Quantitative global phase space analysis of APM

Workshop on

Stability and Instability in Mechanical Systems:

Applications and Numerical Tools

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We look for properties of the phase space of an area preserving map (APM) that help in understanding its qualitative structure providing quantitative data.

Part I:

Local and semi-global analysis.

Part II:

Global analysis.

Clearly, these two parts are related.
Object to study

We consider a one-parameter family of maps

\[ F_\delta : \mathcal{U} \to \mathbb{R}^2, \quad \mathcal{U} \subset \mathbb{R}^2 \text{ domain}, \]

such that

1. \( F_\delta \) analytic in the \((x, y)\)-coordinates of \( \mathcal{U} \),
2. \( \det DF_\delta(x, y) = 1 \), for all \((x, y) \in \mathbb{R}^2 \) and for all \( \delta \in \mathbb{R} \),
3. \( F_\delta \) has a fixed point \( E_0 \) that will be assumed to be at the origin for all \( \delta \in \mathbb{R} \),
4. spec \( DF(E_0) = \{ \lambda, \lambda^{-1} \} \), \( \lambda = \exp(2\pi i\alpha) \), \( \alpha = q/m + \delta \), \( q, m \in \mathbb{Z} \).

For some local results it will be assumed \( \delta \) small enough and irrational.
Hénon map

As an example consider the Hénon map

\[ H_\alpha(x, y) = R_{2\pi \alpha}(x, y - x^2), \quad \alpha \in (0, 1/2) \]

- It has two fixed points:
  - the origin is an elliptic fixed point \( E_0 \),
  - the point \( P_h = (2 \tan(\pi \alpha), 2 \tan^2(\pi \alpha)) \) is a hyperbolic fixed point.
- Reversible with respect to \( y = x^2/2 \) and \( y = \tan(\pi \alpha) x \).
1st part

Local and semi-global analysis

Normal form of APM.
Interpolating flow.
Description of resonances.

Well-known

Inner and outer splitting of separatrices.
Strong resonances.

“New”
Given $F$ as before ($\alpha = q/m + \delta$, $\delta$ irrational small), the Birkhoff Normal Form to order $m$ around $E_0$ can be expressed as

$$\text{BNF}_m(F)(z) = R_{2\pi \frac{q}{m}} \left( e^{2\pi i \gamma(r)} z + \bar{z}^{m-1} \right) + R_{m+1}(z, \bar{z}),$$

where

$$\gamma(r) = \delta + b_1 r^2 + b_2 r^4 + \ldots + b_s r^{2s},$$

being

$$z = x + iy, \quad \bar{z} = x - iy, \quad r = |z|, \quad (\text{complex variables})$$

$$s = \left\lfloor \frac{(m - 1)}{2} \right\rfloor,$$

$b_i \in \mathbb{R}$ are the so-called Birkhoff coefficients,

$R_{m+1}(z, \bar{z})$ denotes the remainder which is of $\mathcal{O}(m + 1)$. 

Quantitative global phase space analysis of APM – p.6/35
1. Effect of other resonances.
To get BNF expression it is assumed that the $m$-order resonance cannot be removed but we have removed the others. It can be seen that in a neighbourhood of the $m$ resonance the effect of the others can be ignored (at least if they are of similar order and in a first order approximation).

2. BNF dynamics reduces to near-the-identity map dynamics.

$$\text{BNF}_m(F)(z) = R_{2\pi \frac{q}{m}} \circ K(z, \bar{z}, \delta)$$

with

$$K(z, \bar{z}, \delta) = \exp(2\pi i \gamma(r)) z + i\bar{z}^{m-1} + R_{m+1}(z, \bar{z}).$$

The $m$-jet of $K$ commutes with the rotation $R_{2\pi \frac{q}{m}}$, hence BNF is dynamically equivalent to the near-the identity map $K$. 

