varepsilon. \tag{3.15}$$

Proof. The hypotheses imply

$$\limsup_{n\to\infty} \int \log^+ |z|\, \mathrm{d}\mu_n < \infty,$$

so we may assume that

$$\sup_{n\in\mathbb{N}} \int \log^+ |z|\, \mathrm{d}\mu_n < \infty. \tag{3.16}$$

Since $1 = o(\log|z|)$ when $|z| \longrightarrow \infty$, we have (μ_n) is tight by Lemma 2.13. Using Prohorov's theorem [Bil99, Theorem 5.1], we can assume, after taking a subsequence if necessary, that there is a probability measure μ_∞ on $\mathbb{C}$ such that μ_n converges weakly to μ_∞. In particular, we know that μ_∞ can integrate $\log^+|z|$, because

$$\int \log^+ |z|\, \mathrm{d}\mu_\infty \leq \limsup_{n\to\infty} \int \log^+ |z|\, \mathrm{d}\mu_n < \infty. \tag{3.17}$$

Set $E_{n,R} \coloneqq \int \mathrm{g}\, \mathrm{d}\mu_{n,R}^{\mathrm{sw}} - \int \mathrm{g}\, \mathrm{d}\mu_n$. In order to prove (3.15) it is enough to show that

$$\limsup_{R\to\infty} \limsup_{n\to\infty} E_{n,R} \leq 0 \tag{3.18}$$

Before the proof of (3.18) we show several auxiliary results. First we have that

$$\lim_{R\to\infty} \limsup_{n\to\infty} \mu_n\{|z| \geq R\} \log R = 0. \tag{3.19}$$

Indeed,

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \mu_n\big\{|z| \geq R\big\} \log R \\ &\leq \mu_\infty\big\{|z| \geq R\big\} \log R \\ &\leq \int_{|z|\geq R} \log^+ |z|\, \mathrm{d}\mu_\infty. \end{aligned}$$

Here the second inequality follows from the Portmanteau theorem [Bil99, Theorem 2.1 (iii)], because μ_n converges weakly to μ_∞ and the set $\big\{|z| \geq R\big\}$ is closed. Since (3.17) implies that the last term vanishes when $R \longrightarrow \infty$, this proves the claim.

Let $L_{n,R}$ and $T_{n,R}$ be the constants appearing in Definition 3.1 for the sweetened truncation of μ_n, $n \geq 1$. There exists $M > 0$ such that

$$\big\{L_{n,R}, T_{n,R} \mid n \geq 1, R > 1\big\} \subseteq [0, M]. \tag{3.20}$$

We see directly from the definitions that $L_{n,R}, T_{n,R} \geq 0$. Also, $T_{n,R}$ is bounded because of (3.16). Since $L_{n,R} \leq T_{n,R} + \log 2$ it is bounded as well.

We have that

$$\limsup_{R\to\infty} \limsup_{n\to\infty} \left((1 - m_{n,R}\eta_{n,R}) \log R - T_{n,R}\right) \leq 0. \tag{3.21}$$

Indeed, since $\log R$ and $L_{n,R}$ are non negative, we see that

$$\begin{aligned}(1-m_{n,R}\eta_{n,R})\log R - T_{n,R} &= \frac{\log R}{\log R + L_{n,R}}\cdot((1-m_{n,R})\log(2R)+T_{n,R}) - T_{n,R}\\ &\leq \big((1-m_{n,R})\log(2R)+T_{n,R}\big) - T_{n,R}\\ &\leq (1-m_{n,R})\log(2R)\\ &\leq \mu_n\big\{|z|\geq R\big\}\log(2R)\end{aligned}$$

We conclude by (3.19).

We next observe that there exists $M'>0$ such that for all $n\geq 1$ and $R>0$

$$(1-m_{n,R}\eta_{n,R})\log R \leq M' \tag{3.22}$$

Holds. This follows from (3.21) and (3.20).

We have the estimate

$$0\leq 1-\eta_{n,R}\leq \frac{M}{\log R}. \tag{3.23}$$

That follows from $1-\eta_{n,R}=\frac{L_{n,R}}{\log R+L_{n,R}}$ and (3.20).

Write $\mathrm{g}=\log^+|z|+\psi$, where $\psi:\mathbb{C}\longrightarrow\mathbb{R}$ is continuous and $\psi=o(\log|z|)$ at infinity. Note that the sequence $(\int\psi\,\mathrm{d}\mu_n)$ is bounded because of (3.16).

We now put together all the previous results. Let $\varepsilon>0$. Since $\psi=o(\log|z|)$, there exists $R_1>1$ such that for all z with $|z|\geq R_0$, we have that $\psi(z)\leq\frac{\varepsilon}{M'}\log|z|$. For $R\geq R_1$ and using the estimate (3.22), we have

$$\begin{aligned}E_{n,R}=&(1-m_{n,R}\eta_{n,R})\int\mathrm{g}\,\mathrm{d}\lambda_{S_R}-\int_{|z|>R}\mathrm{g}\,\mathrm{d}\mu_n-(1-\eta_{n,R})\int_{|z|\leq R}\mathrm{g}\,\mathrm{d}\mu_n\\ =&\big((1-m_{n,R}\eta_{n,R})\log R-T_{n,R}\big)-(1-\eta_{n,R})\int_{|z|\leq R}\log^+|z|\,\mathrm{d}\mu_n\\ &\quad+(1-m_{n,R}\eta_{n,R})\int\psi\,\mathrm{d}\lambda_{S_R}-\int_{|z|>R}\psi\,\mathrm{d}\mu_n-(1-\eta_{n,R})\int_{|z|\leq R}\psi\,\mathrm{d}\mu_n.\\ \leq&\big((1-m_{n,R}\eta_{n,R})\log R-T_{n,R}\big)+\varepsilon-\int_{|z|>R}\psi\,\mathrm{d}\mu_n-(1-\eta_{n,R})\int_{|z|\leq R}\psi\,\mathrm{d}\mu_n.\\ =&\big((1-m_{n,R}\eta_{n,R})\log R-T_{n,R}\big)+\varepsilon-\eta_{n,R}\int_{|z|>R}\psi\,\mathrm{d}\mu_n-(1-\eta_{n,R})\int\psi\,\mathrm{d}\mu_n.\end{aligned}$$

By Lemma 2.13 we know that (μ_n) is ψ-tight. Combining this information with the boundedness of $(\int\psi\,\mathrm{d}\mu_n)$ and the estimates (3.21) and (3.23), we obtain

$$\limsup_{R\to\infty}\limsup_{n\to\infty}E_{n,R}\leq\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, this shows (3.18). □

4. APPROXIMATING MEASURES BY ALGEBRAIC INTEGERS AND BY CAPACITY ONE COMPACT SETS

4.1. **Approximating measures by algebraic integers in the non-compact setting.** The aim of this section is to prove the following result.

Theorem 4.1. *Let* $\mu \in \mathscr{P}_{\log}(\mathbb{C})$*. Then* $\mu \in \mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C})$ *if and only if it is invariant under the complex conjugation and*

$$\int \log |Q| \, \mathrm{d}\mu \geq 0 \quad \textit{for all } Q \in \mathbb{Z}[x] \setminus \{0\}. \tag{4.24}$$

Combining this result with Theorem 2.8 we readily obtain the following consequence for measures with compact support.

Corollary 4.2. *Let* $\mu \in \mathscr{P}_c(\mathbb{C})$*. Then, the following are equivalent:*

1. *there exists a sequence of distinct algebraic integers* (α_n) *such that* $\delta_{O(\alpha_n)}$ *converges properly to* μ*;*
2. *there exists a sequence of distinct algebraic integers* (α_n) *such that* $\delta_{O(\alpha_n)}$ log*-weakly converges to* μ*.*

Hence Theorem 4.1 can be considered as an extension of the Smith–Orloski–Sardari theorem to non-compactly supported measures that can integrate functions of logarithmic growth. This further justifies our claim that the notion of log-weak convergence is the right one for non-compactly supported measures.

On the other hand, weak convergence is not appropriate in this setting because any probability measure that is invariant under the complex conjugation is the weak limit of a sequence of Galois orbits of distinct algebraic integers. In particular such weak limits do not necessarily satisfy condition (4.24), see Section 6.4 for an example in this direction.

Remark 4.3. If (α_n) is a sequence of distinct algebraic integers such that $O(\alpha_n)$ converges log-weakly to μ then $\deg(\alpha_n)$ converges to ∞, as it can be seen using the argument in Remark 2.9.

Proof of Theorem 4.1. First assume that $\mu \in \mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C})$. Then this measure is clearly invariant under the complex conjugation, and we next prove that it satisfies condition (4.24) by adapting to our setting a classical argument appearing for instance in [Ser19, Lemma 1.3.4]. Let (α_n) be a sequence of distinct algebraic integers such that $O(\alpha_n)$ converges log-weakly to μ and denote by $P_n(x) \in \mathbb{Z}[x]$ the minimal polynomial of α_n. Let $Q \in \mathbb{Z}[x]$ be a nonzero polynomial. By Lemma 2.15 we have that

$$\int \log |Q| \mathrm{d}\mu \geq \liminf_n \frac{1}{\deg(\alpha_n)} \sum_{\beta \in O(\alpha_n)} \log |Q(\beta)|.$$

Let $\operatorname{Res}(P_n, Q)$ be the resultant of P_n and Q. We have that $\operatorname{Res}(P_n, Q) = \prod_{\beta \in O(\alpha_n)} Q(\beta)$ because P_n is monic [BG06, Section B.1.13]. Hence

$$\sum_{\beta \in O(\alpha_n)} \log |Q(\beta)| = \log |\operatorname{Res}(P_n, Q)| \geq 0 \quad \text{for all } n \gg 0,$$

the last inequality being due to the fact that $\operatorname{Res}(P_n, Q)$ is a nonzero integer when n is big enough.

Conversely, let $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ invariant under the complex conjugation and satisfying condition (4.24). For $R > 1$ consider the sweetened truncation μ_R^{sw} (Definition 3.1). By Proposition 3.2(3,2) we can apply Theorem 2.8 to μ_R^{sw} to conclude that it can be approximated properly (and in particular log-weakly) by a sequence of Galois orbits of distinct algebraic integers. Then using Proposition 3.2(1) and Lemma 2.16 we can find a diagonal subsequence that converges to μ log-weakly, as desired. □

This result shows that $\mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C})$ is a convex subset of $\mathscr{P}_{\log}(\mathbb{C})$. We next construct an explicit family of measures that form a countable dense subset it. To this end, for each pair P, Q of nonconstant coprime polynomials with integer coefficients we consider the map

$$\varphi_{P,Q} \colon \mathbb{P}^1(\mathbb{C}) \longrightarrow \mathbb{P}^1(\mathbb{C})$$

induced by the rational function $P^{\deg(Q)+1}/Q^{\deg(P)}$. It is finite of degree $\deg(P)(\deg(Q)+1)$.

