The transition to complex-saddle in a Froeschlé-type map

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Consider the map $T : (\psi_1, \psi_2, J_1, J_2) \mapsto (\bar{\psi}_1, \bar{\psi}_2, \bar{J}_1, \bar{J}_2)$ given by

$$
\bar{\psi}_1 = \psi_1 + \delta (\bar{J}_1 + a_2 \bar{J}_2), \quad \bar{\psi}_2 = \psi_2 + \delta (a_2 \bar{J}_1 + a_3 \bar{J}_2),
$$

$$
\bar{J}_1 = J_1 + \delta \sin(\psi_1), \quad \bar{J}_2 = J_2 + \delta \epsilon \sin(\psi_2).
$$

- $T$ is related to the time-$\delta$ map of the flow associated to the Hamiltonian

$$
H(\psi_1, \psi_2, J_1, J_2) = \frac{J_2^2}{2} + \cos \psi_1 + a_2 J_1 J_2 + a_3 \frac{J_2^2}{2} + \epsilon \cos(\psi_2),
$$

- **4 fixed points:** For $\epsilon d > 0$, $d = a_3 - a_2^2$, $|\epsilon| \ll 1$ and $\delta \lesssim 2$

$$
p_1 = (0, 0, 0, 0) \text{ HH}, \quad p_2 = (\pi, 0, 0, 0) \text{ EH}, \quad p_3 = (0, \pi, 0, 0) \text{ HE}, \quad p_4 = (\pi, \pi, 0, 0) \text{ EE}.
$$

→ $T$ models the dynamics at a double resonance, it was derived from BNF around an EE point of a symplectic map in V. Gelfreich, C. Simó & AV, *Dynamics of 4D symplectic maps near a double resonance*, Phys D 243(1), 2013.
Motivation: Transition to complex unstable

→ If $d > 0$ (definite case) the EE point remains EE for all $\epsilon$ and $\delta$.
→ If $d < 0$ (non-definite case) the point $p_4$ suffers a Krein collision at $\epsilon = -(2a_3 - 4d) \pm \sqrt{(2a_3 - 4d)^2 - 4a_3^2} / (2a_3)^2$,

and becomes a complex-unstable point (Hamiltonian-Hopf bifurcation).

Eigenvalues of $DT(p_4)$ for $\delta = 0.5, a_2 = 0.5, a_3 = -0.75$ (hence $d = -1$)
and $\epsilon$ from $-0.01$ (squares) to $-20$. The (first) Krein collision takes place at $\epsilon^c = -4/9$ at a collision angle $\hat{\theta}_K = \arctan(\sqrt{23}/11)$.

→ The CS point has 2D stable/unstable invariant manifolds. → Next plots show their role!

→ The previous considerations also hold for $H$: the eigenvalues collide at the imaginary axis and the 2-dof analogous Hamiltonian-Hopf bifurcation takes place. Later: differences discrete/continuous cases in the splitting of the 2D inv. manifolds.
Motivation: Dynamical consequences

Lyapunov exp. MEGNO, i.e. on $\psi_1 = \psi_2 = 0$: white $\rightarrow$ regular, green $\rightarrow$ mild chaos, black $\rightarrow$ chaos.

Left: $\epsilon = -0.4$. Right: top $\epsilon = -0.44$, bottom: $\epsilon = -0.45$. (Rec: $\epsilon^c = -4/9$.)

$\rightarrow$ Lyapunov inv. curves families, local character of the bifurcation, evolution to global connection,...
**Goal of this work**

We want...

2. Geometry of the 2D invariant manifolds: behaviour of the splitting for the 4D map.

→ But, previously, we review the 2-d.o.f. analogous Hamiltonian-Hopf case.

1. Sokolskii NF.
2. Splitting of the invariant manifolds: Reduction to a 2D near-the identity area-preserving map.

→ Important: How are both cases related?