Quantitative global phase space analysis of APM – p.7/35
(I, \varphi)-Poincaré variables (z = \sqrt{2I} \exp(i\varphi)).

\[ \mathcal{H}_{nr}(I) = \pi \sum_{n=0}^{s} \frac{b_n}{n + 1} (2I)^{n+1} \quad \text{and} \quad \mathcal{H}_r(I, \varphi) = \frac{1}{m} (2I)^{\frac{m}{2}} \cos(m\varphi). \]

Let \( r_* \) such that \( \gamma(r_*) = 0 \), that is \( r_* \approx (-b_0/b_1)^{1/2}, b_0 = \delta \).

The flow \( \phi \) generated by the Hamiltonian

\[ \mathcal{H}(I, \varphi) = \mathcal{H}_{nr}(I) + \mathcal{H}_r(I, \varphi) \]

interpolates \( K \) with an error of order \( m + 1 \) with respect to the \((z, \bar{z})\)-coordinates, that is,

\[ K(I, \varphi) = \phi_{t=1}(I, \varphi) + \mathcal{O}\left(I^{\frac{m+1}{2}}\right). \]

If we assume \( b_1 \neq 0 \) this approximation holds in an annulus centred in the resonance radius \( r_* \) of width \( r_*^{1+\nu} \), for \( \nu > 0 \).
Description of resonances

Generic case: $\alpha = q/m + \delta$, $m > 5$, $\delta$ sufficiently small, $b_1 \neq 0$.

- If $b_1 \delta < 0$ then $F$ has a resonant island of order $m$.
- The resonant zone is determined by two periodic orbits of period $m$ located near two concentric circumferences (in the BNF variables). The closest orbit to the external circumference is elliptic while the one located close to the inner circumference is hyperbolic.
- The width of the resonant island is $O(I^*/4)$, $I_* = -\delta/2b_1$. 

Quantitative global phase space analysis of APM – p.9/35
Computation of 1st. and 2nd. Birkhoff coefficient.

\[ \alpha = 0.21, \quad b_1 \approx -0.0341669659295153 \quad \text{and} \quad r^* \approx 0.540999411522355. \]

Affected by the near-the-identity change of variables of the normal form computation.
For a generic APM such that $\alpha = q/m + \delta$, $\delta < 0$, $b_1 > 0$, $b_2 \neq 0$, the dynamics around an island of the $m$-resonance strip ($m \geq 5$) can be modelled, after suitable scaling, by the time one map of the flow generated by Hamiltonian

$$\mathcal{H}(J, \psi) = \frac{1}{2} J^2 + \frac{c}{3} J^3 - (1 + dJ) \cos(\psi),$$

where

$$c \approx \frac{b_2}{\sqrt{m\pi b_1^{6+m} \delta^{m/4}}}, \quad d \approx \frac{\sqrt{m}}{2\sqrt{\pi b_1^{m-2} \delta^{m/4 - 1}}}.$$ 

In an annulus domain centred at the radius $I_*$ of width $O(I_*^{m/4})$ the above approximation gives an error $O(I_*^\sigma)$, $\sigma = \min\{m/2 - 2, (m + 2)/4\}$. 
• We have described dynamics by terms of a Hamiltonian flow, and hence, by an **integrable approximation**.

• An estimation of how far is an APM to be integrable is given by the splitting of separatrices in a resonance of the phase space. Clearly, this “**distance-to-integrable**” depends on the zone we are studying the map.

In particular, in a resonant chain of islands there are **two splittings** to be considered: the inner $\sigma_-$ and the outer $\sigma_+$ splittings.
Difference inner-outer splittings

$\alpha = 0.212$, 1:5 resonant chain, Hénon map

Decimal logarithm of the inner (blue) and outer (red) splittings of the 1:7 resonance of the Hénon map.
The splittings characterisation

Assumption: \( \sigma \sim A(\log \lambda)^B \exp(-C_r/\log(\lambda)) \cos(C_i/\log(\lambda)), \)
where \( C = 2\pi i \tau, \) with \( \tau \in \mathbb{C} \) the nearest singularity to the real axis of the separatrix \( \{s(t), t \in \mathbb{C}\} \), of the interpolating Hamiltonian.