Given a function $f \colon \mathbb{P}^1(\mathbb{C}) \longrightarrow \mathbb{R}$ we consider its *pushforward* $(\varphi_{P,Q})_* f \colon \mathbb{P}^1(\mathbb{C}) \longrightarrow \mathbb{R}$ defined as

$$(\varphi_{P,Q})_* f(z) = \sum_{w \in \varphi_{P,Q}^{-1}(z)} e_w f(w)$$

with e_w the ramification index of $\varphi_{P,Q}$ at a point w. If f is continuous then this is also the case for its pushforward, by the continuity of the roots of polynomials with respect to variations of their coefficients. Hence for a measure μ on $\mathbb{P}^1(\mathbb{C})$ we can define its *pullback measure* $\varphi_{P,Q}^* \mu$ on $\mathbb{P}^1(\mathbb{C})$ by the rule

$$\int f \, \mathrm{d}(\varphi_{P,Q}^* \mu) := \int (\varphi_{P,Q})_* f \, \mathrm{d}\mu \quad \text{for every continuous } f \colon \mathbb{P}^1(\mathbb{C}) \longrightarrow \mathbb{R}.$$

The supports of these measures are related by $\operatorname{supp}(\varphi_{P,Q}^* \mu) = \varphi_{P,Q}^{-1}(\operatorname{supp}(\mu))$. We have that $\varphi_{P,Q}(\infty) = \infty$, and so if the measure μ is supported on $\mathbb{C} \simeq \mathbb{P}^1(\mathbb{C}) \setminus \{\infty\}$ then this is also the case for its pullback.

Finally we consider the measure on $\mathbb{C}$ defined as

$$\mu_{P,Q} = \frac{(\varphi_{P,Q})^* \lambda_{S_1}}{\deg(P)(\deg(Q)+1)} \tag{4.25}$$

with λ_{S_1} the equilibrium measure of the unit circle.

Proposition 4.4. *We have that $\mu_{P,Q}$ is a probability measure supported on the compact subset*

$$\left\{ z \,\middle|\, |P(z)|^{\deg(Q)+1} = |Q(z)|^{\deg(P)} \right\} \tag{4.26}$$

with potential function $U^{\mu_{P,Q}} = -\max\left\{ \dfrac{\log|P|}{\deg(P)}, \dfrac{\log|Q|}{\deg(Q)+1} \right\}$.

Proof. The support of $\mu_{P,Q}$ is the preimage of S_1 with respect to the map $\varphi_{P,Q}$, which coincides with (4.26). Next set for short $d = \deg(P)$ and $e = \deg(Q)$. Then for each $z \in \mathbb{C}$ we have

$$U^{\mu_{P,Q}}(z) = \int \log|z-w|^{-1} \mathrm{d}\mu_{P,Q}(w) = \frac{-1}{d\,(e+1)} \int \sum_{\varphi_{P,Q}(w)=y} e_w \log|z-w| \, \mathrm{d}\lambda_{S_1}(y). \tag{4.27}$$

We have

$$\sum_{w \in \varphi_{P,Q}^{-1}(z)} e_w \log|z-w| = \log\left| \prod_{w \in \varphi_{P,Q}^{-1}(z)} (z-w)^{e_w} \right| = \log|P(z)^{e+1} - y\, Q(z)^d|$$

and so the formula for the potential follows from (4.27) together with Jensen's formula. □

Theorem 4.5. *For all $P, Q \in \mathbb{Z}[X]$, with P non constant, and P, Q having no common roots, we have that $\mu_{P,Q} \in \mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C})$. Moreover the family*

$$\{\mu_{P,Q} \mid P, Q \in \mathbb{Z}[X] \setminus \mathbb{Z} \text{ with } P \neq Q \text{ monic and irreducible}\} \tag{4.28}$$

is a countable dense subset of $\mathscr{P}^{\mathbb{Z}}_{\log}(\mathbb{C})$.

Proof. Let $P,Q \in \mathbb{Z}[X]\setminus\{0\}$ coprime and take an primitive irreducible polynomial $F\in\mathbb{Z}[x]$ with leading coefficient a and roots $\alpha_1,\dots,\alpha_n$. Using Proposition 4.4 we get

$$\begin{aligned}\int \log|F|\,\mathrm{d}\mu_{P,Q} &= \log|a| - \sum_i U^{\mu_{P,Q}}(\alpha_i)\\ &= \log|a| + \sum_i \max\left\{\frac{\log|P(\alpha_i)|}{\deg(P)}, \frac{\log|Q(\alpha_i)|}{\deg(Q)+1}\right\}\\ &\geq \max\left\{\log|a| + \sum_i \frac{\log|P(\alpha_i)|}{\deg(P)}, \log|a| + \sum_i \frac{\log|Q(\alpha_i)|}{\deg(Q)+1}\right\}\\ &\geq \max\left\{\frac{\log|\operatorname{Res}(F,P)|}{\deg(P)}, \frac{\log|\operatorname{Res}(F,Q)|}{\deg(Q)+1}\right\}\\ &\geq 0,\end{aligned}$$

because $\operatorname{Res}(F,P)$ and $\operatorname{Res}(F,Q)$ are integers that cannot be both zero as P,Q are coprime and F is irreducible. Since $\mu_{P,Q}$ is invariant under complex conjugation, Theorem 4.1 implies that $\mu_{P,Q}\in\mathscr{P}^{\mathbb{Z}}_{\log}(\mathbb{C})$.

Now let $\mu\in\mathscr{P}^{\mathbb{Z}}_{\log}(\mathbb{C})$ and choose a sequence (α_n) of distinct algebraic integers such that $O(\alpha_n)$ converges to μ log-weakly. Let $P_n\in\mathbb{Z}[X]$ be the minimal polynomial of α_n. By Proposition 2.17 we have that

$$-\frac{\log|P_n|}{\deg(P_n)} \text{ converges to } U^{\mu} \text{ in } L^1_{\mathrm{loc}},$$

which implies that

$$-\max\left\{\frac{\log|P_n|}{\deg(P_n)}, \frac{\log|P_{n+1}|}{\deg(P_{n+1})+1}\right\} \text{ converges to } U^{\mu} \text{ in } L^1_{\mathrm{loc}}.$$

By Proposition 2.17 and 4.4 we have that $\mu_{P_n,P_{n+1}}$ converges to μ log-weakly, proving the density of the family (4.28) with respect to the log-weak topology. ☐

Remark 4.6. One could alternatively consider for each pair P,Q in (4.28) the measure $\mu'_{P,Q}$ similarly induced by the rational function $P^{\deg(Q)}/Q^{\deg(P)}$. This gives another countable dense subset of $\mathscr{P}^{\mathbb{Z}}_{\log}(\mathbb{C})$ that might look more natural. The advantage of the measures $\mu_{P,Q}$ is that they are compactly supported, which is important for its later use in the proof of Proposition 7.8.

4.2. **Approximating measures by equilibrium measures of compact sets of capacity one.**

The next result characterizes the measures in $\mathscr{P}_{\log}(\mathbb{C})$ with negative potential as the log-weak closure of the set of equilibrium measures of compact subsets of capacity one.

Theorem 4.7. *Let* $\mu\in\mathscr{P}_{\log}(\mathbb{C})$. *Then the following are equivalent:*

1. *there exists a sequence of compact subsets* $K_n\subseteq\mathbb{C}$ *with* $\operatorname{cap}(K_n)=1$ *such that* μ_{K_n} *converges* log*-weakly to* μ;
2. $U^{\mu}\leq 0$.

In particular, if μ *is invariant under the complex conjugation and* $U^{\mu}\leq 0$ *then* $\mu\in\mathscr{P}^{\mathbb{Z}}_{\log}(\mathbb{C})$.

Proof. We show first that (1) implies (2). Using the monotone convergence theorem we get

$$\begin{aligned} U^{\mu}(z) &= \int \log|z-w|^{-1} \mathrm{d}\mu(w) \\ &= \lim_{M\to\infty} \int \min\left\{M, \log|z-w|^{-1}\right\} \mathrm{d}\mu(w) \\ &= \lim_{M\to\infty} \lim_{n\to\infty} \int \min\left\{M, \log|z-w|^{-1}\right\} \mathrm{d}\mu_{K_n}(w) \\ &\leq \lim_{n\to\infty} U^{\mu_{K_n}}(z), \end{aligned}$$

where the third step follows from the fact that the function $w \mapsto \min\left\{M, \log|z-w|^{-1}\right\}$ is continuous and of logarithmic growth. By Frostman's theorem [Ran95, Theorem 3.3.4(a)]

$$U^{\mu_{K_n}}(z) \leq -\log(\operatorname{cap}(K_n)) = 0 \quad \text{for all } n$$

and so $U^{\mu}(z) \leq 0$ as stated.

Now we prove that (2) implies (1). Using Proposition 3.2(1) and Lemma 2.16 we restrict without loss of generality to the case when μ is supported in a compact subset K. Then we can weakly approximate μ by a sequence of discrete measures

$$\mu_n = \frac{1}{b_n} \sum_{s\in S_n} \delta_s,$$

where S_n is a finite subset of K of cardinality $b_n \in \mathbb{Z}_{>0}$. Then (μ_n) converges properly to μ, and so it also converges log-weakly. By Proposition 2.17 we have that U^{μ_n} converges to U^{μ} in L^1_{loc}, and so $\min\{U^{\mu_n}, 0\}$ converges to $\min\{U^{\mu}, 0\} = U^{\mu}$ in L^1_{loc}.

Setting $P_n(X) = \prod_{\alpha\in S_n}(X-\alpha)$ we have that $U^{\mu_n} = -\log|P_n|/\deg(P_n)$ and so

$$\min\{U^{\mu_n}, 0\} = \min\left\{-\frac{\log|P_n|}{\deg(P_n)}, 0\right\} = U^{\mu_{K_n}}$$

for the compact subset $K_n \coloneqq \left\{z \mid |P_n(z)| \leq 1\right\}$, which is of capacity one because P_n is monic. Applying again Proposition 2.17 we conclude that μ_{K_n} converges log-weakly to μ, as stated.

The last statement follows readily from the Fekete-Szegő theorem combined with Rumely's equidistribution result (Remark 2.7) together with Lemma 2.16. □

Using this result it is easy to find measures in $\mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$ that are not compactly supported. For instance the Fubini-Study form

$$\omega_{\mathrm{FS}} = \frac{i\,\mathrm{d}z\wedge \mathrm{d}\bar{z}}{2\pi\,(|z|^2+1)^2}$$

induces a conjugation invariant measure $\mu_{\mathrm{FS}} \in \mathscr{P}_{\log}(\mathbb{C})$ with potential

$$U^{\mu_{\mathrm{FS}}}(z) = -\frac{1}{2}\log(|z|^2+1) \leq 0,$$

and so Theorem 4.7 gives that $\mu_{\mathrm{FS}} \in \mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$.

Furthermore, combining this result with Theorem 4.5 it is also easy to produce measures in $\mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$ that cannot be approached by equilibrium measures of compact subsets of capacity one, e.g. see example (1.5).

5. Strong duality

Here we present two optimization problems that will be crucial for our study of heights of algebraic points.

Let $\mathrm{g}\colon \mathbb{C} \longrightarrow \mathbb{R}$ be a continuous function that is asymptotically logarithmic at ∞ in the sense of Definition 3.4, and for simplicity fix an enumeration $Q_1, Q_2, Q_3, \dots$ of all nonconstant primitive irreducible polynomials with integer coefficients. We consider the *primal problem* and the *dual problem* respectively defined as

$$\mathcal{P}(\mathrm{g}) = \inf\left\{ \int \mathrm{g}\,\mathrm{d}\mu \;\middle|\; \mu \in \mathscr{P}_{\log}(\mathbb{C}),\ \int \log|Q_n|\,\mathrm{d}\mu \geq 0 \text{ for all } n \in \mathbb{N}\right\} \tag{5.29}$$

and

$$\mathcal{D}(\mathrm{g}) = \sup_{(a_n)} \inf_{z\in\mathbb{C}} \left(\mathrm{g}(z) - \sum_n a_n \log|Q_n(z)|\right), \tag{5.30}$$

the supremum being over the sequences (a_n) in $\mathbb{R}_{\geq 0}$ with $a_n = 0$ for all but a finite number of n's. We refer to the quantities $\mathcal{P}(\mathrm{g})$ and $\mathcal{D}(\mathrm{g})$ as the *optimal values* for these problems.