1. Main idea: Takes NF + interpolating Hamiltonian
2. Differences in the behaviour of the splitting: energy function
2-dof Hamiltonian Hopf (HH): Sokolskii NF

2-dof HH codim 1: Consider a 1-param. family of 2-dof Hamiltonians $H_\nu$ undergoing a HH bifurcation (at the origin).

Concretely: for $\nu > 0$ elliptic-elliptic, $\nu < 0$ complex-saddle.

Analysis of the HH bifurcation $\rightarrow$ Reduction to **Sokolskii NF**:

1. Taylor expansion at 0: $H_\nu = \sum_{k \geq 2} \sum_{j \geq 0} \nu^j H_{k,j}$, where $H_{k,j} \in \mathbb{P}_k$ homogeneous of order $k$.

2. Williamson NF (double purely imaginary eigenvalues):
   
   $H_{2,0} = -\omega (x_2 y_1 - x_1 y_2) + \frac{1}{2} (x_1^2 + x_2^2)$.

3. Use Lie series to order-by-order simplify $H_{2,j}, j > 1$ and $H_{k,j}, k > 2, j > 0$.

   But: **non-semisimple** linear part!

   Then, at each order $(k, j)$, one looks for $G \in \mathbb{P}_k$ s.t.

   $$H_{k,j} + \text{ad}_{H_2}(G) \in \text{Ker } \text{ad}^\top_{H_2}.$$
4. Introducing the Sokolskii coordinates \((dx_1 \wedge dy_1 + dx_2 \wedge dy_2 = dR \wedge dr + d\Theta \wedge d\theta)\)

\[
y_1 = r \cos(\theta), \ y_2 = r \sin(\theta), \ R = (x_1 y_1 + x_2 y_2)/r, \ \Theta = x_2 y_1 - x_1 y_2,
\]

one has \(H_2^\top = -\omega \Theta + \frac{1}{2} r^2\) and

\[
\text{NF}(H_\nu) = -\omega \Gamma_1 + \Gamma_2 + \sum_{\substack{k,l,j \geq 0 \\ k+l \geq 2}} a_{k,l,j} \Gamma_1^k \Gamma_3^l \nu^j.
\]

where

\[
\Gamma_1 = x_2 y_1 - x_1 y_2, \quad \Gamma_2 = (x_1^2 + x_2^2)/2 \quad \text{and} \quad \Gamma_3 = (y_1^2 + y_2^2)/2.
\]

5. Introducing \(\nu = -\delta_\nu^2\), and rescaling \(x_i = \delta_\nu^2 \tilde{x}_i, \ \omega y_i = \delta_\nu \tilde{y}_i, \ i = 1, 2, \omega t = \tilde{t}\), one has

\[
\text{NF}(\tilde{H}_{\delta_\nu}) = -\tilde{\Gamma}_1 + \delta_\nu \left( \tilde{\Gamma}_2 + a \tilde{\Gamma}_3 + \eta \tilde{\Gamma}_3^2 \right) + \mathcal{O}(\delta_\nu^2).
\]

The \(\tilde{\Gamma}_i\) written in terms of the Sokolskii coordinates are given by

\[
\tilde{\Gamma}_1 = \Theta, \quad \tilde{\Gamma}_2 = \frac{1}{2} \left( R^2 + \frac{\Theta^2}{r^2} \right), \quad \text{and} \quad \tilde{\Gamma}_3 = \frac{r^2}{2}.
\]
2-dof HH: invariant manifolds

For \( \nu < 0 \) the origin has stable/unstable inv. manifolds \( W^{s/u}(0) \). Note that

- \( W^{s/u}(0) \) are contained in the zero energy level of \( \text{NF}(\tilde{H}_\delta) \).
- \( \{\tilde{\Gamma}_1, \tilde{\Gamma}_2\} = \{\tilde{\Gamma}_1, \tilde{\Gamma}_3\} = 0 \Rightarrow \tilde{\Gamma}_1 \) is a formal first integral of \( \text{NF}(\tilde{H}_\delta) \).

Hence \( \tilde{\Gamma}_1 = 0 \) on \( W^{s/u}(0) \).

