ENOC'05

A Numerical Exploration of Weakly Dissipative Two-Dimensional Maps

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Goal

To illustrate the effect of adding a weakly dissipation on a conservative map.

- Understand the global behaviour around an elliptic fixed point.
- Describe the structure of the resonances / Geometry of the invariant manifolds.
- Transport properties / Probability of capture in a resonance.
- Identify the regions where the topology of the resonances gives rise to different dynamics.
- Illustrate some “limit” properties when the dissipation goes to zero.

The model

We use in computations a dissipative version of the classical conservative Hénon map. Concretely,

$$H_{\alpha,\epsilon}(x, y) = (1 - \epsilon)H_\alpha(x, y), \tag{1}$$

where

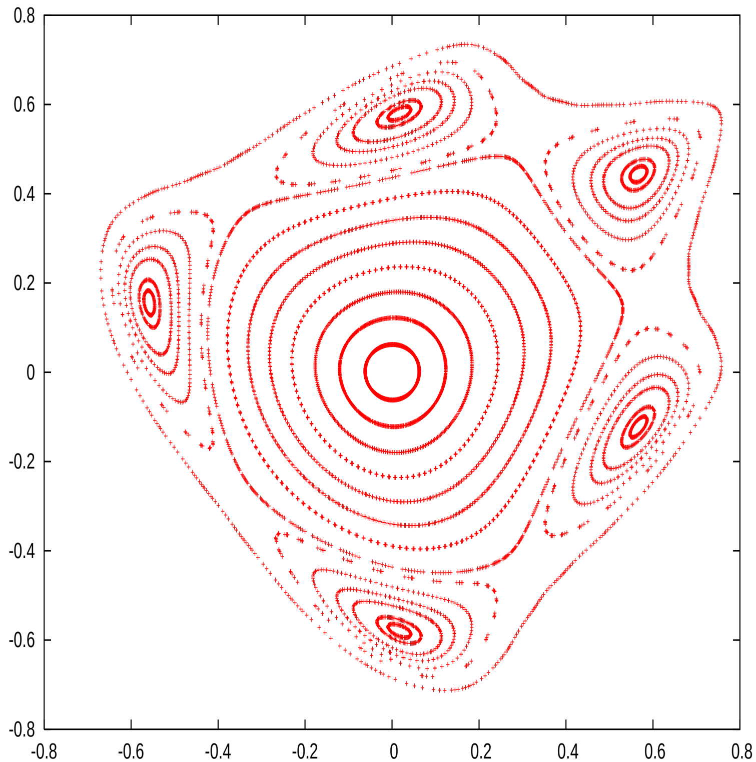
$$H_\alpha : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto R_{2\pi\alpha} \begin{pmatrix} x \\ y - x^2 \end{pmatrix},$$

or $(x, y) \mapsto (1 - ax^2 + y, -bx)$ with suitable $a, b \approx 1$.
Motivation:

- "Simplest" planar map. Composition of two simple reversors.
- Appears when modelling a saddle-node bifurcation.

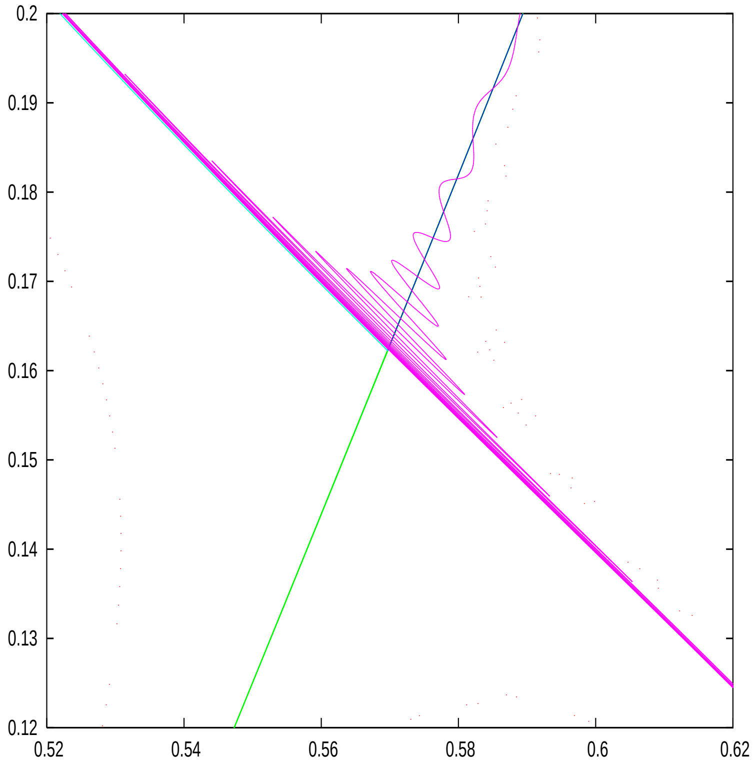
Conservative case ($\epsilon = 0$): global idea

Generically, around an elliptic point it is expected to have an infinite number of resonances characterised by sequences of islands formed by the invariant manifolds of the hyperbolic periodic points and containing an elliptic point. The size of the perturbation from integrable depends on the domain.



