Dynamics of Near-identity Maps and Interpolating Vector Fields

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Interpolating vector fields (IVFs)

Let $f : D \mapsto \mathbb{R}^s$ real analytic on $D \subset \mathbb{R}^s$ open domain. Let $m \geq 0$ and assume that there is $D_0 \subset D$ such that $f^k(D_0) \subset D$ for $0 \leq k \leq m$. Denote $x_k = f^k(x_0)$, $x_0 \in D_0$. There is a unique polynomial $P_m(t; x_0)$ of order m in t such that $P_m(k; x_0) = x_k$ for $0 \leq k \leq m$.

The **interpolating vector field (IVF)** X_m at $x_0 \in D_0$ is the velocity vector of the interpolating curve at $t = 0$, that is, $X_m(x_0) = \partial_t P_m(0, x_0)$.

Useful for theoretical derivations as well as for numerical simulations:

1. Discrete averaging: $X_m(x_0) = \sum_{k=0}^{m} p_{mk} f^k(x_0)$ is a weighted average of the iterates with p_{m0} the Harmonic number and for $k > 1$

$$p_{mk} = (-1)^{k+1} \frac{m+1-k}{k(m+1)} \binom{m+1}{k}.$$

2. Numerics: higher accuracy for symmetric interpolation nodes around x_0. (i.e. we consider p_{2m} s.t. $x_k = p_{2m}(t_k; x_0, \epsilon)$, $\forall t_k = \epsilon k$, $|k| \leq m$.)

IVF-embedding a near-Id map into a flow

1. Consider a one-parameter near-Id family of maps $f_\epsilon(x) = x + \epsilon\, G_\epsilon(x)$, $|\epsilon| < \epsilon_0$, and interpolation nodes $t_k = \epsilon k$. Then the IVF X_{2m} is uniformly bounded in any compact subset of D and [a]

$$f_\epsilon(x) = \Phi^\epsilon_{X_{2m}}(x) + O(|\epsilon|^{2m+1}).$$

2. Refined version of Neishtadt's averaging theorem with explicit v.f. and constants that applies to individual maps: [b]
   If f is ϵ-close-to-Id in a complex δ-neighbourhood of D_0 and $\epsilon/\delta \le 1/6e$, then taking $m = \delta/6e\epsilon$ one has

$$\|\Phi^1_{X_m} - f\|_{D_0} \le 3\epsilon \left(\frac{6(m-1)\epsilon}{\delta} \right)^m .$$

[a] V.Gelfreich & AV, *Interpolating vector fields for near identity maps and averaging*, Nonlinearity 31(9), 2018

[b] V.Gelfreich & AV. *On exponentially accurate approximation of a near the identity map by an autonomous flow.* ArXiv Nov 2024.

IVF-exp. embedding near-Id **symplectic** *maps*

Let f an exact symplectic map ϵ-close-to-Id in $D = D_0 + \delta$ a complex δ-neighbourhood of $D_0 \subset \mathbb{R}^{2d}$. Assume it admits a generating function $G(P,q) = Pq + S(P,q)$ such that S can be analytically continued onto D and denote by $\epsilon = \left\|\nabla S\right\|_D$. As before X_m is the IVF.

Theorem. [a] If $m = \left\lfloor \frac{\delta}{6\mathrm{e}\,\epsilon} - d \right\rfloor \geq 1$, then

$$\|\Phi_{X_m} - f\|_{D_0} \leq 3\,\mathrm{e}^{d+2}\epsilon \exp\left(-\delta/(6\mathrm{e}\,\epsilon)\right).$$

Moreover there is a Hamiltonian interpolating vector field $\hat{X}_m$ such that

$$\|\hat{X}_m - X_m\|_{D_1} \leq 3\,\mathrm{e}^{d+1}\epsilon \exp\left(-\delta/(6\mathrm{e}\,\epsilon)\right),$$

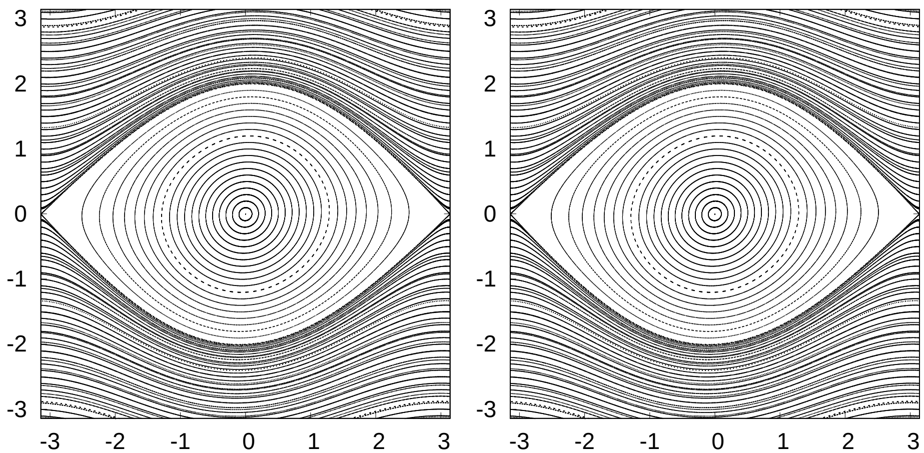
where D_1 is the $\frac{\delta}{2}$-neighbourhood of D_0, and

$$\|\Phi_{\hat{X}_m} - f\|_{D_0} \leq 5\,\mathrm{e}^{d+2}\epsilon \exp\left(-\delta/(6\mathrm{e}\,\epsilon)\right).$$

