Periodic forcing of a 2-dof Hamiltonian undergoing a Hamiltonian-Hopf bifurcation

Dynamics, Bifurcations and Strange Attractors

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Goal of this talk

To show some aspects related to the Hamiltonian-Hopf bifurcation in different contexts. The major interest will be on the behaviour of the splitting of the invariant manifolds for

1. 2-dof Hamiltonian system ← well known
2. 4D symplectic maps
3. 2-dof Hamiltonian + periodic forcing

Let me start with:

1. An overview of the Hamiltonian-Hopf bifurcation for 2-dof Hamiltonian systems
2-dof Hamiltonian Hopf (HH): Sokolskii NF

2-dof HH codim 1: Consider a 1-param. family of 2-dof Hamiltonians $H_\nu$ undergoing a HH bifurcation (at the origin).

**Concretely:** for $\nu > 0$ elliptic-elliptic, $\nu < 0$ complex-saddle.

Analysis of the HH bifurcation $\rightarrow$ Reduction to Sokolskii NF:

1. Taylor expansion at 0: $H_\nu = \sum_{k \geq 2} \sum_{j \geq 0} \nu^j H_{k,j}$, where $H_{k,j} \in P_k$ homogeneous polynomial of order $k$.

2. Williamson NF (double purely imaginary eigenvalues $\pm i\omega$):
   \[ H_{2,0} = -\omega(x_2 y_1 - x_1 y_2) + \frac{1}{2}(x_1^2 + x_2^2). \]

3. Use Lie series to order-by-order simplify $H_{2,j}, j > 1$ and $H_{k,j}, k > 2, j > 0$.
   But: **non-semisimple** linear part!

   Then, at each order $(k, j)$, one looks for $G \in P_k$ s.t.
   \[ H_{k,j} + \text{ad}_{H_2}(G) \in \text{Ker} \text{ad}_{H_2}^\top. \]
4. Introducing the Sokolskii coordinates \((dx_1 \wedge dy_1 + dx_2 \wedge dy_2 = dR \wedge dr + d\Theta \wedge d\theta)\)

\[ y_1 = r \cos(\theta), \quad y_2 = r \sin(\theta), \quad R = (x_1y_1 + x_2y_2)/r, \quad \Theta = x_2y_1 - x_1y_2, \]

one has \(H_2^\top = -\omega\Theta + \frac{1}{2}r^2\) and

\[
\text{NF}(H_\nu) = -\omega \Gamma_1 + \Gamma_2 + \sum_{k,l,j \geq 0, k+l \geq 2} a_{k,l,j} \Gamma_1^k \Gamma_3^l \nu^j,
\]

where

\[
\Gamma_1 = x_2y_1 - x_1y_2, \quad \Gamma_2 = (x_1^2 + x_2^2)/2 \quad \text{and} \quad \Gamma_3 = (y_1^2 + y_2^2)/2.
\]

5. Introducing \(\nu = -\delta_\nu^2\), and rescaling \(x_i = \delta_\nu^2 \tilde{x}_i, \quad \omega y_i = \delta_\nu \tilde{y}_i, \quad i = 1, 2, \quad \omega t = \tilde{t}\), one has

\[
\text{NF}(\tilde{H}_{\delta_\nu}) = -\tilde{\Gamma}_1 + \delta_\nu \left( \tilde{\Gamma}_2 + a\tilde{\Gamma}_3 + \eta\tilde{\Gamma}_3^2 \right) + O(\delta_\nu^2).
\]

The \(\tilde{\Gamma}_i\) written in terms of the Sokolskii coordinates are given by

\[
\tilde{\Gamma}_1 = \tilde{\Theta}, \quad \tilde{\Gamma}_2 = \frac{1}{2} \left( \tilde{R}^2 + \frac{\tilde{\Theta}^2}{\tilde{r}^2} \right), \quad \text{and} \quad \tilde{\Gamma}_3 = \frac{\tilde{r}^2}{2}.
\]
For $\nu < 0$ the origin has stable/unstable inv. manifolds $W_s/u(0)$. Note that
- $W_s/u(0)$ are contained in the zero energy level of $\text{NF}(\tilde{H}_{\delta_{\nu}})$.
- $\{\tilde{\Gamma}_1, \tilde{\Gamma}_2\} = \{\tilde{\Gamma}_1, \tilde{\Gamma}_3\} = 0 \Rightarrow \tilde{\Gamma}_1$ is a formal first integral of $\text{NF}(\tilde{H}_{\delta_{\nu}})$.
  Hence $\tilde{\Gamma}_1 = 0$ on $W_s/u(0)$.

