Abundància d’òrbites periòdiques estables dins dels lòbuls homoclínics

Trobad de Joves Investigadors de la Societat Catalana de Matemàtiques

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I. Introduction.
  What we want to study?

II. Preliminary background.
  The separatrix map (SM). The Simó-Treschev result (2008).

III. Analytical results
  “Symmetric” SM. Main result.

IV. Numerical computations
  Standard map. Hénon map (1:4 resonance).
**Splitting of separatrices + chaotic zone**

Consider an APM $F$ with a hyperbolic fixed point $H$. Generically, the separatrices of $H$ split and create a chaotic zone (CZ) which extends up to the “outermost” invariant curve.
The dynamics within the chaotic zone...

... is not ergodic: “rel. far” from the separatrices there are islands inside CZ.

\[ H_\alpha(x, y) = R_{2\pi\alpha}(x, y - x^2), \alpha = 0.1. \]

Experimental values: 
- \( D_c^H \approx 2.94 \times 10^{-3} \)
- \( D_i^H \approx 2.08 \times 10^{-3} \)

“Fish” interp. Hamiltonian:
- \( D_c^H \approx 2.47 \times 10^{-3} \)
- \( D_i^H \approx 1.85 \times 10^{-3} \)

5-order interp. Hamiltonian:
- \( D_c^H \approx 2.731 \times 10^{-3} \)
- \( D_i^H \approx 2.050 \times 10^{-3} \)

Main idea: SM (and STM aprox.) + Interp. Ham.  

\(^a\) Simó-V. Dynamics in chaotic zones of area preserving maps: close to separatrix and global instability zones. Physica D, 240(8), 2011.
Dynamics within the homoclinic lobes

It looks like chaotic...  

...but, inside the homoclinic lobes, one finds tiny islands of stability:  

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Goal of this work I

Instead of a single APM $F$, we consider a one-parameter family of APMs $F_\epsilon$.

$\rightarrow \quad \epsilon$ – distance-to-integrable parameter

We are interested in the elliptic periodic orbits visiting homoclinic lobes (EPL) of the lowest possible period ("dominant") of $F_\epsilon$ for $\epsilon \ll 1$.

For analytical results: we assume "central symmetry" of $F_\epsilon$ and use the separatrix map (SM) to...

- ... study the abundance of EPL (i.e. the relative measure of the set $E_\epsilon$ of $\epsilon$-parameters for which $F_\epsilon$ has EPL).
- ... describe the pattern of creation/destruction/bifurcation of these EPL in terms of the parameter $\epsilon$.
- ... obtain an (explicit!) accurate estimate of the $m(E_\epsilon)$.

$\rightarrow$ “maybe nice theory”... but, moreover,...
... we want to compare the theoretical results with “real” situations.

To this end, we perform accurate numerical computations to obtain estimates of $m(E_\epsilon)$. In the numerical experiments we will consider as $F_\epsilon$ maps like the standard map (STM) and the Hénon map.

→ Note that a “real” situation does not necessarily fit within “our” theoretical framework (typically, one simplifies the model, uses a perturbative approach,...).
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**Non-symmetric figure-eight**

The figure-eight loops maybe non-symmetric!

**Example of interest:** resonant islands emanating from a fixed elliptic point.

Let $F_\delta$ be a one-parameter family of APMs, $F_\delta(E_0) = E_0$ elliptic f.p.,
dynamics around the $(q : m)$-resonance, $m \geq 5$, $(1 \leq q < m, (q, m) = 1)$.

Spec($DF_\delta(E_0)$) = \{\lambda, \lambda^{-1}\}, \lambda = \exp(2\pi i \alpha), \alpha = q/m + \delta, \delta \in \mathbb{R}.

**Thm.** Under generic assumptions: outer splitting $>\text{ inner splitting.}$

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*Simó-V. Resonant zones, inner and outer splittings in generic and low order resonances of area preserving maps.