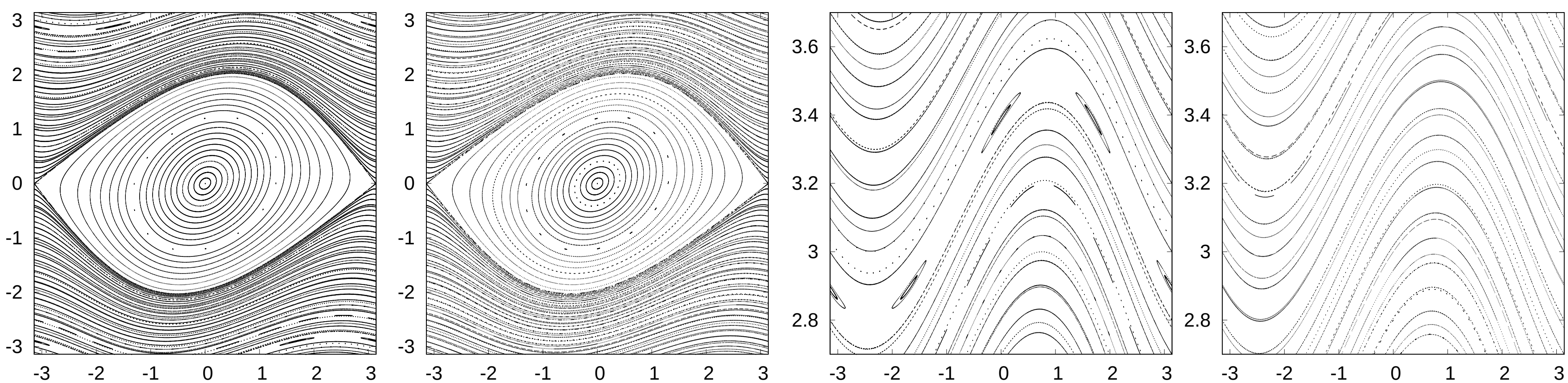
[a] V.Gelfreich & AV. *Nekhoroshev theory and discrete averaging.* ArXiv Nov 2024, submitted to DCDS-A.

Example: Chirikov standard map on $\mathbb{S}^1 \times \mathbb{R}$

$$M_\epsilon : (x, y) \mapsto (\bar{x}, \bar{y}) = (x + \epsilon \bar{y}, y - \epsilon \sin(x)), \quad \epsilon \in \mathbb{R}.$$



$\epsilon = 0.1$, same 200 i.c. Left: 10^3 iterates of M_ϵ. Right: RK78 integration of X_{10} up to $t = 10^3$ plotting every $\Delta t = 0.1$. No visual differences!



Bottom: $\epsilon = 0.5$, left plots for M_ϵ and right plots for X_{10}.

Arnold diffusion: Nekhoroshev estimates

4D symplectic maps – long term dynamics

We study the dynamics of quasi-integrable analytic exact-symplectic maps of $\mathbb{R}^d \times \mathbb{T}^d$

$$F_\varepsilon : \begin{cases} \bar{I} = I + \varepsilon a(I, \varphi), \\ \bar{\varphi} = \varphi + \omega(I) + \varepsilon b(\bar{I}, \varphi) \pmod 1, \end{cases}$$

implicitly defined by the generating function

$$S(\bar{I}, \varphi) = \bar{I}\,\varphi + h_0(\bar{I}) + \varepsilon s(\bar{I}, \varphi), \quad h_0 \text{ convex function, } h_0'(I) = \omega(I),$$

through the relations $I = \partial S / \partial \varphi$, $\bar{\varphi} = \partial S / \partial \bar{I}$.

We want to study the long term (Nekhoroshev) global stability properties of F_ε.

Nekhoroshev estimates

Consider $0 < \varepsilon < \varepsilon_0$ and denote $(I_k, \varphi_k) = F_\varepsilon^k(I_0, \varphi_0)$, $k \in \mathbb{Z}$.

For $d = 1$, the rotational invariant curves divide the 2D phase space and there is no global diffusion if ε is small enough (e.g. Chirikov standard map).

For $d \geq 2$, the complement of KAM d-dimensional discrete tori is connected and trajectories might travel along phase space (Arnold diffusion).

[a] **Nekhoroshev estimate:** $|I_k - I_0| \leq R(\varepsilon)$ when $|k| \leq T(\varepsilon)$, where $R(\varepsilon) \sim \varepsilon^\beta$ and $T(\varepsilon) \sim \exp(c/\varepsilon^\alpha)$ with $\alpha = \beta = 1/(2(d+1))$.

Our main interest is not in the result itself (which is well-known) but in the methodology: we recover this estimate from an explicit construction of the slow variable directly from the iterates of the map (IVFs).

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[a] S.Kuksin and J.Pöschel, *On the inclusion of analytic symplectic maps in analytic Hamiltonian flows and its applications.* Seminar on Dynamical Systems 12:96–116, 1994.

P.Lochak and A.I.Neishtadt, *Estimates of stability time for nearly integrable systems with a quasiconvex Hamiltonian*, Chaos 2, 1992.

Phase space geometry (d=2)

Diffusion along phase space takes place basically along single resonances but double resonances play a key role in an explanation of the Arnold diffusion.