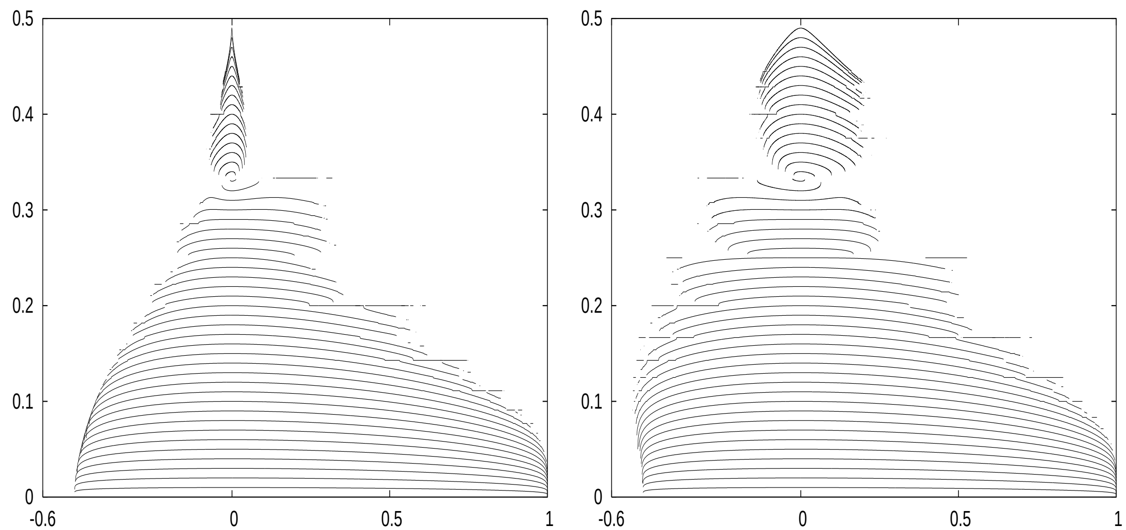
Conservative case: in detail

Moreover, if the map is non-integrable, the invariant manifolds do not coincide and a splitting of the separatrices is created. There are known generic upper bounds of the splitting depending on the parameters of the map.



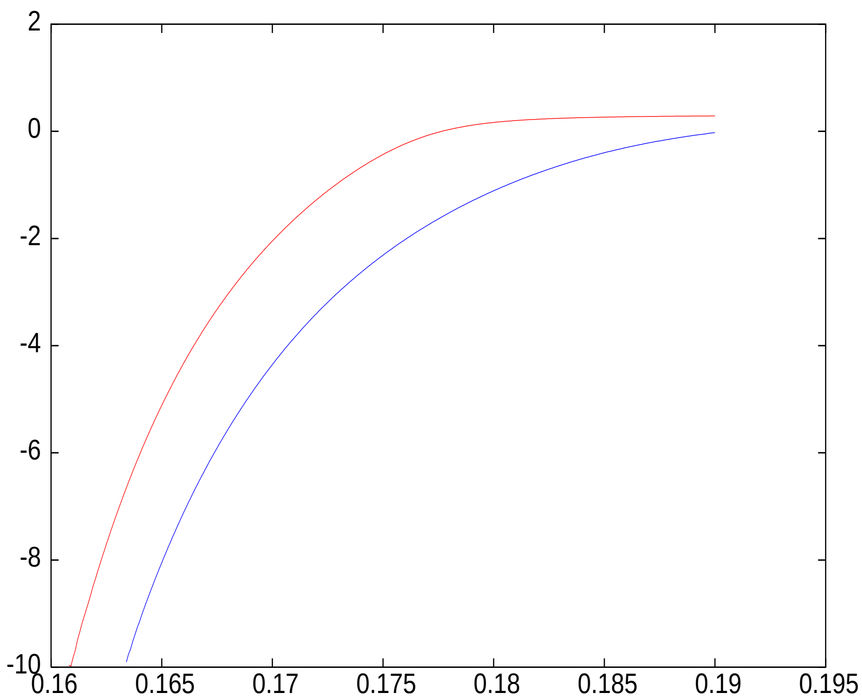
Conservative case: rotation number

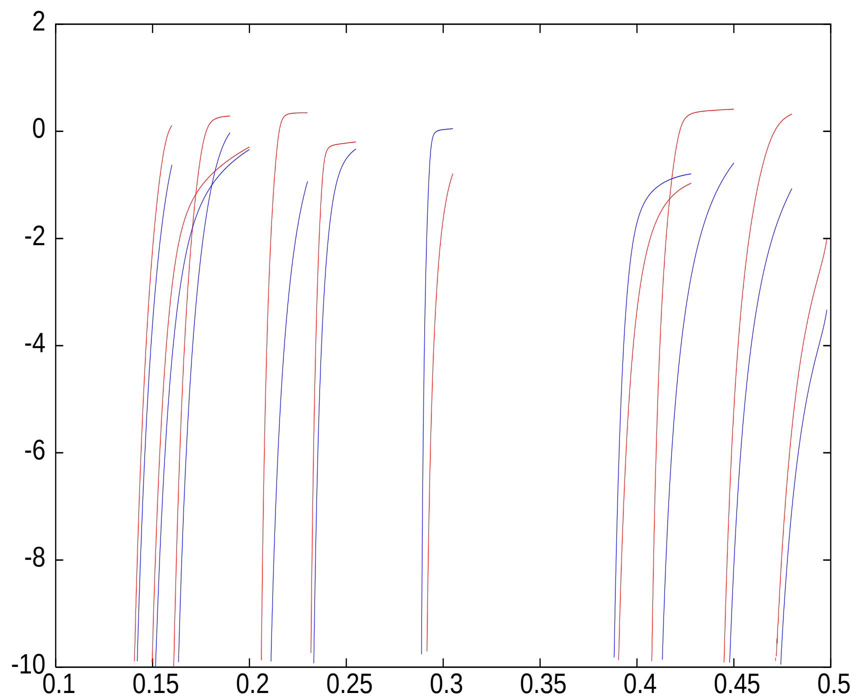
Generically, in a neighbourhood of an elliptic fixed point a twist condition holds, that is, the rotation number is a monotone function of the action. Nevertheless, far from that point the twist condition can violated. Close to the value of the radius where the twist condition does not hold the rotational invariant curves give rise to meandering curves.