Remark 5.1. If g is invariant under the complex conjugation then

$$\mathcal{P}(\mathrm{g}) = \inf\left\{ \int \mathrm{g}\,\mathrm{d}\mu \;\middle|\; \mu \in \mathscr{P}_{\log}^{\mathbb{Z}}(\mathbb{C})\right\}.$$

Indeed, in this case the primal problem can be computed over the measures $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ that are invariant under the complex conjugation and satisfy $\int \log|Q_n|\,\mathrm{d}\mu \geq 0$ for all n, which by Theorem 4.1 coincide with those in $\mathscr{P}_{\log}^{\mathbb{Z}}(\mathbb{C})$.

As explained in Appendix A, this pair of problems is an instance of primal and dual problems in linear optimization (Example A.4). In particular, they satisfy the weak duality property.

Proposition 5.2. *We have* $\mathcal{P}(\mathrm{g}) \geq \mathcal{D}(\mathrm{g})$.

Proof. This is a particular case of Proposition A.5 but can also be checked directly: for all $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ such that $\int \log|Q_n|\,\mathrm{d}\mu \geq 0$ for all n and (a_n) as in (5.30) we have

$$\int \mathrm{g}\,\mathrm{d}\mu \geq \int \Big(\mathrm{g} - \sum_{n\in\mathbb{N}} a_n \log|Q_n|\Big)\,\mathrm{d}\mu \geq \inf_{z\in\mathbb{C}} \Big(\mathrm{g}(z) - \sum_{n\in\mathbb{N}} a_n \log|Q_n(z)|\Big),$$

which gives the inequality. □

Our main result in this section shows that these problems satisfy the strong duality property, which consists in the equality between their optimal values. Its proof is modelled in that of Theorem A.7 for the finite dimensional case.

Theorem 5.3. *We have* $\mathcal{D}(\mathrm{g}) = \mathcal{P}(\mathrm{g}) \in \mathbb{R}$.

Proof. Let $\mathscr{P}'_{\log}(\mathbb{C})$ be the convex set of probability measures on $\mathbb{C}$ that integrate all the functions of the form $\log|Q_i|$. For $n \in \mathbb{N}$ consider the convex subset of $\mathbb{R}^{n+2}$ defined as

$$V_n = \left\{ \Big(\int \mathrm{g}\,\mathrm{d}\mu, \int \log|Q_1|\,\mathrm{d}\mu, \dots, \int \log|Q_n|\,\mathrm{d}\mu \Big) \;\middle|\; \mu \in \mathscr{P}'_{\log}(\mathbb{C})\right\},$$

and for each $\lambda \in \mathbb{R}$ consider also the convex subset of $\mathbb{R}^{n+2}$ defined as

$$W_{n,\lambda} = \{(t, x_1, \dots, x_n) \mid t \leq \lambda,\ x_1, \dots, x_n \geq 0\}.$$

We have that $V_n \cap W_{n,\lambda} \neq \emptyset$ if and only if there exists $\mu \in \mathscr{P}'_{\log}(\mathbb{C})$ such that $\int \mathrm{g}\, \mathrm{d}\mu \leq \lambda$ and $\int \log |Q_i|\, \mathrm{d}\mu \geq 0$, $i = 1, \dots, n$. Now set

$$\lambda_n = \inf\{\lambda \in \mathbb{R} \mid V_n \cap W_{n,\lambda} \neq \emptyset\}.$$

Since g is bounded from below the values $\int \mathrm{g}\, \mathrm{d}\mu$ for $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ are also bounded from below, and so $\lambda_n > -\infty$. On the other hand, the Dirac delta measure $\mu = \delta_z$ for $z \gg 0$ satisfies

$$\int \log |Q_i|\, \mathrm{d}\mu = \log |Q_i(z)| > 0, \quad i = 1, \dots, n. \tag{5.31}$$

In particular $\lambda_n \leq \mathrm{g}(z) < +\infty$. Hence $\lambda_n \in \mathbb{R}$.

Since $\lambda_n - 1/n < \lambda_n$ the convex subsets V_n and $W_{n,\lambda_n - 1/n}$ are disjoint, and so by the hyperplane separation theorem there exists $h \in (\mathbb{R}^{n+2})^\vee \simeq \mathbb{R}^{n+2}$ with $h \neq 0$ such that $h(p) \geq h(p')$ for all $p \in V_n$ and $p' \in W_{n,\lambda_n - 1/n}$. Writing $h = (b, -a_1, \dots, -a_n)$ with $b, a_i \in \mathbb{R}$ these conditions amount to

$$b \int \mathrm{g}\, \mathrm{d}\mu - \sum_{i=1}^{n} a_i \int \log |Q_i|\, \mathrm{d}\mu \geq b\, t - \sum_{i=1}^{n} a_i x_i \tag{5.32}$$

for all $\mu \in \mathscr{P}'_{\log}(\mathbb{C})$, $t \leq \lambda_n - 1/n$ and $x_1, \dots, x_n \geq 0$.

Since x_i can be arbitrarily large this inequality implies that $a_i \geq 0$, $i = 1, \dots, n$, and since t can also be arbitrarily negative we similarly deduce that $b \geq 0$. To exclude the possibility that $b = 0$, consider the case $x_i = 0$, $i = 1, \dots, n$, and note that in this situation (5.32) gives

$$-\sum_{i=1}^{n} a_i \int \log |Q_i|\, \mathrm{d}\mu \geq 0 \quad \text{for all } \mu \in \mathscr{P}'_{\log}(\mathbb{C}).$$

Since $h \neq 0$ we have that $a_i > 0$ for some $i \in \{1, \dots, n\}$, in which case the inequality does not hold for the measure in (5.31). We conclude that $b > 0$, and so we can assume without loss of generality that $b = 1$.

Setting $t = \lambda_n - 1/n$ and $x_1 = \dots = x_n = 0$ in (5.32) we get

$$\int \mathrm{g}\, \mathrm{d}\mu - \sum_{i=1}^{n} a_i \int \log |Q_i|\, \mathrm{d}\mu \geq \lambda_n - \frac{1}{n} \quad \text{for all } \mu \in \mathscr{P}'_{\log}(\mathbb{C}). \tag{5.33}$$

Considering this inequality for $\mu = \delta_z$ with $z \in \mathbb{C} \setminus \overline{\mathbb{Q}}$ we get

$$\mathrm{g}(z) - \sum_{i=1}^{n} a_i \log |Q_i(z)| \geq \lambda_n - \frac{1}{n} \quad \text{for all } z \in \mathbb{C} \setminus \overline{\mathbb{Q}},$$

which extends by density to all $z \in \mathbb{C}$. Hence $\mathcal{D}(\mathrm{g}) \geq \lambda_n - 1/n$. Therefore

$$\limsup_{n \to \infty} \lambda_n \leq \mathcal{D}(\mathrm{g}). \tag{5.34}$$

On the other hand we can choose $\mu_n \in \mathscr{P}'_{\log}(\mathbb{C})$ satisfying

$$\int \mathrm{g}\, \mathrm{d}\mu_n \leq \lambda_n + \frac{1}{n} \quad \text{and} \quad \int \log |Q_i|\, \mathrm{d}\mu_n \geq 0, \quad i = 1, \dots, n. \tag{5.35}$$

By (5.34)

$$\limsup_{n \to \infty} \int \mathrm{g}\, \mathrm{d}\mu_n \leq \mathcal{D}(\mathrm{g}).$$

By Proposition 3.5, for any $\varepsilon > 0$ there exists $R > 1$ such that

$$\limsup_{n\to\infty} \int \mathrm{g}\, \mathrm{d}\mu_{n,R}^{\mathrm{sw}} \leq \mathcal{D}(\mathrm{g}) + \varepsilon,$$

where $\mu_{n,R}^{\mathrm{sw}}$ denotes the sweetened truncation of μ_n (Definition 3.1). Since all the probability measures $\mu_{n,R}^{\mathrm{sw}}$ are supported on the ball $B_R = \{|z| \leq R\}$, up to taking a subsequence we can assume that they converge properly to a probability measure μ_R with support contained in B_R. Then

$$\int \mathrm{g}\, \mathrm{d}\mu_R = \lim_{n\to\infty} \int \mathrm{g}\, \mathrm{d}\mu_{n,R}^{\mathrm{sw}} \leq \mathcal{D}(\mathrm{g}) + \varepsilon,$$

and by Lemma 2.15 and Proposition 3.2(3) we also have

$$\int \log|Q_m|\, \mathrm{d}\mu_R \geq \liminf_{n\to\infty} \int \log|Q_m|\, \mathrm{d}\mu_{n,R}^{\mathrm{sw}} \geq 0 \quad \text{for all } m.$$

Hence μ_R is a candidate for the primal problem and we deduce that $\mathcal{P}(\mathrm{g}) \leq \mathcal{D}(\mathrm{g}) + \varepsilon$. Taking $\varepsilon > 0$ arbitrarily small we get $\mathcal{P}(\mathrm{g}) \leq \mathcal{D}(\mathrm{g})$, and we obtain the equality by combining this with the weak duality property (Proposition 5.2).

The fact that $\mathcal{P}(\mathrm{g})$ is a real number follows from the estimates

$$-\infty < \inf_{z\in\mathbb{C}} \mathrm{g}(z) \leq \mathcal{P}(\mathrm{g}) \leq \int \mathrm{g}\, \mathrm{d}\lambda_{S_1} < \infty.$$

□

Remark 5.4. A related result was proved by Smith for functions on subsets of the real line with super-logarithmic behavior around the point at infinity [Smi24, Theorem 5.11]. We extend this result to the boundary case when the behavior is logarithmic, by slightly modifying the definition of sweetened truncation and using the estimate from Proposition 3.5.

Remark 5.5. As we will see in Section 6.4, there may not exist a measure realizing the optimal value $\mathcal{P}(\mathrm{g})$. On the other hand, when such measure does exist it is not necessarily unique. An example is given by $\mathrm{g}(z) = \log^+|z/2|$: for all $R \in [1,2]$ the equilibrium measure λ_{S_R} lies in $\mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C})$ and attains the optimal value of $\mathcal{P}(\mathrm{g})$.

Remark 5.6. More generally, when $\mathrm{g}(z)$ is *radial* (that is it only depends on $|z|$) and subharmonic, the theory of toric varieties applies, see for instance [BPS14]. In particular [BPS15, Corollary 3.10] implies that the measure λ_{S_1} attains the optimal value of $\mathcal{P}(\mathrm{g})$, and if g is strictly subharmonic then [BPRS19, Theorem 4.18] implies that λ_{S_1} is the unique measure attaining the optimal value.

