Abundance of stable periodic orbits inside homoclinic lobes

Conference on Computational Methods in Dynamics

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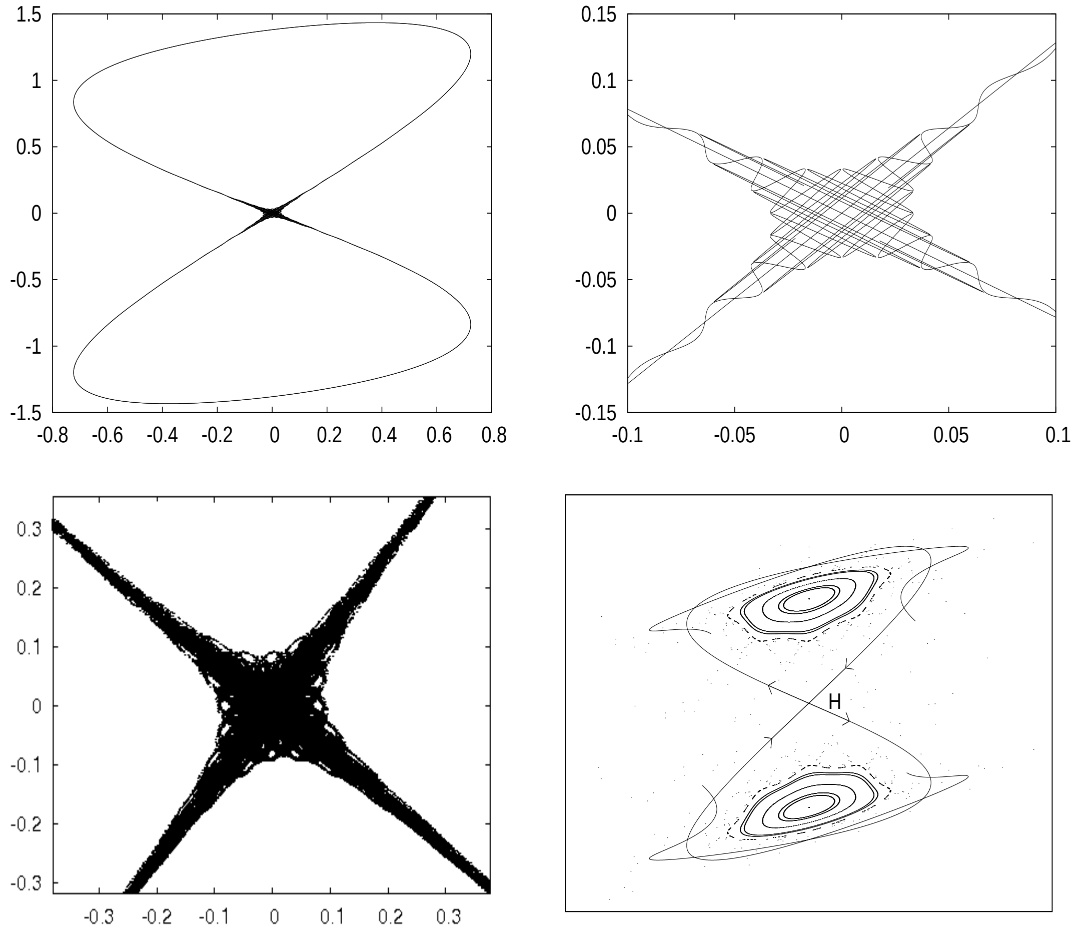
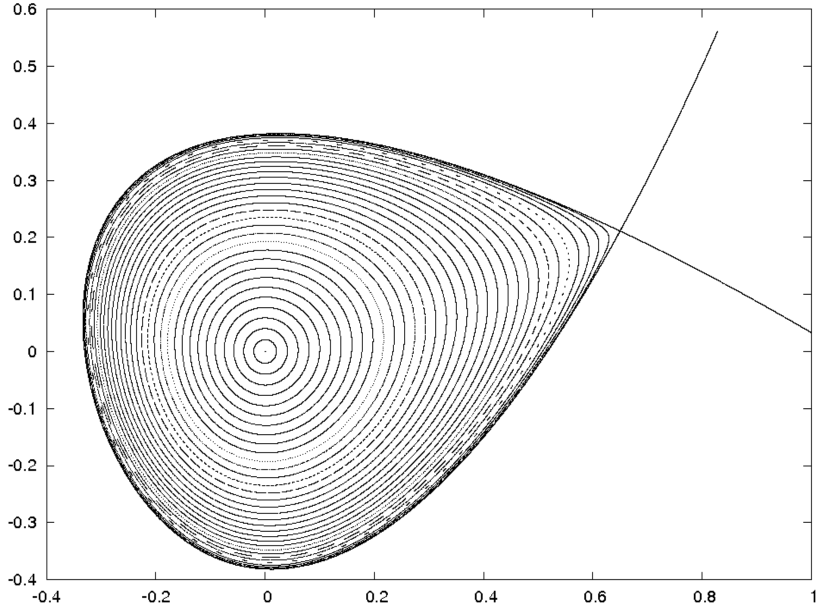
Splitting of separatrices + chaotic zone

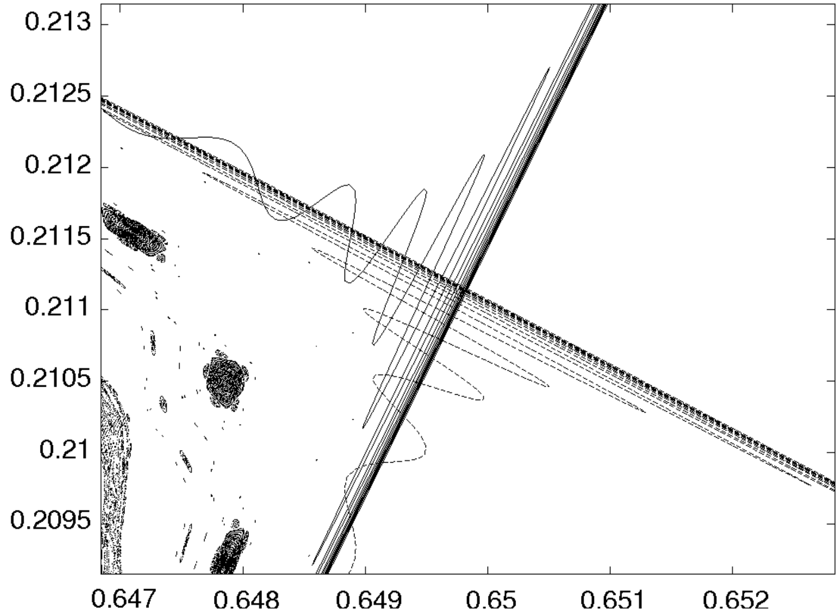
Consider an APM F with a hyperbolic fixed point H. Generically, the separatrices of H split and create a chaotic zone (CZ) which extends up to the "outermost" invariant curve.



The dynamics within the chaotic zone...

... is **not ergodic**: “rel. far” from the separatrices there are islands inside CZ.





