CLOSING THE GAP AROUND THE ESSENTIAL MINIMUM OF HEIGHT FUNCTIONS WITH LINEAR PROGRAMMING

JOSÉ IGNACIO BURGOS GIL, RICARDO MENARES, BINGGANG QU, AND MARTÍN SOMBRA

Abstract. For many common height functions, the explicit determination of the essential minimum is an open problem. We consider a classical method to obtain lower bounds that goes back at least to C.J. Smyth, and a method to obtain upper bounds based on the knowledge of the limit distribution of integral points. We use an infinite dimensional linear programming scheme to show that that both methods agree in the limit, by showing that the principle of strong duality holds in our situation. As a corollary we prove that the essential minimum can be attained by sequences of algebraic integers.

Recent results by A. Smith and B. Orloski–N. Sardari, furnish a characterization of compactly supported measures that can be approximated by complete sets of conjugates of algebraic integers, in terms of infinitely many nonnegativity conditions. We establish an extension of this characterization to measures with non necessarily compact support. As an application of this result and of strong duality, we show that the essential minimum is a computable real number when the Green function used to define the height is computable. We systematically use potential theory for measures that can integrate functions with logarithmic growth.

Contents

1. Introduction 1
2. Potential theory and limit distribution of algebraic integers 9
3. The sweetened truncation 14
4. Approximating measures by algebraic integers and by capacity one compact sets 19
5. Strong duality 21
6. On the essential minimum of heights 25
7. The computability of the essential minimum 33
Appendix A. Duality in linear programming 37
References 40

1. Introduction

1.1. **The essential minimum.** For many common height functions, the explicit determination of the *essential minimum* (see (1.3)) is an open problem. We consider a classical method to obtain lower bounds, that goes back at least to C.J. Smyth, and a method to

J. I. Burgos was partially supported by grants PID2022-142024NB-I00 and CEX2023-001347-S funded by MICIU/AEI/10.13039/501100011033. R. Menares was partially supported by ANID FONDECYT Regular grant 1250734. B. Qu was partially supported by grant PID2022-142024NB-I00 funded by MICIU/AEI/10.13039/501100011033. M. Sombra was partially supported by the MICINN research project PID2019-104047GB-I00, the AGAUR research project 2021-SGR-01468 and the AEI project CEX2020-001084-M of the María de Maeztu program for centers and units of excellence in R&D.

obtain upper bounds based on the knowledge of the limit distribution of integral points used in [BGMRL18]. In this work we use an infinite dimensional linear programming scheme to show that both methods agree in the limit, by showing that the principle of strong duality holds in our situation. This allows us to prove that the essential minimum can be attained by sequences of algebraic integers.

We also establish a characterization, in terms of infinitely many nonnegativity conditions, of probability measures on $\mathbb{C}$ with non necessarily compact support that can be approximated, in a suitable sense, by sequences of complete sets of conjugates of distinct algebraic integers. This characterization, in the case of measures with compact support, was previously established by A. Smith and B. Orloski–N. Sardari. As an application of this result and of strong duality, we show that the essential minimum is a computable real number when the Green function used to define the height is computable. These issues can be considered on any variety X defined over a number field, but in this work we will focus in the cases $X = \mathbb{P}^1$ or $X = \mathbb{A}^1$.

We proceed to state our results precisely. We denote by $\overline{\mathbb{Q}}$ the set of all algebraic numbers. Given $\alpha \in \overline{\mathbb{Q}}$, we denote by $P_\alpha(x) \in \mathbb{Z}[x]$ a primitive irreducible polynomial having α as a root. The polynomial P_α is determined by α up to sign. Its set of roots $O(\alpha)$ is called the *complete set of conjugates* of α.

We say that $\mathrm{g}\colon \mathbb{C} \longrightarrow \mathbb{R}$ is a *Green function*, if it is continuous, invariant under complex conjugation and obeys the asymptotic

$$\mathrm{g}(z) = \log|z| + o(\log|z|), \text{ as } |z| \longrightarrow \infty. \tag{1.1}$$

Such functions correspond to metrics on the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^1$ with certain singularities at infinity (see Section 6.1). We define the height function $\mathrm{h_g}\colon \overline{\mathbb{Q}} \longrightarrow \mathbb{R}$ associated to such g by the rule

$$\mathrm{h_g}(\alpha) = \frac{1}{\#O(\alpha)} \left(\log|a_\alpha| + \sum_{\beta \in O(\alpha)} \mathrm{g}(\beta) \right). \tag{1.2}$$

Here, $a_\alpha \in \mathbb{Z}$ is the leading coefficient of P_α.

Height functions measure the arithmetic complexity of algebraic numbers, due to two key properties: the function $\mathrm{h_g}$ is bounded from below and for any $A, B > 0$ the set

$$\left\{\alpha \in \overline{\mathbb{Q}} \mid \mathrm{h_g}(\alpha) < A,\ \#O(\alpha) < B\right\} \quad \text{is finite.}$$

The *essential minimum* of $\mathrm{h_g}$ is defined as

$$\mathrm{ess}(\mathrm{h_g}) \coloneqq \inf\left\{ \liminf_{n\to\infty} \mathrm{h_g}(\alpha_n) \,\middle|\, (\alpha_n) \subseteq \overline{\mathbb{Q}} \text{ is a sequence of distinct algebraic numbers} \right\}. \tag{1.3}$$

It follows directly from the definitions that $\inf(\mathrm{h_g}) \le \mathrm{ess}(\mathrm{h_g})$. Also, it can be shown that the image of $\mathrm{h_g}$ is dense in $[\mathrm{ess}(\mathrm{h_g}), \infty)$ (see Remark 6.8), so $\mathrm{ess}(\mathrm{h_g})$ coincides with the smallest limit point in $\mathrm{h_g}(\overline{\mathbb{Q}})$.

When $\mathrm{g}(z) = \log^+|z|$, we obtain the Weil (or naïve) height $\mathrm{h}_W = \mathrm{h}_{\log^+}$. Clearly $\mathrm{h}_W \ge 0$ and for any root of unity ζ, it holds $\mathrm{h}_W(\zeta) = 0$. This shows that $\inf(\mathrm{h}_W) = \mathrm{ess}(\mathrm{h}_W) = 0$ and the closure of the image of h_W is $[0, \infty)$.

The essential minimum of a height function plays a role in the subject of equidistribution of small points. Indeed, a generic sequence of algebraic points is *small* if its height converges to the essential minimum. A theorem of Yuan [Yua08, Theorem 3.1], subsuming previous results by many mathematicians, ensures that under an appropriate hypothesis on the essential minimum, small points are equidistributed with respect to a measure depending only on the height under consideration. There are many situations where the hypothesis in Yuan's theorem does not hold, and hence the asymptotic distribution of small points is not known. Very few situations beyond Yuan's theorem are well understood, see [cBS25], [BGPRLS19] and [Küh22].

In the case of $\mathbb{P}^1$ considered here, the hypothesis in Yuan's theorem boils down to $\inf(\mathrm{h_g}) = \mathrm{ess}(\mathrm{h_g})$. However, for many common heights the strict inequality $\inf(\mathrm{h_g}) < \mathrm{ess}(\mathrm{h_g})$ holds and both the explicit determination of $\mathrm{ess}(\mathrm{h_g})$ and the asymptotic distribution of small points, are open problems. An illustrative example is the Zhang-Zagier height, defined as follows. Extend the Weil height to $\mathbb{A}^2(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}}^2$ by $\mathrm{h}^2_W(\alpha,\beta) = \mathrm{h}_W(\alpha) + \mathrm{h}_W(\beta)$. If $X \subsetneq \mathbb{A}^2$ is any subvariety, we define a height h_X on X by $\mathrm{h}_X := (\mathrm{h}^2_W)_{|X}$. Then, a theorem of Zhang [Zha92] ensures that $\mathrm{ess}(\mathrm{h}_X) > 0$ if and only if X does not contain a torsion translate of a toric subvariety. For concreteness, Zagier took $X = \{x + y = 1\} \subset \mathbb{A}^2$ and studied in [Zag93] the resulting height, that we denote by $\mathrm{h_{ZZ}}$. In our framework, identifying X with $\mathbb{A}^1$ by projection on one coordinate, the choice

$$\mathrm{g}(z) = \frac{1}{2}\Big(\log^+|z| + \log^+|1-z|\Big)$$

leads to a height $\mathrm{h_g}$ such that $\mathrm{h_{ZZ}} = 2\,\mathrm{h_g}$. In *loc. cit.*, it is obtained a lower bound for $\mathrm{ess}(\mathrm{h_{ZZ}})$. These results were improved by Doche ([Doc01a], [Doc01b]), who also gives upper bounds for $\mathrm{ess}(\mathrm{h_{ZZ}})$, obtaining

$$\log(1.2817770214) \leq \mathrm{ess}(\mathrm{h_{ZZ}}) \leq \log(1.289735). \tag{1.4}$$

An important height function is the stable Faltings height on elliptic curves. Namely, $\mathrm{h_F}\colon \overline{\mathbb{Q}} \longrightarrow \mathbb{R}$ is defined as

$$\mathrm{h_F}(\alpha) = \mathrm{h_{Fal}}(E_\alpha),$$

where E_α is an elliptic curve with j-invariant $j(E_\alpha) = \alpha$ and $\mathrm{h_{Fal}}(E_\alpha)$ is the stable Faltings height of the elliptic curve E_α. This height is also an explicit multiple of $\mathrm{h_g}$ for an appropriate g, see section 6.5. In [Löb17] is obtained a lower bound that shows $\inf(\mathrm{h_F}) < \mathrm{ess}(\mathrm{h_F})$ and both lower and upper bounds for $\mathrm{ess}(\mathrm{h_F})$ are obtained in [BGMRL18]. In particular, it follows from *loc. cit.* that

$$-0.748629 \leq \mathrm{ess}(\mathrm{h_F}) \leq -0.748622. \tag{1.5}$$

We denote by $\overline{\mathbb{Z}}$ the set of all algebraic integers (recall that $\alpha \in \overline{\mathbb{Q}}$ is an *algebraic integer* if the polynomial P_α can be taken to be monic). In both estimations (1.4) and (1.5), only algebraic integers were used. A surprising consequence of our main result (Theorem B below) and our study of limit distributions of algebraic integers (Theorem C below) is that, even though $\overline{\mathbb{Q}} - \overline{\mathbb{Z}}$ is an infinite set, sequences of algebraic integers are already enough to reach $\mathrm{ess}(\mathrm{h_g})$ from above.

Theorem A (Corollary 6.7 and Corollary 6.9)**.** *Let* $\mathrm{g}\colon \mathbb{C} \longrightarrow \mathbb{R}$ *be a Green function. Then,* $\mathrm{h_g}(\overline{\mathbb{Z}})$ *is dense in* $[\mathrm{ess}(\mathrm{h_g}), \infty)$*. In particular, there exists a sequence* $(\alpha_n) \subseteq \overline{\mathbb{Z}}$ *of distinct algebraic integers such that* $\mathrm{h_g}(\alpha_n)$ *is monotonically decreasing and* $\inf_n \mathrm{h_g}(\alpha_n) = \mathrm{ess}(\mathrm{h_g})$.

A direct consequence of this statement is that the essential minimun of the Faltings height $\mathrm{h_{Fal}}$ can be attained by a sequence of elliptic curves having good reduction everywhere.

1.2. **Strong duality.** The explicit determination of $\mathrm{ess}(\mathrm{h_g})$ from the definition requires to exhibit a *sequence of small points*. That is, a sequence of distinct algebraic numbers (α_n) such that $\lim_{n\to\infty} \mathrm{h_g}(\alpha_n) = \mathrm{ess}(\mathrm{h_g})$. For the Weil height, roots of unity do the job. When g is a radial function, an useful characterization is obtained in [BGPS15], in terms of convex analysis. But for many common heights (such as $\mathrm{h_{ZZ}}$ and $\mathrm{h_F}$) no explicit sequence is known and the essential minimum remains unknown.

We propose in this article a characterization of the essential minimum in terms of the common optimal value of two linear optimization problems which are in duality.

We start by reviewing a method to obtain upper bounds used in [BGMRL18]. The basic idea is to shift from the search of sequences of small points to the determination of measures describing the expected asymptotic distribution of such sequences. In order to make this idea rigorous we introduce a class of measures and a notion of convergence adapted to this situation.

Let $\mathscr{P}_{\log}(\mathbb{C})$ be the space of probability measures μ on $\mathbb{C}$ such that

$$\int \log^+|z|\,\mathrm{d}\mu < \infty.$$

We say a function $f\colon \mathbb{C} \longrightarrow \mathbb{R}$ has *logarithmic growth* if there exist constants A, B such that for all $z \in \mathbb{C}$, we have $|f(z)| \le A + B\log^+|z|$. We say that a sequence of measures $(\mu_n) \subset \mathscr{P}_{\log}(\mathbb{C})$ converges log-*weakly* to $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ if for any continuous function f with logarithmic growth, it holds that

$$\lim_{n\to\infty} \int f\,\mathrm{d}\mu_n = \int f\,\mathrm{d}\mu.$$

To any $\alpha \in \overline{\mathbb{Z}}$ we attach the probability measure

$$\delta_{O(\alpha)} = \delta_{P_\alpha} := \frac{1}{\#O(\alpha)} \sum_{\beta\in O(\alpha)} \delta_\beta.$$

We denote by $\mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$ the subset of $\mathscr{P}_{\log}(\mathbb{C})$ consisting of the limit points of $\{\delta_{O(\alpha)} \mid \alpha \in \overline{\mathbb{Z}}\}$ in the log-weak topology. Below we give structural information on the space $\mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$, but for the moment let us focus on the upper bounds. The basic observation is that for $\alpha \in \overline{\mathbb{Z}}$, the term $\log|a_\alpha|$ in (1.2) vanishes and so $\mathrm{h_g}(\alpha)$ is just the average value of g over $O(\alpha)$. Then, for any measure $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ and sequence (α_n) of distinct algebraic integers such that $\delta_{O(\alpha_n)}$ log-weakly converges to μ, in view of (1.1) we can conclude

$$\lim_{n\to\infty} \mathrm{h_g}(\alpha_n) = \int \mathrm{g}\,\mathrm{d}\mu.$$

Hence,

$$\mathrm{ess}(\mathrm{h_g}) \le \int \mathrm{g}\,\mathrm{d}\mu \text{ and } \int \mathrm{g}\,\mathrm{d}\mu \text{ is in the real closure of } \mathrm{h_g}(\overline{\mathbb{Z}}). \tag{1.6}$$

The best upper bound we can obtain this way is

$$\mathcal{P}(\mathrm{g}) := \inf\left\{ \int \mathrm{g}\,\mathrm{d}\mu \;\middle|\; \mu \in \mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C}) \right\}.$$

Now we describe the method to obtain lower bounds. Let $P_1, \ldots, P_k \in \mathbb{Z}[x]$ be polynomials with integer coefficients and let $a_1, \ldots, a_k$ be non negative real numbers. Set

$$f(z) \coloneqq \mathrm{g}(z) - \sum_{i=1}^{k} a_i \log |P_i(z)|.$$

The basic observation in Smyth's method [Smy81] is that for every $\alpha \in \overline{\mathbb{Q}}$ that is not a zero of any of the polynomials P_i, it holds

$$\mathrm{h_g}(\alpha) \geq \inf_{z\in\mathbb{C}} f(z).$$

Hence, $\mathrm{ess}(\mathrm{h_g}) \geq \inf f$. The best lower bound that we can obtain in this way is

$$\mathcal{D}(\mathrm{g}) \coloneqq \sup \left\{ \inf_{z\in\mathbb{C}} \left(\mathrm{g}(z) - \sum_{i=1}^{k} a_i \log |P_i(z)| \right) \,\middle|\, k \geq 0, a_i \in \mathbb{R}_{\geq 0}, P_i \in \mathbb{Z}[x] \right\}.$$

Our next result is that both $\mathcal{D}(\mathrm{g})$ and $\mathcal{P}(\mathrm{g})$ attain the essential minimum.

Theorem B (Strong duality, Theorem 6.6)**.** *Let* $\mathrm{g}\colon \mathbb{C} \longrightarrow \mathbb{R}$ *be a Green function. Then*

$$\mathcal{D}(\mathrm{g}) = \mathrm{ess}(\mathrm{h_g}) = \mathcal{P}(\mathrm{g}).$$

Assuming g is subharmonic and $\mathrm{g}(z) = \log|z| + a + o(1)$ at infinity, the equality $\mathcal{D}(\mathrm{g}) = \mathrm{ess}(\mathrm{g})$ is the specialization to $\mathbb{P}^1$ of a theorem of Ballaÿ [Bal21, Theorem 1.2]. Our result gives a new proof of this special case.

Our method of proof of Theorem B is based on the interpretation of $\mathcal{P}(\mathrm{g})$ and $\mathcal{D}(\mathrm{g})$ as the optimal values of two linear programming problems which are in duality. They are akin to those considered by Smyth [Smy84], Smith [Smi24] and Orloski–Sardari–Smith [OSS24] in their works on totally real algebraic integers of small trace, which were indeed a major source of inspiration for this part of our work. The *weak duality* $\mathcal{D}(\mathrm{g}) \leq \mathcal{P}(\mathrm{g})$ trivially holds. The content of Theorem B is the statement that the *strong duality* $\mathcal{D}(\mathrm{g}) = \mathcal{P}(\mathrm{g})$ holds.

1.3. **Approximation of measures by algebraic integers.** Smyth in [Smy81] shows that the smallest limit point of the image of the Weil height restricted to totally real algebraic integers is upper bounded by an integral similar to (1.6), for a certain explicit measure μ on the real line. He constructs his μ by first constructing an explicit sequence of points with small asymptotic height and then computing their limit distribution. In this work, in contrast, we require to be able to decide whether a given measure μ belongs to $\mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C})$ without necessarily constructing the sequence of small points. In a recent breakthrough, Smith and Orloski–Sardari obtain an useful characterization of this kind for measures of compact support. We extend their characterization to the case of non necessarily compact support, which allows us to obtain structural information on the space $\mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C})$. We proceed to review the compact support setting and then present our results.

We first remark that for our method on upper bounds to work, the approximation of measures by algebraic integers in the traditional weak-* topology is unsuitable. Indeed, weak convergence allows only for test functions which are continuous and bounded, and our g does not belong to such space. We are thus led to strengthen the required notion of approximation. A first possibility is to consider the *proper convergence*: a sequence of measures (μ_n) properly converges to a measure μ if (μ_n) weakly converges to μ and moreover there exists a compact set K such that $\mathrm{supp}(\mu_n) \subseteq K$ for all n. Also, we denote by $\mathscr{P}_c(\mathbb{C})$ the space of probability measures on $\mathbb{C}$ with compact support, endowed with the topology of the proper convergence. We denote by $\mathscr{P}_c^{\overline{\mathbb{Z}}}(\mathbb{C})$ the subset of $\mathscr{P}_c(\mathbb{C})$ consisting of the limit points of $\left\{\delta_{O(\alpha)} \mid \alpha \in \overline{\mathbb{Z}}\right\}$ in the topology of the proper convergence.

A plethora of measures in $\mathscr{P}_c^{\overline{\mathbb{Z}}}(\mathbb{C})$ can be constructed using potential theory. Indeed, let $K \subseteq \mathbb{C}$ be a compact set which is conjugation invariant and assume that the capacity of K satisfies $\operatorname{cap}(K) = 1$. We denote by μ_K the equilibrium measure of K. Then, combining a landmark result by Fekete and Szegő [FS55] and a theorem of Rumely in [Rum99], it can be shown that $\mu_K \in \mathscr{P}_c^{\overline{\mathbb{Z}}}(\mathbb{C})$ (see Section 2.1). Moreover, among the measures with support in a compact set K with $\operatorname{cap}(K) = 1$ and invariant under complex conjugation, only the equilibrium measure μ_K belongs to $\mathscr{P}_c^{\overline{\mathbb{Z}}}(\mathbb{C})$.