\[ F \text{ APM, } \alpha = q/m + \delta, \delta \text{ sufficiently small, } b_1 \neq 0, m \geq 5. \]
\[ \rightarrow \text{Then, the } m \text{-chain of resonant islands, located at a distance } O(\delta), \text{ verifies:} \]

a) The islands of the resonance have, generically, both splittings different.

b) The outer splitting is larger than the inner one being the difference between the position of the corresponding nearest singularities \( O(\delta^{m/4-1}). \)

c) Neither the inner nor the outer splittings oscillate.
From left to right, it is represented the decimal logarithm of the splitting of the resonances 1:9, 1:8, 1:7, 1:5, 2:9, 2:7, 3:8, 2:5, 3:7 and 4:9, respectively. Each pair of red and blue lines corresponds to the outer and inner splitting, respectively, of a different resonance. Note that in all the cases shown the outer splitting (red) is greater than the inner one (blue). In the $x$-axis it is represented the value of $\alpha$. 
The description of the resonant structure by means of the interpolating Hamiltonian does not hold if $m \leq 4$.

**1:3 resonance:**

$$\mathcal{H}(I, \varphi) = \epsilon I + I^2 + I^{\frac{3}{2}} \cos(3\varphi)$$

- Hyperbolic points at a distance $\mathcal{O}(\epsilon^2)$. Elliptic points at a **finite** distance.
- Outer splitting non-perturvature since the separatrices remain at a finite distance.
- Inner splitting behaves as described in the generic case $m > 4$. 
1:4 resonance: $\mathcal{H}(I, \varphi) = \epsilon I + I^2 + \xi I^2 \cos(4\varphi), \quad \xi < 0$. 

- $\epsilon < 0$, $\xi < -1$ left, $-1 < \xi < 0$ right
- $\epsilon > 0$, $\xi < -1$ left, $-1 < \xi < 0$ right

- Elliptic and hyperbolic points located at a distance $O(\epsilon)$.

- Cases with $\xi < -1$: The splitting **oscillates** and behaves as expected in magnitude in the generic case.

- Case $\epsilon < 0$, $\xi > -1$: The splittings behave as expected in the generic case.
Strong resonances of the Hénon map (I)

1:3 resonance:

\[ \alpha = 0.32 \]

\[ \alpha = 0.34 \]

The outer splitting remains finite

\((\alpha = 1/3 \text{ corresponds to } c = \sqrt{2})\):
Strong resonances of the Hénon map (II)

1:4 resonance: Non-generic!!

- It corresponds to the case $\xi = -1$ in the Hamiltonian above.
- The elliptic point goes to a distance $O(\epsilon^{1/2})$ instead $O(\epsilon)$.
- $H(I, \varphi) = \epsilon I + I^2(1 - \cos(\psi)) + I^3(a + b \cos(\psi) + c \sin(\psi))$.
- Hénon corresponds to $\epsilon < 0$, $a + b > 0$. The inner splitting oscillates and the outer does not. There is a big difference inner-outer splitting magnitude (outer singularity at a distance $O((\epsilon(a + b))^{1/4})$, inner singularity real part distance $2\pi$).
Strong resonances of the Hénon map (III)

Decimal logarithm of the inner (red) and outer (blue) splittings as a function of $\alpha$.

- Big difference in the order of the size of the splittings:
  
  For $\alpha \approx 0.25238741368$ it is $\sigma_+ \approx 2.5238741368 \times 10^{-1}$ and $\sigma_- \approx -2.986620731 \times 10^{-59}$.

- The inner splitting oscillates (“peaks”).
Global analysis

Dynamics in a neighbourhood of any resonance.
Dynamics close to separatrices:
  Separatrix map
  Double separatrix map

Dynamics in Birkhoff zones: Biseparatrix map

\{ “Well-known” but... \}

\{ “New” \}
What we mean by global?

From now on (unless the opposite is stated) it will be assumed that we are interested in dynamics within a region containing a resonant chain of islands.