Since we are in dimension one it is simpler to give a direct proof of these facts. Since g is radial and subharmonic we have that $\varphi(t) := \mathrm{g}(e^t)$ is convex. Note also that since $\lim_{t\to-\infty} \varphi(t) = \mathrm{g}(0)$ is finite, φ is non-decreasing. Then

$$\int \mathrm{g}(z)\, \mathrm{d}\mu = \int \varphi\big(\log|z|\big)\, \mathrm{d}\mu \geq \varphi\left(\int \log|z|\, \mathrm{d}\mu\right) \geq \varphi(0) = \mathrm{g}(1) = \int \inf(z)\, \mathrm{d}\lambda_{S_1},$$

where the first inequality is Jensen's inequality, whereas the second uses that φ is non-decreasing and that $\int \log|z|\, \mathrm{d}\mu \geq 0$.

If we assume further that g is strictly subharmonic then λ_{S_1} is the unique measure in $\mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C})$ attaining the optimal value. In fact, in this case φ is strictly convex and increasing.

If $\mu \in \mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$ is a minimizer, then Jensen's inequality

$$\int \varphi\big(\log|z|\big)\,\mathrm{d}\mu \geq \varphi\left(\int \log|z|\,\mathrm{d}\mu\right)$$

must be an equality, which means $|z|$ must be constant on the support of μ, say $|z| = r$. Then

$$\int \varphi\big(\log|z|\big)\,\mathrm{d}\mu = \varphi\left(\int \log|z|\,\mathrm{d}\mu\right) = \varphi(\log r).$$

Now μ is a minimizer, so $\varphi(\log r) = \varphi(0)$ and hence $r = 1$. Then $\mu \in \mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$ is supported on the unit circle and so $\mu = \lambda_{S_1}$.

6. The essential minimum of height functions

In this section we briefly recall the Arakelov point of view of our height functions and use the strong duality theorem to extract some properties for their essential minima.

6.1. **Arakelov theory on $\mathbb{P}^1$.** Let $\mathbb{P}^1$ be the projective line over $\mathbb{Q}$ equipped with its universal line bundle $\mathcal{O}(1)$. Consider their canonical integral models $\mathbb{P}^1_{\mathbb{Z}}$ and $\mathcal{O}(1)_{\mathbb{Z}}$ together with a continuous metric $\|\cdot\|$ on the holomorphic line bundle $\mathcal{O}(1)_{\mathbb{C}}$ over the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ that is invariant under the complex conjugation. The pair $\overline{\mathcal{O}(1)_{\mathbb{Z}}} = (\mathcal{O}(1)_{\mathbb{Z}}, \|\cdot\|)$ is called a *metrized line bundle* on $\mathbb{P}^1_{\mathbb{Z}}$, and following [BGS94, Section 3] it induces a height function

$$h_{\overline{\mathcal{O}(1)_{\mathbb{Z}}}} \colon \mathbb{P}^1(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}.$$

Let $(x_0 \colon x_1)$ be the homogeneous coordinates of $\mathbb{P}^1$. Then x_1 is a global section of $\mathcal{O}(1)_{\mathbb{C}}$ vanishing at the point at infinity $\infty = (1:0)$. Using the identification $\mathbb{P}^1(\mathbb{C}) \backslash (1:0) \simeq \mathbb{C}$ given by the map $(x_0 \colon x_1) \longmapsto z = x_0/x_1$, the metric $\|\cdot\|$ is encoded by its associated *Green function of continuous type*

$$\mathrm{g} \colon \mathbb{C} \longrightarrow \mathbb{R}, \quad z \longmapsto -\log\|x_1(z\colon 1)\|. \tag{6.36}$$

This is a continuous function that is invariant under the complex conjugation and satisfies the asymptotics $\mathrm{g}(z) = \log|z| + a + o(1)$ as $z \to \infty$ with $a \in \mathbb{C}$. Conversely, every Green function of continuous type defines a continuous and conjugation invariant metric on $\mathcal{O}(1)_{\mathbb{C}}$.

Already in the work of Faltings leading to the proof of Mordell's conjecture [Fal83], it became apparent that for many arithmetic applications one should consider metrics admiting some singularities. In the recent work [YZ26], Yuan and Zhang introduced adelic line bundles on quasi-projective varieties allowing to extend Arakelov geometry to a wide class of singular metrics.

In particular, this allows to consider an adelic line bundle $\overline{L}$ on the affine line $\mathbb{A}^1_{\mathbb{Q}}$ that is defined by the line bundle $\mathcal{O}(1)_{\mathbb{Z}}$ on the compactification $\mathbb{A}^1_{\mathbb{Z}} \subseteq \mathbb{P}^1_{\mathbb{Z}}$, together with a metric $\|\cdot\|$ on $\mathcal{O}(1)_{\mathbb{C}}$ that can be singular at the point at infinity. Following Section 5.3.1 in *loc. cit.* we can associate to this adelic line bundle a height function

$$h_{\overline{L}} \colon \mathbb{A}^1(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}} \longrightarrow \mathbb{R}.$$

The archimedean Green functions associated to these particular adelic line bundles are exactly the Green functions in the sense of Definition 3.4 [YZ26, Theorem 3.6.4]. Within this family of line bundles, $\overline{L}$ is characterized by its Green function g, so for simplicity we denote its height function by h_{g}.

This height function can be described in very concrete terms. Let $\alpha \in \overline{\mathbb{Q}}$ and denote by $P_\alpha \in \mathbb{Z}[x]$ a primitive irreducible polynomial with $P_\alpha(\alpha) = 0$, which is unique up to a sign. Let c_α be the leading coefficient of P_α and $O(\alpha) \subseteq \overline{\mathbb{Q}}$ its set of zeros. The height of α is then given by

$$h_{\mathrm{g}}(\alpha) = \frac{1}{\deg(\alpha)} \left(\log |c_\alpha| + \sum_{\beta \in O(\alpha)} \mathrm{g}(\beta) \right). \tag{6.37}$$

If $\alpha \in \overline{\mathbb{Z}}$ then we can choose P_α monic and so

$$h_{\mathrm{g}}(\alpha) = \frac{1}{\deg(\alpha)} \sum_{\beta \in O(\alpha)} \mathrm{g}(\beta), \tag{6.38}$$

whereas for an arbitrary $\alpha \in \overline{\mathbb{Q}}$ we only have the inequality

$$h_{\mathrm{g}}(\alpha) \geq \frac{1}{\deg(\alpha)} \sum_{\beta \in O(\alpha)} \mathrm{g}(\beta).$$

There is another description of the height of an algebraic number as a sum of local contributions that will be useful later. Let $\mathcal{M}_{\mathbb{Q}}$ be the set of places of $\mathbb{Q}$, that is

$$\mathcal{M}_{\mathbb{Q}} = \{p \in \mathbb{Z} \mid p > 0 \text{ prime}\} \cup \{\infty\}.$$

For each $\nu \in \mathcal{M}_{\mathbb{Q}}$ let $|\cdot|_\nu$ denote either the usual absolute value of $\mathbb{Q}$ if $\nu = \infty$ or the p-adic absolute value of $\mathbb{Q}$ if $\nu = p$ for some prime p. This absolute value extends uniquely to the complete and algebraically closed field $\mathbb{C}_\nu$. Then the height of α can be alternatively written as [BG06, Section 1.5.7]

$$h_{\mathrm{g}}(\alpha) = \frac{1}{\deg(\alpha)} \left(\left(\sum_{p \text{ prime}} \sum_{\beta \in O(\alpha)} \log^+ |\beta|_p \right) + \sum_{\beta \in O(\alpha)} \mathrm{g}(\beta) \right). \tag{6.39}$$

Definition 6.1. Let $h_{\mathrm{g}} \colon \overline{\mathbb{Q}} \longrightarrow \mathbb{R}$ be the height function associated to a Green function g. Its *essential minimum* is defined as

$$\mathrm{ess}(h_{\mathrm{g}}) \coloneqq \inf \left\{ \liminf_{n \to \infty} h_{\mathrm{g}}(\alpha_n) \,\middle|\, (\alpha_n) \text{ is a sequence of distinct algebraic numbers} \right\}.$$

Remark 6.2. To stay as elementary as possible, from now on we will forget about the Arakelov point of view and focus on the Green function g as the source for the height function. Nevertheless this point of view suggests many generalizations of the present work. First, instead of considering the canonical model $(\mathbb{P}^1_{\mathbb{Z}}, \mathcal{O}(1)_{\mathbb{Z}})$ of the pair $(\mathbb{P}^1, \mathcal{O}(1))$ one can consider arbitrary integral models, or even more generally one can consider general adelic line bundles with different metrics over each place of $\mathbb{Q}$. Second, instead of the divisor $[(1:0)]$ of $\mathbb{P}^1$ with respect to which the Green function is defined, one can consider arbitrary divisors. Finally, instead of $\mathbb{P}^1$ one can consider arbitrary curves or even higher dimensional varieties.

6.2. **The essential minimum and linear programming.** In this section we relate the essential minimum of the height function with the linear programming problems $\mathcal{D}(\mathrm{g})$ and $\mathcal{P}(\mathrm{g})$.

Proposition 6.3. *The relation* $\mathrm{ess}(h_{\mathrm{g}}) \geq \mathcal{D}(\mathrm{g})$ *holds true.*

Proof. Recall the product formula $\prod_{\nu\in\mathcal{M}_{\mathbb{Q}}}|\alpha|_\nu = 1$ for $\alpha\in\mathbb{Q}^\times$, which implies that

$$\sum_{\nu\in\mathcal{M}_{\mathbb{Q}}}\sum_{\beta\in O(\alpha)}\log|\beta|_\nu = 0 \quad \text{for } \alpha\in\overline{\mathbb{Q}}^\times \tag{6.40}$$

because $\prod_{\beta\in O(\alpha)}\beta\in\mathbb{Q}$. Let Q_i, $i=1,\dots,n$, be nonzero polynomials with integer coefficients and $a_1,\dots,a_n\in\mathbb{R}_{\geq 0}$. If we show that

$$\operatorname{ess}(h_{\mathrm{g}}) \geq \inf_{z\in\mathbb{C}}\left(\mathrm{g}(z)-\sum_{i=1}^n a_i\log|Q_i(z)|\right), \tag{6.41}$$

then the proposition will be proved.