Smith and Orloski–Sardari provide an elegant characterization of the set $\mathscr{P}_c^{\overline{\mathbb{Z}}}(\mathbb{C})$. The following statement is a weak version of [Smi24, Theorem 1.5] and [OS24a, Theorem 1.2] (in fact, they prove stronger statements in the Fekete–Szegő vein, see Remark 2.9).

Theorem 1.1 (Smith, Orloski–Sardari)**.** *Assume $\mu \in \mathscr{P}_c(\mathbb{C})$ is invariant under complex conjugation. Then, $\mu \in \mathscr{P}_c^{\overline{\mathbb{Z}}}(\mathbb{C})$ if and only if*

$$\int \log |Q| \, \mathrm{d}\mu \geq 0, \text{ for any } Q \in \mathbb{Z}[x].$$

This characterization directly implies that $\mathscr{P}_c^{\overline{\mathbb{Z}}}(\mathbb{C})$ is a convex set and that $\mu_K \in \mathscr{P}_c^{\overline{\mathbb{Z}}}(\mathbb{C})$ for every compact conjugation invariant K with $\operatorname{cap}(K) \geq 1$. On the other hand, Fekete [Fek23] showed that whenever K has $\operatorname{cap}(K) < 1$, there is an open neighborhood U of K such that only finitely many complete sets of conjugates of algebraic integers lie inside U. In particular, $\mu_K \notin \mathscr{P}_c^{\overline{\mathbb{Z}}}(\mathbb{C})$ for such K.

Recall that for any $\mu \in \mathscr{P}_c(\mathbb{C})$, the potential $U^\mu \colon \mathbb{C} \longrightarrow \mathbb{R} \cup \{\infty\}$ is defined as

$$U^\mu(z) = \int \log \frac{1}{|w - z|} \, \mathrm{d}\mu(w). \tag{1.7}$$

Our next result is an extension of Theorem 1.1 beyond measures of compact support. We focus on the space $\mathscr{P}_{\log}(\mathbb{C})$, which is the largest class of probability measures μ on $\mathbb{C}$, with support non necessarily compact such that (1.7) is still a well defined function. Aspects of potential theory in $\mathscr{P}_{\log}(\mathbb{C})$ have been developed for some time, see for example [BLW15],[ORSLW19]. We show that $U^{\mu_n} \longrightarrow U^\mu$ as distributions if and only if $\mu_n \longrightarrow \mu$ log-weakly (combine Proposition 2.14, Lemma 2.4 and Lemma 2.1 in the text), so arguably the log-weak convergence is the right topology on $\mathscr{P}_{\log}(\mathbb{C})$ to do potential theory.

We prove that

Theorem C (Theorem 4.1)**.** *Assume $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ is invariant under complex conjugation. Then, $\mu \in \mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C})$ if and only if*

$$\int \log |Q| \, \mathrm{d}\mu \geq 0 \text{ for any } Q \in \mathbb{Z}[x]. \tag{1.8}$$

As before, we deduce from Theorem C that $\mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C})$ is a convex set.

Even though (1.8) may seem daunting because it refers to infinitely many inequalities, we can construct a countable (log-weakly) dense set of elements in $\mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C})$ as follows. Let $P \neq Q \in \mathbb{Z}[x]$ be two monic and irreducible polynomials. We denote by $\mu_{P,Q}$ the probability measure such that

$$U^{\mu_{P,Q}}(z) = -\log\max\left\{\frac{|P(z)|}{\deg P}, \frac{|Q(z)|}{\deg Q+1}\right\}.$$

It is supported on the compact lemniscate $\{z \in \mathbb{C} \,|\, |P(z)|^{\deg Q+1} = |Q(z)|^{\deg P}\}$ and is nothing but the pullback of the Haar measure on the unit circle by the rational map

$$P^{\deg Q+1}/Q^{\deg P} \colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1.$$

We show that $\mu_{P,Q}$ belongs to $\mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$ and the set of all such measures is dense in $\mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$ with respect to the log-weak topology (Proposition 7.9). This gives a partial answer to (the analogue in our context of) a question of Sarnak alluded to in [OS24b, Section 1.1].

Also, we establish a characterization of the log-weak closure of the set of equilibrium measures of capacity one compact sets as the set of measures with negative potential.

Theorem D (Theorem 4.4). *Let $\mu \in \mathscr{P}_{\log}(\mathbb{C})$. Then, the followings are equivalent*

1. $U^\mu \leq 0$,
2. *there exists a sequence of compact subsets K_n of capacity one, such that $\mu_{K_n} \longrightarrow \mu$* log-*weakly.*

We remark that not all measures satisfying the equivalent conditions in Theorem D have compact support. A first example would be the measure μ_{FS} induced by the Fubini-Study form

$$\omega_{\mathrm{FS}} = \frac{i}{2\pi}\frac{\mathrm{d}z\mathrm{d}\bar{z}}{(|z|^2+1)^2}$$

that is clearly not compactly supported. On the other hand, μ_{FS} has potential

$$U^{\mu_{\mathrm{FS}}}(z) = -\frac{1}{2}\log(1+|z|^2) \leq 0,$$

so $\mu_{\mathrm{FS}} \in \mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$ by Corollary 4.3.

Also, using Theorem D, we can deduce that the measures obtained from the Fekete–Szegő construction and Rumely's theorem do not exhaust all the elements in $\mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$. Indeed, consider the measure μ_{P_0,Q_0} for the choice $P_0(x) = x-1, Q_0(x) = x^2-x+1$. Since $P_0(1/2) = -1/2, Q_0(1/2) = 3/4$, we see that $U^{\mu_{P_0,Q_0}}(1/2) > 0$. By Theorem D, this measure cannot be log-weakly approximated by equilibrium measures of compact sets of capacity one.

In [BGMRL18], it was used the upper bound

$$\mathrm{ess}(\mathrm{h_g}) \leq \inf\left\{\int \mathrm{g}\,\mathrm{d}\mu_K \,\middle|\, K \text{ is a conjugation invariant compact set with } \mathrm{cap}(K) = 1\right\} \tag{1.9}$$

In view of Theorem D and the previous discussion, the closure of the space of conjugation invariant measures of the form μ_K with $\mathrm{cap}(K) = 1$ is strictly contained in $\mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$, so for many functions g the bound (1.9) will not reach $\mathrm{ess}(\mathrm{h_g})$. See Section 6.4 for an example.

1.4. **Computability of the essential minimum.** With regards to the Zhang-Zagier height and the Faltings height discussed at the beginning, a practical algorithm to obtain, in a reasonable time frame, higher accuracy in the estimations (1.4) or (1.5), is not currently known. In practice, after a few iterations, both Smyth's lower bound procedure and the upper bounds procedure described here reach a point where it is unclear how to continue, due to the extremely big size of the search space and the lack of an efficient criterion to find the optimal direction (e.g. see the discussions in [Zag93, Section 3], [Doc01a, Section 5], [Doc01b, p.110] and [BGMRL18, Ancillary file[1]]).

Loosely speaking, a real number λ is *computable* if there exist an algorithm that, for any given precision, after a finite number of steps gives a rational approximation of λ within the required precision. Here, the algorithm is not constrained to run in a reasonable time frame, in any sense. Even so, the set of computable real numbers is countable, so most real numbers are not computable. In view of the previous discussion, this raises the following question: is the essential minimum a computable real number? That is, does there exists an algorithm that, in a finite amount of time, gives us the essential minimum with arbitrary precision?

As a consequence of Theorems B and C, we can show the following computability result.

Theorem E (Theorem 7.6)**.** *Let* $\mathrm{g}\colon \mathbb{C} \longrightarrow \mathbb{R}$ *be a computable Green function. Then* $\mathrm{ess}(\mathrm{h_g})$ *is a computable real number.*

We refer the reader to Section 7 for the precise definitions of computability we use here. The idea is to show that both $\mathcal{D}(\mathrm{g})$ and $\mathcal{P}(\mathrm{g})$ can be taken inside a computable countable set (Proposition 6.10 and Proposition 7.9). Hence, one can set up an algorithm producing an increasing sequence of lower bounds by day and a decreasing sequence of upper bounds by night. When the bounds are less-than-epsilon apart, the procedure has found an approximation of the required precision.

We remark that both upper and lower bounds are required in order to have a halting criterion. On the other hand, the algorithm in our proof depends on a choice of enumeration of the aforementioned countable sets, so it is hopelessly ineffective.

1.5. **Methods and organization of the paper.** In Section 2 we review the preliminaries on potential theory (for non-compactly supported measures) and the theorems on limit distribution of algebraic integers by Fekete–Szegő and by Smith and Orloski–Sardari. We establish basic properties of the log-weak convergence. In particular, we show that we can do diagonal arguments in this topology. We also recall the L^1_{loc} convergence and prove that the log-weak convergence of measures corresponds to the distributional convergence of potentials.

In Section 3 we define the *sweetened truncation* and prove related technical estimates. In the proof of both Theorems B and C we use the technique of sweetened truncation to reduce to the case of compact support. The construction inspired in Smith's concept of sweetened measure [Smi24, Definition 5.5]. The idea is, if $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ and $R > 1$, then the sweetened truncation μ_R^{sw} will be a measure whose support is contained in the ball $\{|z| \le R\}$ and have two key properties: whenever μ satisfies (1.8), then μ_R^{sw} also satisfies (1.8) and $\mu_R^{\mathrm{sw}} \longrightarrow \mu$ log-weakly as $R \longrightarrow \infty$.

In Section 4 we prove Theorem C. Roughly speaking, we use the sweetened truncation to reduce to the case of compact support, where we can use Theorem 1.1, and then take a limit when the radius of the truncation grows. In Section 4.2 we prove Theorem D, where a key ingredient is the L^1_{loc} convergence of the previous section.

In Section 5 we prove a strong duality statement for general continuous functions which are asymptotically logarithmic at infinity.

In Section 6 we introduce the language of Arakelov theory to study heights and relate the results from the previous section to our context. The output is a proof of Theorem B. Also, we review in detail the case of the Faltings height (Section 6.5).

[1]arXiv:1609.00071

In Section 7 we prove Theorem E. After reviewing the necessary concepts from computability theory, the main content of this part is the proof that measures of the form $\mu_{P,Q}$ form a countable log-weak dense subset in $\mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$.

Even though it is not necessary for the proof of Theorem B, for the convenience of the reader we present in Appendix A a coordinate-free formulation of duality in linear optimization that allows to place the optimization problems from Section 5 within a general framework, furnishing a conceptual way to deduce the shape of the dual problem $\mathcal{D}(\mathrm{g})$ from the primal problem $\mathcal{P}(\mathrm{g})$.

Acknowledgments. The authors would like to thank François Ballaÿ, Nuno Hultberg, Norman Levenberg, Mayuresh Londhe, Joaquim Ortega Cerdà, Cristóbal Rojas and Michał Szachniewicz for illustrating discussions. We thank especially Nuno Hultberg for sharing with us the example in Section 6.4. We also thank the organizers of the conference "Arithmetic and Algebraic Geometry week", held at U. Alexandru Ioan Cuza in September 2025, where part of this work was carried.

2. Potential theory and limit distribution of algebraic integers

2.1. Potential theory on the complex plane and the Fekete–Szegő theorem.

Let $\mathscr{P}(\mathbb{C})$ be the set of all probability Borel measures on $\mathbb{C}$ and let $\mathscr{P}_{\log}(\mathbb{C})$ be the subspace of those that can integrate $\log^+|z|$, i.e.

$$\mathscr{P}_{\log}(\mathbb{C}) \coloneqq \left\{ \mu \in \mathscr{P}(\mathbb{C}) \;\middle|\; \int \log^+|z| \,\mathrm{d}\mu < +\infty \right\}.$$

Let $\mathscr{P}_c(\mathbb{C})$ be the set of compactly supported measures. Then $\mathscr{P}_c(\mathbb{C}) \subseteq \mathscr{P}_{\log}(\mathbb{C})$.

Potential theory of measures with non-necessarily compact support has been developed for some time, see for example [BLW15],[ORSLW19]. We recall the basic facts. For any $\mu \in \mathscr{P}_{\log}(\mathbb{C})$, we define its *potential function* as

$$U^\mu \colon \mathbb{C} \longrightarrow \mathbb{R} \cup \{\infty\}, \quad U^\mu(z) \coloneqq \int -\log|w-z| \,\mathrm{d}\mu(w).$$

We justify that U^μ is well defined as follows: since μ can integrate $\log^+|z|$, the integral $\int_{|w-z|\geq 1} -\log|w-z| \,\mathrm{d}\mu(w)$ is absolutely convergent. On the other hand, the monotone convergence theorem ensures that

$$\int_{|w-z|\leq 1} -\log|w-z| \,\mathrm{d}\mu(z) = \lim_{M\to\infty} \int_{|w-z|\leq 1} \min\{M, -\log|w-z|\} \,\mathrm{d}\mu(w)$$

is a well defined element in $\mathbb{R} \cup \{\infty\}$.

Lemma 2.1. *Let $\mu \in \mathscr{P}_{\log}(\mathbb{C})$. Then, $-U^\mu$ is subharmonic. Hence it is locally integrable.*

Proof. The first statement is [BLW15, Lemma 3.2] and the second is a classical result on subharmonic functions. □

To be on the safe side, we next check that Fubini's theorem can be applied to potentials of measures in $\mathscr{P}_{\log}(\mathbb{C})$.

Lemma 2.2. *Let $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ and let $\varphi \in C_c^\infty(\mathbb{C})$. Then*

$$\iint \varphi(z) \log|w-z|^{-1} \mathrm{d}\mu(w)\mathrm{d}\lambda(z) = \iint \varphi(z) \log|w-z|^{-1} \mathrm{d}\lambda(z)\mathrm{d}\mu(w).$$

Proof. It is enough to check that

$$\iint \left|\varphi(z) \log |w-z|^{-1}\right| \mathrm{d}\lambda(z)\mathrm{d}\mu(w) < +\infty \tag{2.10}$$

Since $\log |z|$ is locally integrable, the function $\int |\varphi(z) \log |w-z|^{-1}| \mathrm{d}\lambda(z)$ is a continuous function on $\mathbb{C}$. Clearly it has logarithmic growth. Since $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ the condition (2.10) is satisfied. □

As in the case of compactly supported measures, we recover the measure from the potential.

Lemma 2.3. *Let $\mu \in \mathscr{P}_{\log}(\mathbb{C})$. Then $-\Delta U^{\mu} = 2\pi\mu$ as distributions.*

Proof. Let $\varphi \in C_c^{\infty}(\mathbb{C})$. Then

$$\begin{aligned}
\Delta(-U^{\mu})(\varphi) &= \int -U^{\mu}(z)\Delta(\varphi)(z)\, \mathrm{d}\lambda(z) \\
&= \iint \log|z-w|\Delta(\varphi)(z)\, \mathrm{d}\mu(w)\mathrm{d}\lambda(z) \\
&= \iint \log|z-w|\Delta(\varphi)(z)\, \mathrm{d}\lambda(z)\mathrm{d}\mu(w) \\
&= \int 2\pi\varphi(w)\mathrm{d}\mu(w) \\
&= 2\pi\mu(\varphi).
\end{aligned}$$

Here the third equality is Lemma 2.2 and the fourth equality is the identity $U^{\Delta(\varphi)\lambda} = -2\pi\varphi$. □

We quote a result that will be useful latter.

Lemma 2.4. *Let $f_n, f\colon \mathbb{C} \longrightarrow \mathbb{R} \cup \{-\infty\}$, $n \geq 0$, be subharmonic functions. Then f_n converges to f in L^1_{loc} if and only if f_n converges to f as distributions.*

Proof. [Hör07, Theorem 3.2.13]. □

We next recall the definition of capacity and the Fekete–Szegő theorem. The *energy* of a measure μ as is defined as

$$I(\mu) \coloneqq \int U^{\mu}\, \mathrm{d}\mu = \iint -\log|w-z|\, \mathrm{d}\mu(w)\mathrm{d}\mu(z) \in \mathbb{R}\cup\{-\infty, \infty\}.$$

Let $K \subseteq \mathbb{C}$ be a compact subset, and let $\mathscr{P}(K)$ be the space of all probability measures supported on K. We say K is *polar* if $I(\mu) = +\infty$ for all $\mu \in \mathscr{P}(K)$.

Now assume K is non-polar, then there exists uniquely a measure $\mu_K \in \mathscr{P}(K)$ such that $I(\mu_K) = \inf_{\mu\in\mathscr{P}(K)} I(\mu)$ [Ran95, Theorem 3.7.6]. This μ_K is called the *equilibrium measure* of K, and the *capacity* of K is defined as $\mathrm{cap}(K) \coloneqq e^{-I(\mu_K)}$.

Let α be an algebraic number and $O(\alpha) \coloneqq \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot \alpha$ be the Galois orbit of α, that is the complete set of conjugates of α. Let $S \subseteq \mathbb{C}$ be a subset, we say α is *totally in S* if $O(\alpha) \subseteq S$.

Theorem 2.5 (Fekete–Szegő [Fek23, FS55])**.** *Let $K \subseteq \mathbb{C}$ be a compact subset. Then*

- *if $\mathrm{cap}(K) < 1$, then there exists an open neighborhood U of K, such that there are only finitely many algebraic integers totally in U.*

- *if* $\operatorname{cap}(K) \geq 1$ *and* K *is invariant under complex conjugation, then for any open neighborhood* U *of* K*, there are infinitely many algebraic integers totally in* U*.*

Remark 2.6. In the critical case $\operatorname{cap}(K) = 1$, let (α_n) be a sequence of algebraic integers such that $O(\alpha_n) \subseteq K_{1/n} := \{z \mid |z - w| < 1/n \text{ for some } w \in K\}$. Then the sequence of measures $\delta_{O(\alpha_n)}$ converges properly to the equilibrium measure μ_K [Rum99, Theorem 1].

2.2. **Smith and Orloski–Sardari theorems.** When $\operatorname{cap}(K) \geq 1$, by Fekete–Szegő there will be infinitely many algebraic integers totally contained in any open neighborhood of K and we would like to know what are the possible accumulation measures describing their asymptotic distribution. For $\operatorname{cap}(K) = 1$ this is answered by a result of Rumely as in Remark 2.6, while for $\operatorname{cap}(K) > 1$, this is answered in the recent breakthrough by Smith and Orloski–Sardari.

We say a sequence $(\mu_n) \subseteq \mathscr{P}(\mathbb{C})$ *converges weakly* to μ, if we have for any continuous and compactly supported function $f : \mathbb{C} \longrightarrow \mathbb{R}$,

$$\lim_{n\to\infty} \int f \,\mathrm{d}\mu_n = \int f \,\mathrm{d}\mu.$$

In this notion of convergence we can replace "compactly supported" by "bounded" [Bil97, Lemma 2.2].

For compactly supported measures there is a stronger notion of convergence. Let $\mu \in \mathscr{P}_c(\mathbb{C})$. We say a sequence $(\mu_n) \subseteq \mathscr{P}(\mathbb{C})$ *converges properly* to μ, if $\mu_n \longrightarrow \mu$ weakly and there exists a compact subset $K \subseteq \mathbb{C}$ such that for all n, the support of μ_n is contained in K. Note that this implies that we have for all continuous function $f : \mathbb{C} \longrightarrow \mathbb{R}$,

$$\lim_{n\to\infty} \int f \,\mathrm{d}\mu_n = \int f \,\mathrm{d}\mu. \tag{2.11}$$

Let $(\alpha_n) \subseteq \overline{\mathbb{Q}}$ be a sequence of algebraic numbers. We say $O(\alpha_n)$ converges to μ weakly (resp. properly) if the associated sequence of normalized discrete measures

$$\delta_{O(\alpha_n)} = \frac{1}{\deg(\alpha_n)} \sum_{\beta \in O(\alpha_n)} \delta_\beta$$

converges to μ weakly (resp. properly).