Then, ignoring \( \mathcal{O}(\delta^2) \) terms, \( W^{s/u}(0) \) are given by \( R^2 + ar^2 + \eta r^4 / 2 = 0 \), which is the zero energy level of a **Duffing Hamiltonian system**.

\[
\Rightarrow W^{u/s}(0)\big|_{(R,r)-\text{plane}} \text{ form a figure-eight}
\]

(for \( a < 0, \eta > 0 \); unbounded otherwise!

but only \( r > 0 \) has sense!).

The **2D** \( W^{s/u}(0) \) are rotated around the origin (on \( W^{s/u}(0) \) one has \( \Theta = 0, \dot{\theta} = 1 \)).

For the truncated NF (i.e. ignoring \( \mathcal{O}(\delta^p) \)-terms, \( p > 1 \)) the **2D** stable/unstable inv. manifolds **coincide**. **But:** For the complete 2-dof Hamiltonian **they split**!
2-dof HH: splitting of inv. manifolds

The asymptotic expansion of this splitting has been obtained in J.P. Gaivao, V. Gelfreich, *Splitting of separatrices for the Hamiltonian-Hopf bifurcation with the Swift-Hohenberg equation as an example*, Nonlinearity 24(3), 2011.

\[
\alpha \sim A \delta^B \nu \exp \left( \frac{-\pi}{\sqrt{-a \delta}} \right) \sim A |\text{Re} \lambda|^B \exp \left( \frac{-\pi |\text{Im} \lambda|}{|\text{Re} \lambda|} \right)
\]

Main idea: The exponential part of this formula can be obtained by reducing to a near Id family of analytic APMs + Fontich-Simó thm. (upper bounds are generic!).

Consider \( \Sigma = \{ \theta = 0 \} \) (but in Cartesian coord. to avoid singularities) and

- \( T_{\delta \nu} : \Sigma \to \Sigma \) (Poincaré map of the full 2-dof Hamiltonian) \( \sim \) separatrices split,
- \( T_{\delta \nu}^0 : \Sigma \to \Sigma \) (Poincaré map of the truncated 2-dof Hamiltonian, ignoring \( O(\delta^2) \)) \( \sim \) homoclinic loop.

Then, \( T_{\delta \nu}^0 (R, r, \Theta, \theta) = (\phi^X_{2\pi}, \Theta, \theta \mod 2\pi) \), being \( X \) the vector field

\[
\dot{R} = \delta (ar + \eta r^3), \quad \dot{r} = -\delta R, \quad \text{← Duffing!}
\]

which has a homoclinic solution \( \gamma(t) \) with nearest singularity to the real axis \( \tau = i\pi / 2\sqrt{-a \delta} \) and dominant eigenvalue \( \mu = 2\pi \sqrt{-a \delta} \) (then rescale time by \( \sqrt{-a \delta} \)). But \( T_{\delta \nu}^0 = (\hat{T}_0^{\delta \nu})^2 \), being \( \hat{T}_0^{\delta \nu} \) close to -Id \( \Rightarrow \) use \( \mu / 2 \) instead of \( \mu \) in the exponential part of the upper bound \( C \exp (-2\pi (\text{Im} \tau - \eta) / \mu) \).
2-dof HH: the example

\[ H(\psi_1, \psi_2, J_1, J_2) = \frac{J_1^2}{2} + a_2 J_1 J_2 + a_3 \frac{J_2^2}{2} - \cos(\psi_1) - \epsilon \cos(\psi_2) \]

**Reversibility:** \((\psi_1, \psi_2, J_1, J_2) \in W^u(0)\) then \((-\psi_1, -\psi_2, J_1, J_2) \in W^s(0)\). This suggests to consider \(\Sigma = \{\psi_1 = 0, \psi_2 = 0\}\) and to look for homoclinic points in \(\Sigma\).

\[ a_2 = 0.5, \quad a_3 = -0.75, \quad \epsilon = -0.5 \quad (\epsilon^c = -4/9) \]
Methodology to get a homoclinic point in $\Sigma (I)$

One can locally represent $W^u$ as a series

$$G(s_1, s_2) = \sum_{i+j \geq 0} a_{i,j} s_1^i s_2^j, \quad a_{i,j} \in \mathbb{R}$$

where $s_1, s_2 \in \mathbb{R}$ are (real) local parameters in a fundamental domain (an annulus) $\sim$ parameterisation method.