It is not assumed that the resonance is located close to the elliptic fixed point ($\delta$ arbitrary).
A model away from $E_0$

**Question:** How global are the results obtained before?

Dynamics in an annulus containing a $q : m$ resonance far away of the elliptic point $E_0$ can be studied by means of a **perturbation of an integrable twist map**. After reduce the near integrable twist map to normal form and compute the $m$-th iterate to have a near-the-identity map it can be obtained an interpolating Hamiltonian flow. A straightforward computation gives

$$\mathcal{H}(J, \psi) = \frac{J^2}{2} + \frac{cJ^3}{3} - (1 + dJ) \cos(\psi)$$

that is, **the same Hamiltonian** as the one interpolating the $m$ resonance when located in a neighbourhood of the elliptic fixed point $E_0$.

**BUT** the coefficients $c$ and $d$ are arbitrary (and maybe there are higher order $(J)$ coefficients which play relevant role in dynamics).
A model away from $E_0$: splittings

In particular, it cannot be assumed the outer splitting to be larger than the inner when far from the elliptic point.

$$T_\epsilon(I, \theta) = (I + \epsilon \cos(\theta + \alpha(I)), \theta + \alpha(I))$$

$$\alpha(I) = b_1 I + b_2 I^2$$

2:11 Hénon map

$b_1 = 0.2$
$b_2 = 4$
$\epsilon = 0.14$

$b_1 = 6$
$b_2 = -2$
$\epsilon = 0.14$
Dynamics close to separatrices

We distinguish two cases:

Open map.

Separatrix map

Figure eight.

Double separatrix map
Separatrix map

\[ SM : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + a + b \log(y') \\ y + \sin(2\pi x) \end{pmatrix} \]

- Describes the dynamics in a close neighbourhood of the separatrices emanating from a hyperbolic point \( H \).
- \( a \) is related with a shift needed to get the image in the fundamental domain (“no dynamical relevance”).
- \( b = -1/\log(\lambda) \), \( \lambda \) is the eigenvalue of modulus greater than one of \( H \).
- The \( y \) variable is rescaled by the amplitude of the splitting.
Approximating SM by the Chirikov standard map it is obtained:

→ distance to expect rotational invariant curves:
  → from the stable separatrix: \( d_c \sim \frac{|b|}{k^*} \), \( k^* \) Greene
  → from the hyperbolic point: \( d_{hc}^h \sim \sqrt{\frac{|b|}{k^*}} \)

→ distance to expect islands from the hyperbolic point: \( d_i \sim \sqrt{\frac{b \pi}{2}} \)

Hénon map \((\alpha = 0.1)\), hyperbolic fixed point.

Observed: \( d_{hc}^h \approx 3.2 \times 10^{-3} \), \( d_i \approx 2 \times 10^{-3} \)

Formulas above:

\( P_h \approx (0.64983939, 0.21114562) \)

\( \lambda_+ \approx 1.83785279 \)

\( \sigma \approx 1.19 \times 10^{-5} \)

\( d_{hc}^h \approx 1.12 \times 10^{-2} \), \( d_i \approx 5.5 \times 10^{-3} \)
**Double separatrix map**

\[ DSM : \begin{pmatrix} x \\ y \\ s \end{pmatrix} \mapsto \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{s} \end{pmatrix} = \begin{pmatrix} x + a + b \log |\bar{y}| \pmod{1} \\ y + \nu \bar{s} \sin 2\pi x \\ \operatorname{sign}(\bar{y}) \bar{s} \end{pmatrix} \]

- \(a\) i \(b\) parameters defined as before.
- \(s = 1\) outer separatrix domain \(U\) and \(s = -1\) inner separatrix domain \(D\).
- \(\nu_1 = 1\) and \(\nu_{-1} = A_{-1}/A_1\), where \(A_1\) and \(A_{-1}\) are the amplitudes of the outer and inner splittings respectively of the resonant island.
DSM: invariant curves

- **Inv. curves outside island:** DSM reduce to SM and above formulas hold.