A reformulation of the result by Smith [Smi24, Theorem 1.5] and Orloski–Sardari [OS24a, Theorem 1.2] is

Theorem 2.7 (Smith, Orloski–Sardari)**.** *Assume* $\mu \in \mathscr{P}_c(\mathbb{C})$ *is invariant under complex conjugation. Then the following are equivalent*

(1) *there exists a sequence of distinct algebraic integers* $(\alpha_n) \subseteq \overline{\mathbb{Z}}$ *such that* $O(\alpha_n)$ *converges to* μ *properly,*
(2) $\int \log |Q| \,\mathrm{d}\mu \geq 0$ *for any* $Q \in \mathbb{Z}[x]$*.*

Remark 2.8. Let $\mathrm{h} : \overline{\mathbb{Q}} \longrightarrow \mathbb{R}$ be the standard Weil height function. Suppose $(\alpha_n) \subseteq \overline{\mathbb{Z}}$ is a sequence of distinct algebraic integers such that $O(\alpha_n)$ converges to μ properly, then

$$\lim_{n\to\infty} \mathrm{h}(\alpha_n) = \lim_{n\to\infty} \int \log^+ |z| \,\mathrm{d}\delta_{O(\alpha_n)} = \int \log^+ |z| \,\mathrm{d}\mu < \infty.$$

This forces $\deg(\alpha_n)$ to converge to ∞ because otherwise it would contradict the Northcott property.

Remark 2.9. The original results of Smith and Orloski–Sardari are in fact stronger and are more of Fekete–Szegő style, in the sense that they prove that for any $\varepsilon > 0$ there exists a sequence of algebraic integers (α_n) that are totally contained in

$$\operatorname{supp}(\mu)_\varepsilon \coloneqq \big\{z \in \mathbb{C} \mid |z - w| < \varepsilon \text{ for some } w \in \operatorname{supp}(\mu)\big\}.$$

and such that $O(\alpha_n)$ converges weakly to μ. This is much stronger that just proper convergence. Besides, when μ is supported on $\mathbb{R}$, they prove that the approximating algebraic integers α_n can be chosen to be totally real.

2.3. log**-weak convergence of measures.** We say a function $f \colon \mathbb{C} \longrightarrow \mathbb{R}$ has *logarithmic growth*, if there exist constants A, B such that for all $z \in \mathbb{C}$, we have $|f(z)| \leq A + B \log^+ |z|$.

Definition 2.10. Let $\{\mu, (\mu_n)\} \subset \mathscr{P}_{\log}(\mathbb{C})$. We say μ_n *converges* log*-weakly* to μ, if for any continuous function f with logarithmic growth, we have

$$\lim_{n\to\infty} \int f \,\mathrm{d}\mu_n = \int f \,\mathrm{d}\mu.$$

On the other hand, we say that the sequence (μ_n) is *log-tight* if for all $\varepsilon > 0$, there exists a compact set K such that

$$\int_{K^c} \log^+ |z| \,\mathrm{d}\mu_n(z) < \varepsilon, \qquad \text{for all } n.$$

The following statement follows readily from the definitions.

Lemma 2.11. *Let $\{\mu, (\mu_n)\} \subset \mathscr{P}_{\log}(\mathbb{C})$ such that μ_n converges weakly to μ. Then, the following assertions are equivalent:*

i) The sequence (μ_n) converges log*-weakly to μ*
ii) The sequence (μ_n) is log*-tight*
iii) We have that

$$\lim_{n\to\infty} \int \log^+ |z| \,\mathrm{d}\mu_n(z) = \int \log^+ |z| \,\mathrm{d}\mu(z).$$

We record here the following fact for future use.

Lemma 2.12. *Let $\big\{\mu_n\big\} \subseteq \mathscr{P}(\mathbb{C})$ be a sequence that* log*-weakly converges to μ. Then, for all polynomials $Q(x) \in \mathbb{C}[x]$, we have that*

$$\int \log |Q| \,\mathrm{d}\mu \geq \liminf_n \int \log |Q| \,\mathrm{d}\mu_n.$$

Proof. Using the monotone convergence theorem and (2.11), we have that

$$\begin{aligned}\int \log |Q| \,\mathrm{d}\mu &= \lim_{M\to\infty} \int \max\{-M, \log |Q|\} \,\mathrm{d}\mu \\ &= \lim_{M\to\infty} \lim_{n\to\infty} \int \max\{-M, \log |Q|\} \,\mathrm{d}\mu_n,\end{aligned}$$

the last equality follows from the fact that the function $h^M(z) = \max\{-M, \log |Q(z)|\}$ is continuous of logarithmic growth. We conclude by $h^M(z) \geq \log |Q(z)|$. ☐

It will be useful in what follows to be able to do diagonal arguments when dealing with weak convergence and log-weak convergence.

Lemma 2.13. *Consider $\{\mu, (\mu_n), (\mu_{n,m})\} \subset \mathscr{P}(\mathbb{C})$. Assume that*

- *μ_n converges weakly to μ as $n \longrightarrow \infty$,*
- *for each n, $\mu_{n,m}$ converges weakly to μ_n as $m \longrightarrow \infty$.*

Then, there is a diagonal subsequence μ_{n_i,m_i} that converges weakly to μ as $i \longrightarrow \infty$.

Moreover, the same holds if we consider μ, μ_n and $\mu_{n,m}$ in $\mathscr{P}_{\log}(\mathbb{C})$ and replace weak convergence with log*-weak convergence.*

Proof. Weak convergence on $\mathscr{P}(\mathbb{C})$ is metrizable (e.g. see [Bil99, Theorem 6.8], [Pol84, Example IV.22-23]), so the first assertion follows formally from a diagonal argument in a metric space.

Now we prove the second assertion. Using that for all n, the sequence $(\mu_{n,m})_m$ log-weakly converges to μ_n, we have that there exists $k(n)$ such that for all $m \geq k(n)$, it holds

$$\left| \int \log^+ |z| \, \mathrm{d}\mu_n(z) - \int \log^+ |z| \, \mathrm{d}\mu_{n,m}(z) \right| < \frac{1}{n}.$$

Using the first assertion, we extract from $\{\mu_{n,m} \mid m \geq k(n)\}$ a subsequence $\nu_i = \mu_{n_i,m_i}$ that weakly converges to μ. We remark that we can assume that $n_i < n_{i+1}$ for all i. We will show that the sequence (ν_i) converges log-weakly to μ. By Lemma 2.11, it is enough to show that (ν_i) is log-tight.

Let $\varepsilon > 0$. Since (μ_n) is log-tight, there exists a compact set K such that for all n, we have that $\int_{K^c} \log^+ |z| \, \mathrm{d}\mu_n(z) < \frac{\varepsilon}{3}$. On the other hand, since (ν_i) and (μ_{n_i}) have the same weak limit, there is i_0 such that for all $i \geq i_0$, we have that

$$\left| \int_K \log^+ |z| \, \mathrm{d}\mu_{n_i}(z) - \int_K \log^+ |z| \, \mathrm{d}\nu_i(z) \right| < \frac{\varepsilon}{3}$$

Then, for all $i \geq i_0$,

$$\begin{aligned}\int_{K^c} \log^+ |z| \, \mathrm{d}\nu_i(z) = &\left(\int \log^+ |z| \, \mathrm{d}\nu_i(z) - \int \log^+ |z| \, \mathrm{d}\mu_{n_i}(z) \right) + \int_{K^c} \log^+ |z| \, \mathrm{d}\mu_{n_i}(z) \\ &+ \left(\int_K \log^+ |z| \, \mathrm{d}\mu_{n_i}(z) - \int_K \log^+ |z| \, \mathrm{d}\nu_i(z) \right) \\ < &\frac{1}{n_i} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}\end{aligned}$$

Since $n_i \geq i$, we conclude that for all $i \geq \max\left\{i_0, \frac{3}{\varepsilon}\right\}$, we have that

$$\int_{K^c} \log^+ |z| \, \mathrm{d}\nu_i(z) < \varepsilon,$$

as desired. □

2.4. **Subharmonic functions and L^1_{loc} convergence.** Let $f_n, f \colon \mathbb{C} \longrightarrow \mathbb{R}$, $n \geq 0$, be functions that are locally integrable. We say that the sequence (f_n) converges to f *in* L^1_{loc}, if for any $z \in \mathbb{C}$, there exists an open neighborhood U of x such that

$$\lim_{n \to \infty} \int_U |f_n - f| \, \mathrm{d}\lambda = 0,$$

where λ is the usual Lebesgue measure on the complex plane. In this section we show that the log-weak convergence of measures corresponds exactly to L^1_{loc} convergence of potentials.

Proposition 2.14. *Let $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ and let $(\mu_n) \subset \mathscr{P}_{\log}(\mathbb{C})$ be a sequence. The sequence (μ_n) converges* log*-weakly to μ if and only if the sequence (U^{μ_n}) converges to U^{μ} in L^1_{loc}.*

Proof. We start by the direct implication. By Lemmas 2.1 and 2.4 we need only to show that $U^{\mu_n} \longrightarrow U^{\mu}$ as distributions. Taking $\varphi \in C_c^{\infty}(\mathbb{C})$ and using Lemma 2.2, we have that

$$\begin{aligned}\lim_{n\to\infty}\int \varphi(z)U^{\mu_n}(z)\mathrm{d}\lambda(z) &= \lim_{n\to\infty}\iint \varphi(z)\log|z-w|^{-1}\mathrm{d}\mu_n(w)\mathrm{d}\lambda(z)\\ &= \lim_{n\to\infty}\iint \varphi(z)\log|z-w|^{-1}\mathrm{d}\lambda(z)\mathrm{d}\mu_n(w)\\ &= \iint \varphi(z)\log|z-w|^{-1}\mathrm{d}\lambda(z)\mathrm{d}\mu(w)\\ &= \iint \varphi(z)\log|z-w|^{-1}\mathrm{d}\mu(w)\mathrm{d}\lambda(z)\\ &= \int \varphi(z)U^{\mu}(z)\mathrm{d}\lambda(z).\end{aligned}$$

The third equality follows from the fact that the function

$$w \longmapsto \int_{\mathbb{C}} \varphi(z)\log|z-w|^{-1}\mathrm{d}\lambda(z)$$

is continuous of logarithmic growth and the hypothesis of log-weak convergence.

Now we prove the reverse implication. By Lemma 2.3 we have that $\Delta(-U^{\mu}) = 2\pi\mu$ as distributions. Now assume $U^{\mu_n} \longrightarrow U^{\mu}$ in L^1_{loc}. It follows that $\Delta U^{\mu_n} \longrightarrow \Delta U^{\mu}$ as distributions. We conclude that $\mu_n \longrightarrow \mu$ as distributions. Since $C_c^{\infty}(\mathbb{C})$ is uniformly dense in the space of compactly supported continuous functions, this implies that $\mu_n \longrightarrow \mu$ weakly. By Lemma 2.11 it only remains to be proven that

$$\lim_{n\to\infty}\int \log^+|z|\,\mathrm{d}\mu_n(z) = \int \log^+|z|\,\mathrm{d}\mu(z). \tag{2.12}$$

Let $f\colon \mathbb{C} \longrightarrow \mathbb{R}$ be a smooth subharmonic function such that $|\log^+ - f|$ is bounded and

$$\frac{1}{2\pi}\Delta f = h\mathrm{d}\lambda,$$

where $\mathrm{d}\lambda$ is the Lebesgue measure and h is a smooth function with compact support. Using Lemma 2.2 and the convergence in L^1_{loc}, we have that

$$\begin{aligned}\lim_{n\to\infty}\int f(z)\mathrm{d}\mu_n(z) &= \lim_{n\to\infty}\iint \log|z-w|h(w)\mathrm{d}\lambda(w)\mathrm{d}\mu_n(z)\\ &= \lim_{n\to\infty}\iint \log|z-w|\mathrm{d}\mu_n(z)h(w)\mathrm{d}\lambda(w)\\ &= \lim_{n\to\infty} -\int U^{\mu_n}(w)h(w)\mathrm{d}\lambda(w)\\ &= -\int U^{\mu}(w)h(w)\mathrm{d}\lambda(w)\end{aligned}$$

By a similar reasoning, the last quantity is $\int f(z)\mathrm{d}\mu(z)$. Since $f(z) - \log^+|z|$ is continuous and bounded, by weak convergence we deduce (2.12). □

3. The sweetened truncation

The goal of this section is to define the *sweetened truncation* and prove technical lemmas and propositions that are related to it. They will be used the the next sections.

3.1. **Sweetened truncation.** Let $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ be a probability measure. For any $R > 1$, we want to produce a new probability measure μ_R^{sw}, whose support is contained in the ball $\{|z| \leq R\}$ and such that μ_R^{sw} converges to μ log-weakly as R goes to ∞. The naïve candidate for such a truncation is

$$\mu'_R \coloneqq \mu|_{\{|z|\leq R\}} + (1 - m_R)\lambda_{S_R},$$

where $m_R \coloneqq \mu\left(\{|z| \leq R\}\right)$ and λ_{S_R} is the equilibrium measure on $\{|z| = R\}$. We want to refine the truncation is such a way that, if μ satisfies condition (2) in Theorem 4.1, then the measure μ_R^{sw} still satisfies the same condition. The following definition is inspired in Smith's concept of *sweetened measure*, see [Smi24, Definition 5.5].

Definition 3.1. Let $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ be a probability measure. Let $R > 1$ and set

$$T_R \coloneqq \int_{|z|\geq R} \log^+ |z| \mathrm{d}\mu,\ \ L_R \coloneqq 2(1 - m_R)\log 2 + T_R,\ \ \eta_R \coloneqq \frac{\log R}{\log R + L_R}.$$

We define the sweetened truncation μ_R^{sw} as

$$\mu_R^{\mathrm{sw}} \coloneqq \eta_R \mu|_{\{|z|\leq R\}} + (1 - m_R\eta_R)\lambda_{S_R}. \tag{3.13}$$

Proposition 3.2. *Let $\mu \in \mathscr{P}_{\log}(\mathbb{C})$. Then,*

1. *μ_R^{sw} converges to μ log-weakly as R goes to ∞.*
2. *It holds $U^{\mu_R^{\mathrm{sw}}} \leq \eta_R U^{\mu}$. In particular, if $\int \log|Q| \,\mathrm{d}\mu \geq 0$ for some $Q \in \mathbb{Z}[x]$, then we also have $\int \log|Q| \,\mathrm{d}\mu_R^{\mathrm{sw}} \geq 0$.*
3. *If μ is conjugation invariant, then μ_R^{sw} is conjugation invariant as well.*

We will prove this statement after the following Lemma.

Lemma 3.3. *The naïve truncation μ'_R satisfies $U^{\mu'_R} \leq U^{\mu} + L_R$.*

Proof. Assume $R > 1$. Note that

$$U^{\mu'_R}(z) - U^{\mu}(z) = \int_{|w|>R} \log|z - w| \,\mathrm{d}\mu(w) - (1 - m_R)\log\max\{|z|, R\}.$$

Assume $|z| < R$. Then for all w with $|w| > R$, we have $|z - w| \leq |z| + |w| \leq 2|w|$ and hence

$$\begin{aligned} U^{\mu'_R}(z) - U^{\mu}(z) &= \int_{|w|>R} \log|z - w| \,\mathrm{d}\mu(w) - (1 - m_R)\log R \\ &\leq \int_{|w|>R} \log|2w| \,\mathrm{d}\mu(w) \\ &\leq L_R. \end{aligned}$$

Assume now $|z| \geq R$. Then,

$$
\begin{aligned}
U^{\mu'_R}(z) - U^{\mu}(z) &= \int_{|w|>R} \log|z-w| \,\mathrm{d}\mu(w) - (1-m_R)\log|z| \\
&= \int_{|w|>R} \log\left|1-\frac{w}{z}\right| \mathrm{d}\mu(w) \\
&= \int_{|w|>|z|\geq R} \log\left|1-\frac{w}{z}\right| \mathrm{d}\mu(w) + \int_{|z|\geq|w|>R} \log\left|1-\frac{w}{z}\right| \mathrm{d}\mu(w) \\
&\leq \int_{|w|>|z|\geq R} \log\left|\frac{2w}{R}\right| \mathrm{d}\mu(w) + \int_{|z|\geq|w|>R} \log 2 \,\mathrm{d}\mu(w) \\
&\leq \int_{|w|>R} \log|2w| \,\mathrm{d}\mu(w) + \int_{|w|>R} \log 2 \,\mathrm{d}\mu(w) \\
&= L_R.
\end{aligned}
$$

□

Proof of Proposition 3.2. Item (3) follows readily from the definitions. Now we show (1). It is easy to see from (3.13) that μ_R^{sw} converges to μ weakly as R goes to ∞. So it remains to test against $\log^+|z|$:

$$
\begin{aligned}
&\int \log^+|z| \,\mathrm{d}\mu_R^{\mathrm{sw}} - \int \log^+|z| \,\mathrm{d}\mu \\
&\quad = (1-m_R\eta_R)\log R + (\eta_R - 1)\int_{|z|\leq R} \log^+|z| \,\mathrm{d}\mu - \int_{|z|>R} \log^+|z| \,\mathrm{d}\mu.
\end{aligned}
$$

Clearly the last two terms vanish when R goes to ∞. With regards to the first term, we have that

$$
(1-m_R\eta_R)\log R = \big((1-m_R)\log R + L_R\big)\frac{\log R}{\log R + L_R}. \tag{3.14}
$$

Then, the estimate $0 \leq (1-m_R)\log R \leq T_R$ shows that (3.14) also vanishes as $R \to \infty$.

Now we prove (2). For the first assertion, we use Lemma 3.3

$$
\begin{aligned}
U^{\mu_R^{\mathrm{sw}}} &= \eta_R U^{\mu'_R} + (1-\eta_R)U^{\lambda_{S_R}} \\
&\leq \eta_R U^{\mu} + \eta_R L_R - (1-\eta_R)\log R \\
&= \eta_R U^{\mu}.
\end{aligned}
$$

For the second one, let $a \in \mathbb{Z}$ be the leading coefficient of Q and $\alpha_1, \ldots, \alpha_n$ be the roots of Q. Then

$$
\begin{aligned}
\int \log|Q| \,\mathrm{d}\mu_R^{\mathrm{sw}} &= \log|a| - \sum U^{\mu_R^{\mathrm{sw}}}(\alpha_i) \\
&\geq \eta_R \log|a| - \eta_R \sum U^{\mu}(\alpha_i) \\
&\geq \eta_R \int \log|Q| \,\mathrm{d}\mu \\
&\geq 0.
\end{aligned}
$$

□

3.2. Sweetened truncation and asymptotically logarithmic functions.

Definition 3.4. A function $\mathrm{g}\colon \mathbb{C} \longrightarrow \mathbb{R}$ is called *asymptotically logarithmic at infinity*, if it obeys the asymptotic

$$\mathrm{g}(z) = \log|z| + o(\log|z|), \quad \text{as } |z| \longrightarrow \infty.$$

A *Green function* is a function $\mathrm{g}\colon \mathbb{C} \longrightarrow \mathbb{R}$ which is continuous, invariant under complex conjugation and asymptotically logarithmic at infinity.