Then, ignoring $O(\delta_{\nu}^2)$ terms, $W_s/u(0)$ are given by $R^2 + ar^2 + \eta r^4 / 2 = 0$, which is the zero energy level of a Duffing Hamiltonian system.

$\Rightarrow W^{u/s}(0)|_{(R,r)}$-plane form a figure-eight
(for $a < 0$, $\eta > 0$; unbounded otherwise!
but only $r > 0$ has sense!).

The 2D $W^{s/u}(0)$ are obtained by rotating the right hand side of the figure around the $R$ axis (on $W^{s/u}(0)$ one has $\Theta = 0$, $\dot{\Theta} = 1$).
For the truncated NF (i.e. ignoring $O(\delta^p)$-terms, $p > 1$) the 2D stable/unstable inv. manifolds coincide. **But:** For the complete 2-dof Hamiltonian they split!

1. Consider $\Sigma = \{\theta = 0\}$. The Poincaré map (in Cartesian coord. to avoid singularities) defines a *near-the-Id family of analytic APMs*.

2. The limit vector field is $\dot{R} = \delta (a r + \eta r^3)$, $\dot{r} = -\delta R$, ← Duffing!
   The homoclinic solution $\gamma(t)$ with nearest singularity to the real axis $\tau = i\pi/2\sqrt{-a\delta}$ and dominant eigenvalue $\mu = 2\pi \sqrt{-a\delta}$ (then rescale time by $\sqrt{-a\delta}$).

3. From Fontich-Simó theorem (upper bounds are generic!) it follows

$$\alpha \sim A\delta^B \exp \left( \frac{-\pi}{\sqrt{-a\delta}} \right) \sim A|\text{Re} \lambda|^B \exp \left( \frac{-\pi |\text{Im} \lambda|}{|\text{Re} \lambda|} \right)$$

The asymptotic expansion of this splitting has been obtained in J.P.Gaivao, V.Gelfreich, *Splitting of separatrices for the Hamiltonian-Hopf bifurcation with the Swift-Hohenberg equation as an example*, Nonlinearity 24(3), 2011.
2-dof HH: An example

\[ H(\psi_1, \psi_2, J_1, J_2) = \frac{J_1^2}{2} + a_2 J_1 J_2 + a_3 \frac{J_2^2}{2} - \cos(\psi_1) - \epsilon \cos(\psi_2) \]

**Reversibility:** \((\psi_1, \psi_2, J_1, J_2) \in W^u(0)\) then \((-\psi_1, -\psi_2, J_1, J_2) \in W^s(0)\).

This suggests to consider \(\Sigma = \{\psi_1 = 0, \psi_2 = 0\}\) and to look for homoclinic points in \(\Sigma\).

\(a_2 = 0.5, \ a_3 = -0.75, \ \epsilon = -0.5 \ (\epsilon^c = -4/9)\)
2-dof HH: Checking the behaviour of $\alpha$

$H(\psi_1, \psi_2, J_1, J_2) = \frac{J_1^2}{2} + a_2 J_1 J_2 + a_3 \frac{J_2^2}{2} - \cos(\psi_1) - \epsilon \cos(\psi_2)$

Recall: $\alpha \sim \tilde{A}(|\text{Re } \lambda|)^B \exp \left( \frac{C}{|\text{Re } \lambda|} \right)$, where $C = -\pi |\text{Im } \lambda|$.

For $a_2 = 0.5, a_3 = -0.75$ one gets $C = \sqrt{2\pi}/3 + O(\nu)$.

Fitting function (right plot): $f(x) = A x + B x \log(x) + C$.

$\Rightarrow$ It perfectly fits the behaviour!
Part 2

Up to this point: **2-dof Hamiltonian-Hopf** bifurcation.

1. Everything was “more or less” well-known: Sokolskii NF, geometry of the invariant manifolds, the splitting $\alpha$, ...

2. $\alpha$ behaves as expected for a near-the-identity family of 2D APM.