Double separatrix map (figure-eight)

\[ \begin{pmatrix} x \\ y \\ s \end{pmatrix} \mapsto \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{s} \end{pmatrix} = \begin{pmatrix} x + a_s, \bar{s} + b \log |\bar{y}| \pmod{1} \\ y + \nu, \bar{s} \sin 2\pi x \\ \text{sign}(y) s \end{pmatrix}, \]

- Defined on a domain \( W = U \cup D \) (fundamental domains around the outer/inner separatrices).
- \( a_s, \bar{s} \) suitable “shifts” (reinjection to \( W \)).
- \( b = 1 / \log(\lambda), \lambda \) dominant eigenvalue of \( H \).
- \( y \)-variable rescaled: \( \nu_1 = 1 \) and \( \nu_{-1} = A_{-1}/A_1 \), where \( A_1 \) (resp. \( A_{-1} \)) is the amplitude of the outer (resp. inner) splitting.
A priori stable/unstable cases

Recall that we want to study EPL of $F_\epsilon$, $\epsilon$ dist-to-integr. param., $F_0$ integrable.

**A priori unstable:** $F_0$ has a non-degenerated hyperbolic fixed point $H_0$ s.t. $\lambda(0) > 1$. Then $\lambda(\epsilon) = \lambda(0) + O(\epsilon^r), r > 0$. The separatrices of $H$ form an integrable figure-eight.

![Integrable Figure-Eight](image)

**A priori stable:** $F_0$ has a degenerated fixed point (e.g. we encounter a line of fixed points for $\epsilon = 0$). Then $\lambda(\epsilon) = 1 + O(\epsilon^r), r > 0$.

**Remark:** Islands emanating from a fixed elliptic point $\rightarrow$ a priori stable case. All the examples we deal with fit within the a priori stable framework!
A priori stable/unstable differences

- Size (width) of the homoclinic lobes.
  
  (i) a priori unstable: \( \mathcal{A}_\epsilon = \mathcal{O}(\epsilon^r), \quad r > 0 \)
  
  (ii) a priori stable: \( \mathcal{A}_\epsilon = \mathcal{O}(\exp(-c/\epsilon^r)), \) with \( r, c > 0 \) constants.

- Relation \( F_\epsilon \leftrightarrow \text{SM}_{a,b} \).
  
  (i) a priori unstable: \( a = \mathcal{O}(-\log \epsilon), \quad b = \mathcal{O}(1) \),
  
  (ii) a priori stable: \( a = \mathcal{O}(1/\epsilon^{2r}), \quad b = \mathcal{O}(1/\epsilon^r) \),

Remarks:

- Case (i): \( a, b \) change “independently” (\( a \) changes with \( \epsilon \)).

- Case (ii): Both \( a \) and \( b \) depend on \( \epsilon \). But \( b'(a) \approx \epsilon^r \to 0 \) as \( \epsilon \to 0 \) (i.e. \( a \) changes faster with respect to small variations of \( \epsilon \)).
Simó-Treschev result

$F_\epsilon$ – a priori unstable family of APMs

$E_\epsilon$, $\epsilon < \epsilon_0 << 1$ – set of $\epsilon$-parameters for which $F_\epsilon$ has EPL

Thm. $m(E_\epsilon)$, when $\epsilon_0 \rightarrow 0$, remains greater than a constant $K > 0$ independent of $\epsilon$.  

Comments:

• It does not provide any approximation of $m(E_\epsilon)$.

• It is enough to prove the existence of one EPL for some concrete $a$ and $b$ values of the DSM. Then, using a specific scaling of the SM, one obtains an EPL for values $\epsilon \rightarrow 0$.