To illustrate this we consider the map T_δ defined by the generating function

$$S(\psi_1, \psi_2, J_1, J_2) = \psi_1 \bar{J}_1 + \psi_2 \bar{J}_2 + \delta \mathcal{H}(\psi_1, \psi_2, \bar{J}_1, \bar{J}_2), \text{ where}$$

$$\mathcal{H}(\psi_1, \psi_2, \bar{J}_1, \bar{J}_2) = \frac{J_1^2}{2} + a_2 J_1 J_2 + a_3 \frac{J_2^2}{2} + \cos(\psi_1) + \epsilon \cos(\psi_2),$$

through the relations $J_i = \partial S / \partial \psi_i$, $\bar{\psi}_i = \partial S / \partial \bar{J}_i$, $i = 1, 2$:

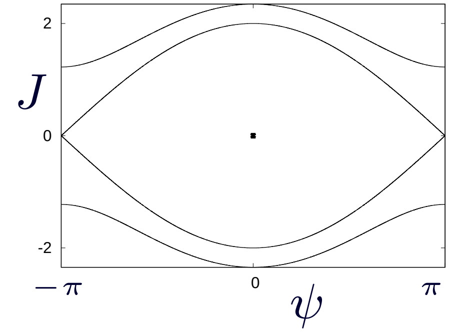
$$T_\delta : \begin{pmatrix} \psi_1 \\ \psi_2 \\ J_1 \\ J_2 \end{pmatrix} \mapsto \begin{pmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \\ \bar{J}_1 \\ \bar{J}_2 \end{pmatrix} = \begin{pmatrix} \psi_1 + \delta(\bar{J}_1 + a_2 \bar{J}_2) \\ \psi_2 + \delta(a_2 \bar{J}_1 + a_3 \bar{J}_2) \\ J_1 - \delta \sin(\psi_1) \\ J_2 - \delta \epsilon \sin(\psi_2) \end{pmatrix}$$

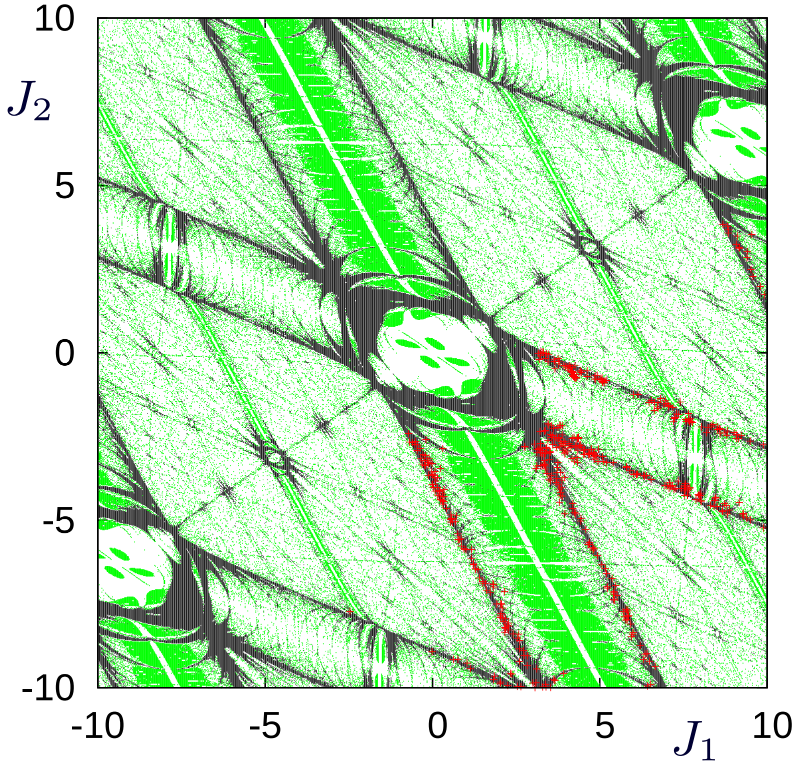
$\mathcal{H}$ resembles to a “two-pendulum” Hamiltonian and T_δ is δ-close to the Id.

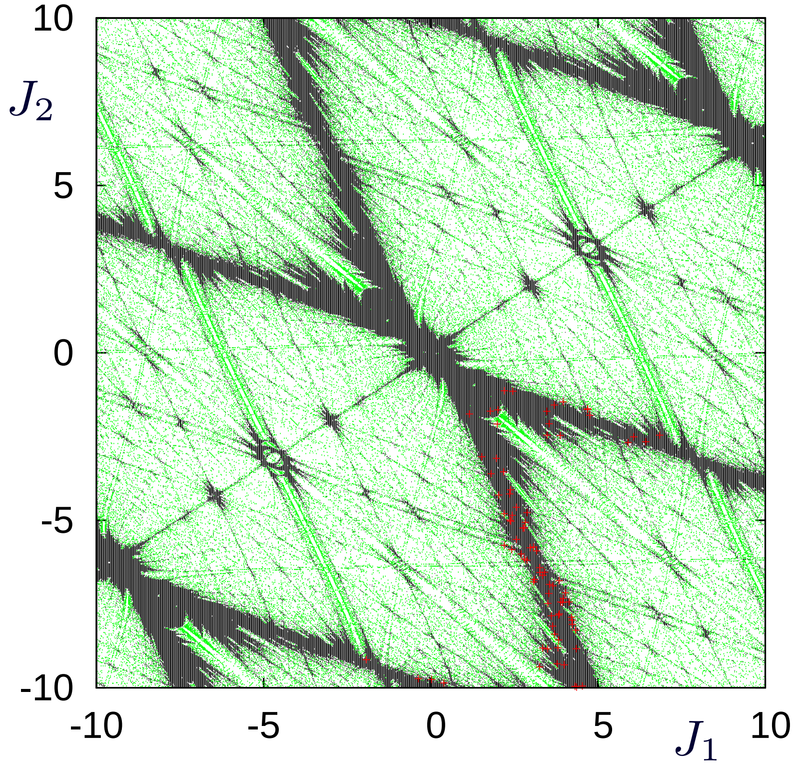
Single resonance: NHIC $\approx$ ric of a pendulum system $\times$ saddle of the other

Double resonance: $\approx (\psi_1, J_1)$-pendulum $\times$ (ψ_2, J_2)-pendulum

Role of double resonances







$\delta = \epsilon = a_2 = 0.5$, $a_3 = 1.25$. Lyap. exp. (megno): **black** $\rightarrow$ chaotic, green $\rightarrow$ weakly chaotic, white $\rightarrow$ regular. Red: Iterates of the point $(0, 0, 4.5, -5.25)$ in a slice of width 5×10^{-3} around $\psi_1 = \psi_2 = 0$ (left plot) and $\psi_1 = \psi_2 = \pi$ (right plot). Total number of iterates=10^{12}.