Such Green functions arise when considering singular metrics on $\mathcal{O}(1)$ (see Remark 6.1). The purpose of this section is to prove the following statement.

Proposition 3.5. *Let $\mathrm{g}\colon \mathbb{C} \longrightarrow \mathbb{R}$ be a continuous function that is asymptotically logarithmic. Let (μ_n) be a sequence of probability measures on $\mathbb{C}$ such that*

$$\limsup_{n\to\infty} \int \mathrm{g}\, \mathrm{d}\mu_n < \infty.$$

Then, for all $\varepsilon > 0$ there exists $R > 1$ such that

$$\limsup_{n\to\infty} \int \mathrm{g}\, \mathrm{d}\mu_{n,R}^{\mathrm{sw}} \leq \limsup_{n\to\infty} \int \mathrm{g}\, \mathrm{d}\mu_n + \varepsilon. \tag{3.15}$$

Let $f\colon \mathbb{C} \longrightarrow \mathbb{R}_{>0}$ be a continuous function. We say that a sequence (μ_n) is *f-tight*, if

$$\lim_{R\to\infty} \sup_{n\in\mathbb{N}} \int_{|z|\geq R} |f|\, \mathrm{d}\mu_n = 0.$$

If we set $f = \log^+|z|$, we recover log-tightness as in Definition 3.16. We say a sequence (μ_n) is *tight*, if it is 1-tight.

The following elementary statement follows directly from the definitions.

Lemma 3.6. *Let $f, h\colon \mathbb{C} \longrightarrow \mathbb{R}_{>0}$ be continuous functions with $h = o(f)$ as $|z| \longrightarrow \infty$. Suppose we have a sequence $(\mu_n) \subseteq \mathscr{P}(\mathbb{C})$ such that*

$$\sup_{n\in\mathbb{N}} \int |f|\, \mathrm{d}\mu_n < +\infty.$$

Then (μ_n) is h-tight.

Proof of Proposition 3.5. The hypotheses imply

$$\limsup_{n\to\infty} \int \log^+|z|\, \mathrm{d}\mu_n < +\infty,$$

so we may assume that

$$\sup_{n\in\mathbb{N}} \int \log^+|z|\, \mathrm{d}\mu_n < +\infty. \tag{3.16}$$

Since $1 = o(\log|z|)$ when $|z| \longrightarrow \infty$, we have (μ_n) is tight by Lemma 3.6. Using Prohorov's theorem [Bil99, Theorem 5.1], we can assume, after taking a subsequence if necessary, that there is a probability measure μ_∞ on $\mathbb{C}$ such that μ_n converges weakly to μ_∞. In particular, we know that μ_∞ can integrate $\log^+|z|$, because

$$\int \log^+|z|\, \mathrm{d}\mu_\infty \leq \limsup_{n\to\infty} \int \log^+|z|\, \mathrm{d}\mu_n < \infty. \tag{3.17}$$

Set $E_{n,R} \coloneqq \int g d\mu_{n,R}^{\mathrm{sw}} - \int g d\mu_n$. In order to prove (3.15) it is enough to show that

$$\limsup_{R\to\infty} \limsup_{n\to\infty} E_{n,R} \leq 0 \tag{3.18}$$

We break the proof of (3.18) in several steps.

Claim 1. We have that

$$\lim_{R\to\infty} \limsup_{n\to\infty} \mu_n\{|z| \geq R\} \log R = 0.$$

Indeed,

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \mu_n\big\{|z| \geq R\big\} \log R \\ &\leq \mu_\infty\big\{|z| \geq R\big\} \log R \\ &\leq \int_{|z|\geq R} \log^+ |z| \, \mathrm{d}\mu_\infty. \end{aligned}$$

Here the second inequality follows from the Portmanteau theorem [Bil99, Theorem 2.1 (iii)], because μ_n converges weakly to μ_∞ and the set $\big\{|z| \geq R\big\}$ is closed. Since (3.17) implies that the last term vanishes when $R \longrightarrow \infty$, this proves the claim.

Claim 2. There exists $M > 0$ such that

$$\{L_{n,R}, T_{n,R} \mid n \geq 1, R > 1\} \subseteq [0, M].$$

We see directly from the definitions that $L_{n,R}, T_{n,R} \geq 0$. Also, $T_{n,R}$ is bounded because of (3.16). On the other hand,

$$\begin{aligned} L_{n,R} &= 2(1 - m_{n,R}) \log 2 + \int_{|z|\geq R} \log^+ |z| \, \mathrm{d}\mu_n \\ &\leq 2 \log 2 + \int \log^+ |z| \, \mathrm{d}\mu_n \\ &\leq 2 \log 2 + \sup_k \int \log^+ |z| \, \mathrm{d}\mu_k < \infty, \end{aligned}$$

by (3.16).

Claim 3. We have that

$$\limsup_{R\to\infty} \limsup_{n\to\infty} \left((1 - m_{n,R}\eta_{n,R}) \log R - T_{n,R}\right) \leq 0.$$

Indeed, using Claim 2 we see that

$$\begin{aligned} (1 - m_{n,R}\eta_{n,R}) \log R - T_{n,R} &= \frac{\log R}{\log R + L_{n,R}} \cdot \left((1 - m_{n,R}) \log(4R) + T_{n,R}\right) - T_{n,R} \\ &\leq \big((1 - m_{n,R}) \log(4R) + T_{n,R}\big) - T_{n,R} \\ &= (1 - m_{n,R}) \log(4R) \end{aligned}$$

We conclude by Claim 1.

Claim 4. There exists $M' > 0$ such that for all $n \geq 1$ and $R > 0$ it holds

$$(1 - m_{n,R}\eta_{n,R}) \log R \leq M'.$$

This follows from Claim 2 and Claim 3.

Claim 5. We have that $0 \leq 1 - \eta_{n,R} \leq \frac{M}{\log R}$.

This follows from $1-\eta_{n,R}=\frac{L_{n,R}}{\log R+L_{n,R}}$ and Claim 2.

Write $\mathrm{g}=\log^+|z|+h$, with $h=o(\log|z|)$ at infinity.

Claim 6. The sequence $(\int h\,\mathrm{d}\mu_n)_{n\geq 1}$ is bounded.

This follows from (3.16) and the fact that there exists a constant $A>0$ such that $|h(z)|\leq A(1+\log^+|z|)$.

Let $\varepsilon>0$. Since $h=o(\log|z|)$, there exists $R_0>1$ such that for all z with $|z|\geq R_0$, we have that $h(z)\leq\frac{\varepsilon}{M'}\log|z|$. For $R\geq R_0$ and using Claim 4, we have

$$\begin{aligned}
E_{n,R}=&(1-m_{n,R}\eta_{n,R})\int \mathrm{g}d\lambda_{S_R}-\int_{|z|\geq R}\mathrm{g}d\mu_n-(1-\eta_{n,R})\int_{|z|\leq R}\mathrm{g}d\mu_n\\
=&\big((1-m_{n,R}\eta_{n,R})\log R-T_{n,R}\big)-(1-\eta_{n,R})\int_{|z|\leq R}\log^+|z|\,\mathrm{d}\mu_n\\
&\qquad+(1-m_{n,R}\eta_{n,R})\int h\,\mathrm{d}\lambda_{S_R}-\int_{|z|\geq R}h\,\mathrm{d}\mu_n-(1-\eta_{n,R})\int_{|z|\leq R}h\,\mathrm{d}\mu_n.\\
\leq&\big((1-m_{n,R}\eta_{n,R})\log R-T_{n,R}\big)+\varepsilon-\int_{|z|\geq R}h\,\mathrm{d}\mu_n-(1-\eta_{n,R})\int_{|z|\leq R}h\,\mathrm{d}\mu_n.\\
=&\big((1-m_{n,R}\eta_{n,R})\log R-T_{n,R}\big)+\varepsilon+\eta_{n,R}\int_{|z|\geq R}h\,\mathrm{d}\mu_n-(1-\eta_{n,R})\int h\,\mathrm{d}\mu_n
\end{aligned}$$

By Lemma 3.6 we know that (μ_n) is h-tight. Combining this information with Claim 3, Claim 4 and Claim 5, we obtain

$$\limsup_{R\to\infty}\limsup_{n\to\infty}E_{n,R}\leq\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, this shows (3.18). □

4. Approximating measures by algebraic integers and by capacity one compact sets

4.1. Approximating measures by algebraic integers in the non-compact setting. The aim of this section is to prove

Theorem 4.1. *Assume $\mu\in\mathscr{P}_{\log}(\mathbb{C})$ is invariant under complex conjugation. Then, $\mu\in\mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C})$ if and only if*

$$\int\log|Q|\,\mathrm{d}\mu\geq 0 \textit{ for any } Q\in\mathbb{Z}[x]. \tag{4.19}$$

It extends the Smith–Orloski–Sardari Theorem 2.7 to non-compactly supported measures that can integrate functions of logarithmic growth. In the setting of non-compactly supported measures, proper convergence does not make sense. Also weak convergence is not appropriate because any probability measure can be written as the weak limit of a sequence of Galois orbits of integers. In particular a weak limit of measures satisfying condition (4.19) does not need to satisfy the same condition (e.g. see the example at the end of Section 6.2). The main result of this section shows that the notion of log-weak convergence is the right one for non-compactly supported measures.

Remark 4.2. Suppose $(\alpha_n) \subseteq \overline{\mathbb{Z}}$ is a sequence of distinct algebraic integers such that $\delta_{O(\alpha_n)}$ converges log-weakly to μ, then $\deg(\alpha_n)$ converges to ∞ by the same reason as Remark 2.8.

Before proving Theorem 4.1 we state a direct consequence.

Corollary 4.3. *Let $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ be a conjugation invariant measure such that $U^{\mu} \leq 0$. Then, μ belongs to $\mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$.*

Proof. The hypothesis $U^{\mu} \leq 0$ readily implies (4.19) in Theorem 4.1. □

Proof of Theorem 4.1. Assume that $\mu \in \mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$. We prove that it satisfies condition (4.19) by an adaptation of a classical argument to our setting (e.g. see [Ser19, Lemma 1.3.4]). Let (α_n) be a sequence of algebraic integers such that $(\delta_{O(\alpha_n)})$ converges log-weakly to μ. We denote by $P_n(x) \in \mathbb{Z}[x]$ the minimal polynomial of α_n. Let $Q \in \mathbb{Z}[x]$ a non zero polynomial. Using Lemma 2.12, we have that

$$\int \log|Q| \mathrm{d}\mu \geq \liminf_n \frac{1}{\deg(\alpha_n)} \sum_{\beta \in O(\alpha_n)} \log|Q(\beta)|.$$

We denote by $\operatorname{Res}(P_n, Q)$ the resultant of P_n and Q. Since P_n is monic, we have that $|\operatorname{Res}(P_n, Q)| = \prod_{\beta \in O(\alpha_n)} |Q(\beta)|$ (e.g. see [BG06, B.1.13]). Hence,

$$\sum_{\beta \in O(\alpha_n)} \log|Q(\beta)| = \log|\operatorname{Res}(P_n, Q)| \geq 0,$$

the last inequality being due to the fact that $\operatorname{Res}(P_n, Q)$ is a non zero integer when n is big enough.

Let now $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ that is invariant under complex conjugation, and that satisfies condition (4.19). For $R > 1$, consider the sweetened truncation μ_R^{sw} as in Definition 3.1. By Proposition 3.2 (2) and (3), we can apply Theorem 2.7 to μ_R^{sw} in order to conclude that it can be approximated properly (in particular log-weakly) by Galois orbits of algebraic integers. Using Proposition 3.2(1) and Lemma 2.13, we can find a diagonal subsequence that converges to μ log-weakly, as desired. □

The classical Fekete–Szegő theorem (Theorem 2.5), plus Rumely's remark on Bilu's equidistribution (Remark 2.6), imply that if $K \subseteq \mathbb{C}$ is a compact subset that is invariant under complex conjugation and has capacity one, then $\mu_K \in \mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$. The Smith–Orloski–Sardari theorem (Theorem 2.7) characterizes all compactly supported ones.

There do exist measures in $\mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$ that are not compactly supported. For instance, the Fubini-Study form

$$\omega_{\mathrm{FS}} = \frac{i}{2\pi} \frac{\mathrm{d}z \mathrm{d}\bar{z}}{(|z|^2+1)^2}$$

induces a conjugation invariant measure $\mu_{\mathrm{FS}} \in \mathscr{P}_{\log}(\mathbb{C})$ with potential

$$U^{\mu_{\mathrm{FS}}}(z) = -\frac{1}{2}\log(|z|^2+1) \leq 0.$$

We deduce $\mu_{\mathrm{FS}} \in \mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$ by Corollary 4.3.

4.2. **Approximating measures by equilibrium measures of compact sets of capacity one.** Here, we prove

Theorem 4.4. *Let $\mu \in \mathscr{P}_{\log}(\mathbb{C})$. Then, the followings are equivalent*
(1) $U^\mu \leq 0$,
(2) *there exists a sequence of compact subsets K_n of capacity one, such that μ_{K_n} converges* log-*weakly to μ.*

This theorem characterizes measures with negative potential as the log-weak closure of the set of equilibrium measures of capacity one compact sets.

Proof of Theorem 4.4. We show first that (1) implies (2). Using the monotone convergence theorem,

$$\begin{aligned} U^\mu(z) &= \int -\log|z-w|\mathrm{d}\mu(w) \\ &= \lim_{M\to\infty}\int \min\{M, -\log|z-w|\}\,\mathrm{d}\mu(w) \\ &= \lim_{M\to\infty}\lim_{n\to\infty}\int \min\{M, -\log|z-w|\}\,\mathrm{d}\mu_{K_n}(w). \end{aligned}$$

In the last step we use that the function $w \mapsto \min\{M, -\log|z-w|\}$ is continuous and of logarithmic growth. We deduce $U^\mu(z) \leq 0$ from

$$\int \min\{M, -\log|z-w|\}\,\mathrm{d}\mu_{K_n}(w) \leq U^{\mu_{K_n}}(z) \leq 0,$$

the last inequality being due to Frostman's theorem [Ran95, Theorem 3.3.4(a)].

Now we prove that (2) implies (1). First, we claim that there exists a sequence of monic polynomials P_n such that $\delta_{O(P_n)}$ converges to μ log-weakly.

Indeed, by Proposition 3.2 (1) and Lemma 2.13, we are reduced to prove this assertion replacing μ by a measure μ' having compact support K. In that case, we can weakly approximate μ' by a sequence of measures (μ_n) with discrete support S_n contained in K and of the form

$$\mu_n = \sum_{s\in S_n}\frac{a_s}{b_n}\delta_s, \quad a_s, b_n \in \mathbb{Z}_{>0}.$$

But then (μ_n) converges properly and we can take $P_n(x) \coloneqq \prod_{s\in S_n}(x-s)^{a_s}$, thus proving our claim.

We denote by $U_n \coloneqq -\frac{\log|P_n|}{\deg(P_n)}$ the potential of $\delta_{O(P_n)}$. Using Proposition 2.14, we have that U_n converges tto U^μ in L^1_{loc}. Let $U_n^- \coloneqq \min\{U_n, 0\}$ be the negative part of U_n. Since $U^\mu \leq 0$, we have that $|U_n^- - U^\mu| \leq |U_n - U^\mu|$, so U_n^- converges to U^μ in L^1_{loc} as well. But $U_n^- = -\frac{\log^+|P_n|}{\deg(P_n)}$ is the potential of the equilibrium measure of the compact set $K_n \coloneqq \{z \mid |P_n(z)| \leq 1\}$, which is of capacity one because P_n is monic. We conclude by another application of Proposition 2.14. □

5. Strong duality

Here we present two optimization problems that will be crucial for our study of heights of algebraic points. They are akin to those considered by Smyth [Smy84] and Smith [Smi24] in their works on totally real algebraic integers of small trace, which were indeed a major source of inspiration for the material in this section.

Let $\mathrm{g}\colon \mathbb{C} \longrightarrow \mathbb{R}$ be a continuous function that is asymptotically logarithmic at ∞ in the sense of Definition 3.4, and for simplicity fix an enumeration $Q_1, Q_2, Q_3, \dots$ of all nonconstant primitive irreducible polynomials with integer coefficients. We consider the *primal problem* and the *dual problem* respectively defined as

$$\mathcal{P}(\mathrm{g}) = \inf \left\{ \int \mathrm{g}\, \mathrm{d}\mu \;\middle|\; \mu \in \mathscr{P}_{\log}(\mathbb{C}), \int \log|Q_n|\, \mathrm{d}\mu \geq 0 \text{ for all } n \in \mathbb{N} \right\} \tag{5.20}$$

and

$$\mathcal{D}(\mathrm{g}) = \sup_{a} \inf_{z\in\mathbb{C}} \left(\mathrm{g}(z) - \sum_{n\in\mathbb{N}} a_n \log|Q_n(z)| \right), \tag{5.21}$$

the supremum being over the sequences $a = (a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_{\geq 0}$ with $a_n = 0$ for all but a finite number of n's. We refer to the quantities $\mathcal{P}(\mathrm{g})$ and $\mathcal{D}(\mathrm{g})$ as the *optimal values* for these problems.

Remark 5.1. If g is invariant under the complex conjugation then

$$\mathcal{P}(\mathrm{g}) = \inf \left\{ \int \mathrm{g}\, \mathrm{d}\mu \;\middle|\; \mu \in \mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C}) \right\}.$$

Indeed, in this case the primal problem can be computed over the measures $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ that are invariant under the complex conjugation and satisfy $\int \log|Q_n|\, \mathrm{d}\mu \geq 0$ for all n, which by Theorem 4.1 coincide with those in $\mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C})$.

As explained in Appendix A, this pair of problems is an instance of primal and dual problems in linear optimization (Example A.4). In particular, they satisfy the weak duality property.

Proposition 5.2. *We have* $\mathcal{P}(\mathrm{g}) \geq \mathcal{D}(\mathrm{g})$.

Proof. This is a particular case of Proposition A.5 but can also be checked directly: for all $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ such that $\int \log|Q_n|\, \mathrm{d}\mu \geq 0$ for all n and $a = (a_n)_{n\in\mathbb{N}}$ as in (5.21) we have

$$\int \mathrm{g}\, \mathrm{d}\mu \geq \int \Big(\mathrm{g} - \sum_{n\in\mathbb{N}} a_n \log|Q_n| \Big)\, \mathrm{d}\mu \geq \inf_{z\in\mathbb{C}} \Big(\mathrm{g}(z) - \sum_{n\in\mathbb{N}} a_n \log|Q_n(z)| \Big),$$

which gives the inequality. □

Our main result in this section shows that these problems satisfy the strong duality property, which consists in the equality between their optimal values. Its proof is modelled in that of Theorem A.7 for the finite dimensional case.

Theorem 5.3. *We have* $\mathcal{D}(\mathrm{g}) = \mathcal{P}(\mathrm{g}) \in \mathbb{R}$.