If $\sum a_i\deg(Q_i)>1$ then

$$\inf_{z\in\mathbb{C}}\left(\mathrm{g}(z)-\sum_{i=1}^n a_i\log|Q_i(z)|\right) = -\infty$$

and the inequality (6.41) is trivially true. So we can assume that

$$\sum a_i\deg(Q_i)\leq 1. \tag{6.42}$$

Let $Z\subseteq\overline{\mathbb{Q}}$ be the union of the set of zeros of the Q_i's. By the product formula (6.40), for all $\alpha\in\overline{\mathbb{Q}}\setminus Z$ we have that

$$h_{\mathrm{g}}(\alpha) = \frac{1}{\deg(\alpha)}\sum_{p \text{ prime}}\sum_{\beta\in O_\nu(\alpha)}\left(\log^+|\beta|_p-\sum_{i=1}^n a_i\log|Q_i(\beta)|_p\right) \\ + \frac{1}{\deg(\alpha)}\sum_{\beta\in O_\nu(\alpha)}\left(\mathrm{g}(\beta)-\sum_{i=1}^n a_i\log|Q_i(\beta)|\right).$$

By (6.42) and the fact that the polynomials Q_i have integer coefficients, for every prime p we have that

$$\sum_{i=1}^n a_i\log|Q_i(\beta)|_p\leq\log^+|\beta|_p.$$

Therefore

$$h_{\mathrm{g}}(\alpha)\geq\frac{1}{\deg(\alpha)}\sum_{\beta\in O(\alpha)}\left(\mathrm{g}(\beta)-\sum_{i=1}^n a_i\log|Q_i(\beta)|\right) \quad \text{for all } \alpha\in\overline{\mathbb{Q}}\setminus Z.$$

This givens the inequality (6.41) and proves the proposition. ☐

Proposition 6.4. *The relation* $\operatorname{ess}(h_{\mathrm{g}})\leq\mathcal{P}(\mathrm{g})$ *holds true.*

Proof. Let $\mu\in\mathscr{P}_{\log}^{\overline{\mathbb{Z}}}$ and take a sequence of distinct algebraic integers (α_n) such that $O(\alpha_n)$ converges log-weakly to μ. Using formula (6.38) and the fact that g is a continuous function of logarithmic growth, we have that

$$\lim_{n\to\infty}h_{\mathrm{g}}(\alpha_n) = \lim_{n\to\infty}\frac{1}{\deg(\alpha_n)}\sum_{p\in O(\alpha_n)}\mathrm{g}(p) = \int\mathrm{g}\,\mathrm{d}\mu. \tag{6.43}$$

Hence $\operatorname{ess}(h_{\mathrm{g}})\leq\inf\left\{\int\mathrm{g}\,\mathrm{d}\mu \,\middle|\, \mu\in\mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C})\right\}=\mathcal{P}(\mathrm{g})$, as stated. ☐

Putting together Propositions 6.3 and 6.4 with Theorem 5.3 we obtain the main theorem of this section.

Theorem 6.5. *Let* g *be a Green function. Then* $\mathcal{D}(\mathrm{g}) = \operatorname{ess}(h_{\mathrm{g}}) = \mathcal{P}(\mathrm{g})$.

A direct consequence of Theorem 6.5 is that the essential minimum can be reached by a sequence of algebraic integers.

Corollary 6.6. *There exists a sequence of distinct algebraic integers* (α_n) *such that*

$$\lim_{n\to\infty} h_{\mathrm{g}}(\alpha_n) = \operatorname{ess}(h_{\mathrm{g}}). \tag{6.44}$$

As a consequence of a result of Szachniewicz based on the theory of globally valued fields [Sza23, Theorem A] it can be shown that $h_{\mathrm{g}}(\overline{\mathbb{Q}})$ is dense in the interval $[\operatorname{ess}(h_{\mathrm{g}}), \infty)$. As another application of our results we show that this is already the case for the set of heights of algebraic integers.

Corollary 6.7. *The set* $h_{\mathrm{g}}(\overline{\mathbb{Z}})$ *is dense in* $[\operatorname{ess}(h_{\mathrm{g}}), \infty)$.

Proof. By (6.43) the set

$$I = \left\{ \int \mathrm{g}\, \mathrm{d}\mu \,\middle|\, \mu \in \mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C}) \right\}$$

is contained in the closure of $h_{\mathrm{g}}(\overline{\mathbb{Z}})$. By Theorem 4.1 it is a convex subset of $\mathbb{R}$, hence an interval. By Theorem 6.5 this interval contains real numbers that are arbitrarily close to $\operatorname{ess}(h_{\mathrm{g}})$, and considering $\mu = \lambda_{S_R}$ for $R \gg 0$ we can also see that it is not bounded above. Hence $(\operatorname{ess}(h_{\mathrm{g}}), \infty) \subseteq I$, which gives the statement. □

Remark 6.8. A direct consequence of Corollary 6.7 is that the sequence of distinct algebraic integers (α_n) in (6.44) can be chosen so that $h_{\mathrm{g}}(\alpha_n)$ is monotonically decreasing.

6.3. **Asymptotic maximal slope.** In this section we prove that in the dual problem we can reduce to linear combinations with rational coefficients. This has two applications. The first one is to relate the essential minimum with another Arakelov theory invariant called the asymptotic maximal slope and the second one is the left computability of the essential minimum as we will see in section 7. Consider the rational dual problems

$$\mathcal{D}_{\mathbb{Q}}(\mathrm{g}) \coloneqq \sup \left\{ \inf_{z\in\mathbb{C}} \left(\mathrm{g}(z) - \sum_{i=1}^{k} a_i \log|Q_i(z)| \right) \,\middle|\, a_i \in \mathbb{Q}_{\geq 0}, Q_i \in \mathbb{Z}[x] \setminus \{0\}, \sum a_i \deg(Q_i) \leq 1 \right\}.$$

and

$$\mathcal{D}'_{\mathbb{Q}}(\mathrm{g}) \coloneqq \sup \left\{ \inf_{z\in\mathbb{C}} \left(\mathrm{g}(z) - \sum_{i=1}^{k} a_i \log|Q_i(z)| \right) \,\middle|\, a_i \in \mathbb{Q}_{\geq 0}, Q_i \in \mathbb{Z}[x] \setminus \{0\}, \sum a_i \deg(Q_i) < 1 \right\}.$$

Clearly $\mathcal{D}(\mathrm{g}) \geq \mathcal{D}_{\mathbb{Q}}(\mathrm{g}) \geq \mathcal{D}'_{\mathbb{Q}}(\mathrm{g})$.

Proposition 6.9. *The equalities* $\mathcal{D}(\mathrm{g}) = \mathcal{D}_{\mathbb{Q}}(\mathrm{g}) = \mathcal{D}'_{\mathbb{Q}}(\mathrm{g})$ *hold.*

Proof. Let $n \geq 1$ and take $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n_{\geq 0}$ and $(Q_1, \dots, Q_n) \in (\mathbb{Z}[x] \setminus \{0\})^n$. Let

$$\varphi_{\mathbf{a}}(z) \coloneqq \mathrm{g}(z) - \sum_{i=1}^{n} a_i \log|Q_i(z)|.$$

Then, it is enough to show that for any $\varepsilon > 0$ and any $\mathbf{a} \in \mathbb{R}^n_{\geq 0}$, there exists $\mathbf{b} \in \mathbb{Q}^n_{\geq 0}$ with $\sum_{i=1}^n b_i \deg Q_i < 1$ such that

$$\inf \varphi_{\mathbf{a}} \leq \inf \varphi_{\mathbf{b}} + \varepsilon. \tag{6.45}$$

Relation (6.45) holds trivially if $\inf \varphi_{\mathbf{a}} = -\infty$ or $\mathbf{a} = \mathbf{0} \in \mathbb{R}^n$, so we can assume $\inf \varphi_{\mathbf{a}} > -\infty$ and $a_i \neq 0$ for all i. Then, necessarily

$$\sum_{i=1}^n a_i \deg Q_i \leq 1. \tag{6.46}$$

Let U be an open and bounded neighborhood of the set of zeroes of $\prod_{i=1}^n Q_i(x)$ such that for all $z \in U$, it holds

$$\inf \varphi_{\mathbf{a}} \leq \mathrm{g}(z) - \sum_{i=1}^n a_i \max\left\{\frac{1}{2}\log|Q_i(z)|, \log|Q_i(z)|\right\} \tag{6.47}$$

Choose $R \geq 1$ big enough so that $\log|Q_i(z)| \geq 0$ for all $|z| \geq R$ and $i = 1, \dots, n$. Set $K \coloneqq U^c \cap \{|z| \leq R\}$. Since K is a compact set, there exist $\mathbf{b} \in \mathbb{Q}^n_{\geq 0}$ such that

$$\frac{a_i}{2} \leq b_i < a_i \text{ for all } i \text{ and } \inf_{x\in K} \varphi_{\mathbf{b}}(x) \geq \inf_{x \in K} \varphi_{\mathbf{a}}(x) - \varepsilon.$$

In particular, in view of (6.46) we have that

$$\sum_{i=1}^n b_i \deg Q_i < 1 \text{ and } \inf_{z\in K} \varphi_{\mathbf{b}}(z) \geq \inf \varphi_{\mathbf{a}} - \varepsilon. \tag{6.48}$$

On the other hand, we claim that

$$\inf_{z \in U} \varphi_{\mathbf{b}}(z) \geq \inf \varphi_{\mathbf{a}}. \tag{6.49}$$

Indeed, due to the choice of the b_i we have

$$\varphi_{\mathbf{b}}(z) \geq \mathrm{g}(z) - \sum_{i=1}^n a_i \max\left\{\frac{1}{2}\log|Q_i(z)|, \log|Q_i(z)|\right\},$$

so the claim follows from (6.47). Finally, for z such that $|z| \geq R$ we have

$$\varphi_{\mathbf{b}}(z) - \varphi_{\mathbf{a}}(z) = \sum_{i=1}^n (a_i - b_i) \log|Q_i(z)| \geq 0,$$

thus $\inf_{|z|\geq R} \varphi_{\mathbf{b}}(z) \geq \inf \varphi_{\mathbf{a}}$. This estimate, together with (6.48) and (6.49), imply (6.45). □

We now consider on $\mathbb{P}^1_{\mathbb{Z}}$ the line bundles $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$. The metric on $\mathcal{O}(1)$ induces metrics on each $\mathcal{O}(n)$. For each global section $s \in \mathrm{H}^0(\mathbb{P}^1_{\mathbb{Z}}, \mathcal{O}(n))$ the sup norm is defined as

$$\|s\|_\infty \coloneqq \sup_{z\in\mathbb{P}^1(\mathbb{C})} \|s(z)\|.$$

For the particular family of adelic line bundles on $\mathbb{A}^1_{\mathbb{Q}}$ introduced in Section 6.1, the *asymptotic maximal slope* of $\overline{L}$ is defined as by

$$\hat{\mu}(\overline{L}) \coloneqq \sup_{\substack{n\geq 1 \\ s\in \mathrm{H}^0(\mathbb{P}^1_{\mathbb{Z}}, \mathcal{O}(n))\setminus\{0\}}} -\frac{\log\|s\|_\infty}{n}.$$

By unfolding the definitions, one can see that

$$\hat{\mu}(\overline{L}) = \mathcal{D}_{\mathbb{Q}}(\mathrm{g}).$$

Hence, as a direct consequence of Theorem 6.5 and Proposition 6.9 we obtain the next result.

Corollary 6.10. *The equality* $\hat{\mu}(\overline{L}) = \operatorname{ess}(h_{\overline{L}})$ *is satisfied.*

Remark 6.11. When g is subharmonic and of continuous type, this equality is a particular case of a result of Ballaÿ [Bal21, Theorem 1.2]. Yuan conjectured a more general version of the result in *loc. cit.*, see [YZ26, Conjecture 5.3.5]. This conjecture has been established in some cases beyond [Bal21, Theorem 1.2], see [QY24, Theorem 1.8] and [YZ26, Theorem 5.3.6].