Conservative case: Splitting

In general, for a generic APM, the inner and the outer splittings of the same island are different. It depends on the derivatives (mainly on the sign of the first two derivatives) of the rotation number with respect to the radial coordinate (that is, on the torsion coefficient and its first derivative). In the plots: $\log_{10}(\text{splitting angle})$ vs α.





Weak Dissipation

Previous considerations:

- For ϵ “big” enough the dynamics collapses to the origin and no resonances outlast the dissipation.
- The periodic points which configure a concrete resonance should be destroyed, when ϵ increases, as a result of saddle-node bifurcation.
- The destruction of a resonance depends on the width and order it has (a priori).
- The resonances should allow to pass more points as ϵ increases.

First critical radius

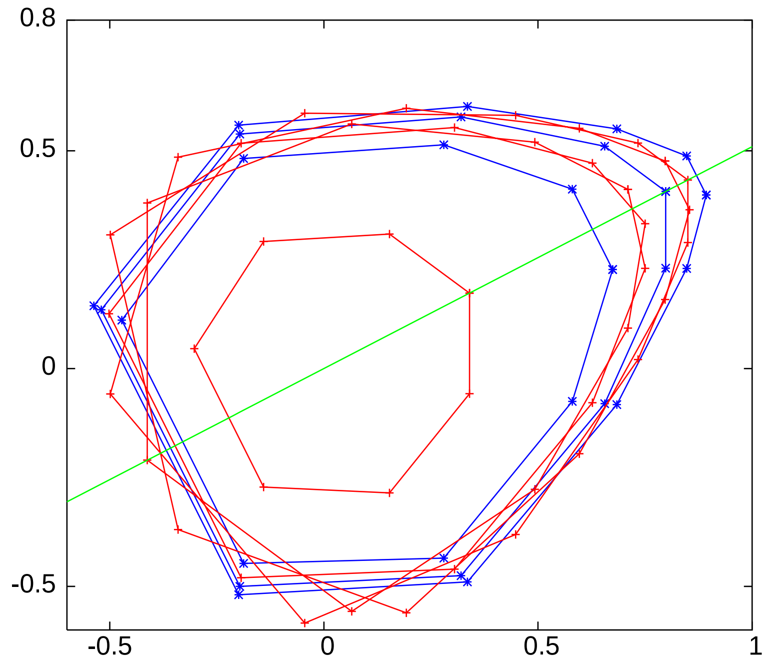
- We expect the small resonances to be destroyed by the dissipation.
- Due to the twist condition, the resonances are arranged by rotation number.
- The width of an m-resonance is of the order $\mathcal{O}(I_*^{m/4})$, where $I_* = -\delta/2b_1$ being $\alpha = q/m + \delta$ and b_1 the first Birkhoff coefficient.

Conclusion:

Close to the origin we can expect a neighbourhood where no resonance survives.

First critical radius: example

$\log_{10}(\epsilon)$	**Res. destroyed**
-6	Inside $B_0(0.27)$
-4.569	(2:19)
-4.625	(1:7)
-3.456	(1:8)
-3.297	(1:9)