Proof. Let $\mathscr{P}'_{\log}(\mathbb{C})$ be the convex set of probability measures on $\mathbb{C}$ that integrate all the functions of the form $\log|Q_i|$. For $n \in \mathbb{N}$ consider the convex subset of $\mathbb{R}^{n+2}$ defined as

$$V_n = \left\{ \left(\int \mathrm{g}\, \mathrm{d}\mu, \int \log|Q_1|\, \mathrm{d}\mu, \dots, \int \log|Q_n|\, \mathrm{d}\mu \right) \;\middle|\; \mu \in \mathscr{P}'_{\log}(\mathbb{C}) \right\},$$

and for each $\lambda \in \mathbb{R}$ consider also the convex subset of $\mathbb{R}^{n+2}$ defined as

$$W_{n,\lambda} = \{ (t, x_1, \dots, x_n) \mid t \leq \lambda,\ x_1, \dots, x_n \geq 0 \}.$$

We have that $V_n \cap W_{n,\lambda} \neq \emptyset$ if and only if there exists $\mu \in \mathscr{P}'_{\log}(\mathbb{C})$ such that $\int \mathrm{g}\,\mathrm{d}\mu \leq \lambda$ and $\int \log|Q_i|\,\mathrm{d}\mu \geq 0$, $i = 1, \dots, n$. Now set

$$\lambda_n = \inf\{\lambda \in \mathbb{R} \mid V_n \cap W_{n,\lambda} \neq \emptyset\}.$$

Since g is bounded from below the values $\int \mathrm{g}\,\mathrm{d}\mu$ for $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ are also bounded from below, and so $\lambda_n > -\infty$. On the other hand, the Dirac delta measure $\mu = \delta_z$ for $z \gg 0$ satisfies

$$\int \log|Q_i|\,\mathrm{d}\mu = \log|Q_i(z)| > 0, \quad i = 1, \dots, n. \tag{5.22}$$

In particular $\lambda_n \leq \mathrm{g}(z) < +\infty$. Hence $\lambda_n \in \mathbb{R}$.

Since $\lambda_n - 1/n < \lambda_n$ the convex subsets V and $W_{n,\lambda_n - 1/n}$ are disjoint, and so by the hyperplane separation theorem there exists $h \in (\mathbb{R}^{n+2})^\vee \simeq \mathbb{R}^{n+2}$ with $h \neq 0$ such that $h(p) \geq h(p')$ for all $p \in V_n$ and $p' \in W_{n,\lambda_n - 1/n}$. Writing $h = (b, -a_1, \dots, -a_n)$ with $b, a_i \in \mathbb{R}$ these conditions amount to

$$b \int \mathrm{g}\,\mathrm{d}\mu - \sum_{i=1}^n a_i \int \log|Q_i|\,\mathrm{d}\mu \geq b\,t - \sum_{i=1}^n a_i x_i \tag{5.23}$$

for all $\mu \in \mathscr{P}'_{\log}(\mathbb{C})$, $t \leq \lambda_n - 1/n$ and $x_1, \dots, x_n \geq 0$.

Since x_i can be arbitrarily large this inequality implies that $a_i \geq 0$, $i = 1, \dots, n$, and since t can also be arbitrarily negative we similarly deduce that $b \geq 0$. To exclude the possibility that $b = 0$, consider the case $x_i = 0$, $i = 1, \dots, n$, note that in this situation (5.23) gives

$$-\sum_{i=1}^n a_i \int \log|Q_i|\,\mathrm{d}\mu \geq 0 \quad \text{for all } \mu \in \mathscr{P}'_{\log}(\mathbb{C}).$$

Since $h \neq 0$, we have $a_i > 0$ for some $i \in \{1, \dots, n\}$, in which case the inequality does not hold for the measure in (5.22). We conclude that $b > 0$, and so we can assume without loss of generality that $b = 1$.

Setting $t = \lambda_n - 1/n$ and $x_1 = \cdots = x_n = 0$ in (5.23) we get

$$\int \mathrm{g}\,\mathrm{d}\mu - \sum_{i=1}^n a_i \int \log|Q_i|\,\mathrm{d}\mu \geq \lambda_n - \frac{1}{n} \quad \text{for all } \mu \in \mathscr{P}'_{\log}(\mathbb{C}). \tag{5.24}$$

Considering this inequality for $\mu = \delta_z$ with $z \in \mathbb{C} \setminus \overline{\mathbb{Q}}$ we get

$$\mathrm{g}(z) - \sum_{i=1}^n a_i \log|Q_i(z)| \geq \lambda_n - \frac{1}{n} \quad \text{for all } z \in \mathbb{C} \setminus \overline{\mathbb{Q}},$$

which extends by density to all $z \in \mathbb{C}$. Hence $\mathcal{D}(\mathrm{g}) \geq \lambda_n - 1/n$. Therefore

$$\limsup_{n\to\infty} \lambda_n \leq \mathcal{D}(\mathrm{g}). \tag{5.25}$$

On the other hand we can choose $\mu_n \in \mathscr{P}'_{\log}(\mathbb{C})$ satisfying

$$\int \mathrm{g}\,\mathrm{d}\mu_n \leq \lambda_n + \frac{1}{n} \quad \text{and} \quad \int \log|Q_i|\,\mathrm{d}\mu_n \geq 0, \quad i = 1, \dots, n. \tag{5.26}$$

By (5.25)

$$\limsup_{n\to\infty} \int \mathrm{g}\,\mathrm{d}\mu_n \leq \mathcal{D}(\mathrm{g}).$$

By Proposition 3.5, for any $\varepsilon > 0$ there exists $R > 1$ such that

$$\limsup_{n\to\infty} \int \mathrm{g}\, \mathrm{d}\mu^{\mathrm{sw}}_{n,R} \le \mathcal{D}(\mathrm{g}) + \varepsilon,$$

where $\mu^{\mathrm{sw}}_{n,R}$ denotes the sweetened truncation of μ_n to the ball $B_R = \{z \in \mathbb{C} \mid |z| \le R\}$ (Definition 3.1). Since all the probability measures $\mu^{\mathrm{sw}}_{n,R}$ are supported on B_R, up to taking a subsequence we can assume that they converge properly to a probability measure μ_R with support contained in B_R. Then

$$\int \mathrm{g}\, \mathrm{d}\mu_R = \lim_{n\to\infty} \int \mathrm{g}\, \mathrm{d}\mu^{\mathrm{sw}}_{n,R} \le \mathcal{D}(\mathrm{g}) + \varepsilon,$$

and by Lemma 2.12 and Proposition 3.2(2) we also have

$$\int \log|Q_n|\, \mathrm{d}\mu_R \ge \liminf_{n\to\infty} \int \log|Q|\, \mathrm{d}\mu^{\mathrm{sw}}_{n,R} \ge 0 \quad \text{for all } n \in \mathbb{N}.$$

Hence $\mu_R \in$ is a candidate for the primal problem and we deduce $\mathcal{P}(\mathrm{g}) \le \mathcal{D}(\mathrm{g}) + \varepsilon$. Taking $\varepsilon > 0$ arbitrarily small we get $\mathcal{P}(\mathrm{g}) \le \mathcal{D}(\mathrm{g})$, and we obtain the equality by combining this with the weak duality property (Proposition 5.2).

To see that $\mathcal{P}(\mathrm{g})$ is a real number we use the estimates

$$-\infty < \inf_{z\in\mathbb{C}} \mathrm{g}(z) \le \mathcal{P}(\mathrm{g}) \le \int \mathrm{g}\, \mathrm{d}\mu_R < \infty.$$

□

Remark 5.4. A related result was proved by Smith for functions on subsets of the real line with super-logarithm behavior around the point at infinity [Smi24, Theorem 5.11]. The proof is also similar to *loc. cit.* and consists of two ingredients: the finite dimensional strong duality and the passage from finite dimensional to infinite dimensional by a limiting process.

The finite dimensional strong duality part shares the same spirit of the proof of Theorem A.7 in the appendix, which is the standard proof of strong duality for finite dimensional linear programming. But here we also need some suitable modifications due to the fact that here we deal with specific infinite dimensional vectors spaces and cones that are not necessarily closed.

The passage to the limit part is carried out by the technique of sweetened truncation and related estimations, especially Proposition 3.5.

Remarks 5.5.

i) As we will see in Section 6.4, there may not exist a measure realizing the optimal value $\mathcal{P}(\mathrm{g})$. On the other hand, when such measure does exist, it needs not to be unique. An example is $\mathrm{g}(z) = \log^+|z/2|$. In fact, let λ_r be the equilibrium measure on the circle $\{|z| = r\}$. Then for any $r \in [1,2]$ we have that $\lambda_r \in \mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$ and it attains the optimal value of $\mathcal{P}(\mathrm{g})$.

ii) More generally, when $\mathrm{g}(z)$ is radial (i.e. $\mathrm{g}(z) = \mathrm{g}(|z|)$ for all z) and g is also subharmonic, the measure λ_1 attains the optimal value of $\mathcal{P}(\mathrm{g})$. If we assume further that g is smooth and strictly subharmonic, then this measure is also unique. We will not give details here, but these assertions can be proved based on the study of toric Arakelov geometry established in [BGPS14, BGPS15, BGPRLS19].

6. On the essential minimum of heights

In this section we recall the Arakelov point of view of the height functions on $\mathbb{P}^1$ and use the strong duality theorem to extract some properties for the essential minimum of such heights.

6.1. **Arakelov theory on $\mathbb{P}^1$.** Let $\mathbb{P}^1_{\mathbb{Z}}$ denote the projective line over $\operatorname{Spec}(\mathbb{Z})$ and let $\overline{\mathcal{L}} = (\mathcal{O}(1), \|\cdot\|)$ be the metrized line bundle on $\mathbb{P}^1_{\mathbb{Z}}$ consisting of the line bundle $\mathcal{O}(1)$ plus the extra data of a continuous norm on the holomorphic line bundle $\mathcal{O}(1)(\mathbb{C})$ over the Riemann sphere $\mathbb{P}^1(\mathbb{C})$, that is invariant under complex conjugation. To such metrized line bundle one associates a height function $\mathrm{h}_{\overline{\mathcal{L}}}$ [BGS94], see also [Yua12, Section 8,9] for a survey.

Let $(x_0 \colon x_1)$ be homogeneous coordinates of $\mathbb{P}^1$. Then x_1 is a global section of $\mathcal{O}(1)$. Using the identification $\mathbb{P}^1(\mathbb{C})\backslash(1:0) \cong \mathbb{C}$, given by $(x_0 \colon x_1) \mapsto x = x_0/x_1$, the data of the metric $\|\cdot\|$ is encoded in the so called Green function (of continuous type)

$$\mathrm{g} \colon \mathbb{C} \longrightarrow \mathbb{R}, \quad x \longmapsto -\log\|x_1(x\colon 1)\|. \tag{6.27}$$

This Green function is continuous, invariant under complex conjugation, and there is a neighborhood U of the point $(1\colon 0) \in \mathbb{P}^1(\mathbb{C})$ such that $\mathrm{g}(x) - \log(x)$ extends continuously to the point $(1\colon 0)$. Clearly this condition is stronger than the condition in Definition 3.4. In fact any Green function of continuous type defines a continuous norm on $\mathcal{O}(1)$, thus the data of a continuous metric and the data of a Green function of continuous type are equivalent.

Remark 6.1. Already in the work of Faltings leading to the proof of Mordell's conjecture [Fal83], it became apparent that for many arithmetic applications one should not restrict one self to continuous metrics but also allow for some singularities. In the recent work [YZ26], Yuan and Zhang have introduced adelic line bundles on quasi-projective vartieties extending the notion of heights of points to a very wide class of singular metrics. If we assume that the metric $\|\cdot\|$ is singular at the point $(1\colon 0)$ in such a way that $\overline{\mathcal{L}}$ defines an adelic line bundle over the quasi-projective variety $\mathbb{A}^1 \subset \mathbb{P}^1$ then the associated Green function g satisfies Definition 3.4. The converse is also true, if g is a Green function as in Definition 3.4, then the associated metrized line bundle is an adelic metrized line bundle over $\mathbb{A}^1$ in the sense of [YZ26].

Following Remark 6.1, from now on we will assume that the metrized line bundle $\overline{\mathcal{L}} = (\mathcal{O}(1), \|\cdot\|)$ defines an adelic line bundle over $\mathbb{A}^1$ and that g is the associated Green function given by (6.27). Hence it is a Green function as in Definition 3.4. The height function $\mathrm{h}_{\overline{\mathcal{L}}}$ only depends on g and will be denoted as h_{g} unless we want to stress the Arakelov point of view.

This height function can be described in very concrete terms. Let $\alpha \in \overline{\mathbb{Q}}$ and let $P_\alpha \in \mathbb{Z}[x]$ be a primitive irreducible polynomial with $P_\alpha(\alpha) = 0$. The polynomial P_α is unique up to a sign. Denote by $O(\alpha) \subset \overline{\mathbb{Q}}$ the set of zeros of P_α and let a_α be the leading coefficient of P_α. The height of α is given by

$$\mathrm{h}_{\mathrm{g}}(\alpha) = \frac{1}{\deg(\alpha)}\left(\log|a_\alpha| + \sum_{\beta\in O(\alpha)} \mathrm{g}(\beta)\right). \tag{6.28}$$

If $\alpha \in \overline{\mathbb{Z}}$ then, we can choose P_α monic. Therefore

$$\mathrm{h}_{\mathrm{g}}(\alpha) = \frac{1}{\deg(\alpha)} \sum_{\beta\in O(\alpha)} \mathrm{g}(\beta), \qquad \alpha \in \overline{\mathbb{Z}}. \tag{6.29}$$

While for an arbitrary algebraic number we only have the inequality

$$\mathrm{h_g}(\alpha) \geq \frac{1}{\deg(\alpha)} \sum_{\beta \in O(x)} \mathrm{g}(\beta), \qquad \alpha \in \overline{\mathbb{Q}}.$$

There is another description of the height of an algebraic number as a sum of local contributions that will be useful latter. Let $\mathcal{M}_{\mathbb{Q}}$ be the set of places of $\mathbb{Q}$, that is

$$\mathcal{M}_{\mathbb{Q}} = \{p \in \mathbb{Z} \mid p > 0 \text{ prime}\} \cup \{\infty\}.$$

For each $\nu \in \mathcal{M}_{\mathbb{Q}}$ let $|\cdot|_\nu$ denote either the usual absolute value of $\mathbb{Q}$ if $\nu = \infty$ or the p-adic absolute value if $\nu \neq \infty$. The absolute value $|\cdot|_\nu$ extends uniquely to the field $\mathbb{C}_\nu$. For each ν fix an embedding $j_\nu \colon \overline{\mathbb{Q}} \to \mathbb{C}_\nu$ and write

$$|\alpha|_\nu = |j_\nu(\alpha)|_\nu, \quad \alpha \in \overline{\mathbb{Q}}$$

for a choice of absolute value in $\overline{\mathbb{Q}}$ that extends $|\cdot|_\nu$.

The alternative description of the height of α is [BG06, Section 1.5.7]

$$\mathrm{h_g}(\alpha) = \frac{1}{\deg(\alpha)} \left(\left(\sum_{p \text{ prime}} \sum_{\beta \in O(\alpha)} \log^+ |\beta|_p \right) + \sum_{\beta \in O(\alpha)} \mathrm{g}(\beta) \right). \tag{6.30}$$

Definition 6.2. Let $\mathrm{h_g}$ be the height function associated with a Green function g. The *essential minimum* of $\mathrm{h_g}$ is

$$\mathrm{ess}(\mathrm{h_g}) \coloneqq \inf \left\{ \liminf_{n\to\infty} \mathrm{h_g}(\alpha_n) \,\middle|\, (\alpha_n) \subseteq \overline{\mathbb{Q}} \text{ is a sequence of distinct algebraic numbers} \right\}.$$

Remark 6.3. To be as elementary as possible, from now on we will forget about the Arakelov point of view and focus on the Green function g as the source for the height function. Nevertheless the Arakelov point of view already points out towards many generalizations of the present work. First, instead of considering the canonical model $\mathbb{P}^1_{\mathbb{Z}}$ of $\mathbb{P}^1_{\mathbb{Q}}$ one can consider arbitrary models of $\mathbb{P}^1_{\mathbb{Q}}$ over $\mathrm{Spec}(\mathbb{Z})$ or even more generally one can consider general adelic line bundles, with different metrics over each place of $\mathbb{Q}$. Second, instead of the open set $\mathbb{A}^1 \subset \mathbb{P}^1$ and the divisor $[(1:0)]$ of $\mathbb{P}^1$ with respect to which the Green function is defined, one can consider arbitrary divisors of $\mathbb{P}^1$. Third, instead of $\mathbb{P}^1$ one can consider arbitrary curves or even higher dimensional varieties.

6.2. **The essential minimum and linear programming.** In this section we relate the essential minimum of the height function with the linear programming problems $\mathcal{D}(\mathrm{g})$ and $\mathcal{P}(\mathrm{g})$.

Proposition 6.4. *The relation* $\mathrm{ess}(\mathrm{h_g}) \geq \mathcal{D}(\mathrm{g})$ *holds true.*

Proof. Recall the product formula

$$\prod_{\nu \in \mathcal{M}_{\mathbb{Q}}} |\alpha|_\nu = 1, \qquad \alpha \in \mathbb{Q}^\times,$$

wich implies that, for $\alpha \in \overline{\mathbb{Q}}^\times$,

$$\sum_{\nu \in \mathcal{M}_{\mathbb{Q}}} \sum_{\beta \in O(\alpha)} \log |\beta|_\nu = 0, \tag{6.31}$$

because $\prod_{\beta\in O(\alpha)}\beta \in \mathbb{Q}$. Let Q_i, $i=1,\dots,n$ be polynomials with integer coefficients and $a_1,\dots,a_n\in\mathbb{R}_{\geq 0}$. If we show that

$$\operatorname{ess}(\mathrm{h_g}) \geq \inf_{z\in\mathbb{C}}\left(\mathrm{g}(z)-\sum_{i=1}^{n} a_i \log|Q_i(z)|\right), \tag{6.32}$$

then the proposition will be proved.

If $\sum a_i \deg(Q_i) > 1$, then

$$\inf_{z\in\mathbb{C}}\left(\mathrm{g}(z)-\sum_{i=1}^{n} a_i \log|Q_i(z)|\right) = -\infty$$

and equation (6.32) is trivially true. So we can assume that

$$\sum a_i \deg(Q_i) \leq 1. \tag{6.33}$$

Write Z for the set of zeroes of the polynomials Q_i. This is a finite set. Then, by the product formula, for every $\alpha\in\overline{\mathbb{Q}}\setminus Z$,

$$\mathrm{h_g}(\alpha) = \frac{1}{\deg(\alpha)}\sum_{p\text{ prime}}\sum_{\beta\in O(\alpha)}\left(\log^+|\beta|_p - \sum_{i=1}^{n} a_i\log|Q_i(\beta)|_p\right) + \frac{1}{\deg(\alpha)}\sum_{\beta\in O(\alpha)}\left(\mathrm{g}(\beta)-\sum_{i=1}^{n} a_i\log|Q_i(\beta)|\right).$$

By (6.33), and the fact that the polynomials Q_i have integer coefficients, we obtain, for every prime p,

$$\sum_{i=1}^{n} a_i\log|Q_i(\beta)|_p \leq \log^+|\beta|_p.$$

Therefore

$$\mathrm{h_g}(\alpha) \geq \frac{1}{\deg(\alpha)}\sum_{\beta\in O(\alpha)}\left(\mathrm{g}(\beta)-\sum_{i=1}^{n} a_i\log|Q_i(\beta)|\right), \quad \alpha\in\overline{\mathbb{Q}}\setminus Z.$$

This implies that equation (6.32) holds, proving the proposition. □

Proposition 6.5. *The relation* $\operatorname{ess}(\mathrm{h_g}) \leq \mathcal{P}(\mathrm{g})$ *holds true.*

Proof. Let $\mu\in\mathscr{P}_{\log}^{\overline{\mathbb{Z}}}$ and take a sequence of distinct algebraic integers $(\alpha_n)_{n\geq 1}\subseteq\overline{\mathbb{Z}}$ such that $\delta_{O(\alpha_n)}$ converges in the log-weak topology to μ. Using formula (6.29) and the fact that g is a continuous function of logarithmic growth, we have that

$$\lim_{n\to\infty}\mathrm{h_g}(\alpha_n) = \lim_{n\to\infty}\frac{1}{\deg(\alpha_n)}\sum_{p\in O(\alpha_n)}\mathrm{g}(p) = \int \mathrm{g}\,\mathrm{d}\mu.$$

Hence,

$$\operatorname{ess}(\mathrm{h_g}) \leq \inf_{\mu\in\mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C})}\int \mathrm{g}\,\mathrm{d}\mu.$$

□

Putting together Propositions 6.4 and 6.5 with Theorem 5.3 we obtain the main theorem of this section.