6.4. **An example with no minimizer.** In this subsection we present an example due to Nuno Hultberg, where the infimum

$$\inf_{\mu\in\mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})} \int \mathrm{g}\,\mathrm{d}\mu$$

is not attained by any measure $\mu_0 \in \mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$. Consider the automorphism $f\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1$, $x \longmapsto x^{-1}+2$. In homogeneous coordinates is given by $(x_0\colon x_1) \longmapsto (x_1+2x_0\colon x_0)$ under the identification $(x:1) = (x_0\colon x_1)$. Let $\overline{\mathcal{L}} = \big(\mathcal{O}(1), \|\cdot\|_{\mathrm{can}}\big)$, where the canonical metric $\|\cdot\|_{\mathrm{can}}$ is the metric whose Green function is $\log^+$. Consider the hermitian line bundle $f^*\overline{\mathcal{L}}$. By the projection formula $h_{f^*\overline{\mathcal{L}}} = h_{\overline{\mathcal{L}}} \circ f$, we see that

$$\operatorname{ess}(h_{f^*\overline{\mathcal{L}}}) = \operatorname{ess} h_{\overline{\mathcal{L}}} = 0.$$

The Green function associated with $f^*\overline{\mathcal{L}}$ is $\mathrm{g}(z) = \log^+|z^{-1}+2| + \log|z|$. In particular, Theorem 6.5 applies to $h_{f^*\overline{\mathcal{L}}}$.

Assume there exists $\mu_0 \in \mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$ such that $\int \mathrm{g}\,\mathrm{d}\mu_0 = \operatorname{ess}(h_{\mathrm{g}}) = 0$. Then there exists a sequence of distinct algebraic integers (α_n) such that $\delta_{O(\alpha_n)}$ converges to μ_0 in the log-weak sense. Therefore $h_{\mathrm{g}}(\alpha_n)$ converges to $\int \mathrm{g}\,\mathrm{d}\mu_0 = 0$. Using the projection formula and Bilu's equidistribution theorem [Bil97, Theorem 1.1], we see that $\delta_{O(\alpha_n)}$ must converge weakly to $f^*\mu_{S^1}$, which forces $\mu_0 = f^*\mu_{S^1}$. However,

$$\begin{aligned}\int \log^+|z^{-1}+2| + \log|z|\,\mathrm{d}f^*\mu_{S^1}(z) &= \int \log^+|t| + \log|t-2|^{-1}\mathrm{d}\mu_{S^1}(t)\\ &= -\int \log|t-2|\mathrm{d}\mu_{S^1}(t)\\ &= -\log 2 \neq 0,\end{aligned}$$

which is a contradiction.

The preceeding argument also shows that $f^*\mu_{S^1} \notin \mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$, even though μ_{S^1} does belong to $\mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$. To see this more concretely, consider the polynomial

$$P_n(x) = \frac{x^{n+1}-2x^n+1}{x-1} \in \mathbb{Z}[x]$$

and let α_n be a root of P_n. Then $f^{-1}(\alpha_n)$ is an algebraic integer and the sequence $h_{\mathrm{g}}(f^{-1}(\alpha_n)) = h_{\overline{\mathcal{L}}}(\alpha_n)$, $n \geq 1$, converges to zero when n goes to ∞. If we look at the Galois orbit of $f^{-1}(\alpha_n)$, there are $n-1$ conjugates of $f^{-1}(\alpha_n)$ that are close to the support of $f^*\mu_{S^1}$, yet the remaining conjugate goes to infinity at the speed of 2^n. Therefore $\delta_{O(f^{-1}(\alpha_n))}$ converges weakly to $f^*\mu_{S^1}$ but does not converge log-weakly. The situation is similar to Autissier's counterexample [Aut06] showing that logarithmic equidistribution of small points is not true in general.

In this example, it also happens that the essential minimum is strictly smaller than

$$\inf\left\{\int \mathrm{g}\,\mathrm{d}\mu_K \;\middle|\; K \text{ is a conjugation invariant compact set of capacity one}\right\}. \tag{6.50}$$

Indeed,

$$\mathrm{g}(z) = \log^+|z^{-1}+2| + \log|z| = \log|z| + \log 2 + o(1) \quad \text{as } |z| \longrightarrow \infty.$$

Let $\mu_\infty \coloneqq \frac{1}{2\pi}\Delta \mathrm{g} = f^*\mu_{S^1}$. Then $\mathrm{g}(z) = -U^{\mu_\infty} + \log(2)$ and for any probability measure μ,

$$\int \mathrm{g}\,\mathrm{d}\mu = \log 2 + \int -U^{\mu_\infty}\,\mathrm{d}\mu = \log 2 + \int -U^{\mu}\,\mathrm{d}\mu_\infty.$$

If μ is equilibrium measure of a capacity one set, then $-U^\mu \geq 0$, and we will have $\int \mathrm{g}\,\mathrm{d}\mu \geq \log 2$. So (6.50) cannot be the true essential minimum (which is 0).

6.5. **The essential minimum of Faltings' height.** Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and consider the associated modular curve Y, which is defined over $\mathbb{Q}$. Note that $Y(\overline{\mathbb{Q}})$ is in bijection with the set of isomorphism classes of elliptic curves over $\overline{\mathbb{Q}}$. Let X be the compactification of Y. There is a line bundle M_{12} on X such that $\mathrm{H}^0(X, M_{12}^{\otimes n}) \simeq M_{12n}(\Gamma,\mathbb{Q})$ as Hecke modules, where the latter is the $\mathbb{Q}$-space of modular forms of weight $12n$ and level 1 with rational Fourier coefficients.

We have a canonical integral model $(\mathcal{X}/\mathbb{Z}, \mathcal{M}_{12})$ of $(X/\mathbb{Q}, M_{12})$. Note that $\mathrm{H}^0(\mathcal{X}, n\mathcal{M}_{12}) \simeq \mathcal{M}_{12n}(\Gamma,\mathbb{Z})$ as Hecke modules, where the latter is the abelian group of modular forms of weight $12n$ and level 1 with integral Fourier coefficients.

The Petersson metric $\|\cdot\|_{\mathrm{Pet}}$ on $M_{12}(\mathbb{C})$ is defined as $\|f\|_{\mathrm{Pet}}(\tau) \coloneqq |f(\tau)|\big(4\pi \operatorname{Im}\tau\big)^6$, i.e. taking $\big(4\pi\operatorname{Im}\tau\big)^6$ as metric weight, where f is any modular form of weight 12 and level 1 and $\tau \in \mathbb{H} \coloneqq \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$ is a point in the Poincaré upper half plane. The function $\|f\|_{\mathrm{Pet}}(\tau)$ is invariant under the action of Γ and descents to a function on $Y(\mathbb{C})$. We also define the L_2-norm on $M_{12}(\mathbb{C})$ by

$$\|f\|_{L^2} \coloneqq \int_{Y(\mathbb{C})} \|f\|_{\mathrm{Pet}}\,\mathrm{d}\mu_{\mathrm{hyp}}.$$

Here, μ_{hyp} is the hyperbolic measure on $Y(\mathbb{C})$ normalized such that it is a probability measure.

Let $\overline{\mathcal{M}_{12}} \coloneqq (\mathcal{M}_{12}, \|\cdot\|_{\mathrm{Pet}})$. Then, the Faltings height can be brought to our framework by the relation [BMR18, Section 2.1]

$$h_{\mathrm{F}}(\alpha) = \frac{1}{12} h_{\overline{\mathcal{M}_{12}}}(\alpha).$$

Remark 6.12. This height is in fact the *stable* Faltings height of the elliptic curve of j-invariant α, and can also be defined using the Hodge bundle equipped a canonical metric [YZ26, Theorem 5.5.1].

The j-function induces an isomorphism $j\colon \mathcal{X} \cong \mathbb{P}^1_{\mathbb{Z}}$ over $\mathbb{Z}$, where

- the cusp point corresponds to ∞,
- the automorphic bundle $\mathcal{M}_{12}$ corresponds to $\mathcal{O}(1)$,
- the modular discriminant $\Delta \in \mathrm{H}^0(\mathcal{X}, \mathcal{M}_{12})$ corresponds to the section $x_1 \in \mathrm{H}^0(\mathbb{P}^1_{\mathbb{Z}}, \mathcal{O}(1))$.

Let $\|\cdot\|_{\mathrm{hyp}}$ be the metric on $\mathcal{O}(1)(\mathbb{C})$ corresponding to the Petersson metric on $M_{12}(\mathbb{C})$ and let $\mathrm{g}_{\mathrm{hyp}}\colon \mathbb{C} \longrightarrow \mathbb{R}$ be the induced Green function. By the previous identifications it is given by

$$\mathrm{g}_{\mathrm{hyp}}(z) = -\log \|\Delta(\tau_z)\|_{\mathrm{Pet}},$$

where $\tau_z \in \mathbb{H}$ is sent to z by j. Note that we have the asymptotic estimate [BMR18, Section 3.2]

$$\mathrm{g}_{\mathrm{hyp}}(z) = \log|z| - 6\log\log|z| + O(1), \quad \text{as } |z| \longrightarrow \infty,$$

so $\mathrm{g}_{\mathrm{hyp}}$ is a Green function in the sense of Definition 3.4. Thus Theorem 6.5 and Corollary 6.10 apply and we get

Theorem 6.13.

$$\sup_{\substack{n\geq 1\\ s\in \mathrm{H}^0(\mathbb{P}^1_{\mathbb{Z}},\mathcal{O}(n))\setminus\{0\}}} -\frac{\log\|s\|_{\mathrm{hyp},\infty}}{n} = 12\,\mathrm{ess}(h_{\mathrm{F}}) = \inf_{\mu\in \mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})} \int \mathrm{g}_{\mathrm{hyp}}\,\mathrm{d}\mu.$$

Theorem 6.14. *Let $\mathscr{P}^{\overline{\mathbb{Z}}}_{\log}\big(\mathcal{Y}(\mathbb{C})\big)$ be the set of probability measures that can be approximated* log*-weakly by Galois orbits in $\mathcal{Y}(\overline{\mathbb{Z}})$. Then*

$$\begin{aligned}\sup_{\substack{n\geq 1\\ f\in \mathcal{M}_{12n}(\Gamma,\mathbb{Z})\setminus\{0\}}} -\frac{\log\|f\|_{\mathrm{Pet},\infty}}{n} &= \sup_{\substack{n\geq 1\\ f\in \mathcal{M}_{12n}(\Gamma,\mathbb{Z})\setminus\{0\}}} -\frac{\log\|f\|_{L^2}}{n}\\ &= 12\,\mathrm{ess}(h_{\mathrm{F}})\\ &= \inf_{\mu\in \mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathcal{Y}(\mathbb{C}))} \int -\log\|\Delta\|_{\mathrm{Pet}}\,\mathrm{d}\mu.\end{aligned}$$

Proof. The first equality follows from [CGS21, Lemma 3.4.5], where it is proved that the distorsion of sup-norm and L^2-norm is subexponential. The rest are just a reformulation of Theorem 6.13. ☐

Remark 6.15. In [BMR18], the authors used

$$\sup_{\substack{n\geq 1\\ f\in \mathcal{M}_{12n}(\Gamma,\mathbb{Z})\setminus\{0\}}} -\frac{\log\|f\|_{\mathrm{Pet},\infty}}{n} \leq 12\,\mathrm{ess}(h_{\mathrm{F}}) \leq \inf_{\mathrm{cap}(K)=1} \int \mathrm{g}_{\mathrm{hyp}}\,\mathrm{d}\mu_K$$

to give numerical estimates of $\mathrm{ess}(h_{\mathrm{F}})$. Here we have proved that the lower bound indeed reaches $\mathrm{ess}(h_{\mathrm{F}})$, while for the upper bound to reach $\mathrm{ess}(h_{\mathrm{F}})$, we may need to consider more general measures $\mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$ than just equilibrium measures of compact sets of capacity one.