Lochak approach steps

The role of *double resonances* ($d = 2$) is emphasized in the *Lochak-Neishtadt* approach to proof the Nekhoroshev estimates. The map F_ϵ is the isoenergetic Poincaré return map of a $(d+1)$-dof analytic Hamiltonian

$$\hat{H}(\hat{I}, \hat{\psi}, \epsilon) = \hat{H}_0(\hat{I}) + \epsilon \hat{H}_1(I, \hat{\phi}, \epsilon), \text{ where}$$
$$\hat{I} = (I, I_3), \hat{\psi} = (\psi, \psi_3), \hat{w}(\hat{I}) = (w(I), 1), \text{ and } \hat{H}_0(\hat{I}) = \hat{\omega}(\hat{I}) \cdot \hat{I}.$$

1. Construct a covering of the action space by open neighbourhoods of a finite number (depending on ϵ) of unperturbed tori bearing periodic motions (maximum resonances).
2. Normalize the Hamiltonian around a periodic orbit: by successive changes of variables (averaging procedure) the non-resonant terms of H can be annihilated within an exponentially small error $\rightsquigarrow$ slow observable
3. Use convexity to guarantee exponential stability in the neighbourhood.

Indirect procedure: The evaluation of the local (in each domain of the covering) slow observable (to measure diffusion) requires a transformation to NF.

IVFs – ”Our Lochak-like approach”

Note that, for a map $F_\varepsilon = F_0 + \mathcal{O}(\varepsilon)$, $F_0(I,\varphi) = (I, \varphi + \omega(I))$, if $n\omega(I_*) \in \mathbb{Z}^d$ for some $n \in \mathbb{N}$ and $I_* \in \mathbb{R}^d$ then $I = I_*$ is a torus invariant by F_0 foliated by invariant n-periodic orbits. Note that near I_* the map F_ε^n becomes close-to-the-identity.

We proof of the Nekhoroshev estimates: [a]

1. using the approximation of a close-to-Id map by an autonomous Hamiltonian flow with an exponential small error.
2. constructing an approximating vector field using discrete averaging and interpolating vector fields (IVFs): it is explicit in terms of iterates of the map, can be easily implemented numerically and avoids changes of variables.

[a] V.Gelfreich & AV. *Nekhoroshev theory and discrete averaging.* ArXiv Nov 2024, submitted to DCDS-A.

Exploring (Arnold) diffusion: qualitative and quantitative description

Diffusion - general picture

Consider F_ε near-integrable 4D map. We need a slow observable (adiabatic invariant) h_m easy to compute from the iterates of the map, and accurate enough (with exp. small error) to measure diffusion. IVFs provide a way to obtain a Hamiltonian vector field $\hat{X}_m$, with energy h_m that is preserved for long times. Indeed:

1. **Near a double resonance:** Closer to a tori bearing periodic orbits of short period n, the distance-to-Id of the lift f_ε^n of the near-integrable map F_ε^n becomes smaller. Hence, h_m^n is well-preserved for a much larger number of iterates.

2. **Single resonances:** For double resonances of different enough order, hence with large n, h_m^n is badly preserved since f_ε^n is far-from-Id. This is responsible of the fast drift along single resonances typically observed.

IVFs- “Poincaré” sections to visualize dynamics

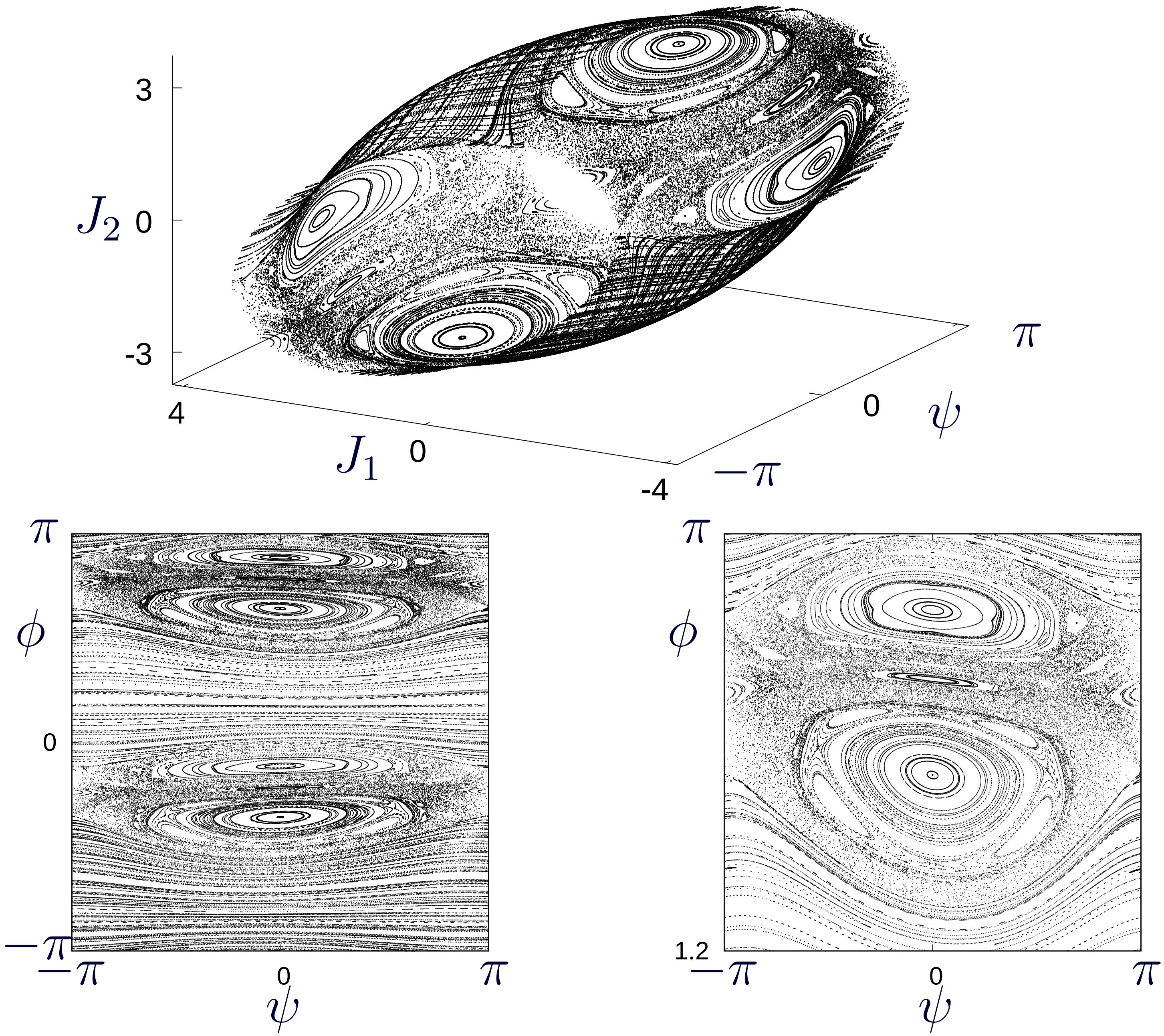
Let $g : \mathbb{R}^m \to \mathbb{R}$ smooth s.t. $\Sigma = \{x \in \mathbb{R}^m : g(x) = 0\}$ is a smooth hyper-surface of codimension one. Take $x_0 \in D_0$ and iterate $x_{k+1} = f_\epsilon(x_k)$. Assume that $g(x_k)g(x_{k+1}) \leq 0$ (crossing). If the limit vector field G_0 is (locally) transversal to Σ then, for ϵ small enough, there is a unique $t_k \in [0, \epsilon]$ such that $g(\Phi^{t_k}_{X_n}(x_k)) = 0$.