• This scaling holds because $b$ is indep. of $\epsilon$ (a priori unstable)

scaling idea: $\epsilon_2 = \epsilon_1/\lambda^{1/r} \Rightarrow a(\epsilon_2) \approx a(\epsilon_1) \mod 1$

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Central symmetry

We assume that $F_\epsilon$ commutes with the central symmetry with respect to $H_\epsilon$. This implies:

1. The figure-eight loops are symmetric.
2. The lowest possible period for an EPL is $\hat{p} = 4$.

Non-symmetric $\hat{p} = 3$ EPL                  Symmetric $\hat{p} = 4$ ($p = 2$) EPL
We can then identify both domains of definition of the DSM and consider a simple model

\[
\text{SM}_{a,b} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x + a + b \log |y_1| \\ y + \sin(2\pi x) \end{pmatrix}
\]

**Motivation:** For generic (non-strong) res. islands emanating from an elliptic fixed point, the “lack of symmetry” is detected in a “second order” approximation of the dynamics, which can be described by the Hamiltonian:

\[
\mathcal{H}(J, \psi) = \frac{1}{2} J^2 + \frac{c}{3} J^3 - (1 + dJ) \cos(\psi), \quad c = \mathcal{O}(\delta^{m/4}), \quad d = \mathcal{O}(\delta^{m/4-1}).
\]

**Rec:** If the multiplier of the elliptic point is \( \alpha = q/m + \delta \), the \( m \)-resonant islands are located at \( I_* = \mathcal{O}(\delta) \) and have a width \( \mathcal{O}(\delta^{m/4}) \). Then \( J, \psi \) are adapted coordinates around the \( m \)-island. Strong resonances have also been studied in some of the cited papers.
Main result

Assume $F_\epsilon$ a priori stable + central symmetry  
$\Rightarrow$ we use $SM_{a,b}$ to describe dynamics within the homoclinic lobes.

**Idea:** For a fixed $b$ we look for the measure of the set of maps (depending on $a \in [0, 1)$) having EPL of period $p = 2$ ($\hat{p} = 4$).

**Thm.** For a fixed $b$, let $\sum \Delta a$ denote the sum of the lengths of the intervals $\Delta a = (a_-, a_+)$ such that for $a \in \Delta a$ the separatrix map $SM_{a,b}$ has a $p = 2$ EPL. Then,  
$$
\lim_{b \to +\infty} \sum \Delta a = \frac{1}{2\pi^2} \approx 0.05066.\
$$

**Rec:** $a = a(\epsilon)$ and $b = b(\epsilon)$, but $a$ changes quickly!

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Transversality: EPL strips & \((a, b)\)-curve of \(F_\epsilon\)

- \(b\) large enough (integrable limit, \(b = \mathcal{O}(1/\epsilon)\)).
- Each \(p = 2\) EPL strip is related to different periodic \(P\) trajectory of \(F_\epsilon\).
- \(F_\epsilon\) defines a curve \(C\) which intersects transversally the EPL strips.
Overlapping

Each periodic $P$ trajectory of $F_\epsilon$ gives two $a$-intervals of EPL. For $P$ rel. small, elementary overlaps between these $a$-intervals occur. Skipping these overlaps: $\lim_{b \to +\infty} \sum \Delta a = \frac{1}{2\pi^2} (1/2 + \log(3/2)) \approx 0.04587$. Additional overlaps of tiny intervals related to large period are disregarded.

**Numerical check for the SM:** $x$-axis: $-\log(b)$, $y$-axis: $\sum \Delta a$. 

![Graph with overlapping](image1.png)

![Graph removing “all” overlappings](image2.png)
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Standard map & $p = 2$ EPL

$$\text{STM}_\epsilon : (x, y) \rightarrow (\bar{x}, \bar{y} = (x + \epsilon \bar{y}, y + \epsilon \sin(x)))$$

It commutes with the central symmetry (the figure-eight loops are symmetric).

To obtain EPL intervals we continue w.r.t. $\epsilon$ periodic trajectories of the form:
Standard map: $\epsilon$-intervals of EPL

We consider $\epsilon \in (0.7256, 1.18303)$ and we...

1. scan for initial conditions inside the homoclinic lobe (the central symmetry helps!),

2. refine them (Newton method) to obtain a periodic (typically highly hyperbolic!) trajectories,

3. continue them to obtain different EPL intervals.