$H_\alpha(x, y) = R_{2\pi\alpha}(x, y - x^2), \alpha = 0.1.$

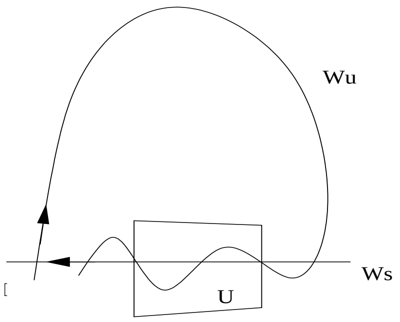
Experimental values:	$D_c^H \approx 2.94 \times 10^{-3}$	$D_i^H \approx 2.08 \times 10^{-3}$
“Fish” interp. Hamiltonian:	$D_c^H \approx 2.47 \times 10^{-3}$	$D_i^H \approx 1.85 \times 10^{-3}$
5-order interp. Hamiltonian:	$D_c^H \approx 2.731 \times 10^{-3}$	$D_i^H \approx 2.050 \times 10^{-3}$

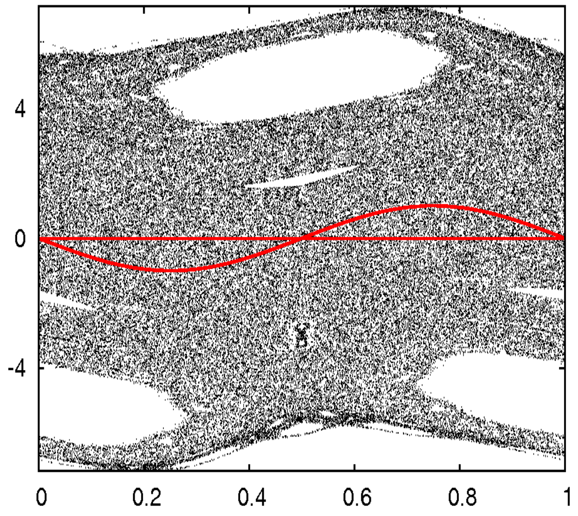
Main idea: SM (and STM aprox.) + Interp. Ham. [a]

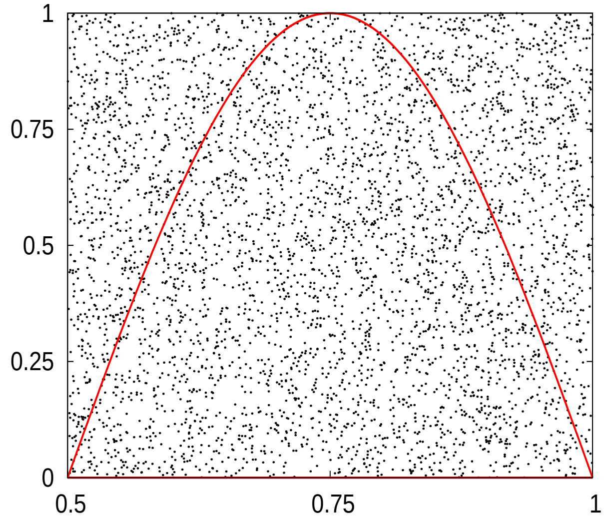
[a] Simó-V. *Dynamics in chaotic zones of area preserving maps: close to separatrix and global instability zones.* Physica D, 240(8), 2011.

Dynamics within the homoclinic lobes

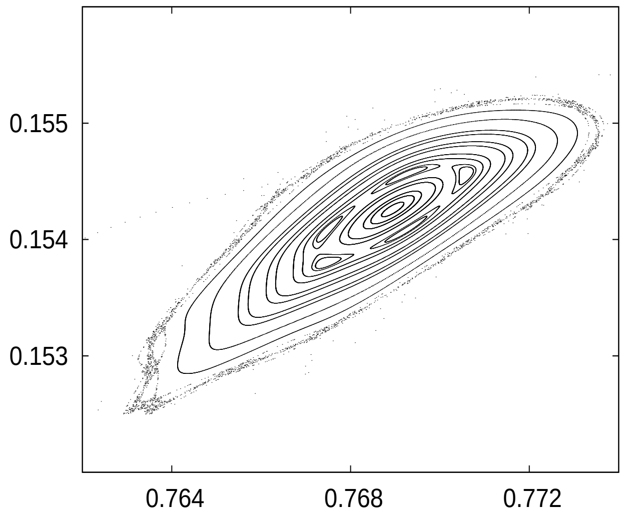
Apparently chaotic...







...but, inside the homoclinic lobes, one finds **tiny** islands of stability: [a]



[a] Simó-Treschev, *Stability islands in the vicinity of separatrices of near-integrable symplectic maps*, Disc. Cont. Dyn. Sys. B, 10(2,3), 2008

Goal of this work I

Instead of a single APM F, we consider a **one-parameter family of APMs** F_ϵ.

$\longrightarrow$ ϵ – distance-to-integrable parameter

We are interested in the **elliptic periodic orbits visiting homoclinic lobes (EPL)** of the **lowest possible period** ("dominant") of F_ϵ for $\epsilon << 1$.

For **analytical** results: we assume "central symmetry" of F_ϵ and use the *separatrix map* (SM) to...

(concrete details later...)

- ... study the *abundance* of EPL (i.e. the relative measure of the set E_ϵ of ϵ-parameters for which F_ϵ has EPL).
- ... describe the pattern of creation/destruction/bifurcation of these EPL in terms of the parameter ϵ.
- ... obtain an (explicit!) accurate estimate of the $m(E_\epsilon)$.

$\rightarrow$ "maybe nice theory"... **but, moreover,...**

Goal of this work II

... we want **to compare** the theoretical results with **“real” situations**.

To this end, we perform **accurate numerical computations** to obtain estimates of $m(E_\epsilon)$. The numerical experiments we will consider as F_ϵ the standard map (STM) and the Hénon map.

$\longrightarrow$ Note that a “real” situation does not necessarily fit within “our” theoretical framework (typically, one simplifies the model, use a perturbative approach,...).

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Non-symmetric figure-eight

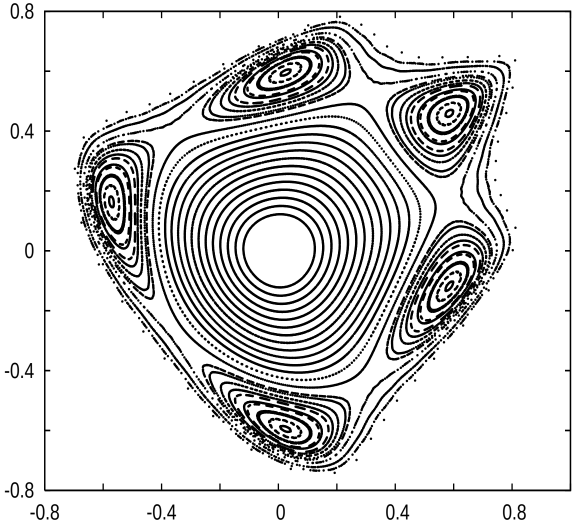
The figure-eight loops maybe non-symmetric!