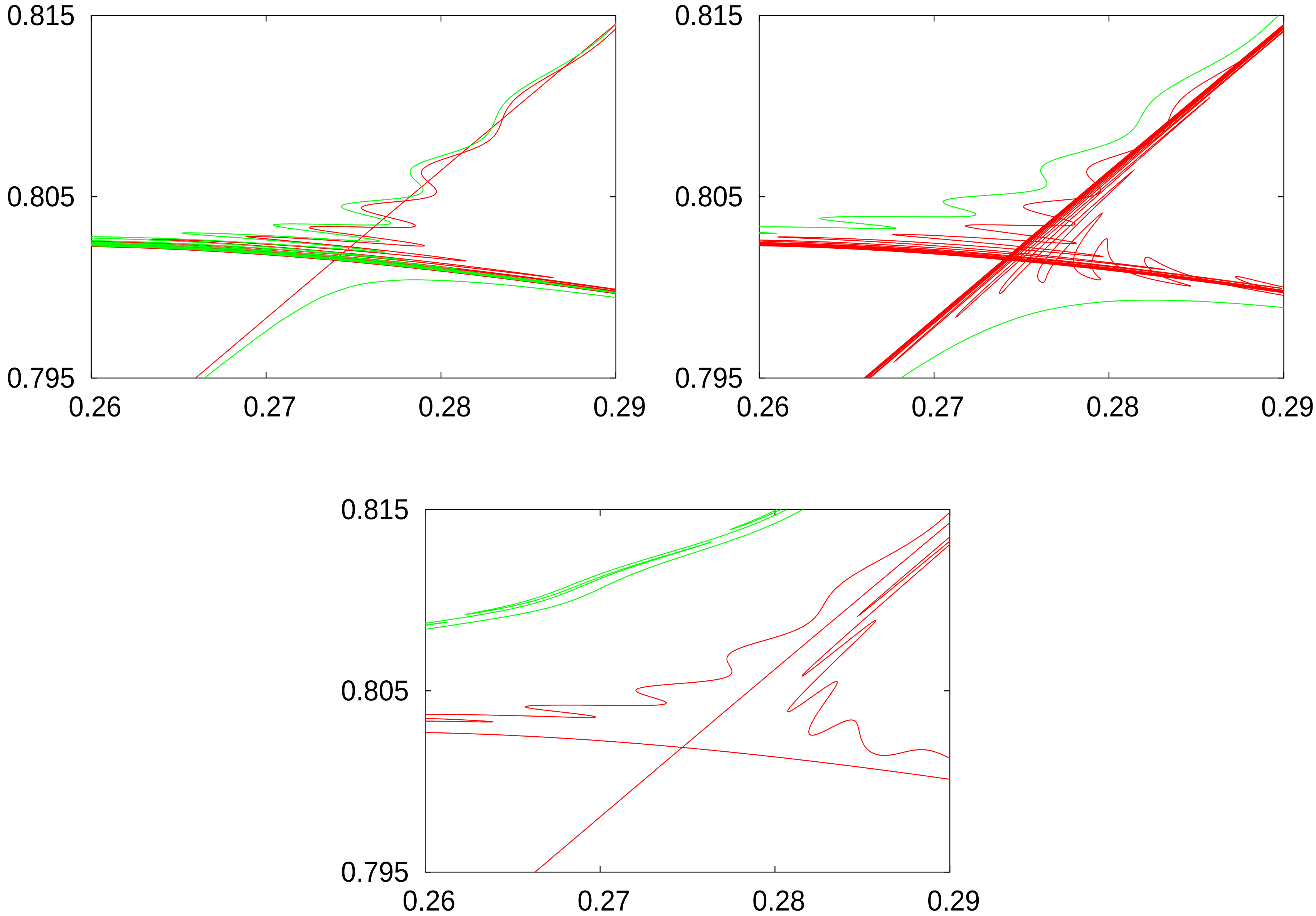
Resonances: (1:7), (1:8), (2:17), (1:9), (2:19) and (1:10) ($\alpha = 0.15$).

Second critical radius

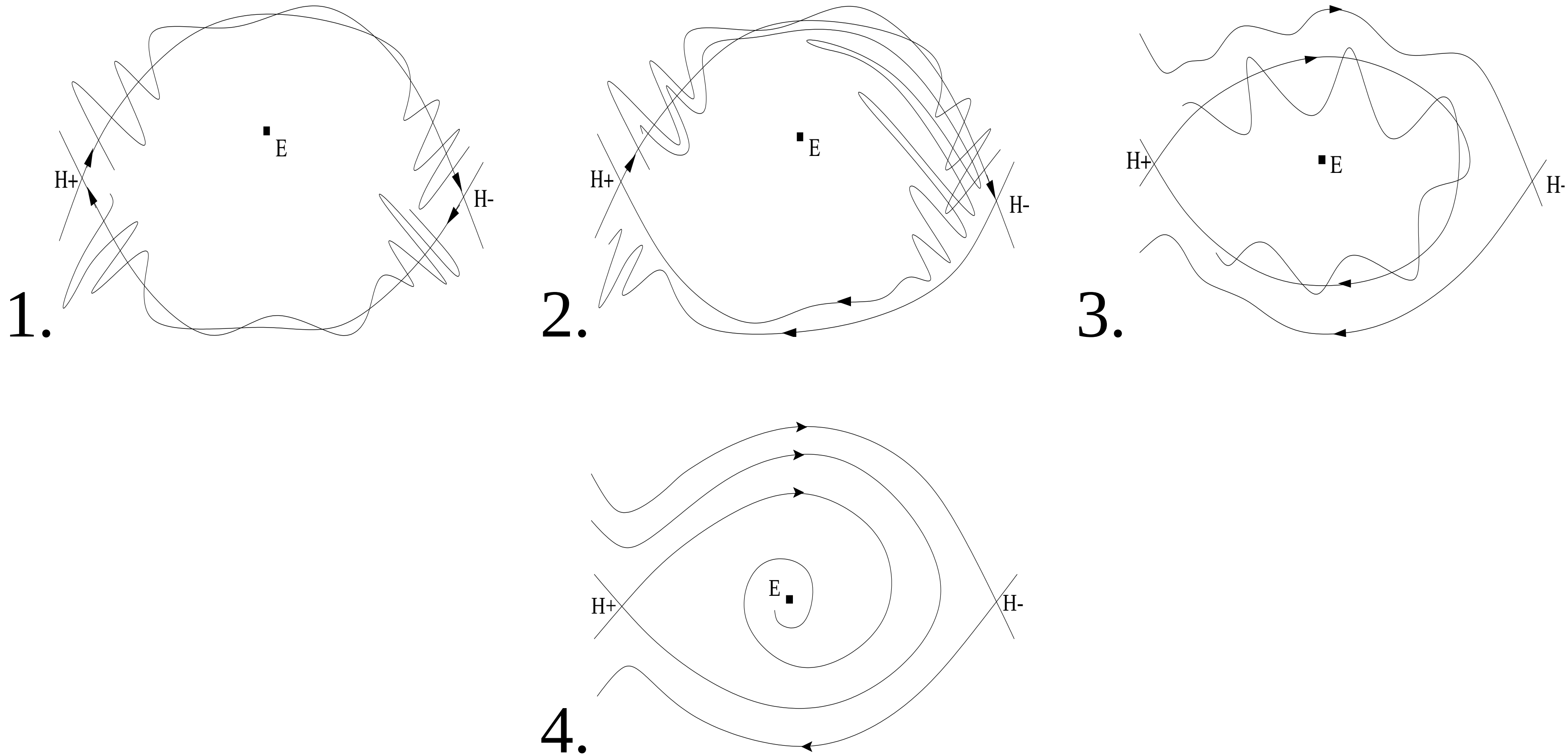
- Outside the first critical radius we find some resonances that have survived the dissipation.
- The structure of these resonances has changed:
  - The elliptic points become attractor foci
  - The position of the invariant manifolds allows to pass more points

We can determine two different “types” of resonances depending on the existence or destruction of homoclinic orbits. We need first to understand the evolution of a resonance better.