$\longrightarrow$ Plot $y_k = \Phi^{t_k}_{X_n}(x_k)$ instead of (any other projection of) x_k.

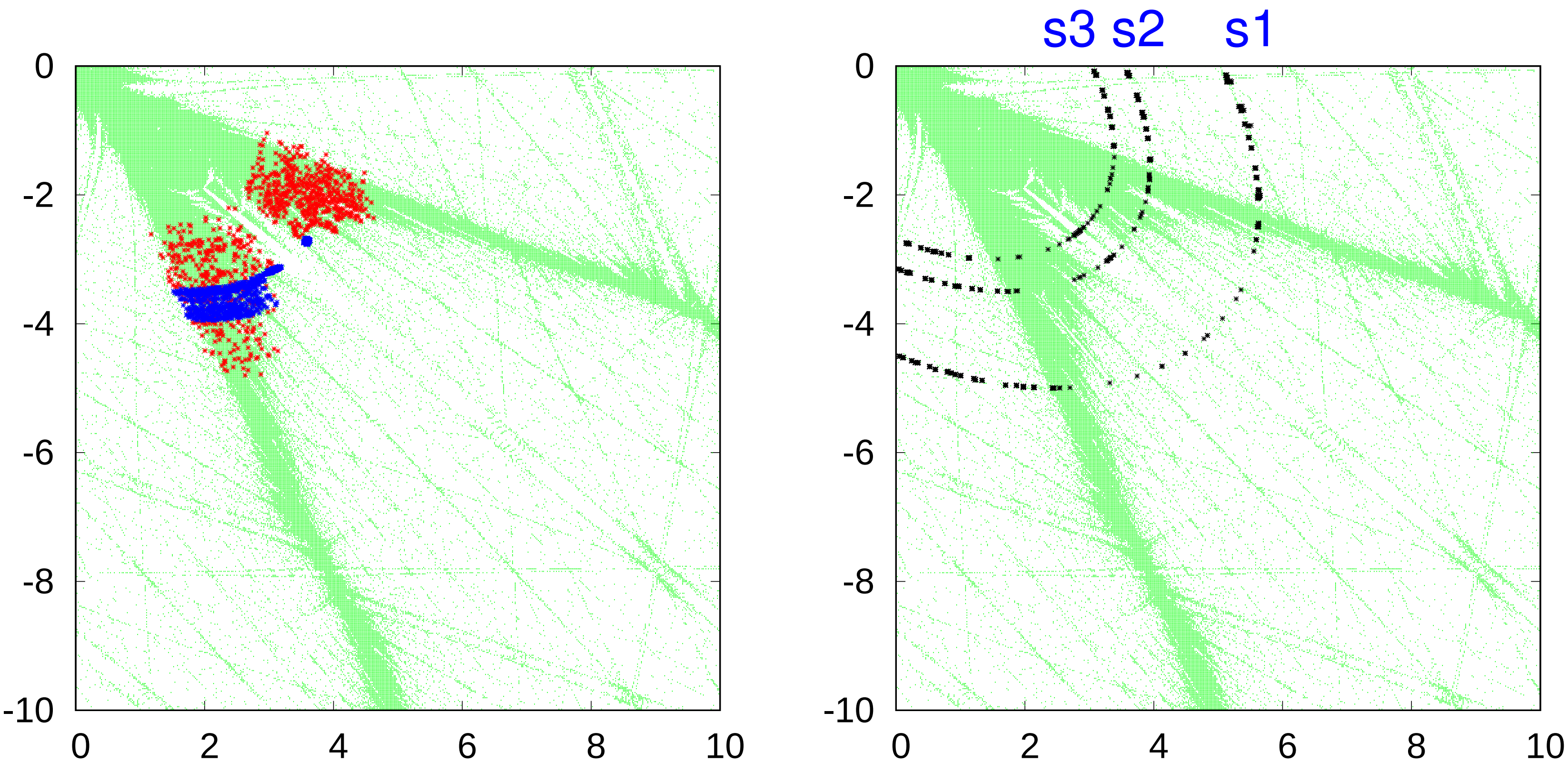
Example (visualizing 4D near-Id discrete dynamics): For a map like T_δ, obtained as a discretization of $H = J_1^2/2 + a_2 J_1 J_2 + a_3 J_2^2/2 + V(\psi)$, $\Sigma = \{\psi_1 = \psi_2\}$ is a transversal section (if $|\delta|$ small enough). On a moderate time scale the iterates of $x_0 \in \mathbb{T}^2 \times \mathbb{R}^2$ remain close to the “energy” surface $M^m_E = \{x : h_m(x) = E\}$, where $E = h_m(x_0)$.