Let me now continue with:

2. A brief excursion to 4D symplectic maps undergoing a HH bifurcation
Consider the map $T : (\psi_1, \psi_2, J_1, J_2) \mapsto (\bar{\psi}_1, \bar{\psi}_2, \bar{J}_1, \bar{J}_2)$ given by

\[
\begin{align*}
\bar{\psi}_1 &= \psi_1 + \delta(\bar{J}_1 + a_2 \bar{J}_2), & \bar{\psi}_2 &= \psi_2 + \delta(a_1 \bar{J}_1 + a_3 \bar{J}_2), \\
\bar{J}_1 &= J_1 + \delta \sin(\psi_1), & \bar{J}_2 &= J_2 + \delta \epsilon \sin(\psi_2).
\end{align*}
\]

- $T$ is related to the time-$\delta$ map of the flow associated to the Hamiltonian

\[
H(\psi_1, \psi_2, J_1, J_2) = \frac{J_1^2}{2} + \cos \psi_1 + a_2 J_1 J_2 + a_3 \frac{J_2^2}{2} + \epsilon \cos(\psi_2),
\]

- **4 fixed points**: For $\epsilon d > 0$, $d = a_3 - a_2^2$, $|\epsilon| \ll 1$ and $\delta \lesssim 2$

\[
\begin{align*}
p_1 &= (0, 0, 0, 0) \text{ HH}, & p_2 &= (\pi, 0, 0, 0) \text{ EH}, & p_3 &= (0, \pi, 0, 0) \text{ HE}, & p_4 &= (\pi, \pi, 0, 0) \text{ EE}.
\end{align*}
\]

$T$ models the dynamics at a double resonance, it was derived from BNF around an EE point of a symplectic map in V. Gelfreich, C. Simó & AV, *Dynamics of 4D symplectic maps near a double resonance*, Phys D 243(1), 2013.
Motivation: Transition to complex unstable

→ If $d > 0$ (definite case) the EE point remains EE for all $\epsilon$ and $\delta$.
→ If $d < 0$ (non-definite case) the point $p_4$ suffers a Krein collision at

$$\epsilon = \left(- (2a_3 - 4d) \pm \sqrt{(2a_3 - 4d)^2 - 4a_3^2}\right) / (2a_3^2),$$

and becomes a complex-unstable point (Hamiltonian-Hopf bifurcation).

Eigenvalues of $DT(p_4)$ for $\delta = 0.5$, $a_2 = 0.5$, $a_3 = -0.75$ (hence $d = -1$)

and $\epsilon$ from $-0.01$ (squares) to $-20$. The (first) Krein collision takes place at $\epsilon^c = -4/9$ at a collision angle $\hat{\theta}_K = \arctan(\sqrt{23}/11)$.

→ The CS point has 2D stable/unstable invariant manifolds.
One can compute $W^{u/s}(0)$ and a homoclinic point $p_h \in \Sigma = \{\psi_1 = \psi_2 = 0\}$ (similarly to the 2-dof case).
We compute the volume of a $4D$ parallelootope defined by two pairs of vectors tangent to $W^u$ and $W^s$ at $p_h \in \Sigma$:

$G(s_1, s_2)$ - local parameterisation

$(s^h_1, s^h_2)$ - local parameters s.t. $T^N(s^h_1, s^h_2) = p_h$, $N > 0$.

1. Consider the vectors:

$\tilde{v}_1 = (\frac{\partial G}{\partial s_1})(s^h_1, s^h_2)$, $\tilde{v}_2 = (\frac{\partial G}{\partial s_2})(s^h_1, s^h_2) \leftarrow$ tangent to $W^u(0)$

2. Transport these vectors under $T^N$ to $p_h$ and consider, by the reversibility,

$\tilde{v}_3 = R(\tilde{v}^{ph}_1)$, $\tilde{v}_4 = R(\tilde{v}^{ph}_2) \leftarrow$ tangent to $W^s(0)$

3. Finally, normalize them $v_j = \frac{\tilde{v}^{ph}_j}{||\tilde{v}^{ph}_j||}$, $j = 1, \ldots, 4$ and define

$$V = \det(v_1, v_2, v_3, v_4)$$

Question: How does $V$ behave as $\epsilon \rightarrow \epsilon^c$?
**$T$: Behaviour of the splitting volume $V$**

\[ T: \quad \bar{\psi}_1 = \psi_1 + \delta(\bar{J}_1 + a_2\bar{J}_2), \quad \bar{\psi}_2 = \psi_2 + \delta(a_2\bar{J}_1 + a_3\bar{J}_2), \]
\[ \bar{J}_1 = J_1 - \delta \sin(\psi_1), \quad \bar{J}_2 = J_2 - \delta \epsilon \sin(\psi_2). \]