Evolution: $\alpha = 0.17$; $\log_{10}(\epsilon) = -6, -5.4, -4$, res.(1:7)



A possible scenario



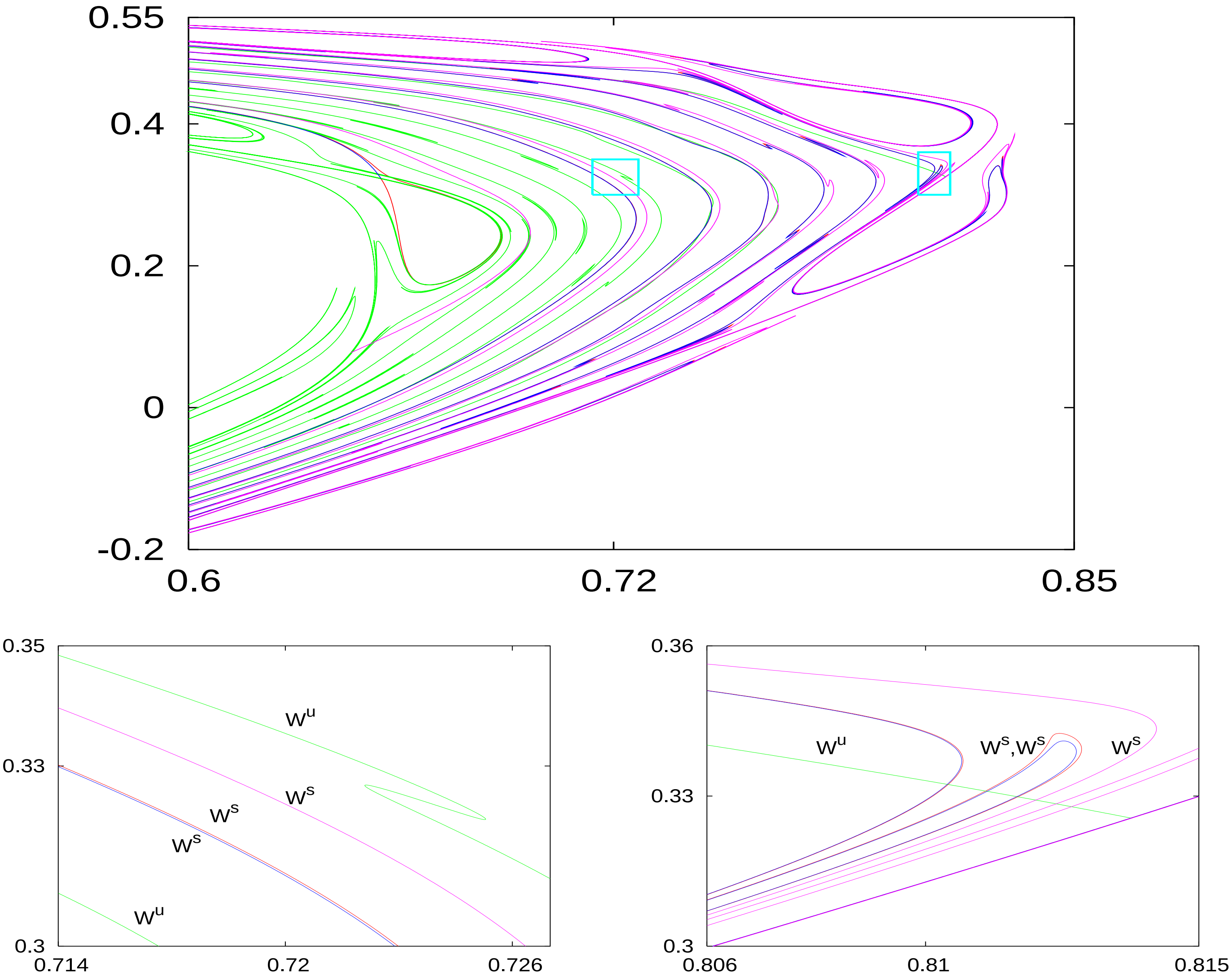
Flow type resonances

- When all the homoclinic orbits are destroyed by the dissipation (but not very close to homoclinic tangency) the dynamics in a resonance can be approximated by a flow.
- The probability of capture in a flow type resonance depends on the strips that determine the invariant manifolds of the hyperbolic point.