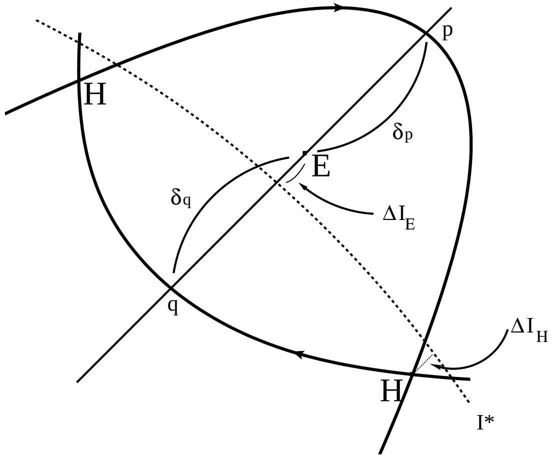
Example of interest: resonant islands emanating from a fixed elliptic point.

Let F_δ be a one-parameter family of APMs, $F_\delta(E_0) = E_0$ elliptic f.p.,

dynamics around the $(q : m)$-resonance, $m \geq 5$, $(1 \leq q < m, (q, m) = 1)$.

Spec$(DF_\delta)(E_0) = \{\lambda, \lambda^{-1}\}$, $\lambda = \exp(2\pi i\alpha)$, $\alpha = q/m + \delta$, $\delta \in \mathbb{R}$.





"Outer splitting $\leftrightarrow$ p"

"Inner splitting $\leftrightarrow$ q"

Thm. Under generic assumptions: outer splitting $>$ inner splitting. [a]

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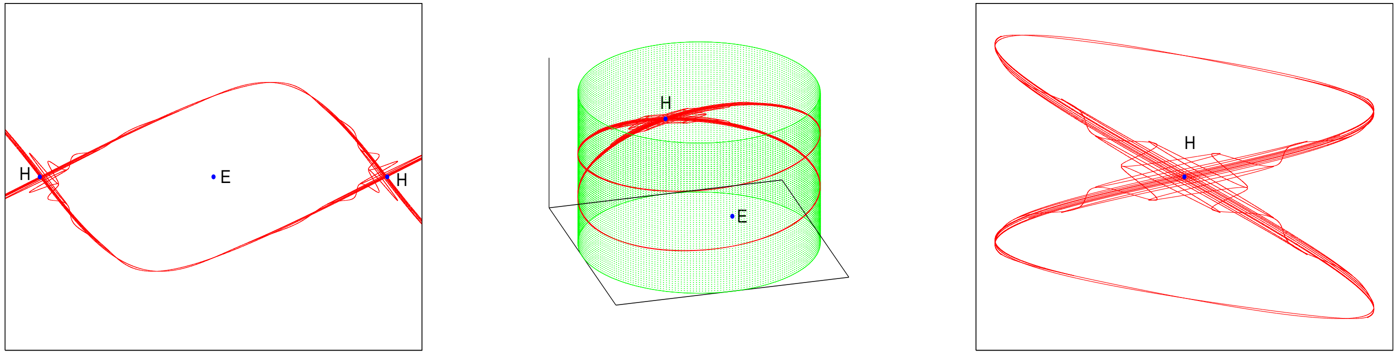
[a] Simó-V. *Resonant zones, inner and outer splittings in generic and low order resonances of area preserving maps.* Nonlinearity 22, 5:1191–1245, 2009.

Double separatrix map (figure-eight)



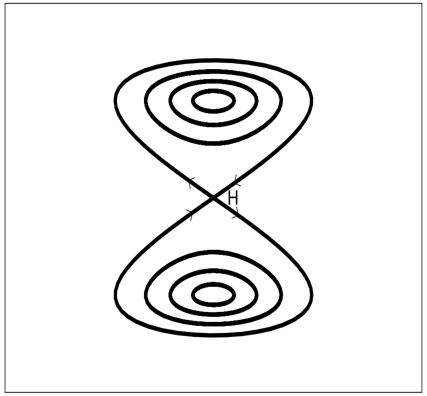
$$\text{DSM}: \begin{pmatrix} x \\ y \\ s \end{pmatrix} \longmapsto \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{s} \end{pmatrix} = \begin{pmatrix} x + a_{s,\bar{s}} + b \log|\bar{y}| \pmod 1 \\ y + \nu_{\bar{s}} \sin 2\pi x \\ \text{sign}(y)\, s \end{pmatrix},$$

- Defined on a domain $\mathcal{W} = \mathcal{U} \cup \mathcal{D}$ (around the outer/inner separatrices).
- $a_{s,\bar{s}}$ suitable “shifts” (reinjection to $\mathcal{W}$).
- $b = 1/\log(\lambda)$, λ dominant eigenvalue of H.
- y-variable rescaled: $\nu_1 = 1$ and $\nu_{-1} = A_{-1}/A_1$, where A_1 (resp. A_{-1}) is the amplitude of the outer (resp. inner) splitting.

A priori stable/unstable cases

Recall that we want to study EPL of F_ϵ, ϵ dist-to-integr. param., F_0 integrable.

***A priori* unstable:** F_0 has a non-degenerated hyperbolic fixed point H_0 s.t. $\lambda(0) > 1$. Then $\lambda(\epsilon) = \lambda(0) + \mathcal{O}(\epsilon^r)$, $r > 0$. The separatrices of H form an integrable figure-eight.



***A priori* stable:** F_0 has a degenerated fixed point (e.g. we encounter a line of fixed points for $\epsilon = 0$). Then $\lambda(\epsilon) = 1 + \mathcal{O}(\epsilon^r)$, $r > 0$.

Remark: Islands emanating from a fixed elliptic point $\longrightarrow$ *a priori stable* case.

All the examples we deal with fit within the *a priori stable* framework!

A priori stable/unstable differences

- Size (width) of the homoclinic lobes.

  (i) *a priori unstable*: $\mathcal{A}_\epsilon = \mathcal{O}(\epsilon^r)$, $r > 0$

  (ii) *a priori stable*: $\mathcal{A}_\epsilon = \mathcal{O}(\exp(-c/\epsilon^r))$, with $r, c > 0$ ctants.

- Relation $F_\epsilon \longleftrightarrow \mathrm{SM}_{a,b}$.

  (i) *a priori unstable*: $a = \mathcal{O}(-\log \epsilon)$, $b = \mathcal{O}(1)$,

  (ii) *a priori stable*: $a = \mathcal{O}(1/\epsilon^{2r})$, $b = \mathcal{O}(1/\epsilon^r)$,

Remarks:

- Case (i): a, b change "independently" (a changes with ϵ).
- Case (ii): Both a and b depend on ϵ. But $b'(a) \approx \epsilon^r \to 0$ as $\epsilon \to 0$ (i.e. a changes faster with respect to small variations of ϵ).