Theorem 6.6. *Let* g *be a Green function. Then,*

$$\mathcal{D}(\mathrm{g}) = \mathrm{ess}(\mathrm{h_g}) = \mathcal{P}(\mathrm{g}).$$

A direct consequence of Theorem 6.6 is that the essential minimum can be reached by a sequence of algebraic integers.

Corollary 6.7. *There exists a sequence of distinct algebraic integers* $(\alpha_n) \subseteq \overline{\mathbb{Z}}$ *such that* $\mathrm{h_g}(\alpha_n)$ *is monotonically decreasing and* $\inf_n \mathrm{h_g}(\alpha_n) = \mathrm{ess}(\mathrm{h_g})$.

Remark 6.8. The fact that $\mathrm{h_g}(\overline{\mathbb{Q}})$ is dense in $[\mathrm{ess}(\mathrm{h_g}), \infty)$ can be obtained as a consequence of Szachniewicz [Sza23, Theorem A] that uses the theory of globally valued fields. As another consequence of our previous results, here we can refine this statement to $\overline{\mathbb{Z}}$.

Corollary 6.9. *Let* $\mathrm{g}\colon \mathbb{C} \longrightarrow \mathbb{R}$ *be a Green function. Then,* $\mathrm{h_g}(\overline{\mathbb{Z}})$ *is dense in* $[\mathrm{ess}(\mathrm{h_g}), \infty)$.

Proof. By (1.6), the set

$$M = \left\{ \int \mathrm{g}\,\mathrm{d}\mu \,\middle|\, \mu \in \mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C}) \right\}$$

is contained in the closure of $\mathrm{h_g}(\overline{\mathbb{Z}})$. From Theorem 4.1 we deduce that M is a convex subset of $\mathbb{R}$, hence an interval. In view of Theorem 6.6, such interval contains $(\mathrm{ess}(\mathrm{h_g}), \infty)$, as desired. □

6.3. **Asymptotic maximal slope.** In this section we prove that in the dual problem we can reduce to linear combinations with rational coefficients. This has two applications. The first one is to relate the essential minimum with another Arakelov theory invariant called the assymptotic maximal slope and the second one is the left computability of the essential minimum as we will see in section 7. Consider the rational dual problems

$$\mathcal{D}_{\mathbb{Q}}(\mathrm{g}) \coloneqq \sup \left\{ \inf_{z\in\mathbb{C}} \Big(\mathrm{g}(z) - \sum_{i=1}^{k} a_i \log|Q_i(z)|\Big) \,\middle|\, a_i \in \mathbb{Q}_{\geq 0}, Q_i \in \mathbb{Z}[x], \sum a_i \deg(Q_i) \leq 1 \right\}.$$

and

$$\mathcal{D}'_{\mathbb{Q}}(\mathrm{g}) \coloneqq \sup \left\{ \inf_{z\in\mathbb{C}} \Big(\mathrm{g}(z) - \sum_{i=1}^{k} a_i \log|Q_i(z)|\Big) \,\middle|\, a_i \in \mathbb{Q}_{\geq 0}, Q_i \in \mathbb{Z}[x], \sum a_i \deg(Q_i) < 1 \right\}.$$

Clearly $\mathcal{D}(\mathrm{g}) \geq \mathcal{D}_{\mathbb{Q}}(\mathrm{g}) \geq \mathcal{D}'_{\mathbb{Q}}(\mathrm{g})$.

Proposition 6.10. *The equalities* $\mathcal{D}(\mathrm{g}) = \mathcal{D}_{\mathbb{Q}}(\mathrm{g}) = \mathcal{D}'_{\mathbb{Q}}(\mathrm{g})$ *hold.*

Proof. Let $n \geq 1$ and take $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n_{\geq 0}$ and $(Q_1, \dots, Q_n) \in (\mathbb{Z}[x])^n$. Let

$$\varphi_{\mathbf{a}}(z) \coloneqq \mathrm{g}(z) - \sum_{i=1}^{n} a_i \log|Q_i(z)|.$$

Then, it is enough to show that for any $\varepsilon > 0$ and any $\mathbf{a} \in \mathbb{R}^n_{\geq 0}$, there exists $\mathbf{b} \in \mathbb{Q}^n_{\geq 0}$ with $\sum_{i=1}^n b_i \deg Q_i < 1$ such that

$$\inf \varphi_{\mathbf{a}} \leq \inf \varphi_{\mathbf{b}} + \varepsilon. \tag{6.34}$$

Relation (6.34) holds trivially if $\inf \varphi_{\mathbf{a}} = -\infty$ or $\mathbf{a} = \mathbf{0} \in \mathbb{R}^n$, so we can assume $\inf \varphi_{\mathbf{a}} > -\infty$ and $a_i \neq 0$ for all i. Then, necessarily

$$\sum_{i=1}^{n} a_i \deg Q_i \leq 1. \tag{6.35}$$

Let U be an open and bounded neighborhood of the set of zeroes of $\prod_{i=1}^{n} Q_i(x)$ such that for all $z \in U$, it holds

$$\inf \varphi_{\mathbf{a}} \leq \mathrm{g}(z) - \sum_{i=1}^{n} a_i \max\left\{\frac{1}{2}\log|Q_i(z)|, \log|Q_i(z)|\right\} \tag{6.36}$$

Choose $R \geq 1$ big enough so that $\log|Q_i(z)| \geq 0$ for all $|z| \geq R$ and $i = 1, \dots, n$. Set $K := U^c \cap \{|z| \leq R\}$. Since K is a compact set, there exist $\mathbf{b} \in \mathbb{Q}^n_{\geq 0}$ such that

$$\frac{a_i}{2} \leq b_i < a_i \text{ for all } i \text{ and } \inf_{x \in K} \varphi_{\mathbf{b}}(x) \geq \inf_{x \in K} \varphi_{\mathbf{a}}(x) - \varepsilon.$$

In particular, in view of (6.35) we have that

$$\sum_{i=1}^{n} b_i \deg Q_i < 1 \text{ and } \inf_{z \in K} \varphi_{\mathbf{b}}(z) \geq \inf \varphi_{\mathbf{a}} - \varepsilon. \tag{6.37}$$

On the other hand, we claim that

$$\inf_{z \in U} \varphi_{\mathbf{b}}(z) \geq \inf \varphi_{\mathbf{a}}. \tag{6.38}$$

Indeed, due to the choice of the b_i we have

$$\varphi_{\mathbf{b}}(z) \geq \mathrm{g}(z) - \sum_{i=1}^{n} a_i \max\left\{\frac{1}{2}\log|Q_i(z)|, \log|Q_i(z)|\right\},$$

so the claim follows from (6.36). Finally, for z such that $|z| \geq R$ we have

$$\varphi_{\mathbf{b}}(z) - \varphi_{\mathbf{a}}(z) = \sum_{i=1}^{n} (a_i - b_i) \log|Q_i(z)| \geq 0,$$

thus $\inf_{|z| \geq R} \varphi_{\mathbf{b}}(z) \geq \inf \varphi_{\mathbf{a}}$. This estimate, together with (6.37) and (6.38), imply (6.34). □

We now consider on $\mathbb{P}^1_{\mathbb{Z}}$ the line bundles $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$. The metric on $\mathcal{O}(1)$ induces metrics on each $\mathcal{O}(n)$. For each global section $s \in \mathrm{H}^0(\mathbb{P}^1_{\mathbb{Z}}, \mathcal{O}(n))$ the sup norm is defined as

$$\|s\|_\infty := \sup_{z \in \mathbb{P}^1(\mathbb{C})} \|s(z)\|.$$

The *asymptotic maximal slope* of $\overline{\mathcal{L}}$ is defined as by

$$\hat{\mu}(\overline{\mathcal{L}}) := \sup_{\substack{n \geq 1 \\ s \in \mathrm{H}^0(\mathbb{P}^1_{\mathbb{Z}}, \mathcal{O}(n))}} -\frac{\log\|s\|_\infty}{n}.$$

By unfolding the definitions, one can see that

$$\hat{\mu}(\overline{\mathcal{L}}) = \mathcal{D}_{\mathbb{Q}}(\mathrm{g}),$$

which implies the next result.

Corollary 6.11. *The equality* $\hat{\mu}(\overline{\mathcal{L}}) = \mathrm{ess}(\mathrm{h}_{\overline{\mathcal{L}}})$ *is satisfied.*

Remark 6.12. When g is subharmonic and $\mathrm{g}(z) = \log|z| + a + o(1)$ at infinity, this equality is the specialization to $\mathbb{P}^1$ of a result of Ballaÿ [Bal21, Theorem 1.2]. Yuan conjectured a more general version of the result in *loc. cit.*, see [YZ26, Conjecture 5.3.5]. This conjecture has been established in some cases beyond [Bal21, Theorem 1.2], see [QY24, Theorem 1.8] and [YZ26, Theorem 5.3.6].

6.4. **An example with no minimizer.** In this subsection we present an example due to Nuno Hultberg, where the infimum

$$\inf_{\mu \in \mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})} \int \mathrm{g}\, \mathrm{d}\mu$$

is not attained by any measure $\mu_0 \in \mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$. Consider the automorphism $f\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1$, $x \longmapsto x^{-1}+2$. In homogeneous coordinates is given by $(x_0\colon x_1) \longmapsto (x_1+2x_0\colon x_0)$ under the identification $(x:1) = (x_0\colon x_1)$. Let $\overline{\mathcal{L}} = \big(\mathcal{O}(1), \|\cdot\|_{\mathrm{can}}\big)$, where the canonical metric $\|\cdot\|_{\mathrm{can}}$ is the metric whose Green function is $\log^+$. Consider the hermitian line bundle $f^*\overline{\mathcal{L}}$. By the projection formula $\mathrm{h}_{f^*\overline{\mathcal{L}}} = \mathrm{h}_{\overline{\mathcal{L}}} \circ f$, we see that

$$\mathrm{ess}(\mathrm{h}_{f^*\overline{\mathcal{L}}}) = \mathrm{ess}\, \mathrm{h}_{\overline{\mathcal{L}}} = 0.$$

The Green function associated with $f^*\overline{\mathcal{L}}$ is $\mathrm{g}(z) = \log^+|z^{-1}+2| + \log|z|$. In particular, Theorem 6.6 applies to $\mathrm{h}_{f^*\overline{\mathcal{L}}}$.

Assume there exists $\mu_0 \in \mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$ such that $\int \mathrm{g}\, \mathrm{d}\mu_0 = \mathrm{ess}(\mathrm{h}_{\mathrm{g}}) = 0$. Then there exists a sequence of distinct algebraic integers (α_n) such that $\delta_{O(\alpha_n)}$ converges to μ_0 in the log-weak sense. Therefore $\mathrm{h}_{\mathrm{g}}(\alpha_n)$ converges to $\int \mathrm{g}\, \mathrm{d}\mu_0 = 0$. Using the projection formula and Bilu's equidistribution theorem [Bil97, Theorem 1.1], we see that $\delta_{O(\alpha_n)}$ must converge weakly to $f^*\mu_{S^1}$, which forces $\mu_0 = f^*\mu_{S^1}$. However,

$$\begin{aligned}\int \log^+|z^{-1}+2| + \log|z|\, \mathrm{d}f^*\mu_{S^1}(z) &= \int \log^+|t| + \log|t-2|^{-1}\mathrm{d}\mu_{S^1}(t)\\ &= -\int \log|t-2|\mathrm{d}\mu_{S^1}(t)\\ &= -\log 2 \neq 0,\end{aligned}$$

which is a contradiction.

The preceeding argument also shows that $f^*\mu_{S^1} \notin \mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$, even though μ_{S^1} does belong to $\mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$. To see this more concretely, consider the polynomial

$$P_n(x) = \frac{x^{n+1}-2x^n+1}{x-1} \in \mathbb{Z}[x]$$

and let α_n be a root of P_n. Then $f^{-1}(\alpha_n)$ is an algebraic integer and the sequence $\mathrm{h}_{\mathrm{g}}(f^{-1}(\alpha_n)) = \mathrm{h}_{\overline{\mathcal{L}}}(\alpha_n)$, $n \geq 1$, converges to zero when n goes to ∞. If we look at the Galois orbit of $f^{-1}(\alpha_n)$, there are $n-1$ conjugates of $f^{-1}(\alpha_n)$ that are close to the support of $f^*\mu_{S^1}$, yet the remaining conjugate goes to infinity at the speed of 2^n. Therefore $\delta_{O(f^{-1}(\alpha_n))}$ converges weakly to $f^*\mu_{S^1}$ but does not converge log-weakly. The situation is similar to Autissier's counterexample [Aut06] showing that logarithmic equidistribution of small points is not true in general.

In this example, it also happens that the essential minimum is strictly smaller than

$$\inf\left\{\int \mathrm{g}\,\mathrm{d}\mu_K \,\middle|\, K \text{ is a conjugation invariant compact set of capacity one}\right\}. \tag{6.39}$$

Indeed,

$$\mathrm{g}(z) = \log^+|z^{-1}+2| + \log|z| = \log|z| + \log 2 + o(1) \quad \text{as } z \longrightarrow \infty.$$

Let $\mu_\infty := \frac{1}{2\pi}\Delta\mathrm{g} = f^*\mu_{S^1}$. Then $\mathrm{g}(z) = -U^{\mu_\infty} + \log(2)$ and for any probability measure μ,

$$\int \mathrm{g}\,\mathrm{d}\mu = \log 2 + \int -U^{\mu_\infty}\,\mathrm{d}\mu = \log 2 + \int -U^{\mu}\,\mathrm{d}\mu_\infty.$$

If μ is equilibrium measure of a capacity one set, then $-U^\mu \geq 0$, and we will have $\int \mathrm{g}\,\mathrm{d}\mu \geq \log 2$. So (6.39) cannot be the true essential minimum (which is 0).

6.5. **The essential minimum of Faltings' height.** Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and consider the associated modular curve Y, which is defined over $\mathbb{Q}$. Note that $Y(\overline{\mathbb{Q}})$ is in bijection with the set of isomorphism classes of elliptic curves over $\overline{\mathbb{Q}}$. Let X be the compactification of Y. There is a line bundle M_{12} on X such that $\mathrm{H}^0(X, M_{12}^{\otimes n}) \simeq M_{12n}(\Gamma,\mathbb{Q})$ as Hecke modules, where the latter is the $\mathbb{Q}$-space of modular forms of weight $12n$ and level 1 with rational Fourier coefficients.

We have a canonical integral model $(\mathcal{X}/\mathbb{Z}, \mathcal{M}_{12})$ of $(X/\mathbb{Q}, M_{12})$. Note that $\mathrm{H}^0(\mathcal{X}, n\mathcal{M}_{12}) \simeq \mathcal{M}_{12n}(\Gamma,\mathbb{Z})$ as Hecke modules, where the latter is the abelian group of modular forms of weight $12n$ and level 1 with integral Fourier coefficients.

The Petersson metric $\|\cdot\|_{\mathrm{Pet}}$ on $M_{12}(\mathbb{C})$ is defined as $\|f\|_{\mathrm{Pet}}(\tau) := |f(\tau)|\big(4\pi\operatorname{Im}\tau\big)^6$, i.e. taking $\big(4\pi\operatorname{Im}\tau\big)^6$ as metric weight, where f is any modular form of weight 12 and level 1 and $\tau \in \mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ is a point in the Poincaré upper half plane. The function $\|f\|_{\mathrm{Pet}}(\tau)$ is invariant under the action of Γ and descents to a function on $Y(\mathbb{C})$. We also define the L_2-norm on $M_{12}(\mathbb{C})$ by

$$\|f\|_{L^2} := \int_{Y(\mathbb{C})} \|f\|_{\mathrm{Pet}}\,\mathrm{d}\mu_{\mathrm{hyp}}.$$

Here, μ_{hyp} is the hyperbolic measure on $Y(\mathbb{C})$ normalized such that it is a probability measure.

Let $\overline{\mathcal{M}_{12}} := (\mathcal{M}_{12}, \|\cdot\|_{\mathrm{Pet}})$. Then, the Faltings height can be brought to our framework by the relation [BGMRL18, Section 2.1]

$$\mathrm{h}_{\mathrm{F}}(\alpha) = \frac{1}{12}\,\mathrm{h}_{\overline{\mathcal{M}_{12}}}(\alpha).$$

Remark 6.13. This height is in fact the *stable* Faltings height of the elliptic curve of j-invariant α, and can also be defined using the Hodge bundle equipped a canonical metric [YZ26, Theorem 5.5.1].

The j-function induces an isomorphism $j\colon \mathcal{X} \cong \mathbb{P}^1_{\mathbb{Z}}$ over $\mathbb{Z}$, where

- the cusp point corresponds to ∞,
- the automorphic bundle $\mathcal{M}_{12}$ corresponds to $\mathcal{O}(1)$,
- the modular discriminant $\Delta \in \mathrm{H}^0(\mathcal{X}, \mathcal{M}_{12})$ corresponds to the section $x_1 \in \mathrm{H}^0(\mathbb{P}^1_{\mathbb{Z}}, \mathcal{O}(1))$.

Let $\|\cdot\|_{\mathrm{hyp}}$ be the metric on $\mathcal{O}(1)(\mathbb{C})$ corresponding to the Petersson metric on $M_{12}(\mathbb{C})$ and let $\mathrm{g}_{\mathrm{hyp}}\colon \mathbb{C} \longrightarrow \mathbb{R}$ be the induced Green function. By the previous identifications it is given by

$$\mathrm{g}_{\mathrm{hyp}}(z) = -\log \|\Delta(\tau_z)\|_{\mathrm{Pet}},$$

where $\tau_z \in \mathbb{H}$ is sent to z by j. Note that we have the asymptotic estimate [BGMRL18, Section 3.2]

$$\mathrm{g}_{\mathrm{hyp}}(z) = \log|z| - 6\log\log|z| + O(1), \quad \text{as } |z| \longrightarrow \infty,$$

so $\mathrm{g}_{\mathrm{hyp}}$ is a Green function in the sense of Definition 3.4. Thus Theorem 6.6 and Corollary 6.11 apply and we get

Theorem 6.14.

$$\sup_{\substack{n\geq 1\\ s\in \mathrm{H}^0(\mathbb{P}^1_{\mathbb{Z}},\mathcal{O}(n))}} -\frac{\log\|s\|_{\mathrm{hyp},\infty}}{n} = 12\,\mathrm{ess}(\mathrm{h_F}) = \inf_{\mu\in\mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})} \int \mathrm{g}_{\mathrm{hyp}}\,\mathrm{d}\mu.$$

Theorem 6.15. *Let $\mathscr{P}^{\overline{\mathbb{Z}}}_{\log}\big(\mathcal{Y}(\mathbb{C})\big)$ be the set of probability measures that can be approximated* log*-weakly by Galois orbits in $\mathcal{Y}(\overline{\mathbb{Z}})$. Then*

$$\begin{aligned}\sup_{\substack{n\geq 1\\ f\in\mathcal{M}_{12n}(\Gamma,\mathbb{Z})}} -\frac{\log\|f\|_{\mathrm{Pet},\infty}}{n} &= \sup_{\substack{n\geq 1\\ f\in\mathcal{M}_{12n}(\Gamma,\mathbb{Z})}} -\frac{\log\|f\|_{L^2}}{n}\\ &= 12\,\mathrm{ess}(\mathrm{h_F})\\ &= \inf_{\mu\in\mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathcal{Y}(\mathbb{C}))}\int -\log\|\Delta\|_{\mathrm{Pet}}\,\mathrm{d}\mu.\end{aligned}$$

Proof. The first equality follows from [CGS21, Lemma 3.4.5], where it is proved that the distorsion of sup-norm and L^2-norm is subexponential. The rest are just a reformulation of Theorem 6.14. □

Remark 6.16. In [BGMRL18], the authors used

$$\sup_{\substack{n\geq 1\\ f\in\mathcal{M}_{12n}(\Gamma,\mathbb{Z})}} -\frac{\log\|f\|_{\mathrm{Pet},\infty}}{n} \leq 12\,\mathrm{ess}(\mathrm{h_F}) \leq \inf_{\mathrm{cap}(K)=1}\int \mathrm{g}_{\mathrm{hyp}}\,\mathrm{d}\mu_K$$

to give numerical estimates of $\mathrm{ess}(\mathrm{h_F})$. Here we have proved that the lower bound indeed reaches $\mathrm{ess}(\mathrm{h_F})$, while for the upper bound to reach $\mathrm{ess}(\mathrm{h_F})$, we may need to consider more general measures $\mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$ than just equilibrium measures of compact sets of capacity one.