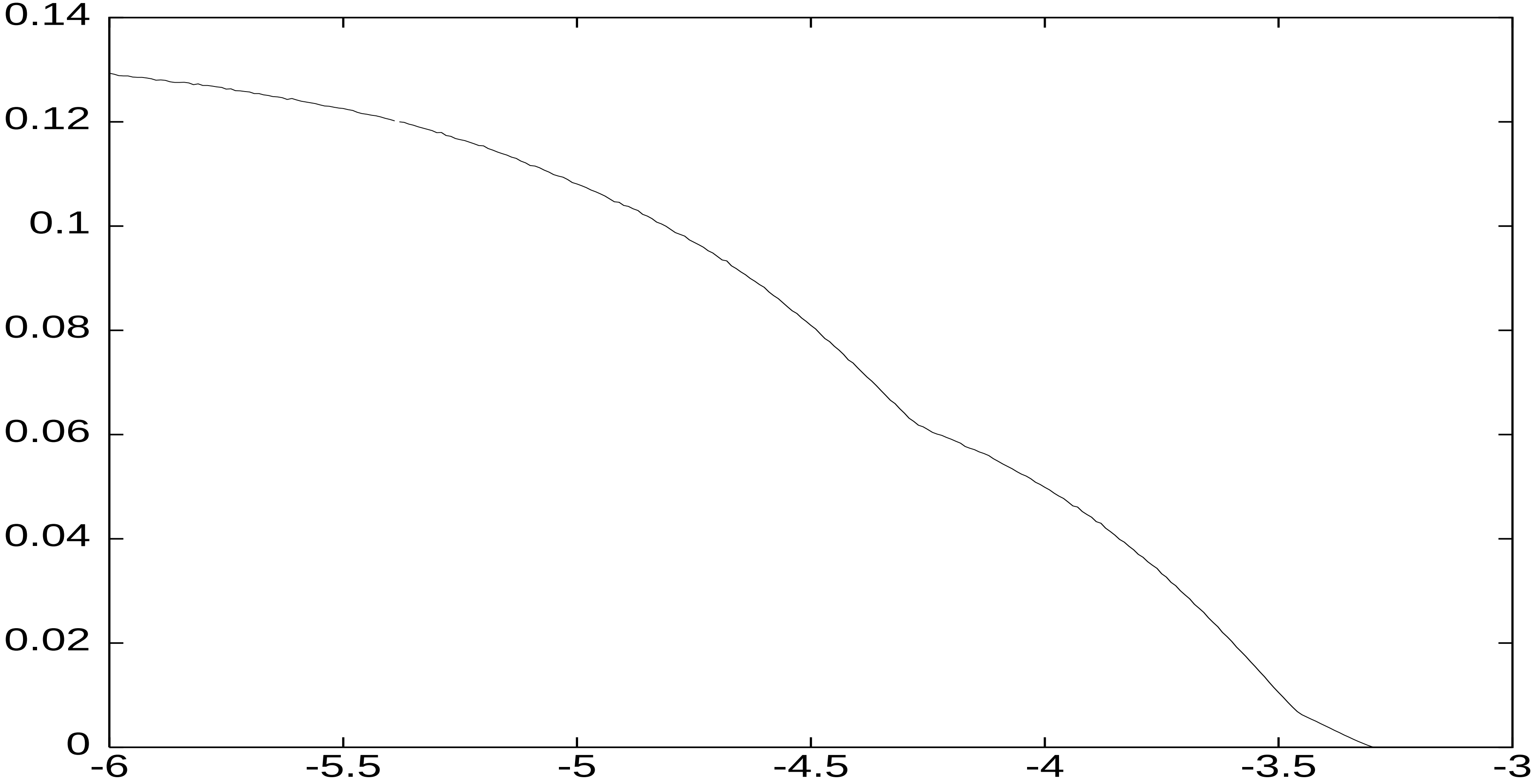
We want...

- ...to describe how the strips travel along the phase space.
- ...to determine how the probability of capture changes depending on ϵ and on the map we have. In particular, which is the limit of this probability when $\epsilon \searrow 0$?