Theorem 6.6 specializes to

Theorem 6.16. *The essential minimum of Faltings' height can be attained by a sequence of elliptic curves with good reduction everywhere.*

In fact, Theorem 6.6 says that we can approach the essential minimum of Faltings' height by elliptic curves with integral j-invariants hence having potentially good reduction everywhere [Sil09, Chapter VII. Proposition 5.5]. So after a finite field extension they become good reduction everywhere.

7. The computability of the essential minimum

For a particular height function, it is not obvious how to compute its essential minimum. As a consequence of the strong duality Theorem 6.5, we can construct both a decreasing sequence and an increasing sequence converging to the essential minimum, thus showing that this invariant is a computable real number. The obtained algorithm is far from being practical, but might be a first step in the search of an effective procedure for this problem.

To make these ideas precise we will use the theory of computability. We refer the reader to [PR89] for the preliminaries on this theory.

7.1. **Computability.**

We first recall the notion of computable number in one of its equivalent definitions.

Definition 7.1. A real number r is *computable* if there exists a Turing machine that given any rational number $\varepsilon > 0$ produces a rational number q with $|r-q| < \varepsilon$. A complex number is *computable* if both its real and imaginary part are computable.

The set of computable numbers is a countable subfield of $\mathbb{C}$ that contains all algebraic numbers, and as a consequence most real numbers are non-computable. Nevertheless it is very difficult to give a concrete non-computable number. Examples of such numbers are Chaitin's constants, which are associated to Turing's halting problem.

There are weaker notions of computability.

Definition 7.2. A real number r is *left computable* if there is a Turing machine that given a natural number n produces a rational number x_n such that $\sup_n x_n = r$. Similarly r is *right computable* if there is a Turing machine that given n produces a rational number y_n such that $\inf_n x_n = r$.

Lemma 7.3. *A real number is computable if and only if it is both left and right computable.*

Proof. Assume that r is computable. Then there is a Turing machine that given n produces a rational number z_n with $|r - z_n| < 1/n$. The sequences $x_n = z_n - 1/n$ and $y_n = z_n + 1/n$ show that r is both left and right computable.

Now assume that r is both right and left computable and $0 < \varepsilon \in \mathbb{Q}$. Then there is a Turing machine that given a natural number N produces $\alpha_N = \max_{n \leq N} x_n$ and $\beta_N = \min_{n \leq N} y_n$. When $\beta_N - \alpha_N < \varepsilon$ we have reached the desired precision and the algorithm stops. □

Definition 7.4. A function $f\colon \mathbb{N} \longrightarrow \mathbb{N}$ is *computable* if there is a Turing machine that given input n produces the output $f(n)$.

The Green functions that will be well suited for our computations are those which are computable in appropriate compact sets and have effective asymptotics.

Definition 7.5. A Green function $\mathrm{g}\colon \mathbb{C} \longrightarrow \mathbb{R}$ is *computable* if it satisfies the conditions:

1. for any rational rectangle E (i.e. $E = \{z \in \mathbb{C} \colon a_1 \leq \mathrm{Re}(z) \leq b_1, a_2 \leq \mathrm{Im}(z) \leq b_2\}$ for rational numbers $a_1 < b_1, a_2 < b_2$) the restriction $\mathrm{g}|_E \colon E \longrightarrow \mathbb{R}$ is computable in the sense of [PR89, Section 0.3, Definition A];
2. there exists a computable function $f\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that

$$\big|\mathrm{g}(z) - \log|z|\big| \leq \frac{1}{n}\log|z| \quad \text{for all } z \in \mathbb{C} \text{ with } |z| > f(n).$$

The following is our main result in this section.

Theorem 7.6. *Let* g *be a computable Green function. Then* $\mathrm{ess}(h_{\mathrm{g}})$ *is a computable real number.*

The proof will be carried out in the next two sections, where we will see that the essential minimum is both left and right computable.

7.2. **Left computability.** By Proposition 6.9 we have

$$\mathrm{ess}(h_{\mathrm{g}}) = \sup_{\Lambda} \inf_{z\in\mathbb{C}} \Big(\mathrm{g}(z) - \sum_{i=1}^{k} a_i \log|Q_i(z)|\Big)$$

with

$$\Lambda = \Big\{(Q_1,\dots,Q_k,a_1,\dots,a_k) \,\Big|\, k\in\mathbb{N}, Q_i \in \mathbb{Z}[x]\setminus\{0\}, a_i\in\mathbb{Q}_+ \text{ and } \sum_{i=1}^{k} a_i \deg(Q_i) < 1\Big\}.$$

Since the index set Λ can be effectively enumerated, the left computability of $\mathrm{ess}(h_g)$ is a direct consequence of the next statement.

Proposition 7.7. *Assume that* g *is a computable Green function. Let* $(Q_1,\dots,Q_k,a_1,\dots,a_k)\in\Lambda$ *and set* $\varphi = \mathrm{g} - \sum_{i=1}^{k} a_i \log|Q_i|$. *Then* $\inf_{z\in\mathbb{C}}\varphi(z)$ *is a computable real number.*

Proof. Since g is a computable Green function and $\sum a_i \deg(Q_i) < 1$, we can determine a rational rectangle E such that the infimum of φ is attained in E. Similarly we can determine $M\in\mathbb{N}$ such that there is $z\in E$ with $\varphi(z)\le M$ and so

$$\inf_{z\in\mathbb{C}}\varphi(z) = \inf_{z\in E}\min\{\varphi(z),M\}.$$

Then this infimum is computable because the restriction of $\min\{\varphi,M\}$ to E is a computable function [PR89, Section 0.6, Theorem 7]. ☐

7.3. **Right computability.** For the right computability, we look at the other side of the strong duality property. By Theorem 6.5 and Theorem 4.5 we have

$$\mathrm{ess}(h_{\mathrm{g}}) = \inf_{(P,Q)\in\Theta} \int \mathrm{g}\, \mathrm{d}\mu_{P,Q}$$

for the index set

$$\Theta = \big\{(P,Q) \,\big|\, P,Q\in\mathbb{Z}[x]\setminus\mathbb{Z} \text{ with } P\neq Q \text{ monic and irreducible}\big\},$$

and where for each $(P,Q)\in\Theta$ we denote by $\mu_{P,Q}\in\mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$ the measure in (4.25).

Since the index set Θ can also be effectively enumerated, the right computability of the essential minimum follows from the next result.

Proposition 7.8. *Assume that* g *is a computable Green function and let* $(P,Q)\in\Theta$. *Then the integral* $\int \mathrm{g}\,\mathrm{d}\mu_{P,Q}$ *is computable.*

Proof. Set for short $d=\deg(P)$ and $e=\deg(Q)$ and consider the function

$$\rho\colon [0,1]\longrightarrow\mathbb{R},\quad \rho(\theta) = \frac{((\varphi_{P,Q})_*\mathrm{g})(e^{2\pi i\theta})}{d\,(e+1)}.$$

For each $\theta \in [0,1]$ consider also the polynomial $S_\theta = P^{\deg(Q)+1} - e^{2\pi i\theta} Q^{\deg(P)} \in \mathbb{C}[X]$. Then

$$\rho(\theta) = \frac{1}{d\,(e+1)} \sum_{S_\theta(w)=0} e_w \, \mathrm{g}(w). \tag{7.51}$$

There exists an effective $R \in \mathbb{Q}_{>0}$ such that $\varphi_{P,Q}^{-1}(S^1) \subseteq [-R,R] \times i[-R,R]$. Since g restricted to this rectangle is computable by definition and the process of finding the complete set of roots of polynomials is computable by [Spe69] (see also [BCR25, Proposition 3.13]) we have that ρ is a computable function, and so the Riemann integral

$$\int_0^1 (\varphi_{P,Q})_* \mathrm{g}\,(e^{2\pi i\theta})\,\mathrm{d}\theta$$

is computable [PR89, Section 0.5, Theorem 5]. $\square$

Appendix A. Duality in linear programming

In this appendix we present a coordinate-free formulation of duality in linear optimization and recall the proof of the strong duality property in the finite dimensional case. This allows to place the optimization problems from Section 5 within a general framework and to give a more conceptual approach to their duality properties.

Let E, F be two real vector spaces equipped with a pairing $\langle -,- \rangle \colon E \times F \longrightarrow \mathbb{R}$. Let $E \longrightarrow F^\vee$ and $F \longrightarrow E^\vee$ be the induced linear maps between these spaces and their duals, that we respectively denote by $x \longmapsto x^\dagger$ and $y \longmapsto y^\dagger$. They are defined by setting

$$x^\dagger(y) = y^\dagger(x) = \langle x, y\rangle \quad \text{for all } x \in E \text{ and } y \in F.$$

Consider two convex cones $\sigma \subset E$ and $\tau \subset F$. Their duals are the convex cones

$$\sigma^\vee = \{u \in E^\vee \mid u(x) \geq 0 \text{ for all } x \in \sigma\}, \quad \tau^\vee = \{v \in F^\vee \mid v(y) \geq 0 \text{ for all } y \in \tau\}.$$

Let also $u_0 \in E^\vee$ and $v_0 \in F^\vee$.

Definition A.1. The *primal problem* and the *dual problem* for this datum are the optimization problems respectively given by

$$\mathcal{P} = \inf\{u_0(x) \mid x \in \sigma,\ x^\dagger - v_0 \in \tau^\vee\}, \quad \mathcal{D} = \sup\{v_0(y) \mid y \in \tau,\ u_0 - y^\dagger \in \sigma^\vee\}.$$

We refer to the quantities $\mathcal{P}$ and $\mathcal{D}$ as the *optimal values* of these problems.

Remark A.2. The role of the primal and the dual problems can be exchanged: denote by $\mathcal{P}^{\mathrm{op}}$ and $\mathcal{D}^{\mathrm{op}}$ the primal and dual problems that arise when swapping the vector spaces E and F and considering the cones $-\tau \subset F$ and $-\sigma \subset E$ together with the functionals $-v_0 \in F^\vee$ and $-u_0 \in E^\vee$. Then it can be easily verified that $\mathcal{P} = -\mathcal{D}^{\mathrm{op}}$ and $\mathcal{D} = -\mathcal{P}^{\mathrm{op}}$.

The classical problems in linear programming are a particular case of this framework.

Example A.3. Set $E = \mathbb{R}^m$ and $F = \mathbb{R}^n$ and given $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ consider the pairing and functionals defined by

$$(x,y) \longmapsto \langle x,y\rangle = x^{\mathrm{T}} A\, y, \quad x \longmapsto u_0(x) = b^{\mathrm{T}} x, \quad y \longmapsto v_0(y) = c^{\mathrm{T}} y$$

together with the cones $\sigma = \mathbb{R}^m_{\geq 0}$ and $\tau = \mathbb{R}^n_{\geq 0}$. Then the associated primal and dual problems boil down to the usual forms

$$\mathcal{P} = \inf\{b^{\mathrm{T}} x \mid x \geq 0,\ A^{\mathrm{T}} x \geq c\}, \quad \mathcal{D} = \sup\{c^{\mathrm{T}} y \mid y \geq 0,\ A\, y \leq b\},$$

where $\leq$ and $\geq$ on these real vector spaces means that these inequalities hold coordinate-wise.