Simó-Treschev result

F_ϵ – a priori unstable family of APMs

E_ϵ, $\epsilon < \epsilon_0 << 1$ – set of ϵ-parameters for which F_ϵ has EPL

Thm. $m(E_\epsilon)$, when $\epsilon_0 \to 0$, remains greater than a constant $K > 0$ independent of ϵ. [a]

Comments:

- It does not provide any approximation of $m(E_\epsilon)$.
- It is enough to proof the existence of one EPL for some concrete a and b values of the DSM. Then, using a specific scaling of the SM, one obtains an EPL for values $\epsilon \to 0$.
- This scaling holds because b is indep. of ϵ (a priori unstable) scaling idea: $\epsilon_2 = \epsilon_1/\lambda^{1/r} \Rightarrow a(\epsilon_2) \approx a(\epsilon_1)$ (mod 1)

[a] Simó-Treschev, *Stability islands in the vicinity of separatrices of near-integrable symplectic maps*, Disc. Cont. Dyn. Sys. B, 10(2,3), 2008

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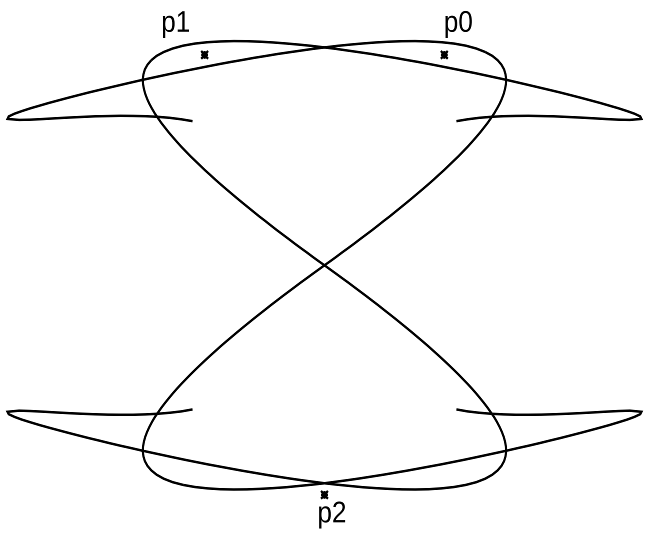
Central symmetry

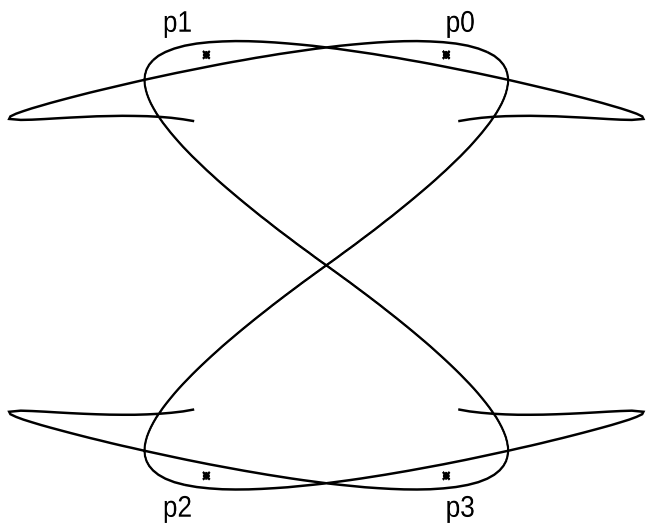
We assume that F_ϵ commutes with the central symmetry with respect to H_ϵ.

This implies:

1. The figure-eight loops are symmetric.
2. The lowest possible period for an EPL is $\hat{p} = 4$.



Non-symmetric $\hat{p} = 3$ EPL



Symmetric $\hat{p} = 4$ ($p = 2$) EPL

DSM $\longrightarrow$ "symmetric SM"

We can then identify both domains of definition of the DSM and consider a simple model

$$\mathrm{SM}_{a,b}: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x + a + b\log|y_1| \\ y + \sin(2\pi x) \end{pmatrix}$$

Motivation: For generic (non-strong) res. islands emanating from an elliptic fixed point, the "lack of symmetry" is detected in a "second order" approximation of the dynamics, which can be described by the Hamiltonian:

$$\mathcal{H}(J,\psi) = \frac{1}{2}J^2 + \frac{c}{3}J^3 - (1+dJ)\cos(\psi),\ c = \mathcal{O}(\delta^{\frac{m}{4}}),\ d = \mathcal{O}(\delta^{\frac{m}{4}-1}).$$

Rec: If the multiplier of the elliptic point is $\alpha = q/m + \delta$, the m-resonant islands are located at $I_* = \mathcal{O}(\delta)$ and have a width $\mathcal{O}(\delta^{m/4})$. Then J, ψ are adapted coordinates around the m-island.

Main result

Assume F_ϵ a priori stable + central symmetry

$\Rightarrow$ we use SM$_{a,b}$ to describe dynamics within the homoclinic lobes.

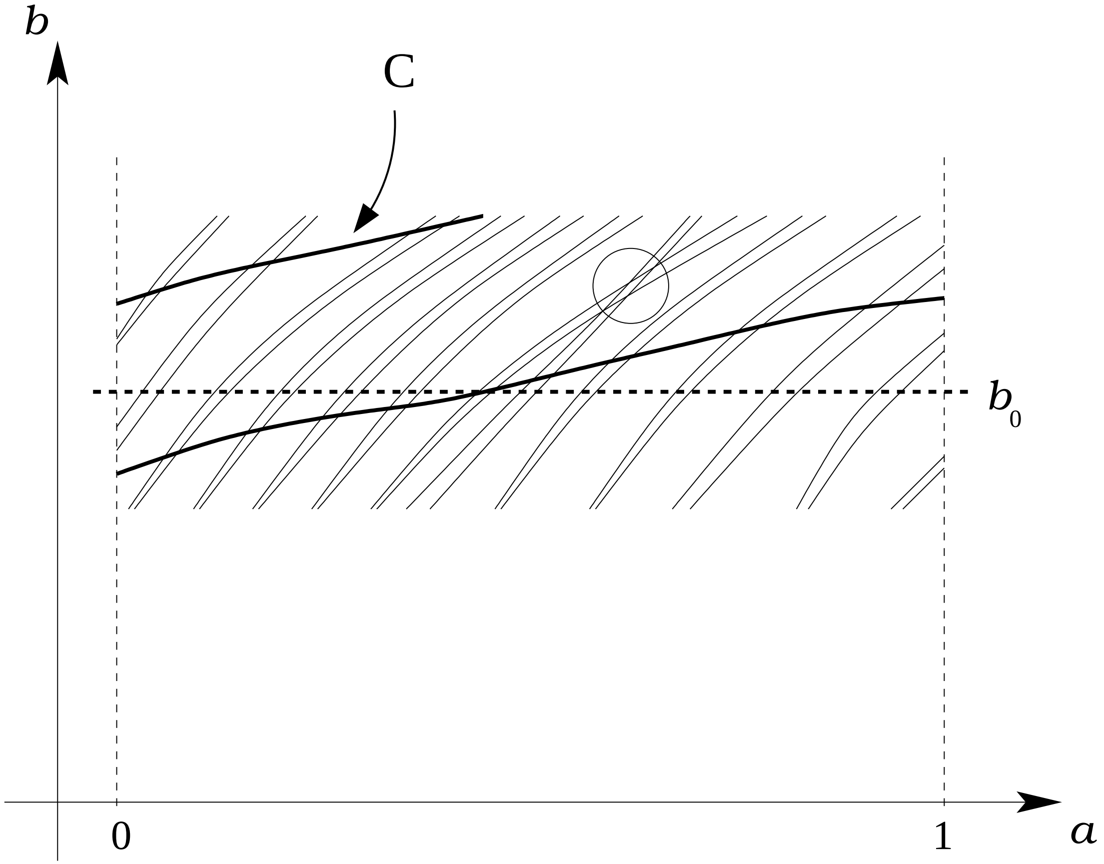
Idea: For a fixed b we look for the measure of the set of maps (depending on $a \in [0,1)$) having EPL of period $p = 2$ ($\hat{p} = 4$).