For E large enough, one has $M^n_E \cong \mathbb{T}^3$. Then $\psi = \psi_1 = \psi_2$, $\phi = \arg(J_1 + iJ_2)$ are convenient coordinates on $\Sigma \cap M^n_E \cong \mathbb{T}^2$.

T_δ, $\delta = 0.35$, 400 *i.c. on* $\Sigma \cap \{h_{10} = 4\}$, 500 *it*



Turning at a resonant crossing

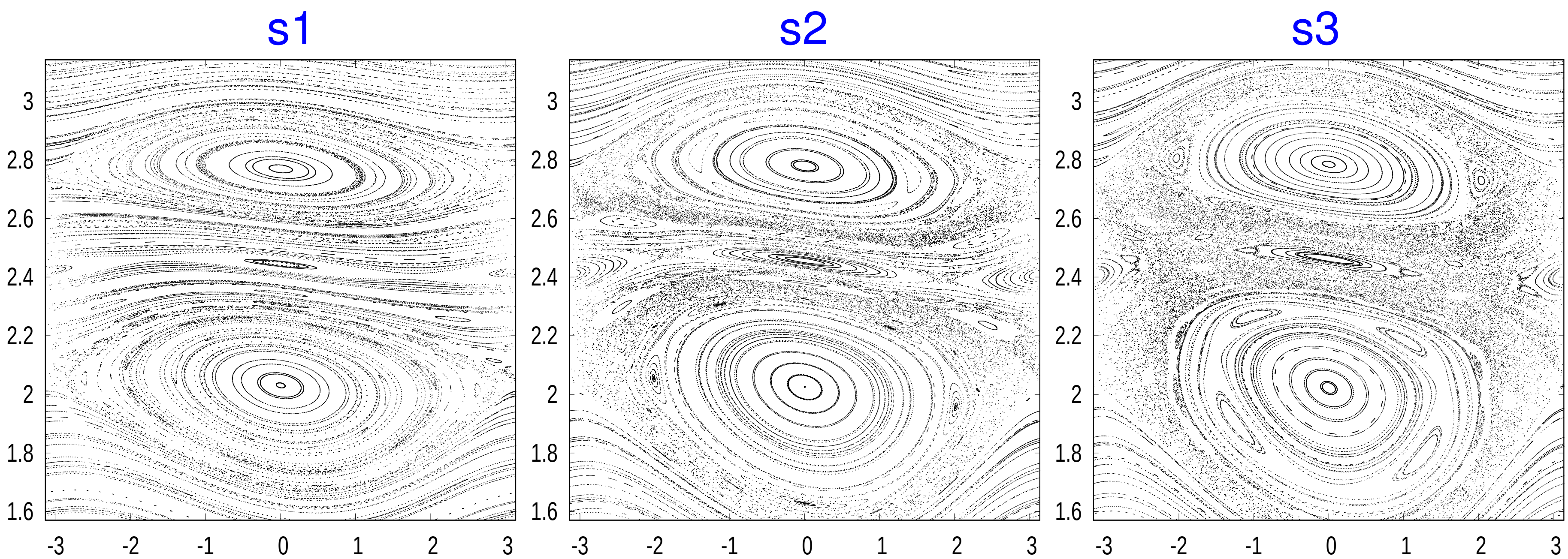


T_δ, $\delta = 0.4$. Left: IC $(3, 3, 2.136447, -3.904401)$ near $J_1 + a_2 J_2 \approx 0$.

We perform around 10^8 (resp. 10^{10}) iterates and show in blue (resp. red) iterates on $\Sigma = \{\psi_1 = \psi_2\}$ with $|\psi_1 - \pi| < 0.35$. Similar for most orbits.

Right: Energy levels (s1 and s2 above the level of the crossing observed).

“Poincaré” sections & last “RIC”



J_1	$\tilde{h}^1_{11}$	tori?
2.5	12.327	Y s1
2.0	7.889	Y
1.75	6.041	Y s2
1.625	5.209	N
1.5	4.439	N s3

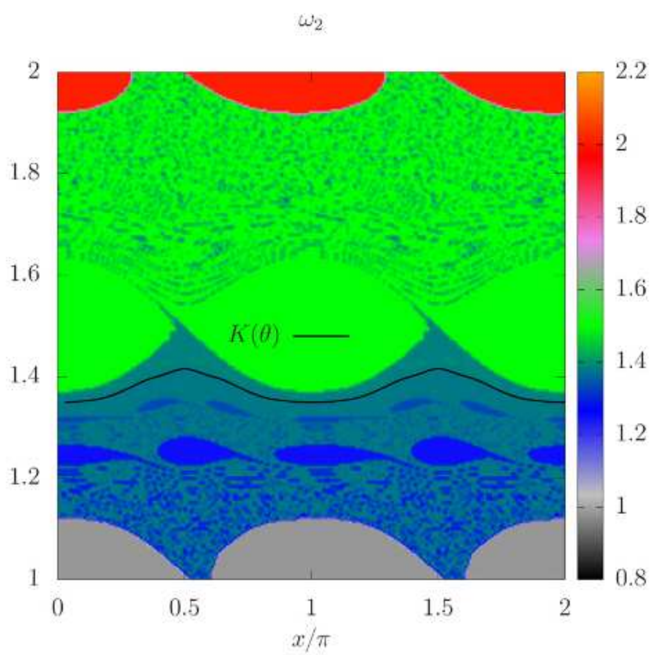
Approaching the HH-point (with $h = 0$) of the double resonance the projection “Poincaré” maps become more chaotic. The last “rotational invariant curve” is at $h \approx h(\pi, \pi, J_1, -a_2 J_1) \approx 5.209$. It corresponds to $J_1 \approx 1.625$. Numerical simulations detect passages for $1.37 \lesssim J_1 \lesssim 1.5$.