Then, one can propagate the local representation and get the invariant manifolds (e.g. using Taylor integrator).

Main steps:

1. Compute the local parameterisation of $W^u$ (order by order).

2. Truncate it to order $N$ and look for $r_\star$ (radius in $(s_1, s_2)$) such that the invariance equation is verified up to a given tolerance $\text{tol}$. The points on the circle of radius $r_\star$ can be parameterised by an angle $\theta$. 


Methodology to get a homoclinic point in $\Sigma$ (II)

3. To compute $\theta$ s.t. parameterises a point on $\Sigma$ we proceed as follows:

(a) **Discretize** $\theta$: $\{\theta_i\}_{i=1,\ldots,1000}$.

Each $\theta_i$ gives an initial condition $\rightarrow$ integrate (Taylor method).

(b) **Integrate** each i.c. up to $\{\psi_2 = 0\}$.

**Problem:** $\{\psi_2 = 0\}$ is crossed many times before we arrive to $\Sigma$!!

We proceed as follows:

i. We fix a number $m$ and we integrate up to the $m$ crossing with $\psi_2 = 0$.

Hence, for each $i$ we obtain a point on $\psi_2 = 0$. Denote by $\psi_{1,i}$ the corresponding coordinate of this point.

ii. If for a concrete $i$ one has $\psi_{1,i} \psi_{1,i-1} < 0$ then we look for $\theta \in (\theta_{i-1}, \theta_i)$ such that $\psi_1 = 0$ in $\Sigma$ (e.g. secant method).

Otherwise, if there is not $i$ verifying this last condition, we increase $m$.

$\implies$ We get a homoclinic point on $\Sigma$ (first intersection!).
2-dof HH: Computing the splitting

Using the last methodology one obtains \((s^h_1, s^h_2)\) corresponding a homoclinic point \(p_h\) on \(\Sigma\) (at the first intersection!). \(\sim\) The homoclinic orbit was shown in the last plot!

To measure the splitting angle \(\alpha\) at \(p_h\):

1. **Compute a basis of** \(T_{X_h^0}(W_{loc}^u(0)) \sim v_t^0 = \frac{\partial G}{\partial s_1}(s^h_1, s^h_2), v_{vf}^0\)

2. **Transport the vectors to** \(\Sigma \sim v^\Sigma_t, v^\Sigma_{vf}\) (integrating variational eqs.)
   These vectors form a basis of \(T_{p_h}(W^u(0))\).

3. **Compute an orthogonal basis of** \(T_{p_h}(W^u(0)) \sim w_1, v_{vf}^\Sigma\)

4. **Compute the splitting angle.** By reversibility, from \(w_1 \in T_{p_h}(W^u(0))\)
   we obtain a vector \(w_2 \in T_{p_h}(W^s(0))\). Then,
   \[
   \alpha = \text{angle}(w_1, w_2)
   \]
2-dof HH: Checking the behaviour of $\alpha$

$$H(\psi_1, \psi_2, J_1, J_2) = \frac{J_1^2}{2} + a_2 J_1 J_2 + a_3 \frac{J_2^2}{2} - \cos(\psi_1) - \epsilon \cos(\psi_2)$$

Left: $\log(\alpha)$ vs. $\epsilon - \epsilon^c$. Right: $\text{Re}(\lambda) \log(\alpha)$ vs. $\text{Re}(\lambda)$.

Recall: $\alpha \sim \tilde{A}(|\text{Re} \lambda|)^B \exp \left( \frac{C}{|\text{Re} \lambda|} \right)$, where $C = -\pi |\text{Im} \lambda|$.