The strips

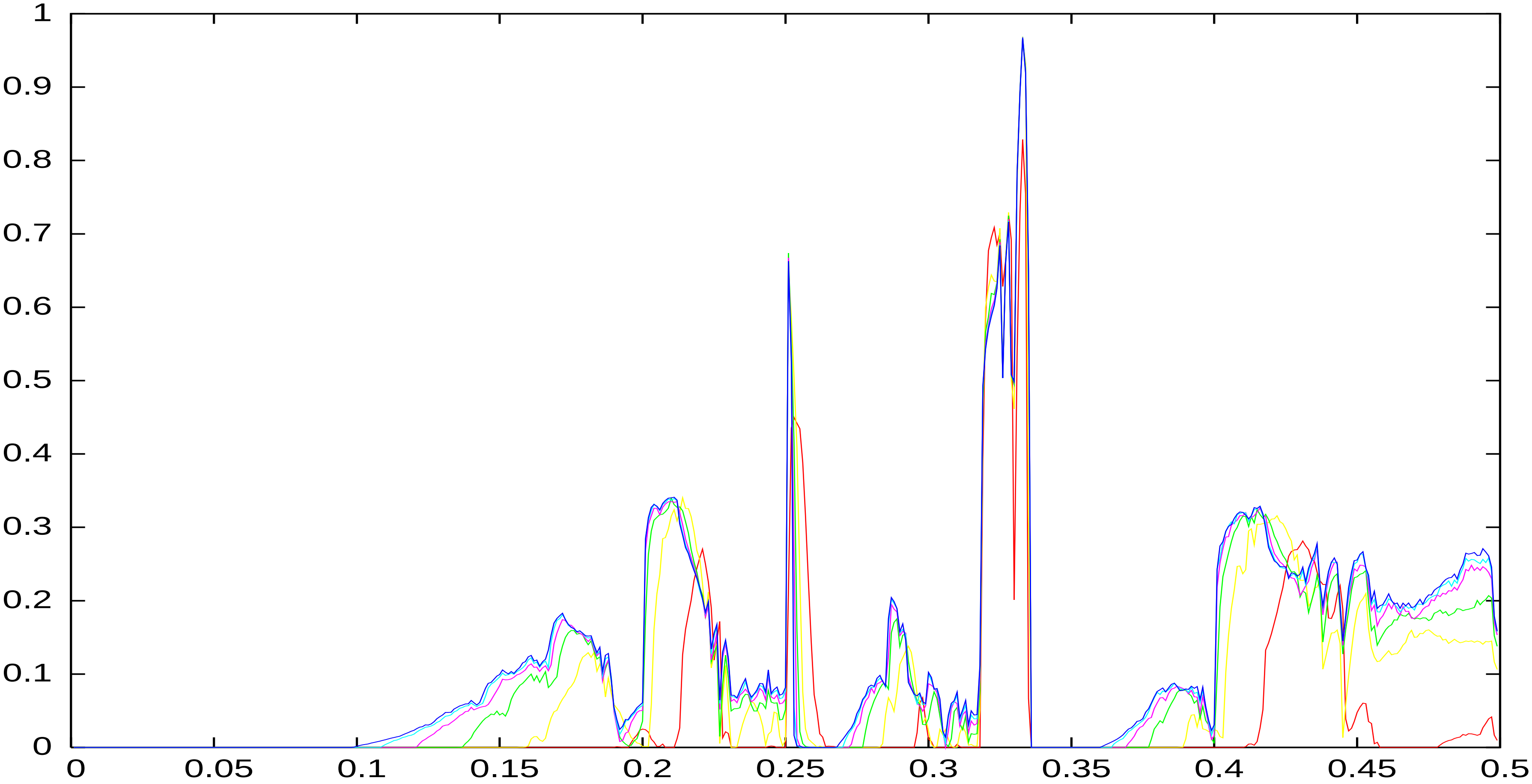

0.55
0.4
0.2
0
-0.2
0.6
0.72
0.85
0.35
0.33
0.3
0.714
0.72
0.726
W^u
W^s
W^s
W^s
W^u
0.36
0.33
0.3
0.806
0.81
0.815
W^u
W^s,W^s
W^s

Points captured by resonances - ϵ



x-axis: $\log_{10}(\epsilon)$ **y-axis:** ratio of number of points captured by the foci of resonances and number of points that do not leave the ball of radius 0.97 $\alpha = 0.15$