Dynamics and chaos in dissipative maps

(numerical simulations leading to open theoretical problems)

Near-conserv. dynamics – coexistence attractors

Motivating example: The *spin-orbit* problem is a simplified model for the rotational dynamics of a satellite orbiting around a central planet and rotating around an internal spin-axis. This is a non-autonomous periodic in time ode system that, considering the time orbital period map, reduces to the so-called spin-orbit map.



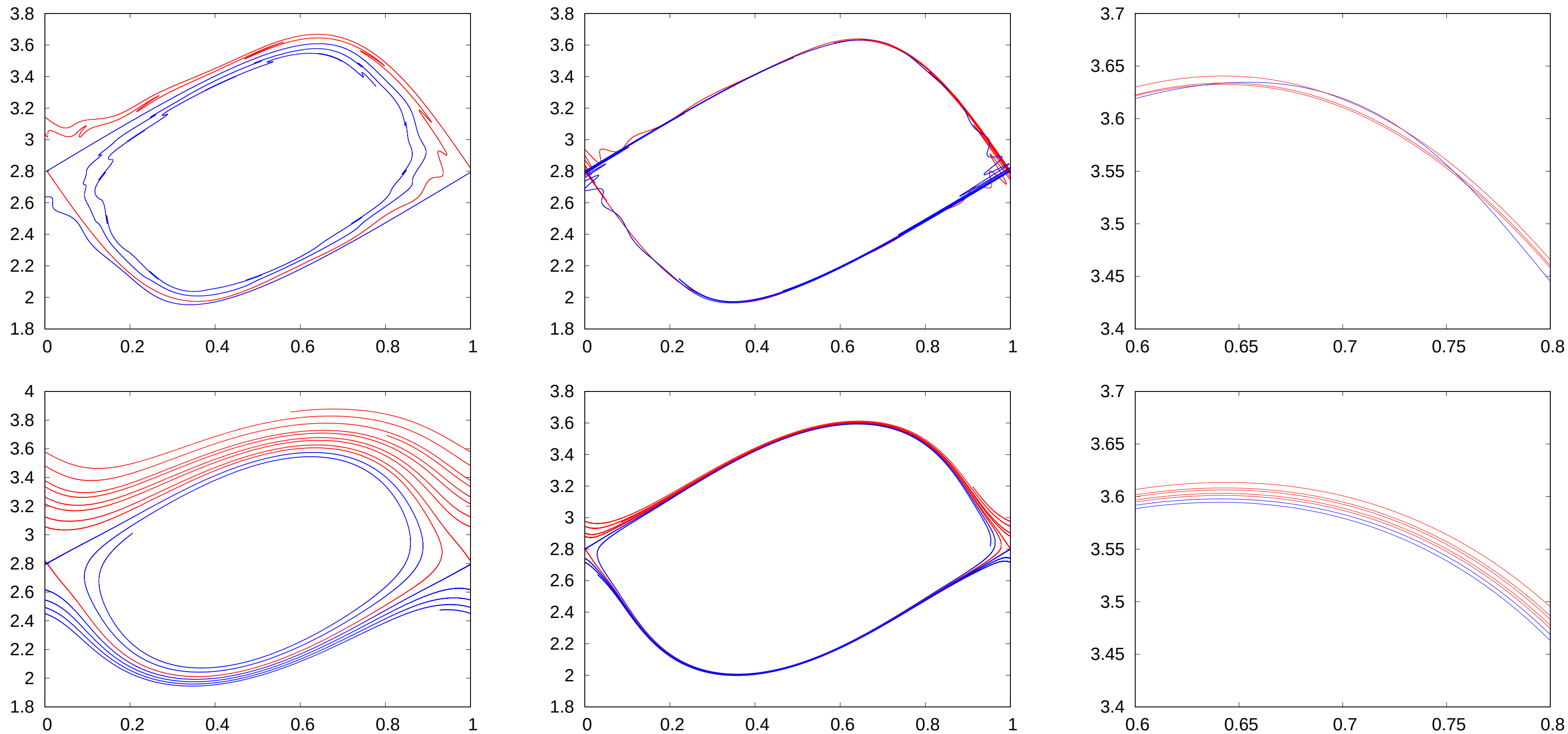
For some parameters, there is a coexistence an attracting invariant curve and many periodic attracting points. To study the dynamics in nearby resonances one uses resonant BNF around the inv. curve. A major question concerns the probability of capture by each attractor.

R. Calleja, A. Celletti, J. Gimeno, R. de la Llave. KAM quasi-periodic tori for the dissipative spin–orbit problem. CNNS 106, 2022.

C.Simó and AV. Planar Radial Weakly-Dissipative Diffeomorphisms. Chaos 20(4), 2010.