Theorem 6.7 specializes to

Theorem 6.17. *The essential minimum of Faltings' height can by attained by a sequence of elliptic curves with good reduction everywhere.*

In fact, Theorem 6.7 says that we can approach the essential minimum of Faltings' height by elliptic curves with integral j-invariants hence having potentially good reduction everywhere [Sil09, Chapter VII. Proposition 5.5]. So after a finite field extension they become good reduction everywhere.

7. The computability of the essential minimum

For a particular height function, it is not obvious how to compute its essential minimum. As a consequence of the strong duality Theorem 6.6, we can construct both a decreasing sequence and an increasing sequence converging to the essential minimum, thus showing that this invariant is a computable real number. The obtained algorithm is far from being practical, but might be a first step in the search of an effective procedure for this problem.

To make the ideas precise, we will use the theory of computability and we refer the reader to [PER89] for preliminaries on computability.

7.1. Computability.

We first recall the notion of computable number, in one of its equivalent definitions.

Definition 7.1. A real number r is *computable* if there exists a Turing machine that given any rational number $\varepsilon > 0$ produces a rational number q with $|r - q| < \varepsilon$. A complex number is *computable* if both its real and imaginary part are computable.

The set of computable numbers is a countable subfield of $\mathbb{C}$ that contains all algebraic numbers and as a consequence, most real numbers are non-computable. Nevertheless it is very difficult to give a concrete non-computable number. Examples of such numbers are Chaitin's constants, which are associated to Turing's halting problem.

There are weaker notions of computability.

Definition 7.2. A real number r is *left computable* if there is a Turing machine that given a natural number n produces a rational number x_n such that $\sup_n x_n = r$. Similarly r is *right computable* if there is a Turing machine that given n produces a rational number y_n such that $\inf_n x_n = r$.

Lemma 7.3. *A real number is computable if and only if it is both left and right computable.*

Proof. Assume that r is computable. Then there is a Turing machine that given n produces a rational number z_n with $|r - z_n| < 1/n$. The sequences $x_n = z_n - 1/n$ and $y_n = z_n + 1/n$ show that r is left and right computable.

Now assume that r is both right and left computable and $0 < \varepsilon \in \mathbb{Q}$. Then there is a Turing machine that given a natural number N produces $\alpha_N = \max_{n \leq N} x_n$ and $\beta_N = \min_{n \leq N} y_n$. When $\beta_N - \alpha_N < \varepsilon$ we have reached the desired precision and the algorithm stops. □

Definition 7.4. A function $f\colon \mathbb{N} \longrightarrow \mathbb{N}$ is *computable* if there is a Turing machine that given input n produces output $f(n)$.

The Green functions that will be well suited for our computations are those computable in appropriate compact sets and having effective asymptotics.

Definition 7.5. Let $\mathrm{g}\colon \mathbb{C} \longrightarrow \mathbb{R}$ be a Green function as in 3.4. It is said to be *computable* if it satisfies the following conditions:

(1) for any rational rectangle E (i.e. $E = \{z \in \mathbb{C} \colon a_1 \leq \mathrm{Re}(z) \leq b_1, a_2 \leq \mathrm{Im}(z) \leq b_2\}$ for rational numbers $a_1 < b_1, a_2 < b_2$), the restriction $\mathrm{g}\colon E \longrightarrow \mathbb{R}$ is computable in the sense of [PER89, Section 0.3, Definition A],

(2) there is a computable function $f\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that for all $z \in \mathbb{C}$ with $|z| > f(n)$

$$\big|\mathrm{g}(z) - \log|z|\big| \leq \frac{1}{n}\log|z|.$$

The following is our main result in this section.

Theorem 7.6. *Let* g *be a computable Green function. Then* $\mathrm{ess}(\mathrm{h_g})$ *is a computable real number.*

The proof will be carried out in the next two sections, where we will see that the essential minimum is both left and right computable.

7.2. **Left computability.** By Proposition 6.10 we have

$$\mathrm{ess}(\mathrm{h_g}) = \sup_{\Lambda} \inf_{z\in\mathbb{C}} \Big(\mathrm{g}(z) - \sum_{i=1}^{k} a_i \log|Q_i(z)|\Big),$$

with

$$\Lambda = \Big\{(Q_1,\dots,Q_k,a_1,\dots,a_k) \;\Big|\; k\in\mathbb{N}, Q_i\in\mathbb{Z}[x], a_i\in\mathbb{Q}_+ \text{ and } \sum_{i=1}^{k} a_i \deg(Q_i) < 1\Big\}.$$

Since the index set Λ can be effectively enumerated, the left computability of $\mathrm{ess}(\mathrm{h}_g)$ is a direct consequence of the next statement.

Proposition 7.7. *Assume that* g *is a computable Green function. Let* $(Q_1,\dots,Q_k,a_1,\dots,a_k)\in\Lambda$ *and set* $\varphi = \mathrm{g} - \sum_{i=1}^{k} a_i \log|Q_i|$. *Then,* $\inf_{z\in\mathbb{C}}\varphi(z)$ *is a computable real number.*

Proof. Since g is a computable Green function and $\sum a_i\deg(Q_i) < 1$, we can determine a rational rectangle E such that the infimum of φ is attained in E. Similarly we can determine $M\in\mathbb{N}$ such that that there is $z\in E$ with $\varphi(z)\le M$. Then

$$\inf_{z\in\mathbb{C}}\varphi(z) = \inf_{z\in E}\min\{\varphi(z), M\}.$$

Then this infimum is computable [PER89, Section 0.6, Theorem 7], because the restriction of $\min\{\varphi, M\}$ to E is a computable function. □

7.3. **Right computability.** For the right computability, we look at the other side of the strong duality property, namely

$$\mathrm{ess}(\mathrm{h_g}) = \inf\Big\{\int \mathrm{g}\,\mathrm{d}\mu \;\Big|\; \mu\in\mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})\Big\}.$$

Since the set $\mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$ is not countable, our first step is to show that it admits a countable subset that is dense with respect to the log-weak topology.

Consider the index set

$$\Theta = \big\{(P,Q) \;\big|\; P,Q\in\mathbb{Z}[x] \text{ with } P\neq Q \text{ monic irreducible polynomials}\big\}.$$

For $(P,Q)\in\Theta$ we denote by $\varphi_{P,Q}\colon\mathbb{P}^1\longrightarrow\mathbb{P}^1$ the map induced by the rational function $P^{\deg(Q)+1}/Q^{\deg(P)}$. This is a finite and proper map of degree $\deg(P)(\deg(Q)+1)$.

Let $f:\mathbb{C}\longrightarrow\mathbb{R}$ be a function. We denote by $(\varphi_{P,Q})_* f:\mathbb{C}\longrightarrow\mathbb{R}$ the *push-forward* of f by $\varphi_{P,Q}$, defined as

$$(\varphi_{P,Q})_* f(z) = \sum_{w|\varphi_{P,Q}(w)=z} e_w f(w),$$

where e_w is the ramification index of $\varphi_{P,Q}$ at w.

We remark that due the properness of $\varphi_{P,Q}$, the push-forward operation preserves the space of continuous and compactly supported functions. Hence, if μ is a measure on $\mathbb{C}$, we can define the pullback measure $\varphi_{P,Q}^{*}\mu$ by the rule

$$\int f \mathrm{d}(\varphi_{P,Q}^{*}\mu) := \int (\varphi_{P,Q})_{*} f \mathrm{d}\mu,$$

where $f : \mathbb{C} \longrightarrow \mathbb{R}$ is continuous and compactly supported.

Finally, we consider the probability measure $\mu_{P,Q}$ defined as

$$\mu_{P,Q} = \frac{(\varphi_{P,Q})^{*}\mu_{S^1}}{\deg(P)(\deg(Q)+1)},$$

with μ_{S^1} the Haar measure on the unit circle.

Proposition 7.8. *Let $(P,Q) \in \Theta$. Then $\mu_{P,Q}$ is a probability measure supported on the compact set*

$$\left\{ x \,\middle|\, |P(x)|^{\deg(Q)+1} = |Q(x)|^{\deg(P)} \right\} \tag{7.40}$$

with potential function $U^{\mu_{P,Q}} = \min\left\{ -\frac{\log|P|}{\deg(P)}, -\frac{\log|Q|}{\deg(Q)+1} \right\}$.

Proof. The support of $\mu_{P,Q}$ is the preimage of S^1 with respect to the map $\varphi_{P,Q}$, which coincides with (7.40). On the other hand, set $d = \deg(P)$ and $e = \deg(Q)$. Then for $z \in \mathbb{C}$ we have

$$U^{\mu_{P,Q}}(z) = -\int \log|z-w| \,\mathrm{d}\mu_{P,Q}(w) = \frac{-1}{d\,(e+1)} \int \sum_{\varphi_{P,Q}(w)=y} e_w \log|z-w| \,\mathrm{d}\mu_{S^1}(y) \tag{7.41}$$

We have

$$\sum_{\varphi_{P,Q}(w)=y} e_w \log|z-w| = \log\left| \prod_{\varphi_{P,Q}(w)=y} (z-w)^{e_w} \right| = \log|P(z)^{e+1} - yQ(z)^d|$$

and the expression formula for the potential follows from (7.41) applying Jensen's formula. □

Proposition 7.9. *The family of probability measures $\{\mu_{P,Q} \mid (P,Q) \in \Theta\}$ is a countable dense subset of $\mathscr{P}_{\log}^{\overline{\mathbb{Z}}}(\mathbb{C})$.*

Proof. Let $(P,Q) \in \Theta$ and take an irreducible primitive polynomial $F \in \mathbb{Z}[x]$ with leading coefficient $a \in \mathbb{Z} \setminus \{0\}$ and roots $\alpha_1, \ldots, \alpha_n$. Using Proposition 7.8 we get

$$\begin{aligned}
\int \log|F| \,\mathrm{d}\mu_{P,Q} &= \log|a| - \sum_i U^{\mu_{P,Q}}(\alpha_i) \\
&= \log|a| - \sum_i \max\left\{ \frac{\log|P(\alpha_i)|}{\deg(P)}, \frac{\log|Q(\alpha_i)|}{\deg(Q)+1} \right\} \\
&\geq \max\left\{ \sum_i \frac{\log|P(\alpha_i)|}{\deg(P)}, \sum_i \frac{\log|Q(\alpha_i)|}{\deg(Q)+1} \right\} \\
&= \max\left\{ \frac{\log|\mathrm{Res}(P,F)|}{\deg(P)}, \frac{\log|\mathrm{Res}(Q,F)|}{\deg(Q)+1} \right\} \\
&\geq 0
\end{aligned}$$

where the last inequality follows from the fact that $\mathrm{Res}(P,F)$ and $\mathrm{Res}(Q,F)$ are integers that cannot be both zero because F, P, Q are irreducible and P,Q are coprime. Hence $\mu_{P,Q} \in \mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$.

Now let $\mu \in \mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$, and let (α_n) be a sequence of distinct algebraic integers with $\delta_{O(\alpha_n)} \longrightarrow \mu$ log-weakly, and denote by P_n the minimal polynomial of α_n. Then by Proposition 2.14, we have

$$-\frac{\log|P_n|}{\deg(P_n)} \longrightarrow U^{\mu} \quad \text{in } L^1_{\mathrm{loc}}.$$

Hence

$$\min\left\{-\frac{\log|P_n|}{\deg(P_n)}, -\frac{\log|P_{n+1}|}{\deg(P_{n+1})+1}\right\} \longrightarrow U^{\mu} \quad \text{in } L^1_{\mathrm{loc}},$$

which gives $\mu_{P_n,P_{n+1}} \longrightarrow \mu$ log-weakly by Proposition 2.14 again. This proves that the family $\{\mu_{P,Q}\}$ is dense under the log-weak topology. □

Remark 7.10. One could alternatively consider for each $(P,Q) \in \Theta$ the probability measure $\mu'_{P,Q}$ similarly induced by the rational function $P^{\deg(Q)}/Q^{\deg(P)}$. This gives another countable dense subset of $\mathscr{P}^{\overline{\mathbb{Z}}}_{\log}(\mathbb{C})$ and might look more natural than $\{\mu_{P,Q}\}$. The advantage of the measures $\mu_{P,Q}$ is that they are compactly supported, which is important in the proof of Proposition 7.11.

By Proposition 7.9, we have

$$\mathrm{ess}(\mathrm{h_g}) = \inf_{(P,Q)\in\Theta} \int \mathrm{g}\, \mathrm{d}\mu_{P,Q}.$$

Hence the right computability of the essential minimum follows by the next result.

Proposition 7.11. *Let $(P,Q) \in \Theta$. Assume that* g *is a computable Green function. Then, the integral* $\int \mathrm{g}\, \mathrm{d}\mu_{P,Q}$ *is computable.*

Proof. Set for short $d = \deg(P)$ and $e = \deg(Q)$ and consider the function

$$\rho\colon [0,1] \longrightarrow \mathbb{R}, \quad \rho(\theta) = \frac{((\varphi_{P,Q})_*\mathrm{g})(e^{2\pi i\theta})}{d\,(e+1)}.$$

For each $\theta \in [0,1]$ consider also the polynomial $S_\theta = P^{\deg(Q)+1} - e^{2\pi i\theta}Q^{\deg(P)} \in \mathbb{C}[x]$. Then

$$\rho(\theta) = \frac{1}{d\,(e+1)} \sum_{S_\theta(w)=0} e_w \mathrm{g}(w). \tag{7.42}$$

There exists an effective $0 < R \in \mathbb{Q}$ such that $\varphi^{-1}_{P,Q}(S^1) \subseteq [-R,R] \times i[-R,R]$. Since g restricted to $[-R,R] \times i[-R,R]$ is computable by definition, and the process of finding the complete set of roots of polynomials is computable by [Spe69], ρ is a computable function and hence the Riemann integral

$$\int_0^1 \Big((\varphi_{P,Q})_*\mathrm{g}\Big)(e^{2\pi i\theta})\, \mathrm{d}\theta$$

is computable [PER89, Section 0.5, Theorem 5]. □

Appendix A. Duality in linear programming

In this appendix we present a coordinate-free formulation of duality in linear optimization and recall the proof of the strong duality property in the finite dimensional case. This allows to place the optimization problems from Section 5 within a general framework and to give a more conceptual approach to their duality properties.

Let E, F be two real vector spaces equipped with a pairing $\langle -, - \rangle \colon E \times F \longrightarrow \mathbb{R}$. Let $E \longrightarrow F^\vee$ and $F \longrightarrow E^\vee$ be the induced linear maps between these spaces and their duals, that we respectively denote by $x \longmapsto x^\dagger$ and $y \longmapsto y^\dagger$. They are defined by setting

$$x^\dagger(y) = y^\dagger(x) = \langle x, y \rangle \quad \text{for all } x \in E \text{ and } y \in F.$$

Consider two convex cones $\sigma \subset E$ and $\tau \subset F$. Their duals are the convex cones

$$\sigma^\vee = \{u \in E^\vee \mid u(x) \geq 0 \text{ for all } x \in \sigma\}, \quad \tau^\vee = \{v \in F^\vee \mid v(y) \geq 0 \text{ for all } y \in \tau\}.$$

Let also $u_0 \in E^\vee$ and $v_0 \in F^\vee$.

Definition A.1. The *primal problem* and the *dual problem* for this datum are the optimization problems respectively given by

$$\mathcal{P} = \inf\{u_0(x) \mid x \in \sigma,\ x^\dagger - v_0 \in \tau^\vee\}, \quad \mathcal{D} = \sup\{v_0(y) \mid y \in \tau,\ u_0 - y^\dagger \in \sigma^\vee\}.$$

We refer to the quantities $\mathcal{P}$ and $\mathcal{D}$ as the *optimal values* of these problems.

Remark A.2. The role of the primal and the dual problems can be exchanged: denote by $\mathcal{P}^{\mathrm{op}}$ and $\mathcal{D}^{\mathrm{op}}$ the primal and dual problems that arise when swapping the vector spaces E and F and considering the cones $-\tau \subset F$ and $-\sigma \subset E$ together with the functionals $-v_0 \in F^\vee$ and $-u_0 \in E^\vee$. Then it can be easily verified that $\mathcal{P} = -\mathcal{D}^{\mathrm{op}}$ and $\mathcal{D} = -\mathcal{P}^{\mathrm{op}}$.

The classical problems in linear programming are a particular case of this framework.

Example A.3. Set $E = \mathbb{R}^m$ and $F = \mathbb{R}^n$ and given $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ consider the pairing and functionals defined by

$$(x, y) \longmapsto \langle x, y \rangle = x^{\mathrm{T}} A\, y, \quad x \longmapsto u_0(x) = b^{\mathrm{T}} x, \quad y \longmapsto v_0(y) = c^{\mathrm{T}} y$$

together with the cones $\sigma = \mathbb{R}^m_{\geq 0}$ and $\tau = \mathbb{R}^n_{\geq 0}$. Then the associated primal and dual problems boil down to the usual forms

$$\mathcal{P} = \inf\{b^{\mathrm{T}} x \mid x \geq 0,\ A^{\mathrm{T}} x \geq c\}, \quad \mathcal{D} = \sup\{c^{\mathrm{T}} y \mid y \geq 0,\ A\, y \leq b\},$$

where $\leq$ and $\geq$ on these real vector spaces means that these inequalities hold coordinate-wise.

The problems from Section 5 also fit within this framework. Denote by $\mathscr{M}_{\log}(\mathbb{C})$ and $\mathscr{S}_{\log}(\mathbb{C})$ the cone and the vector space generated by $\mathscr{P}_{\log}(\mathbb{C})$. The elements of $\mathscr{M}_{\log}(\mathbb{C})$ are the measures on $\mathbb{C}$ that integrate the function $\log^+ |z|$, whereas those of $\mathscr{S}_{\log}(\mathbb{C})$ are the differences of these measures.

We also let $\mathscr{P}'_{\log}(\mathbb{C})$ be the set of probability measures $\mu \in \mathscr{P}_{\log}(\mathbb{C})$ that integrate the functions $\log |Q|$ for all $Q \in \mathbb{Z}[x]$, and we respectively denote by $\mathscr{M}'_{\log}(\mathbb{C})$ and $\mathscr{S}'_{\log}(\mathbb{C})$ the cone and vector space generated by this set of probability measures.