Thm. For a fixed b, let $\sum \Delta a$ denote the sum of the lengths of the intervals $\Delta a = (a_-, a_+)$ such that for $a \in \Delta a$ the separatrix map SM$_{a,b}$ has a $p = 2$ EPL. Then,

$$\lim_{b \to +\infty} \sum \Delta a = \frac{1}{2\pi^2} \approx 0.05066.$$

Rec: $a = a(\epsilon)$ and $b = b(\epsilon)$, but a changes quickly!

Transversality: EPL strips & (a, b)-curve of F_ϵ



- b large enough (integrable limit, $b = \mathcal{O}(1/\epsilon)$).
- Each $p = 2$ EPL strip is related to different periodic P trajectory of F_ϵ.
- F_ϵ defines a curve $\mathcal{C}$ which intersects **transversally** the EPL strips.

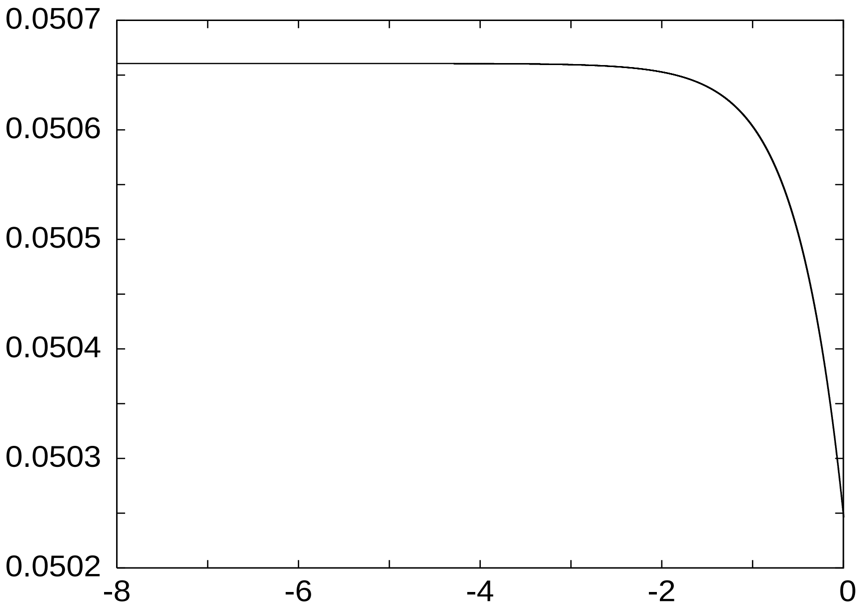
Overlapping

Each periodic P trajectory of F_ϵ gives two a-intervals of EPL.

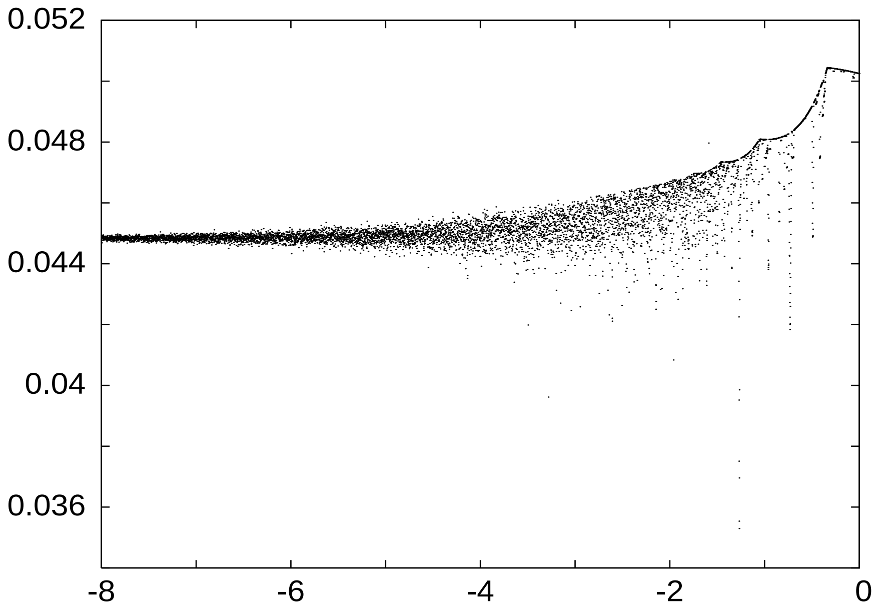
For P rel. small, elementary overlaps between these a-intervals occur.

Skipping these overlaps: $\lim_{b\to+\infty} \sum \Delta a = \frac{1}{2\pi^2}(1/2 + \log(3/2)) \approx 0.04587$.

Numerical check: x-axis: $-\log(b)$, y-axis: $\sum \Delta a$.