For $a_2 = 0.5$, $a_3 = -0.75$ one gets $C = \sqrt{2\pi}/3 + O(\nu)$ (Sokolskii NF).

Fitting function (right plot): $f(x) = A x + B x \log(x) + C$.

$\sim$ It perfectly fits the behaviour!
Up to this point: **2-dof Hamiltonian-Hopf** bifurcation.

1. Everything was “more or less” well-known: Sokolskii NF, geometry of the invariant manifolds, the splitting $\alpha$, ...

2. $\alpha$ behaves as expected for a near-the-identity family of 2D APM.

   Guiding example: $H = \frac{J_1^2}{2} + a_2 J_1 J_2 + a_3 \frac{J_2^2}{2} - \cos(\psi_1) - \epsilon \cos(\psi_2)$.

**Now: 4D discrete Hamiltonian-Hopf!**

Guiding example: the 4D symplectic map $T$ given by

$$
\bar{\psi}_1 = \psi_1 + \delta (\bar{J}_1 + a_2 \bar{J}_2), \quad \bar{\psi}_2 = \psi_2 + \delta (a_2 \bar{J}_1 + a_3 \bar{J}_2), \\
\bar{J}_1 = J_1 + \delta \sin(\psi_1), \quad \bar{J}_2 = J_2 + \delta \epsilon \sin(\psi_2).
$$

The origin undergoes a HH bif. and 2D stable/unstable manifolds are born. **Question:** Behaviour of the splitting of the 2D inv. manifolds?
Planning:

**First**: Numerical exploration of $T$.
- Computation of the invariant manifolds.
- Behaviour of the splitting.
- A naive justification of the behaviour observed.

**After**: General theoretical results on splitting of inv. manifolds for the 4D HH.
- Upper bounds from a suitable energy function.
One can compute $W_{u/s}^u(0)$ and \( p_h \in \Sigma = \{ \psi_1 = \psi_2 = 0 \} \) (similarly to the 2-dof case).
**T: Splitting volume V**

We compute the volume of a 4D paralleloptope defined by two pairs of vectors tangent to $W^u$ and $W^s$ at $p_h \in \Sigma$:

$G(s_1, s_2)$ - local parameterisation

$(s^h_1, s^h_2)$ - local parameters s.t. $T^N(s^h_1, s^h_2) = p_h, N > 0$.

1. Consider the vectors:

$$\tilde{v}_1 = (\partial G/\partial s_1)(s^h_1, s^h_2), \quad \tilde{v}_2 = (\partial G/\partial s_2)(s^h_1, s^h_2) \quad \text{← tangent to } W^u(0)$$

2. Transport these vectors under $T$ to $p_h$ and consider, by the reversibility,

$$\tilde{v}_3 = R(\tilde{v}^{ph}_1), \quad \tilde{v}_4 = R(\tilde{v}^{ph}_2) \quad \text{← tangent to } W^s(0)$$

3. Finally, normalize them $v_j = \tilde{v}^{ph}_j / \|\tilde{v}^{ph}_j\|, \quad j = 1, \ldots, 4$ and define

$$V = \det(v_1, v_2, v_3, v_4)$$

**Question:** How does $V$ behave as $\epsilon \to \epsilon_c$?
$T: \text{Behaviour of } V$

$$T: \quad \bar{\psi}_1 = \psi_1 + \delta(\bar{J}_1 + a_2 \bar{J}_2), \quad \bar{\psi}_2 = \psi_2 + \delta(a_2 \bar{J}_1 + a_3 \bar{J}_2),$$

$$\bar{J}_1 = J_1 + \delta \sin(\psi_1), \quad \bar{J}_2 = J_2 + \delta \epsilon \sin(\psi_2).$$

Fixed $a_2, a_3$ one has $\epsilon^c = \epsilon^c(a_2, a_3)$. The (Krein) collision angle $\hat{\theta}_K$ depends on $\delta$.