We compute the volume of a 4D parallelootope defined by two pairs of vectors tangent to $W^u$ and $W^s$ at $p_h \in \Sigma$.

\[ a_2 = 0.5, \quad a_3 = -0.75 \sim \epsilon^c = -4/9, \quad \delta = 0.5 \sim \theta_k = \arctan(\sqrt{23}/11)/2\pi \quad (\in \mathbb{R} \setminus \mathbb{Q}). \]

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Left: $\log V$ vs. $\epsilon$. Right: $h|\log(V)|$ vs. $h$ ($h = \log(\lambda)$).
Consider (generic) symplectic map $F_{\delta_t, \epsilon_t}$ in $\mathbb{R}^4$ that undergoes a HH bif.

Rec: $\delta_t$: Collision angle $\hat{\theta}_K = 2\pi (q/m + \delta_t)$. $\epsilon_t$: Relative distance to the bifurcation.

Different (naive) important aspects:

1. “Two” exp. small effects: one within the Hamiltonian itself (already studied!), the other measures the “Hamiltonian-map distance”.


3. The Hamiltonian part is known ⇒ only necessary to measure the second effect. But: We have a “privilegiated direction” (the time!) ⇒ we will use an energy function $\psi$ to measure the splitting in that direction (instead of using the splitting potential or the Melnikov vector which measures both effects together).
Towards a sharp upper bound of the splitting

Let $F_{\epsilon_t}$ be a family of symplectic maps s.t. at $\epsilon_t = 0$ undergoes a HH bifurcation. The inv. manifolds $W^{u/s}(0)$ are given by $u(\alpha, t)$ and $v(\alpha, t)$ resp., where $(\alpha, t) \in [t_0, t_0 + h) \times S^1$.

Under reasonable conditions: Define $\psi(\alpha, t) = E(u(\alpha, t))$, $E$ analytic energy function. Then:

(i) **Rational Krein collision.** Let $\theta_K = p/q$, with $(p, q) = 1$. Then, there exists $\epsilon_t^0 > 0$ s.t. for $\epsilon_t < \epsilon_t^0$

$$|\psi(\alpha, t)| \leq K \exp(-C/h), \quad C, K > 0.$$

(ii) **Irrational Krein collision.** Let $\theta_K \in \mathbb{R} \setminus \mathbb{Q}$. Then, $\psi$ is bounded by a function that is exponentially small in a parameter $\gamma$, s.t. $\gamma \searrow 0$ when $h \searrow 0$. Moreover, the dominant harmonic $k(h)$ of $\psi$ changes infinitely many times as $h \to 0$. 
Intrinsic geometry plays a role

The theory is not fully satisfactory because (at the moment!) we can’t explain:

1. all the different changes in slope observed.
2. when the changes take place.

Maximum $\alpha_M$ and minimum $\alpha_m$ angle of splitting.

$h \log(|Q|)$ vs $h$, being $Q = V$ (red), $\alpha_M$ (blue), $\alpha_m$ (magenta).

Let me consider a “similar” but somehow “easier” setting:

3. Periodically perturbed Hamiltonian-Hopf
The system

We consider

\[ H(x, y, t) = H_0(x, y) + \epsilon H_1(x, y, t), \quad x, y \in \mathbb{R}^2, \]

where

1. \( H_0 \) is the truncated 2-dof Sokolskii NF up to order 4

\[ H_0(x, y) = \Gamma_1 + \nu (\Gamma_2 - \Gamma_3 + \Gamma_3^2), \]

where \( \Gamma_1 = x_1y_2 - x_2y_1, \Gamma_2 = (x_1^2 + x_2^2)/2 \) and \( \Gamma_3 = (y_1^2 + y_2^2)/2. \)

2. \( H_1 \) is periodic in \( t \). We choose \( H_1 = y_1^5 \frac{1}{5-\sin(\gamma t)}. \)

Then: \( F_1 = \Gamma_1 \) and \( F_2 = \Gamma_2 - \Gamma_3 + \Gamma_3^2 \) are first integrals of the unperturbed system \( H_0 \). We can use them to measure the splitting when perturbing.
The unperturbed system

The 2-dimensional \( W^{u/s}(0) \) of the **unperturbed system** (\( \epsilon = 0 \)):

1. Are contained in the level \( F_{1}^{-1}(0) \) and \( F_{2}^{-1}(0) \).

2. Intersect the Poincaré section \( \Sigma = \max\{y_{1}^{2} + y_{2}^{2}\} \) in the curve \( x_{1} = 0, y_{1}^{2} + y_{2}^{2} = 2 \).