with overlapping



removing “all” overlappings

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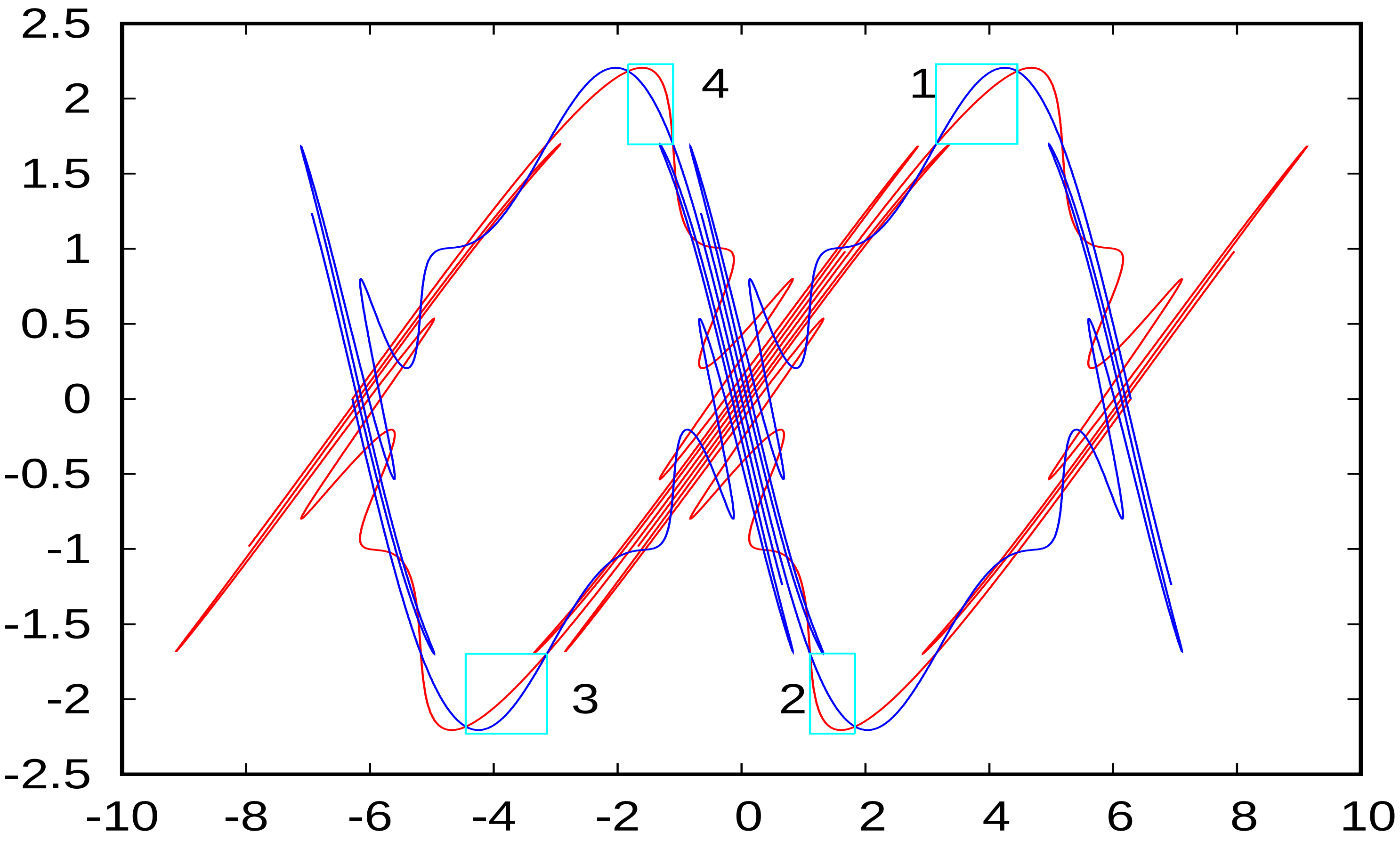
Standard map & $p = 2$ EPL

$$\text{STM}_\epsilon : \ (x, y) \to (\bar{x}, \bar{y} = (x + \epsilon\bar{y}, y + \epsilon \sin(x))$$

It commutes with the central symmetry (the figure-eight loops are symmetric).

To obtain EPL intervals we continue w.r.t. ϵ periodic trajectories of the form:

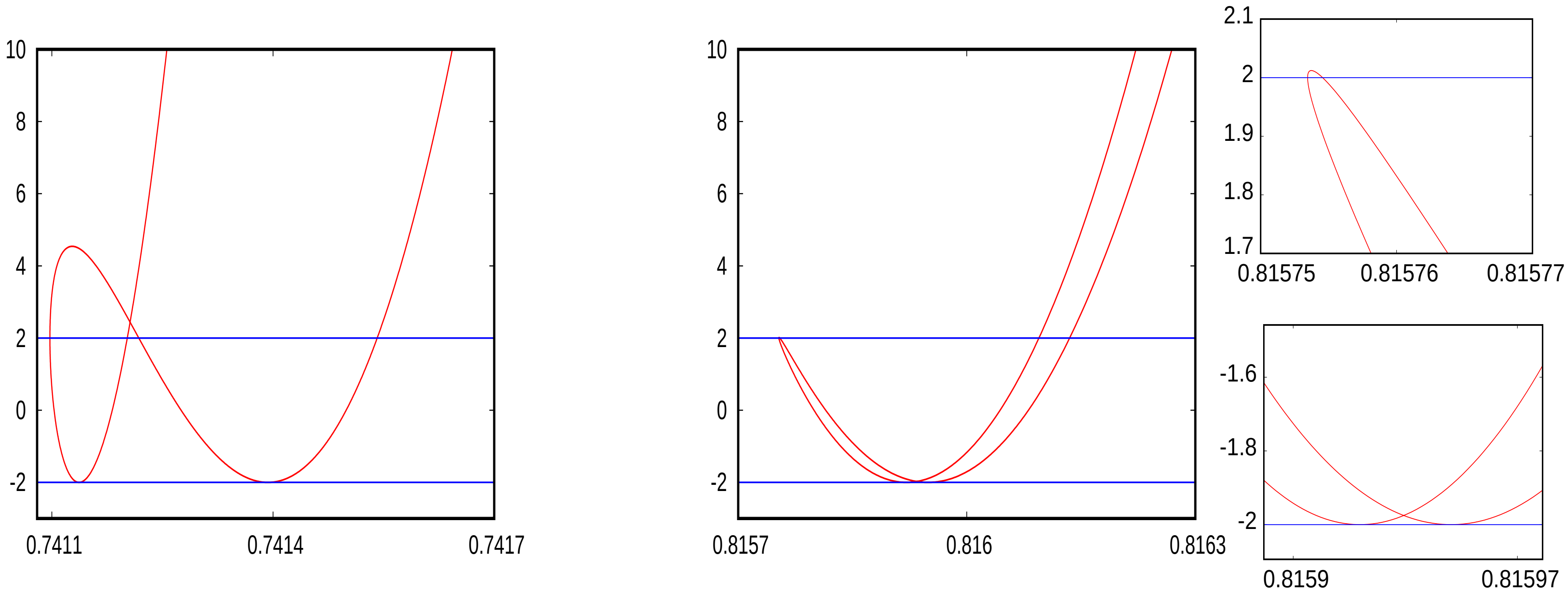


Standard map: ϵ-intervals of EPL

We consider $\epsilon \in (0.7256, 1.18303)$ and we...

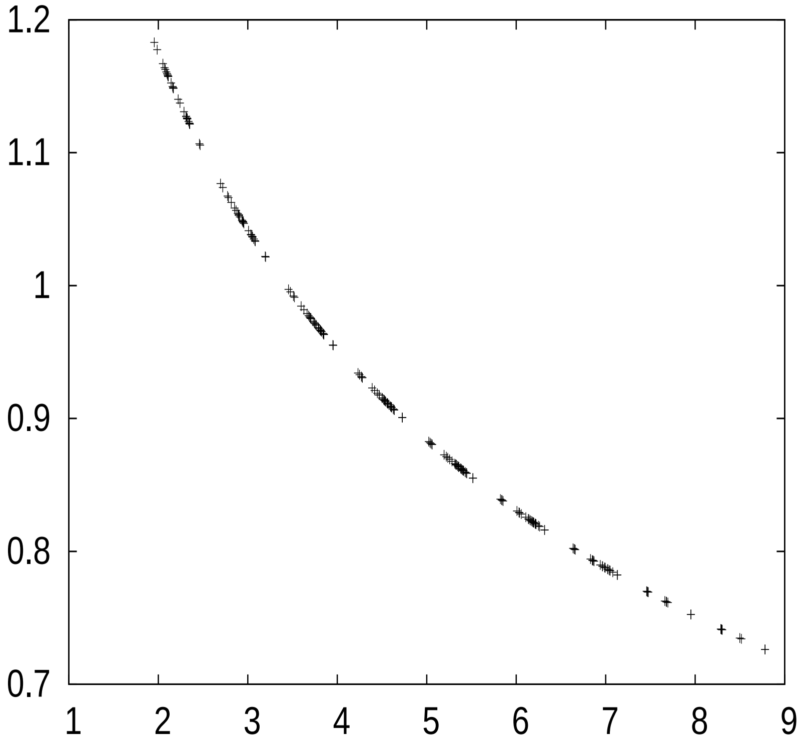
1. scan for initial conditions inside the homoclinic lobe (the central symmetry helps!),
2. refine them (Newton method) to obtain a periodic (tipically highly hyperbolic!) trajectories,
3. continue them to obtain different EPL intervals.