$a_2 = 0.5, a_3 = -0.75 \leadsto \epsilon^c = -4/9. \quad \delta = 0.5 \leadsto \theta_K = \arctan(\sqrt{23}/11)/2\pi \in \mathbb{R} \setminus \mathbb{Q}$.

Left: $\log V$ vs. $\epsilon$. Right: $h|\log(V)|$ vs. $h$ ($h = \log(\lambda)$).
Naive explanation of the behaviour of $V$

Consider a (generic) symplectic map $F$ in $\mathbb{R}^4$ undergoing a HH bif. Discrete HH bif. $\leadsto$ codim 2 bif $\leadsto$ Let $\delta_t, \epsilon_t$ be the unfolding parameters. 

$\delta_t$: Collision angle $\hat{\theta}_K = 2\pi(q/m + \delta_t)$.

$\epsilon_t$: Measures the relative distance to the critical parameter.

Different (naive) important aspects:

1. “Two” exp. small effects: one within the Hamiltonian itself (already studied!), the other measures the “map-Hamiltonian distance”.


3. The Hamiltonian part is known $\Rightarrow$ only necessary to measure the second effect. But: We have a “privilegiated direction” (the time!) $\Rightarrow$ we will use an energy function to measure the splitting in that direction (instead of using the splitting potential or the Melnikov vector which measures both effects together).
Towards a sharp upper bound of the splitting (I)

Idea: It is enough to measure the “Hamiltonian-map distance”.

Let \( F_{\epsilon_t} \) be a family of symplectic maps s.t. at \( \epsilon_t = 0 \) undergoes a HH bifurcation. The inv. manifolds \( W^{u/s}(0) \) are given by \( u(\alpha, t) \) and \( v(\alpha, t) \) resp., where \( (\alpha, t) \in [t_0, t_0 + h) \times S^1 \). This defines FD’s \( D^{u/s} \).

Main result: Assume that

(H1) There exists an energy function \( E \), i.e. such that \( E \circ F_{\epsilon_t} = E \), defined in a neighbourhood of the fundamental domain \( D^s \) such that \( E(v(\alpha, t)) = 0 \). Moreover we assume that \( E \) and \( v(\alpha, t) \) can be analytically extended to a neighbourhood of \( W^u(0) \) within \( D^u \) (by iteration of \( F_{\epsilon_t}^{-1} \)).

We define the splitting function:

\[
\psi(\alpha, t) = E(u(\alpha, t))
\]
Towards a sharp upper bound (II)

(H2) There is a (limit) vector field

\[ \dot{x} = f(x), \quad x \in \mathbb{R}^2, \]

such that \( f \) is analytic, it possesses a hyperbolic saddle fixed point and a homoclinic orbit \( \sigma(t) \) associated to it, and satisfies that compact pieces of the real invariant manifolds of \( F_{\epsilon_t} \) are \( \epsilon_t \)-close to an embedding of \( \mathbb{S}^1 \times \{\sigma(t), \ t \in \mathbb{R}\} \) for \( \epsilon_t > 0 \) small enough.

(H3) \( F_{\epsilon_t} \) can be extended analytically to a neighbourhood of

\[ \{\alpha \in \mathbb{C}/2\pi \mathbb{Z}, |\text{Im} \alpha| < \rho\} \times \{\sigma(t), |\text{Im} t| < \tau\} \]

for some \( 0 < \tau < \tau_0 \) and \( 0 < \rho < \rho_0 \).
Towards a sharp upper bound (Result)

Under (H1), (H2) and (H3)...

(i) **Rational Krein collision.** Let $\theta_K = p/q$, with $(p, q) = 1$. Then, there exists $\epsilon_t^0 > 0$ s.t. for $\epsilon_t < \epsilon_t^0$

$$|\psi(\alpha, t)| \leq K \exp(-C/h), \quad C, K > 0.$$ 

(ii) **Irrational Krein collision.** Let $\theta_K \in \mathbb{R} \setminus \mathbb{Q}$. Then, $\psi$ is bounded by a function that is exponentially small in a parameter $\gamma$, s.t. $\gamma \searrow 0$ when $h \nearrow 0$. Moreover, the dominant harmonic $k(h)$ of $\psi$ changes infinitely many times as $h \to 0$.