Example A.4. Set

$$E = \mathscr{S}'_{\log}(\mathbb{C}) \quad \text{and} \quad F = \mathbb{R} \oplus \bigoplus_{n\in\mathbb{N}} \mathbb{R}.$$

Fix an enumeration $Q_1, Q_2, Q_3, \dots$ of all nonconstant integer polynomials and recall that the elements of F are the tuples $a = (a_0, a_1, a_2, \dots)$ with $a_n = 0$ for all but a finite number of n's. We then consider the pairing $E \times F \longrightarrow \mathbb{R}$ defined by

$$(\mu, a) \longmapsto a_0 \int \mathrm{d}\mu + \sum_{n\in\mathbb{N}} a_n \int \log|Q_n| \, \mathrm{d}\mu.$$

We also consider the convex cones defined as

$$\sigma = \mathscr{M}'_{\log}(\mathbb{C}) \subset E \quad \text{and} \quad \tau = \{(a_0, a_1, \dots, a_n, \dots) \mid a_n \geq 0 \text{ for all } n > 0\} \subset F.$$

Let $\mathrm{g} \colon \mathbb{C} \longrightarrow \mathbb{R}$ be a continuous function that is asymptotically logarithmic at ∞ in the sense of Definition 3.4, and define the functionals $u_0 \in E^\vee$ and $v_0 \in F^\vee$ as

$$u_0(\mu) = \int \mathrm{g} \, \mathrm{d}\mu \quad \text{and} \quad v_0(a) = a_0.$$

The associated primal problem amounts to the minimization $\mathcal{P}(\mathrm{g}) = \inf_\mu \int \mathrm{g}\, \mathrm{d}\mu$ over the measures $\mu \in \mathscr{M}'_{\log}(\mathbb{C})$ such that

$$a_0 \Big(\int \mathrm{d}\mu - 1 \Big) + \sum_{n\in\mathbb{N}} a_n \int \log|Q_n| \, \mathrm{d}\mu \geq 0$$

for all $a_0 \in \mathbb{R}$ and $a_n \in \mathbb{R}_{\geq 0}$, $n \in \mathbb{N}$, with $a_n = 0$ for all but a finite number of n's. Since a_0 is arbitrary, this forces $\mu \in \mathscr{P}'_{\log}(\mathbb{C})$. Hence, this minimization is over the probability measures $\mu \in \mathscr{P}'_{\log}(\mathbb{C})$ such that $\int \log|Q_n| \, \mathrm{d}\mu \geq 0$ for all n, which coincide with those in $\mathscr{P}_{\log}(\mathbb{C})$ satisfying the same condition. Hence

$$\mathcal{P}(\mathrm{g}) = \inf \Big\{ \int \mathrm{g} \, \mathrm{d}\mu \,\Big|\, \mu \in \mathscr{P}_{\log}(\mathbb{C}), \int \log|Q_n| \, \mathrm{d}\mu \geq 0 \text{ for all } n \in \mathbb{N} \Big\}$$

as in (5.20). Similarly, the associated dual problem is the maximization $\mathcal{D}(\mathrm{g}) = \sup_a a_0$ over $a \in \tau$ such that

$$\int \mathrm{g} \, \mathrm{d}\mu - a_0 \int \mathrm{d}\mu - \sum_{n\in\mathbb{N}} a_n \int \log|Q_n| \, \mathrm{d}\mu \geq 0 \quad \text{for all } \mu \in \mathscr{M}'_{\log}(\mathbb{C}). \tag{A.43}$$

This is equivalent to the inequality $\mathrm{g}(z) - \sum_{n\in\mathbb{N}} a_n \log|Q_n(z)| \geq a_0$ for all $z \in \mathbb{C}$, as it can be seen by considering (A.43) for the Dirac delta measures $\mu = \delta_z$ for all $z \in \mathbb{C} \setminus \overline{\mathbb{Q}}$. Hence

$$\mathcal{D}(\mathrm{g}) = \sup_{a\in\tau} \inf_{x\in\mathbb{C}} \Big(\mathrm{g}(x) - \sum_{n\in\mathbb{N}} a_n \log|Q_n(x)| \Big),$$

in agreement with (5.21).

The weak duality property is the fact that the optimal value of the primal problem bounds above that of the dual, and follows readily from the definitions.

Proposition A.5. *We have $\mathcal{P} \geq \mathcal{D}$.*

Proof. For all $x \in \sigma$ and $y \in \tau$ such that $x^\dagger - v_0 \in \tau^\vee$ and $u_0 - y^\dagger \in \sigma^\vee$ we have $u_0(x) \geq \langle x, y\rangle \geq v_0(y)$, which gives the inequality. □

The strong duality property is the equality between the optimal values of the primal and the dual problem. We end this appendix by recalling the proof of this property in the finite dimensional situation.

Definition A.6. The *feasibility set* of the primal problem $\mathcal{P}$ and the dual problem $\mathcal{D}$ are

$$S_{\mathcal{P}} = \{x \in \sigma \mid x^{\dagger} - v_0 \in \tau^{\vee}\} \subset E \quad \text{and} \quad S_{\mathcal{D}} = \{y \in \tau \mid u_0 - y^{\dagger} \in \sigma^{\vee}\} \subset F.$$

We say that $\mathcal{P}$ (respectively $\mathcal{D}$) is *feasible* if $S_{\mathcal{P}} \neq \emptyset$ (respectively if $S_{\mathcal{D}} \neq \emptyset$), and we say that $\mathcal{P}$ (respectively $\mathcal{D}$) is *bounded* if the set $\{u_0(x) \colon x \in S_{\mathcal{P}}\}$ is bounded below (respectively if the set $\{v_0(y) \colon y \in S_{\mathcal{D}}\}$ is bounded above).

We also say that $\mathcal{P}$ (respectively $\mathcal{D}$) is *attained* if there exists $x \in S_{\mathcal{P}}$ such that $\mathcal{P} = u_0(x)$ (respectively if there exists $y \in S_{\mathcal{D}}$ such that $\mathcal{D} = v_0(y)$).

The primal problem is feasible and bounded if and only if $\mathcal{P} \in \mathbb{R}$, and similarly for the dual problem. Moreover, the weak duality property shows that if one of these problems is feasible then the other is bounded.

Theorem A.7. *Assume that E and F are finite dimensional vector spaces and that σ and τ are closed convex cones. The following conditions are equivalent:*

(1) *the primal problem $\mathcal{P}$ is feasible and bounded,*
(2) *the dual problem $\mathcal{D}$ is feasible and bounded.*

If any of these conditions holds then $\mathcal{P} = \mathcal{D} \in \mathbb{R}$ and both $\mathcal{P}$ and $\mathcal{D}$ are attained.

Proof. First assume that (1) holds and consider the closed convex subsets of $\mathbb{R} \times F^{\vee}$ defined as

$$V = \{(u_0(x), x^{\dagger}) \mid x \in \sigma\} \quad \text{and} \quad W_{\lambda} = \{(t, v + v_0) \mid t \leq \lambda,\ v \in \tau^{\vee}\}, \quad \lambda \in \mathbb{R}.$$

For each $\lambda \in \mathbb{R}$ we have that $V \cap W_{\lambda} \neq \emptyset$ if and only if there exists $x \in S_{\mathcal{P}}$ such that $u_0(x) \leq \lambda$.

We have the decomposition $W_{\lambda} = \mathbb{R}_{\leq 0} \times \tau^{\vee} + (p, v_0)$. Hence considering the closed convex cone and the point respectively defined as

$$C = V - \mathbb{R}_{\leq 0} \times \tau = \{(u_0(x) - t, x^{\dagger} - v) : t \in \mathbb{R}_{\leq 0},\ x \in \sigma,\ v \in \tau^{\vee}\} \quad \text{and} \quad p_{\lambda} = (\lambda, v_0),$$

the condition $V \cap W_{\lambda} \neq \emptyset$ turns out to be equivalent to $p_{\lambda} \in C$. Since C is a closed cone, this condition on $\lambda \in \mathbb{R}$ is closed, and it is also nonempty and bounded below because $\mathcal{P}$ is feasible and bounded. Thus setting $\lambda_0 = \inf\{\lambda \mid p_{\lambda} \in C\}$ we have

$$\mathcal{P} = \lambda_0 = \min\{\lambda \mid p_{\lambda} \in C\} \in \mathbb{R}.$$

In particular $\mathcal{P}$ is attained.

Now let $\lambda < \lambda_0$. By the point-cone separation theorem there exists $h \in (\mathbb{R} \times F^{\vee})^{\vee} = \mathbb{R} \oplus F$ such that $h|_C \geq 0$ and $h(p_{\lambda}) < 0$, which implies that $h|_V > h|_{W_{\lambda}}$. Hence writing $h = (b, -y)$ with $b \in \mathbb{R}$ and $y \in F$ we have

$$b\, u_0(x) - \langle x, y\rangle > b\, t - v(y) - v_0(y) \quad \text{for all } t \leq \lambda, x \in \sigma, v \in \tau^{\vee}. \tag{A.44}$$

Specializing (A.44) to $x = x_1 \in S_{\mathcal{P}} \subset \sigma$ and $v = v_1 = x_1^{\dagger} - v_0 \in \tau^{\vee}$ we get $b\, u_0(x_1) > b\, t$, which implies that $b > 0$ because t can be an arbitrarily large negative number.

We assume without loss of generality that $b = 1$. Then (A.44) specialized to $t = \lambda$ becomes

$$u_0(x) - \langle x, y\rangle > \lambda - v(y) - v_0(y) \quad \text{for all } x \in \sigma, v \in \tau^{\vee}. \tag{A.45}$$

Specializing this inequality to $x = 0$ gives $v(y) \geq 0$ for all $v \in \tau^{\vee}$, whereas taking instead $v = 0$ gives $(u_0 - y^{\dagger})(x) \geq 0$ for all $x \in \sigma$. Since τ is assumed to be closed, $y \in \tau$ and $u_0 - y^{\dagger} \in \sigma^{\vee}$, and so the dual problem $\mathcal{D}$ is feasible. We also have that $\mathcal{D}$ is bounded because $\mathcal{P}$ is feasible, thus proving the condition (2). Moreover (A.45) specialized to $x = 0$ and $v = 0$ gives

$$\mathcal{D} \geq v_0(y) > \lambda,$$

and since λ can be arbitrarily close to $\mathcal{P}$ we obtain $\mathcal{P} \leq \mathcal{D}$. Combining with the weak duality property (Proposition A.5) we conclude that $\mathcal{P} = \mathcal{D}$, as stated.

Finally, the case when (2) holds reduces to the previous one using Remark A.2. □

Remark A.8. The proof of Theorem 5.3 follows this approach, with suitable modifications due to the fact that there we deal with some specific infinite dimensional vectors spaces and cones that are not necessarily closed.

References

[Aut06] Pascal Autissier. Sur une question d'équirépartition de nombres algébriques. *Comptes Rendus Mathematique*, 342(9):639–641, 2006.

[Bal21] François Ballaÿ. Successive minima and asymptotic slopes in Arakelov geometry. *Compos. Math.*, 157(6):1302–1339, 2021.

[BG06] Enrico Bombieri and Walter Gubler. *Heights in Diophantine geometry*, volume 4 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2006.

[BGMRL18] José Ignacio Burgos Gil, Ricardo Menares, and Juan Rivera-Letelier. On the essential minimum of Faltings' height. *Math. Comp.*, 87(313):2425–2459, 2018.

[BGPRLS19] José Ignacio Burgos Gil, Patrice Philippon, Juan Rivera-Letelier, and Martín Sombra. The distribution of Galois orbits of points of small height in toric varieties. *Amer. J. Math.*, 141(2):309–381, 2019.

[BGPS14] José Ignacio Burgos Gil, Patrice Philippon, and Martín Sombra. Arithmetic geometry of toric varieties. Metrics, measures and heights. *Astérisque*, 360:vi+222, 2014.

[BGPS15] José Ignacio Burgos Gil, Patrice Philippon, and Martín Sombra. Successive minima of toric height functions. *Ann. Inst. Fourier (Grenoble)*, 65(5):2145–2197, 2015.

[BGS94] Jean-Benoît Bost, Henri Gillet, and Christophe Soulé. Heights of projective varieties and positive Green forms. *J. Amer. Math. Soc.*, 7(4):903–1027, 1994.

[Bil97] Yuri Bilu. Limit distribution of small points on algebraic tori. *Duke Math. J.*, 89(3):465–476, 1997.

[Bil99] Patrick Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons, Inc., New York, second edition, 1999.

[BLW15] Thomas Bloom, Norman Levenberg, and Frank Wielonsky. Logarithmic potential theory and large deviation. *Comput. Methods Funct. Theory*, 15(4):555–594, 2015.

[cBS25] François Ballaÿ and Martín Sombra. Approximation of adelic divisors and equidistribution of small points, 2025. URL: `https://arxiv.org/abs/2407.14978`, `arXiv:2407.14978`.

[CGS21] Ted Chinburg, Quentin Guignard, and Christophe Soulé. On the slopes of the lattice of sections of hermitian line bundles. *Journal of Number Theory*, 228:294–341, 2021.

[Doc01a] Christophe Doche. On the spectrum of the Zhang-Zagier height. *Math. Comp.*, 70(233):419–430, 2001.

[Doc01b] Christophe Doche. Zhang-Zagier heights of perturbed polynomials. *J. Théor. Nombres Bordeaux*, 13(1):103–110, 2001.

[Fal83] Gerd Faltings. Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. *Invent. Math.*, 73(3):349–366, 1983.

[Fek23] Michael Fekete. Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten. *Math. Z.*, 17(1):228–249, 1923.

[FS55] Michael Fekete and Gábor Szegő. On algebraic equations with integral coefficients whose roots belong to a given point set. *Math. Z.*, 63:158–172, 1955.

[Hör07] Lars Hörmander. *Notions of convexity*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2007. Reprint of the 1994 edition.

[Küh22] Lars Kühne. Points of small height on semiabelian varieties. *J. Eur. Math. Soc. (JEMS)*, 24(6):2077–2131, 2022. `doi:10.4171/jems/1125`.

[Löb17] Steffen Löbrich. A gap in the spectrum of the Faltings height. *J. Théor. Nombres Bordeaux*, 29(1):289–305, 2017.

[ORSLW19] Ramón Angel Orive Rodríguez, Joaquín Francisco Sánchez Lara, and Franck Wielonsky. Equilibrium problems in weakly admissible external fields created by pointwise charges. *J. Approx. Theory*, 244:71–100, 2019.

[OS24a] Bryce Joseph Orloski and Naser Talebizadeh Sardari. Limiting distributions of conjugate algebraic integers, 2024. arXiv:2302.02872.

[OS24b] Bryce Joseph Orloski and Naser Talebizadeh Sardari. A quantitative converse of fekete's theorem, 2024. arXiv:2304.10021.

[OSS24] Bryce Joseph Orloski, Naser Talebizadeh Sardari, and Alexander Smith. New lower bounds for the schur-siegel-smyth trace problem. *Math. Comput.*, 94(354):2005–2040, 2024.

[PER89] Marian Boykan Pour-El and J. Ian Richards. *Computability in analysis and physics*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1989.

[Pol84] David Pollard. *Convergence of stochastic processes*. Springer Series in Statistics. Springer-Verlag, New York, 1984.

[QY24] Binggang Qu and Hang Yin. Arithmetic Demailly approximation theorem. *Adv. Math.*, 458:Paper No. 109961, 24, 2024.

[Ran95] Thomas Ransford. *Potential theory in the complex plane*, volume 28 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1995.

[Rum99] Robert Rumely. On Bilu's equidistribution theorem. In *Spectral problems in geometry and arithmetic (Iowa City, IA, 1997)*, volume 237 of *Contemp. Math.*, pages 159–166. Amer. Math. Soc., Providence, RI, 1999.

[Ser19] Jean-Pierre Serre. Distribution asymptotique des valeurs propres des endomorphismes de Frobenius [d'après Abel, Chebyshev, Robinson,...]. *Astérisque*, 414:Exp. No. 1146, 379–425, 2019. Séminaire Bourbaki. Vol. 2017/2018. Exposés 1136–1150.

[Sil09] Joseph Hillel Silverman. *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics*. Springer, Dordrecht, second edition, 2009.

[Smi24] Alexander Smith. Algebraic integers with conjugates in a prescribed distribution. *Ann. of Math. (2)*, 200(1):71–122, 2024.

[Smy81] Christopher James Smyth. On the measure of totally real algebraic integers. II. *Math. Comp.*, 37(155):205–208, 1981.

[Smy84] Christopher James Smyth. Totally positive algebraic integers of small trace. *Ann. Inst. Fourier (Grenoble)*, 34(3):1–28, 1984.

[Smy81] Christopher James Smyth. On the measure of totally real algebraic integers. *J. Austral. Math. Soc. Ser. A*, 30(2):137–149, 1980/81.

[Spe69] Ernst Specker. The fundamental theorem of algebra in recursive analysis. In *Constructive Aspects of the Fundamental Theorem of Algebra (Proc. Sympos., Zürich-Rüschlikon, 1967)*, pages 321–329. Wiley-Interscience [A Division of John Wiley & Sons, Ltd.], London-New York-Sydney, 1969.

[SUZ97] L. Szpiro, E. Ullmo, and S. Zhang. équirépartition des petits points. *Invent. Math.*, 127(2):337–347, 1997. doi:10.1007/s002220050123.

[Sza23] Michał Szachniewicz. Existential closedness of $\overline{\mathbb{Q}}$ as a globally valued field via arakelov geometry, 2023. arXiv:2306.06275.

[Yua08] Xinyi Yuan. Big line bundles over arithmetic varieties. *Invent. Math.*, 173(3):603–649, 2008. doi:10.1007/s00222-008-0127-9.

[Yua12] Xinyi Yuan. Algebraic dynamics, canonical heights and Arakelov geometry. In *Fifth International Congress of Chinese Mathematicians. Part 1, 2*, volume 51, pt. 1, 2 of *AMS/IP Stud. Adv. Math.*, pages 893–929. Amer. Math. Soc., Providence, RI, 2012.

[YZ26] Xinyi Yuan and Shou-Wu Zhang. *Adelic Line Bundles on Quasi-Projective Varieties*. Annals of Mathematics Studies. Princeton University Press, 2026.

[Zag93] Don Zagier. Algebraic numbers close to both 0 and 1. *Math. Comp.*, 61(203):485–491, 1993.

[Zha92] Show-Wu Zhang. Positive line bundles on arithmetic surfaces. *Ann. of Math. (2)*, 136(3):569–587, 1992.

José Ignacio Burgos Gil:
Instituto de Ciencias Matemáticas (CSIC-UAM-UCM-UCM3), Calle Nicolás Cabrera 15, Campus UAM, Cantoblanco, 28049 Madrid, Spain
Email address: `burgos@icmat.es`

Ricardo Menares:
Pontificia Universidad Católica de Chile, Facultad de Matemáticas, Vicuña Mackenna 4860, Santiago, Chile
Email address: `rmenares.v@gmail.com`

Binggang Qu:
Instituto de Ciencias Matemáticas (CSIC-UAM-UCM-UCM3), Calle Nicolás Cabrera 15, Campus UAM, Cantoblanco, 28049 Madrid, Spain
Email address: `bingang.qu@icmat.es`

Martín Sombra:
Institució Catalana de Recerca i Estudis Avançats, Passeig Lluís Companys 23, 08010 Barcelona, Spain
Departament de Matemàtiques i Informàtica, Universitat de Barcelona, Gran Via 585, 08007 Barcelona, Spain
Centre de Recerca Matemàtica, Edifici C, Campus Bellaterra, 08193 Bellaterra, Spain
Email address: `martin.sombra@icrea.cat`