$\longrightarrow$ 223 different ϵ-intervals.



Standard map: a-intervals of EPL

Using $a = a(\epsilon) \approx \frac{\log A(\epsilon)}{\log \lambda(\epsilon)}$ (we ignore $\mathcal{O}(1/\epsilon)$ terms!) we obtain the a-intervals.



a-interval	$m_L(E_{b(\epsilon)})$
$[2.5, 3.3]$	0.06619105
$[3.3, 4.1]$	0.07210729
$[4.1, 4.9]$	0.06797864
$[4.9, 5.7]$	0.07159551
$[5.7, 6.5]$	0.08013797
$[6.5, 7.3]$	0.07146606

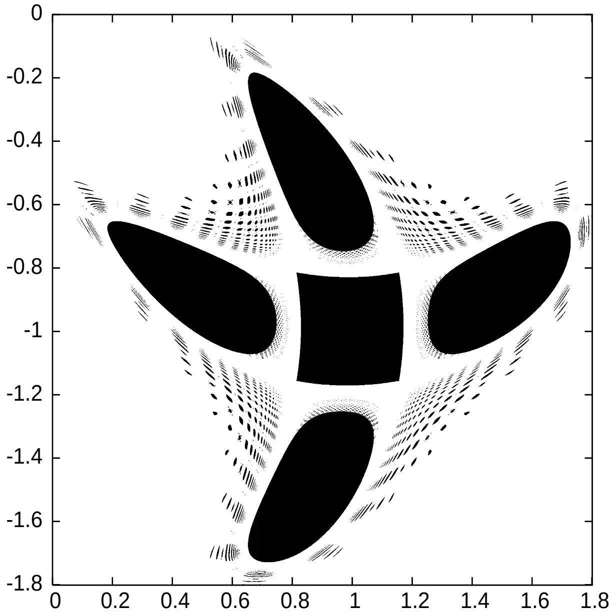
Figure: x-axis: a (without mod 1), y-axis: ϵ, each point corresponds to an EPL a-interval.

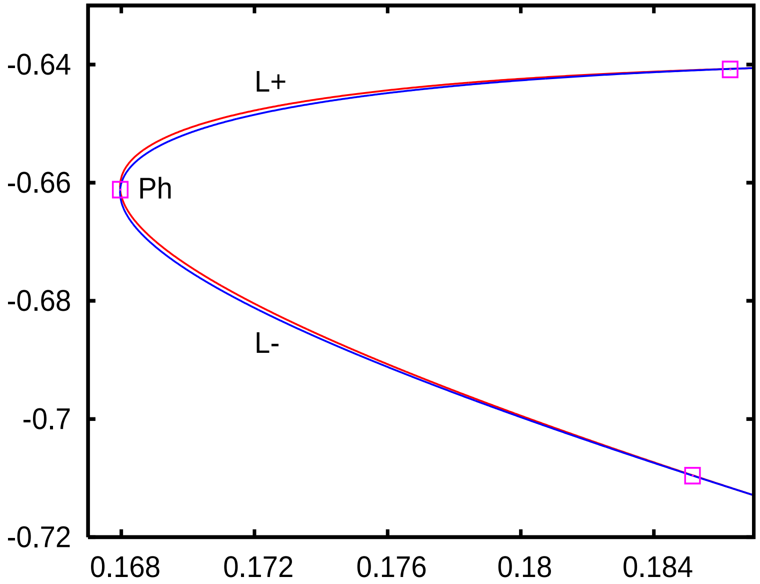
Table: $\sum \Delta a$ for each fundamental interval.

Hénon map

$$H_c : (x, y) \mapsto (c(1 - x^2) + 2x + y, -x)$$

- We focus on the $(1:4)$ resonant islands arising for $c > 1$ (strong resonance!).
- Completely **non-symmetric!**: e.g. for $c = 1.015$ the inner splitting $\mathcal{O}(10^{-54})$ and the outer $\mathcal{O}(10^{-1})$.