**Idea:** Bounding the Fourier coefficients of $\psi$, one gets

$$|\psi(\alpha, t)| < K \sum_{(k, n) \in \mathbb{Z}^2^*} \exp\left(-2\pi |n - \theta_0 k| \tau/h - |k| \rho\right).$$

Then we look for $k = k_*(h) > 0$ s.t. the dominant coefficient $\beta_{k_*}$ in the exponential bound is minimum (different cases according to the properties of $\theta_K$).
Map $T$: fit of the volume $V$ (I)

$$
T: \quad \bar{\psi}_1 = \psi_1 + \delta(J_1 + a_2 J_2), \quad \bar{\psi}_2 = \psi_2 + \delta(a_2 J_1 + a_3 J_2),
\quad \bar{J}_1 = J_1 + \delta \sin(\psi_1), \quad \bar{J}_2 = J_2 + \delta \epsilon \sin(\psi_2).
$$

$a_2 = 0.5$, $a_3 = -0.75 \sim \epsilon^c = -4/9$.

We look for the dominant coefficients $\beta_k(h)$. They depend on $\theta_K$ and $h = \log(\lambda) = \mathcal{O}(\sqrt{|\epsilon - \epsilon^c|})$. We fix $\theta_K = \arctan(\sqrt{23/11})/2\pi$.

Left: first five dominant exponents $\beta_k$ as a function of $h$. Right: values of $k_*$ corresponding to the minimum exponent $\beta_k$. Both in log–log scale.
Map $\mathcal{T}$: fit of the volume $V$ (II)

- We have $k_*=1, 15, 46, 107, 703, 2002, 9307, 25919, \ldots$ as $h \to 0$.
- The values of $k_*$ are related to the approximants of $\theta_K \approx 0.06543462308$: $1/15, 3/46, 4/61, 7/107, 39/596, 46/703, 85/1299, 131/2002, \ldots$
- Not all the approximants produce a change of $k_*(h)$ as $h \to 0$, only those that are smaller than $\theta_K$ play a role (except the first one $1/15 > \theta_0$).
- The length of the interval in $h$ where $k_*(h)$ dominates depends on the CFE($\theta_K$) = $[15, 3, 1, 1, 5, 1, 1, 3, 1, 2, \ldots]$, but also on the constants in front of the exponential terms of $V$ (terms with larger $\beta_k$ can dominate for finite $h > 0$!!)

**Conclusion:** The numerical fit data show that the different slopes observed are related to the different values $k_*(h)$ obtained $\sim$ OK!!!
1. **Other aspects** related to the HH bifurcation for 4D maps have been also investigated (preprint).

For example:

(a) Structure of the Lyapunov families of invariant curves (analytic results on: the detachment of the Lyapunov families, analysis of the rational and irrational collision angle $\theta_k$ cases, stability of the inv. curves, ...).

![Graphs showing the detachment of the Lyapunov families of invariant curves for $T$: $\epsilon = -0.1$, $-0.4$ and $-0.5$ ($\epsilon^c = -4/9$).](image)

Detachment of the Lyapunov families of invariant curves for $T$:

$\epsilon = -0.1$, $-0.4$ and $-0.5$ ($\epsilon^c = -4/9$).
(b) Possible diffusive patterns **through and around** the double resonance.

Left: Positive definite case \( (\delta = \epsilon = a_2 = 0.5 \text{ and } a_3 = 1.25) \).
Centre/Right: Non-definite case \( (\delta = \epsilon = a_2 = 0.5 \text{ and } a_3 = -0.75) \).

2. **Many open questions**: Theorem of splitting for a family of 4D maps? Separatrix return map? Diffusive properties (quantitative data)?

...but this is left for future works...
Thanks for your attention!!