- **Inv curves inside island:**

  **IDEA:** Both inner and outer separatrices play a role.

Assume that the dynamics of \( F \) inside the “pendulum” like island is modelled by the time one flow of an interpolating Hamiltonian \( \mathcal{H}(J, \phi) \). Let \( J = J_m \) be the action on the separatrix in the inner domain and \( J_M \) be the action on the separatrix in the outer domain. Put

\[
f = \frac{\nabla H(J_m)}{\nabla H(J_M)}.\]

→ Then, a distance \( d \) measured with respect the outer separatrix becomes a distance \( f d \) with respect the inner one.
1:4 resonance Hénon \((H_c(x, y) = ((1 - x^2)c + 2x + y, -x), c = 1.015)\).

\[\lambda \approx 1.1284291, \sigma_+ \sim 10^{-54}, \sigma_- \sim 10^{-1}.\]

Outside (inner) island: \(d_c \approx 10^{-52}\).

Inside. Interpolating flow of \(H_4^c\) given by
\[H(x, y) = H_0 + \delta H_1 + \delta^2 H_2,\]
with \(\delta = 2\pi\alpha - \pi/2\) and

\[
\begin{align*}
H_0 &= x^2y^2 - x^4y - xy^4 + x^6/3 + 2x^3y^3 + y^6/3 - x^5y^2 - x^2y^5 - 5x^4y^4/6, \\
H_1 &= -2x^2 - 2y^2 + 2x^2y + 2xy^2 - x^4 - 2x^3y - y^4 + x^5 - 2x^3y^2 + 2x^2y^3 + 2x^5y - 5x^4y^2/3 + 13x^2y^4/3, \\
H_2 &= -2x^3 + 4xy^2 - x^4/3 - 4x^3y + x^2y^2/2 - 4y^4/3.
\end{align*}
\]

The value of \(\nabla H\) in the maximum (outer zone) of the separatrices oscillates between 0.0086 and 0.0098 depending on the island considered. On the other hand, the corresponding value in the minimum (inner zone) is \(\approx 0.00066\). Then \(f\) is between 13 and 15 which coincides with what is observed in the figure.
Let $F$ be an APM having an elliptic fixed point with rotation number $\alpha = q/m + \delta$, $q, m \in \mathbb{Z}$, $\delta \in \mathbb{R} \setminus \mathbb{Q}$.

Denote by $b_1 \in \mathbb{R}$ the first Birkhoff coefficient of the normal form of $F$ around the elliptic point and assume $b_1 \delta < 0$.

Then, for $|\delta|$ small enough, the width of the chaotic outer zone is larger than the width of the inner one if, and only if, $\text{sign} b_1 \cdot \text{sign} b_2 > 0$. Both amplitudes of the stochastic layer are of the same order of magnitude of the outer splitting.
Let $F$ be an APM. A **Birkhoff zone of instability** is a rotational non-contractile annulus without rotational invariant curves.

Assume we are interested in the dynamics between two concentric chains of islands. Let $d$ denote the distance between them. A simple model is given by the **biseparatrix map**

$$BSM : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} u + \alpha + \beta_1 \log(v') - \beta_2 \log(d - v') \\ v + \sin(2\pi u) \end{pmatrix}$$

$\beta_1 = 1/\log(\lambda), \beta_2 = 1/\log(\mu), \lambda$ and $\mu$ eigenvalues of modulus greater than one associated to the hyperbolic points of each chain of islands.

**Just qualitative but...**
BSM: twist case

Chirikov standard map with \( k = 0.16 \).

For the corresponding BSM model:

\[
\beta_1 = \beta_2 \approx 1.0365 \\
(\lambda = \mu \approx 2.624248)
\]

Amplitude splitting outer island \( \approx 1.2 \times 10^{-2} \)

Amplitude splitting inner island \( \approx 1.5 \times 10^{-2} \)

Distance between the islands \( \approx 0.424 \)

\[ \Rightarrow d \text{ between } 28.2 \text{ and } 35.4 \]
For APM it can be zones without rotational invariant curves but where the twist vanished. $F_b(x, y) = (\bar{x}, \bar{y}) = (x + \epsilon(y^2 - b), y + \epsilon \sin x)$

$$BSM : \begin{pmatrix} u \\ v \end{pmatrix} \rightarrow \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} u + \alpha + \beta_1 \log(v') + \beta_2 \log(d - v') \\ v + \sin(2\pi u) \end{pmatrix}$$
The End

Thank you!