Near-conserv. dynamics – IVFs prob(capture)

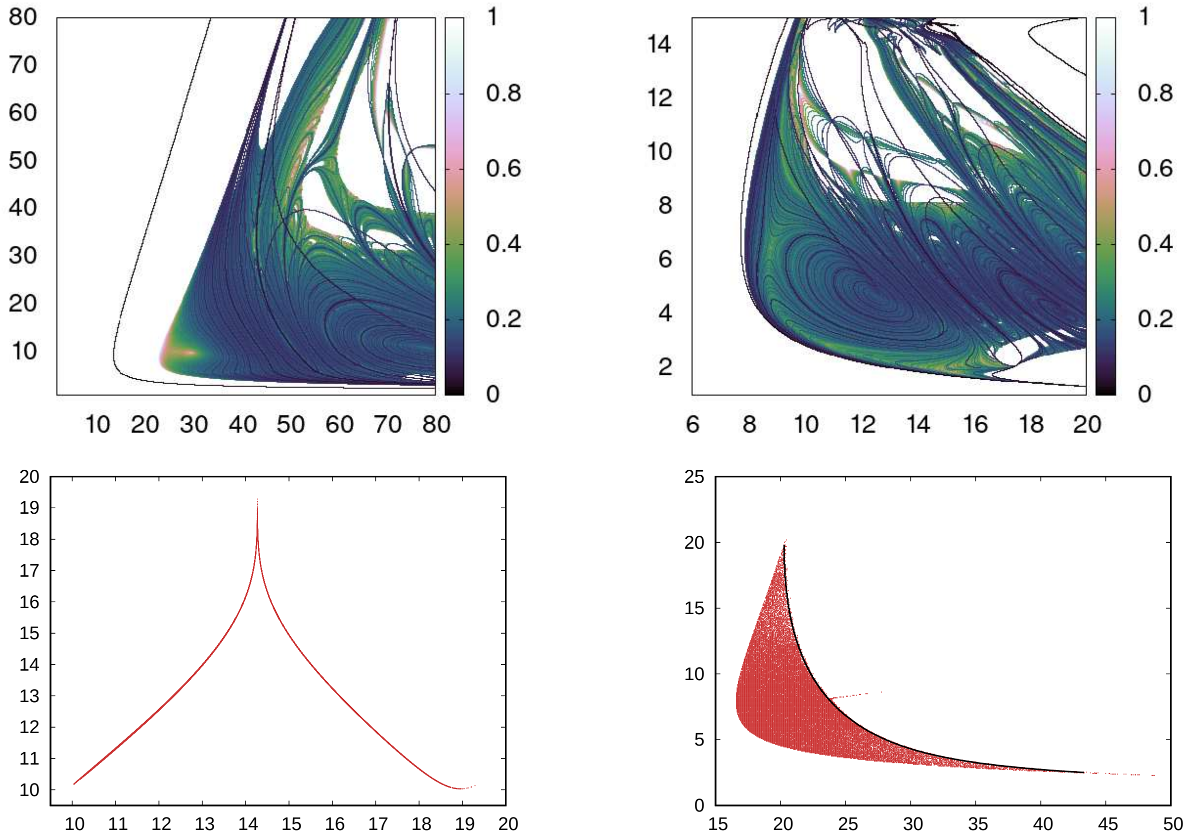
Diss. Std. map $M_{\epsilon,\delta} : (x, y) \mapsto (\bar{x}, \bar{y}) = (x + \delta\bar{y}, (1-\epsilon)y - \delta\sin(2\pi x) + c), \quad \epsilon \in \mathbb{R}$. We consider $\delta \approx 3.57 \times 10^{-1}$, $\omega \approx 6.18 \times 10^{-1}$ and $\epsilon = 10^{-2}$ (left), 10^{-3} (center/right).



The origin is an attracting focus. Preliminary numerical explorations indicates that the probability of capture by the focus can be defined as the ratio between the entrance/exit strips (one can avoid homoclinics). Note that this implies $\lim_{\epsilon\to 0} P_{\text{capture}} > 0$. Theoretical justification is missed!

IVFs – Discrete Lorenz attractors

Lorenz map: $\bar{x} = x + \delta(\sigma(y - x)),\ \bar{y} = y + \delta(\bar{x}(\rho - z) - y),\ \bar{z} = z + \delta(\bar{x}y - 8z/3).$ [a]



For δ small, we use IVF to compute kneading diagram, (ρ, σ)-parameter space (top right $\delta = 0.001$, top left $\delta = 0.06$), reduce dynamics to 1D-“Poincaré maps” (bottom left, $\delta = 0.001$), and compute the region with pseudohyperbolic discrete Lorenz-like attractors (bottom right, $\delta = 0.01$).

Convert numerical evindence into a direct proof for the existence of discrete Lorenz attractors?

[a] A.Kazakov, A.Murillo, AV, K.Zaichikov, “Numerical study of discrete Lorenz-like attractors.” Reg. Chaotic Dyn. 29(1), 2024.

Conclusions

- IVFs – a numerical tool to study near-Id dynamics:
  1. IVFs explicitly relate discrete near-Id maps with the dynamics of the average vector fields obtained from suspension+averaging construction (discrete averaging).
  2. IVFs allow to compute the slowest variable at any point of the phase (useful for visualizations/quantitative simulations of diffusion) from simulations in original system variables.
  3. We can use IVFs to compute "Poincaré maps" directly from the discrete dynamics. This is a useful tool to investigate chaos in conservative/dissipative near-Id discrete systems.

  In particular, we have used IVFs to investigate the *key role of double resonances* in the (Arnold) diffusion process of 4D symplectic maps.

- IVFs – analytical tool to study near-Id dynamics:

  The relation of IVFs with discrete averaging allow to obtain optimal and explicit theoretical results: bounds of the error of the flow approximation, bounds of the error produced by the projections onto "Poincaré surfaces", an exponential embedding of a symplectic near-Id map into a Hamiltonian flow and Nekhoroshev estimates for near-integrable maps.

Thanks for your attention!!