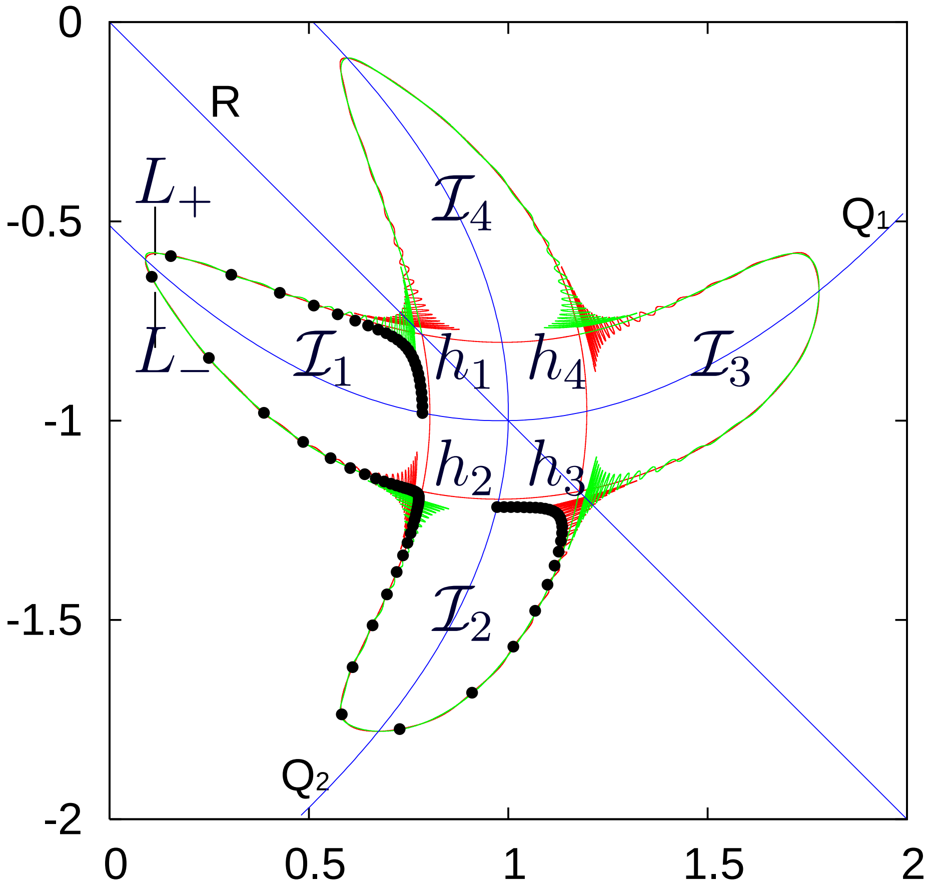


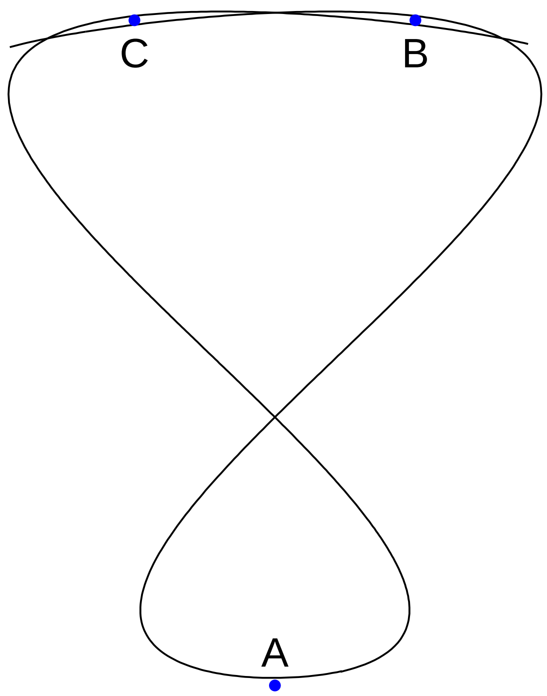


Hénon map: EPL type

Dominant EPL are $\hat{p} = 3$ EPL (non-symmetric, do not visit all hom. lobes).

H_c is reversible w.r.t $R:\ y = -x$ and $Q_1:\ y = c(x^2-1)/2 - x$.





Example: $m = 93$ (i.e. $P = 4m = 742$), we represent iterates of H_c^4.

Hénon map: c-intervals

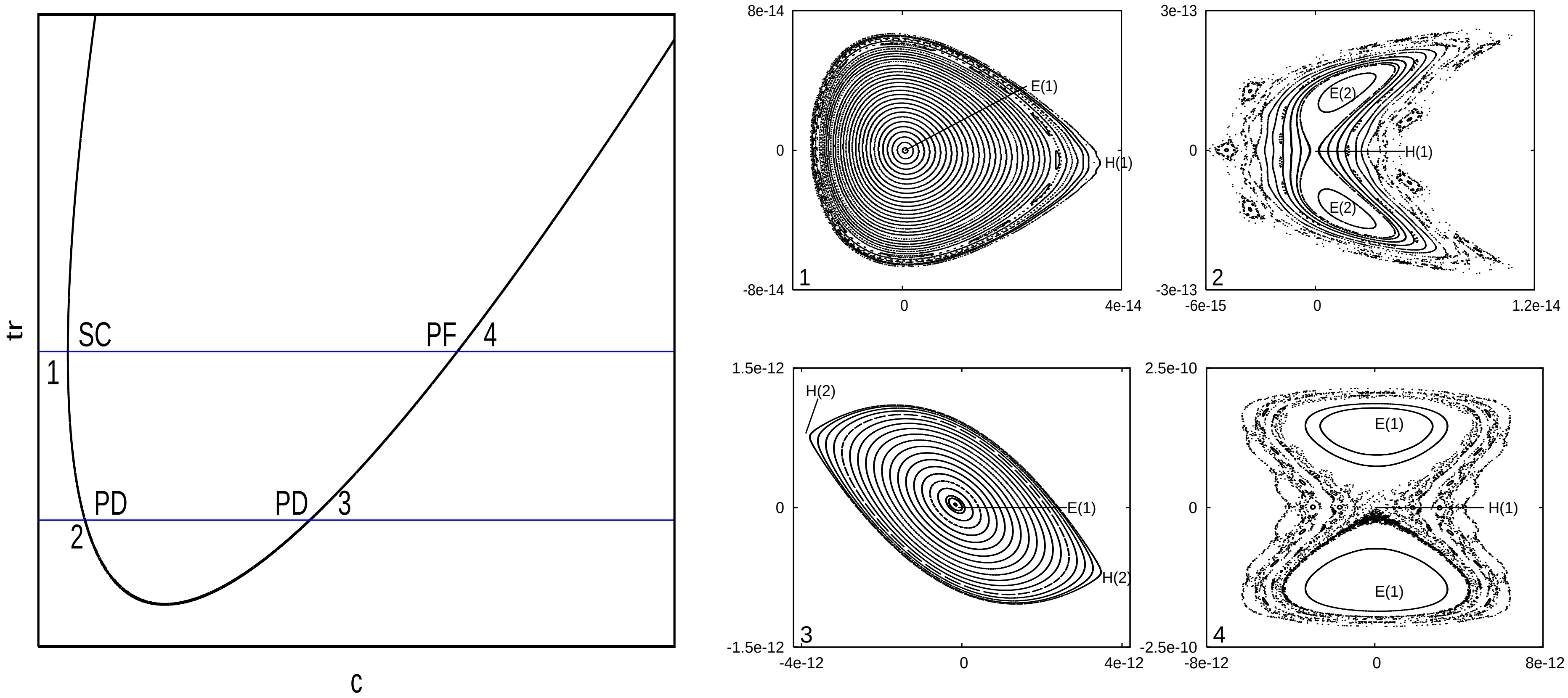
- For $c = 1.02$, we scan for p.o. of the previous type with $P < 1200$
- We found 274896 p.o.
- We continue those with $Tr(DH_c^P) < 10^8$ (2367 initial conditions).
- Numerically observed: each i.c. gives at most two c-intervals.
- We found a total amount of 1989 different c-intervals of stability.
- Sum of the lengths $\approx 7.216 \times 10^{-8}$.
- One pair of c-intervals overlap. Length of the overlapping $\approx 7.82 \times 10^{-12}$.
- Length of the largest (shortest) c-interval obtained $\approx 0.82 \times 10^{-9}$ ($\approx 2.1 \times 10^{-20}$).

$\longrightarrow$ Qualitative agreement but **not** quantitative (far from 5% of the set of parameters).

Hénon map: continuation pattern

General observed pattern: period-doubling bifurcations (non-symmetric!).

Tiny islands (the largest islands, of size 10^{-9}, with shortest period $P = 678$).



$P = 742$ periodic orbit (shown before).

Final comments

Possible explanations for the “non-completely” quantitative agreement in the examples:

- SM$_{a,b}$ only considers first harmonic of the oscillation between W^u/W^s.
- Slope of the EPL strips for the range of parameters considered.
- Approximated relation of a with the parameter of the family (STM example).
- Non-symmetric case: proper model DSM.
- Specific type of EPL considered in the Hénon map example.

Thanks for your attention!!