$\rightarrow$ **223 different $\epsilon$-intervals.** $Tr(DT^P)$ is plotted as a function of $\epsilon$. 

![Graphs showing $Tr(DT^P)$ as a function of $\epsilon$.]
Standard map: $\alpha$-intervals of EPL

Using $a = a(\epsilon) \approx \frac{\log A(\epsilon)}{\log \lambda(\epsilon)}$ (we ignore $O(1/\epsilon)$ terms!) we obtain the $\alpha$-intervals.

![Graph showing $\alpha$ and $\epsilon$ intervals.]

<table>
<thead>
<tr>
<th>$\alpha$-interval</th>
<th>$m_L(E_b(\epsilon))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[2.5, 3.3]</td>
<td>0.06619105</td>
</tr>
<tr>
<td>[3.3, 4.1]</td>
<td>0.07210729</td>
</tr>
<tr>
<td>[4.1, 4.9]</td>
<td>0.06797864</td>
</tr>
<tr>
<td>[4.9, 5.7]</td>
<td>0.07159551</td>
</tr>
<tr>
<td>[5.7, 6.5]</td>
<td>0.08013797</td>
</tr>
<tr>
<td>[6.5, 7.3]</td>
<td>0.07146606</td>
</tr>
</tbody>
</table>

Figure: $x$-axis: $\alpha$ (without mod 1), $y$-axis: $\epsilon$, each point corresponds to an EPL $\alpha$-interval.

Table: $\sum \Delta \alpha$ for each fundamental interval.
Hénon map

\[ H_c : (x, y) \mapsto (c(1 - x^2) + 2x + y, -x) \]

- We focus on the \((1 : 4)\) resonant islands arising for \(c > 1\) (strong resonance!).
- Completely **non-symmetric!**: e.g. for \(c = 1.015\) the inner splitting \(\mathcal{O}(10^{-54})\) and the outer \(\mathcal{O}(10^{-1})\).
Dominant EPL are $\hat{p} = 3$ EPL (non-symmetric, do not visit all hom. lobes).

$H_c$ is reversible w.r.t $R: y = -x$ and $Q_1: y = c(x^2 - 1)/2 - x$.

Example: $P = 742$ (we represent $m = 93$ iterates of $H_c^4$).
Hénon map: \(c\)-intervals

- For \(c = 1.02\), we scan for p.o. of the previous type.
- We find 274896 p.o. with \(P < 1200\).
- We continue those with \(Tr(DH_c^P) < 10^8\) (2367 initial conditions).
- Numerically observed: each i.c. gives at most two \(c\)-intervals.
- We find a total amount of 1989 different \(c\)-intervals of stability in 
  \([c_m, c_M] = [1.0198, 1.02]\).
- Sum of the lengths \(\approx 7.216 \times 10^{-8}\).
- One pair of \(c\)-intervals overlap. Length of the overlapping \(\approx 7.82 \times 10^{-12}\).
- Length of the largest (shortest) \(c\)-interval obtained \(\approx 0.82 \times 10^{-9}\)
  \((\approx 2.1 \times 10^{-20})\).

\[\rightarrow\] Qualitative agreement but **not** quantitative.
Hénon map: continuation pattern

General observed pattern: period-doubling bifurcations (non-symmetric!).
Tiny islands (the largest islands, of size $10^{-9}$, with shortest period $P = 678$).

$P = 742$ periodic orbit (shown before).
Final comments

Possible explanations for the “non-completely” quantitative agreement in the examples:

- $\text{SM}_{a,b}$ only considers first harmonic of the oscillation between $\frac{W^u}{W^s}$.
- Slope of the EPL strips for the range of parameters considered.
- Approximated relation of $a$ with the parameter of the family (STM example).
- Non-symmetric case: proper model DSM.
- Specific type of EPL considered in the Hénon map example.
Moltes gràcies per la vostre atenció!!