The problems from Section 5 also fit within this framework. Denote by $\mathscr{M}_{\log}(\mathbb{C})$ and $\mathscr{S}_{\log}(\mathbb{C})$ the cone and the vector space generated by $\mathscr{P}_{\log}(\mathbb{C})$. The elements of $\mathscr{M}_{\log}(\mathbb{C})$ are the measures on $\mathbb{C}$ that integrate the function $\log^+|z|$, whereas those of $\mathscr{S}_{\log}(\mathbb{C})$ are the differences of these measures.

We also let $\mathscr{P}'_{\log}(\mathbb{C})$ be the set of probability measures $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ that integrate the functions $\log|Q|$ for all $Q \in \mathbb{Z}[x]$, and we respectively denote by $\mathscr{M}'_{\log}(\mathbb{C})$ and $\mathscr{S}'_{\log}(\mathbb{C})$ the cone and vector space generated by this set of probability measures.

Example A.4. Set

$$E = \mathscr{S}'_{\log}(\mathbb{C}) \quad \text{and} \quad F = \mathbb{R} \oplus \bigoplus_{n\in\mathbb{N}} \mathbb{R}.$$

Fix an enumeration $Q_1, Q_2, Q_3, \dots$ of all nonconstant integer polynomials and recall that the elements of F are the tuples $a = (a_0, a_1, a_2, \dots)$ with $a_n = 0$ for all but a finite number of n's. We then consider the pairing $E \times F \longrightarrow \mathbb{R}$ defined by

$$(\mu, a) \longmapsto a_0 \int \mathrm{d}\mu + \sum_{n\in\mathbb{N}} a_n \int \log|Q_n| \, \mathrm{d}\mu.$$

We also consider the convex cones defined as

$$\sigma = \mathscr{M}'_{\log}(\mathbb{C}) \subset E \quad \text{and} \quad \tau = \{(a_0, a_1, \dots, a_n, \dots) \mid a_n \geq 0 \text{ for all } n > 0\} \subset F.$$

Let $\mathrm{g}\colon \mathbb{C} \longrightarrow \mathbb{R}$ be a continuous function that is asymptotically logarithmic at ∞ in the sense of Definition 3.4, and define the functionals $u_0 \in E^\vee$ and $v_0 \in F^\vee$ as

$$u_0(\mu) = \int \mathrm{g} \, \mathrm{d}\mu \quad \text{and} \quad v_0(a) = a_0.$$

The associated primal problem amounts to the minimization $\mathcal{P}(\mathrm{g}) = \inf_\mu \int \mathrm{g} \, \mathrm{d}\mu$ over the measures $\mu \in \mathscr{M}'_{\log}(\mathbb{C})$ such that

$$a_0 \Big(\int \mathrm{d}\mu - 1 \Big) + \sum_{n\in\mathbb{N}} a_n \int \log|Q_n| \, \mathrm{d}\mu \geq 0$$

for all $a_0 \in \mathbb{R}$ and $a_n \in \mathbb{R}_{\geq 0}$, $n \in \mathbb{N}$, with $a_n = 0$ for all but a finite number of n's. Since a_0 is arbitrary, this forces $\mu \in \mathscr{P}'_{\log}(\mathbb{C})$. Hence, this minimization is over the probability measures $\mu \in \mathscr{P}'_{\log}(\mathbb{C})$ such that $\int \log|Q_n| \, \mathrm{d}\mu \geq 0$ for all n, which coincide with those in $\mathscr{P}_{\log}(\mathbb{C})$ satisfying the same condition. Hence

$$\mathcal{P}(\mathrm{g}) = \inf \Big\{ \int \mathrm{g} \, \mathrm{d}\mu \,\Big|\, \mu \in \mathscr{P}_{\log}(\mathbb{C}), \int \log|Q_n| \, \mathrm{d}\mu \geq 0 \text{ for all } n \in \mathbb{N} \Big\}$$

as in (5.29). Similarly, the associated dual problem is the maximization $\mathcal{D}(\mathrm{g}) = \sup_a a_0$ over $a \in \tau$ such that

$$\int \mathrm{g} \, \mathrm{d}\mu - a_0 \int \mathrm{d}\mu - \sum_{n\in\mathbb{N}} a_n \int \log|Q_n| \, \mathrm{d}\mu \geq 0 \quad \text{for all } \mu \in \mathscr{M}'_{\log}(\mathbb{C}). \tag{A.52}$$

This is equivalent to the inequality $\mathrm{g}(z) - \sum_{n\in\mathbb{N}} a_n \log|Q_n(z)| \geq a_0$ for all $z \in \mathbb{C}$, as it can be seen by considering (A.52) for the Dirac delta measures $\mu = \delta_z$ for all $z \in \mathbb{C} \setminus \overline{\mathbb{Q}}$. Hence

$$\mathcal{D}(\mathrm{g}) = \sup_{a\in\tau} \inf_{x\in\mathbb{C}} \Big(\mathrm{g}(x) - \sum_{n\in\mathbb{N}} a_n \log|Q_n(x)| \Big),$$

in agreement with (5.30).

The weak duality property is the fact that the optimal value of the primal problem bounds above that of the dual, and follows readily from the definitions.

Proposition A.5. *We have* $\mathcal{P} \geq \mathcal{D}$.

Proof. For all $x \in \sigma$ and $y \in \tau$ such that $x^\dagger - v_0 \in \tau^\vee$ and $u_0 - y^\dagger \in \sigma^\vee$ we have $u_0(x) \geq \langle x, y\rangle \geq v_0(y)$, which gives the inequality. ☐

The strong duality property is the equality between the optimal values of the primal and the dual problem. We end this appendix by recalling the proof of this property in the finite dimensional situation.

Definition A.6. The *feasibility set* of the primal problem $\mathcal{P}$ and the dual problem $\mathcal{D}$ are

$$S_{\mathcal{P}} = \{x \in \sigma \mid x^\dagger - v_0 \in \tau^\vee\} \subset E \qquad \text{and} \qquad S_{\mathcal{D}} = \{y \in \tau \mid u_0 - y^\dagger \in \sigma^\vee\} \subset F.$$

We say that $\mathcal{P}$ (respectively $\mathcal{D}$) is *feasible* if $S_{\mathcal{P}} \neq \emptyset$ (respectively if $S_{\mathcal{D}} \neq \emptyset$), and we say that $\mathcal{P}$ (respectively $\mathcal{D}$) is *bounded* if the set $\{u_0(x) \colon x \in S_{\mathcal{P}}\}$ is bounded below (respectively if the set $\{v_0(y) \colon y \in S_{\mathcal{D}}\}$ is bounded above).

We also say that $\mathcal{P}$ (respectively $\mathcal{D}$) is *attained* if there exists $x \in S_{\mathcal{P}}$ such that $\mathcal{P} = u_0(x)$ (respectively if there exists $y \in S_{\mathcal{D}}$ such that $\mathcal{D} = v_0(y)$).

The primal problem is feasible and bounded if and only if $\mathcal{P} \in \mathbb{R}$, and similarly for the dual problem. Moreover, the weak duality property shows that if one of these problems is feasible then the other is bounded.

Theorem A.7. *Assume that E and F are finite dimensional vector spaces and that σ and τ are closed convex cones. The following conditions are equivalent:*

1. *the primal problem $\mathcal{P}$ is feasible and bounded;*
2. *the dual problem $\mathcal{D}$ is feasible and bounded.*

If any of these conditions holds then $\mathcal{P} = \mathcal{D} \in \mathbb{R}$ and both $\mathcal{P}$ and $\mathcal{D}$ are attained.

Proof. First assume that (1) holds and consider the closed convex subsets of $\mathbb{R} \times F^\vee$ defined as

$$V = \{(u_0(x), x^\dagger) \mid x \in \sigma\} \qquad \text{and} \qquad W_\lambda = \{(t, v + v_0) \mid t \leq \lambda,\ v \in \tau^\vee\} \qquad \text{for } \lambda \in \mathbb{R}.$$

For each $\lambda \in \mathbb{R}$ we have that $V \cap W_\lambda \neq \emptyset$ if and only if there exists $x \in S_{\mathcal{P}}$ such that $u_0(x) \leq \lambda$.

We have the decomposition $W_\lambda = \mathbb{R}_{\leq 0} \times \tau^\vee + (p, v_0)$. Hence considering the closed convex cone and the point respectively defined as

$$C = V - \mathbb{R}_{\leq 0} \times \tau = \{(u_0(x) - t, x^\dagger - v) : t \in \mathbb{R}_{\leq 0},\ x \in \sigma,\ v \in \tau^\vee\} \qquad \text{and} \qquad p_\lambda = (\lambda, v_0),$$

the condition $V \cap W_\lambda \neq \emptyset$ turns out to be equivalent to $p_\lambda \in C$. Since C is a closed cone, this condition on $\lambda \in \mathbb{R}$ is closed, and it is also nonempty and bounded below because $\mathcal{P}$ is feasible and bounded. Thus setting $\lambda_0 = \inf\{\lambda \mid p_\lambda \in C\}$ we have

$$\mathcal{P} = \lambda_0 = \min\{\lambda \mid p_\lambda \in C\} \in \mathbb{R}.$$

In particular $\mathcal{P}$ is attained.

Now let $\lambda < \lambda_0$. By the point-cone separation theorem there exists $h \in (\mathbb{R} \times F^\vee)^\vee = \mathbb{R} \oplus F$ such that $h|_C \geq 0$ and $h(p_\lambda) < 0$, which implies that $h|_V > h|_{W_\lambda}$. Hence writing $h = (b, -y)$ with $b \in \mathbb{R}$ and $y \in F$ we have

$$b\, u_0(x) - \langle x, y\rangle > b\, t - v(y) - v_0(y) \quad \text{for all } t \leq \lambda, x \in \sigma, v \in \tau^\vee. \tag{A.53}$$

Specializing (A.53) to $x = x_1 \in S_{\mathcal{P}} \subset \sigma$ and $v = v_1 = x_1^{\dagger} - v_0 \in \tau^{\vee}$ we get $b\, u_0(x_1) > b\, t$, which implies that $b > 0$ because t can be an arbitrarily large negative number.

We assume without loss of generality that $b = 1$. Then (A.53) specialized to $t = \lambda$ becomes

$$u_0(x) - \langle x, y\rangle > \lambda - v(y) - v_0(y) \quad \text{for all } x \in \sigma, v \in \tau^{\vee}. \tag{A.54}$$

Specializing this inequality to $x = 0$ gives $v(y) \geq 0$ for all $v \in \tau^{\vee}$, whereas taking instead $v = 0$ gives $(u_0 - y^{\dagger})(x) \geq 0$ for all $x \in \sigma$. Hence $u_0 - y^{\dagger} \in \sigma^{\vee}$, and since τ is assumed to be closed we also have that $y \in \tau$, and so the dual problem $\mathcal{D}$ is feasible. We also have that $\mathcal{D}$ is bounded because $\mathcal{P}$ is feasible, thus proving the condition (2). Moreover (A.54) specialized to $x = 0$ and $v = 0$ gives

$$\mathcal{D} \geq v_0(y) > \lambda,$$

and since λ can be arbitrarily close to $\mathcal{P}$ we obtain $\mathcal{P} \leq \mathcal{D}$. Combining with the weak duality property (Proposition A.5) we conclude that $\mathcal{P} = \mathcal{D}$, as stated.

Finally, the case when (2) holds we reduce to the previous situation using Remark A.2. ☐

Remark A.8. The proof of Theorem 5.3 follows this approach, with suitable modifications due to the fact that there we deal with some specific infinite dimensional vectors spaces and cones that are not necessarily closed.

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