3. Are foliated by the 1-parameter family of **homoclinic orbits** given by
   \[
   (R_{1}(t) \cos(\psi), R_{1}(t) \sin(\psi), -R_{2}(t) \cos(\psi), -R_{2}(t) \sin(\psi)),
   \]
   where
   \[
   \psi = t + \psi_{0}, \psi_{0} \in [0, 2\pi) \text{ initial phase}
   \]
   \[
   R_{1}(t) = \sqrt{2} \operatorname{sech}(\nu t) \tanh(\nu t), \quad R_{2}(t) = \sqrt{2} \operatorname{sech}(\nu t).
   \]
   \( \implies \text{Singularities at } t = (2m + 1)i\pi/2\nu, m \in \mathbb{Z}. \)
Adding an autonomous perturbation

Consider \( H_1(x, y) = y_1^5 \). Question: Splitting behaviour w.r.t \( \nu \)?

Taking suitable i.c. on \( W^u \) and on \( W^s \) we propagate them until \( \Sigma \).

Let \( \theta = \arctan(y_2/y_1) \). We fit numerically \( F_1^{W^{u/s}}(\theta) = \sum_{k=1}^{6} a_{k}^{u/s}(\nu)e^{ik\theta} \).

Then we compute the difference \( \Delta F_1 = F_1^{W^u} - F_1^{W^s} \).

![Graph](image)

Left: Log of the amplitudes \( A_i \) of the 6 main harmonics of \( \Delta F_1 \) (vs. \( \nu \)). Right: Fit of \( \nu \log(A_1) \) by 
\[
 f(\nu) = a\nu + b\nu \log(\nu) + c, \text{ gives } c \approx \pi/2 \text{ and } b \approx -5.
\]

Melnikov prediction: \( \Delta F_1 = O(\nu^{-5} \exp(-\pi/2\nu)) \).
Adding a non-autonomous perturbation

Consider $H_1(x, y, t) = y_1^5/(5 - \cos(\gamma t))$, $\gamma = (\sqrt{5} - 1)/2$, $\epsilon = 10^{-3}$.

**Same question:** Splitting behaviour w.r.t $\nu$?

Taking suitable i.c. on $W^u$ and on $W^s$, depending on the initial values of $\psi_1$ and the phase of $\gamma t$, we propagate them up to $\Sigma$.

Similar to what was done before we compute $\Delta F_i, i = 1, 2$, (i.e., the splitting function) which depend on two angles, and compute the nodal curves (i.e. the zero level curves) of $\Delta F_i, i = 1, 2$.

**Remark:** Intersections of the two nodal curves $\leftrightarrow$ homoclinic trajectories.
Behaviour:
\[ \Delta F_i \approx \sum A_j(\nu) \exp(-\beta_j \pi/2\nu) \]
where \( \beta_j \) contain suitable combinations of the angles.

Remark: For the displayed values of \( \nu \Delta F_2 \) is “almost” equal. The nodal lines “almost coincide”. 

The \( F_1 \)-difference of the invariant manifolds
resonance \((k, l)\)
dominant term
\[ k\theta - \hat{t} \approx 0 \]
\[ \Delta F_1 \Delta F_2 \]
left: \((0,1), (0,1)\)
right: \((1,1), (0,1)\)
Nodal curves II

resonance dominant term
\[ k\theta - l\hat{t} \approx 0 \]

\[ \Delta F_1 \Delta F_2 \]
left: (1,1), (1,1)
right: (2,1), (1,1)

\[ \Delta F_1 \Delta F_2 \]
left: (2,1), (2,1)
right: (5,3), (2,1)
Conclusions:

- Several bifurcations are observed.
- The changes in the nodal lines of $\Delta F_1$ and $\Delta F_2$ occur for different $\nu$.
- The dominant terms of $\Delta F_1$ and $\Delta F_2$ coincide for values in between the changes.
Future work

1. Dynamics close to separatrices? Derive a return **separatrix map**.
   Ingredients:
   (a) The **splitting function**.
   (b) **Flight times** (in $\hat{t}$ and $\hat{\theta}$) from $\Sigma$ to $\Sigma$.

2. Use the separatrix map to obtain **quantitative** information about distance to maximal tori, to secondary resonances,... from the separatrices.

3. Analyse the **diffusion** properties by performing accurate computations in different regimes.

Thanks for your attention!!