Points captured by resonances - α

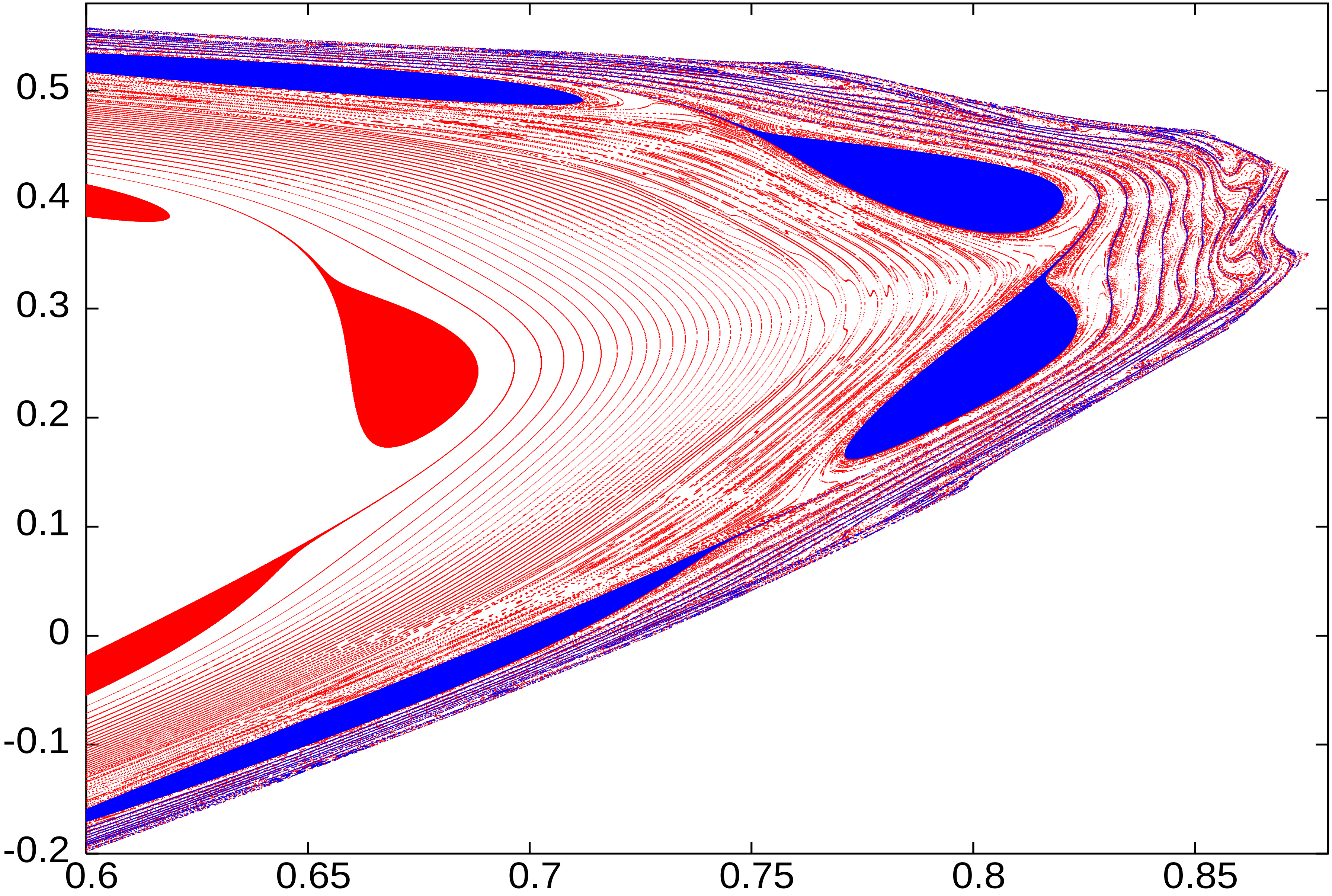


x-axis: α **y-axis:** ratio of the number of points captured by the foci of the islands and the ones that do not escape; **curves:** values of $\epsilon :\ 10^{-k},\ k = 2$ (red), $\ldots, 7$ (blue).

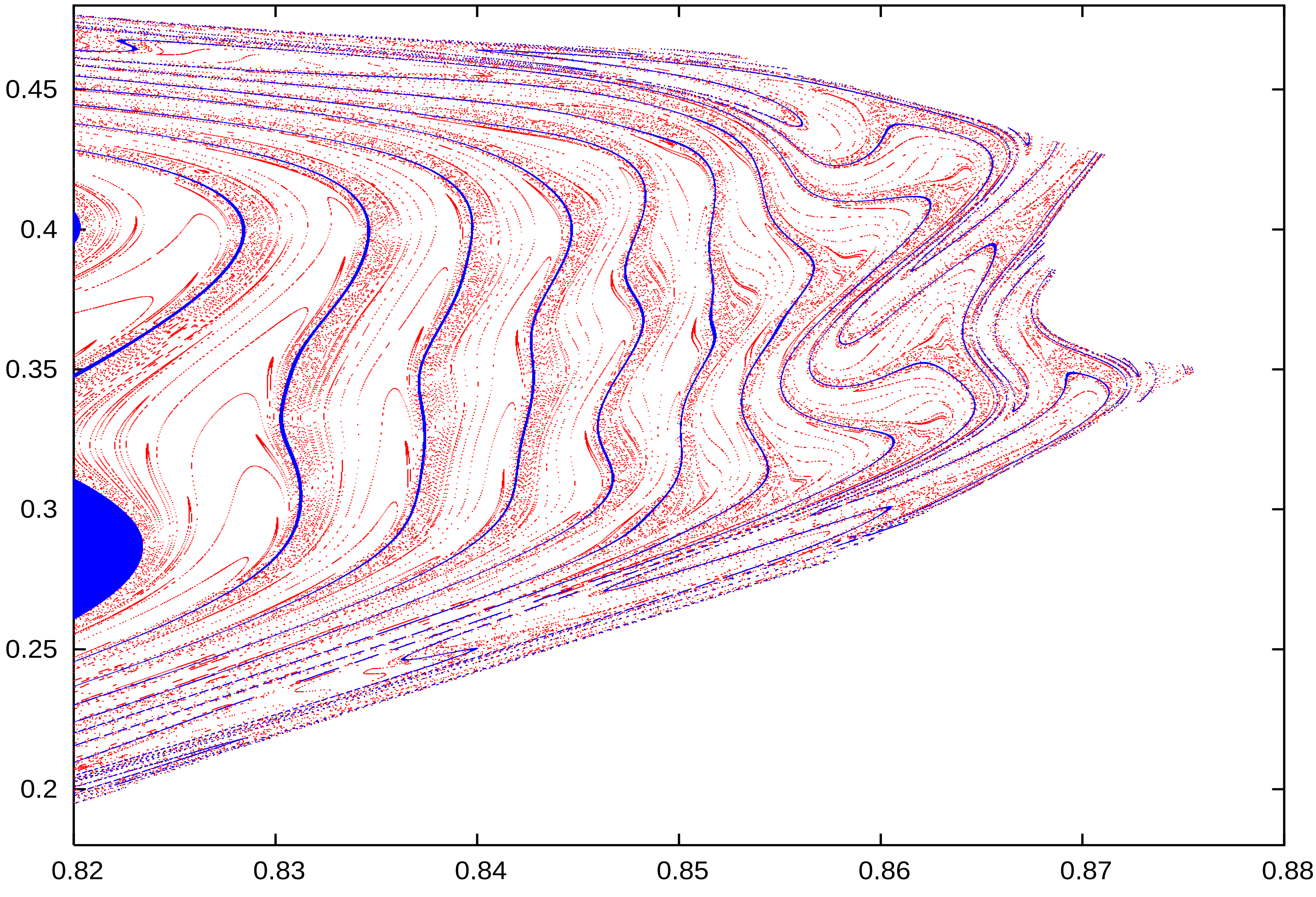
Future

- It is necessary to do a theoretical approach and generalise what is observed for the Hénon map to a general map.
- In particular, it is necessary to develop models to understand the dynamics around each type of resonances:
  - Flow type: NF + Approx. by flow C.Simó and A.V. (in progress).
  - Homoclinic type: Deal with the diffeomorphism directly. It requires suitable models (in progress).
- To prove the conjecture to be stated later.
- Generalisations to higher dimensions.

A final picture...



And a magnification



Conjecture

Assume that ϵ is small but sufficiently large to do not have homoclinic points in a resonance.

- The measure of the points captured by the resonance when approaching the conservative case, assuming that no homoclinic points are created, is the sum of the measure of the islands and the measure of the strips (both can be approximated using normal forms).

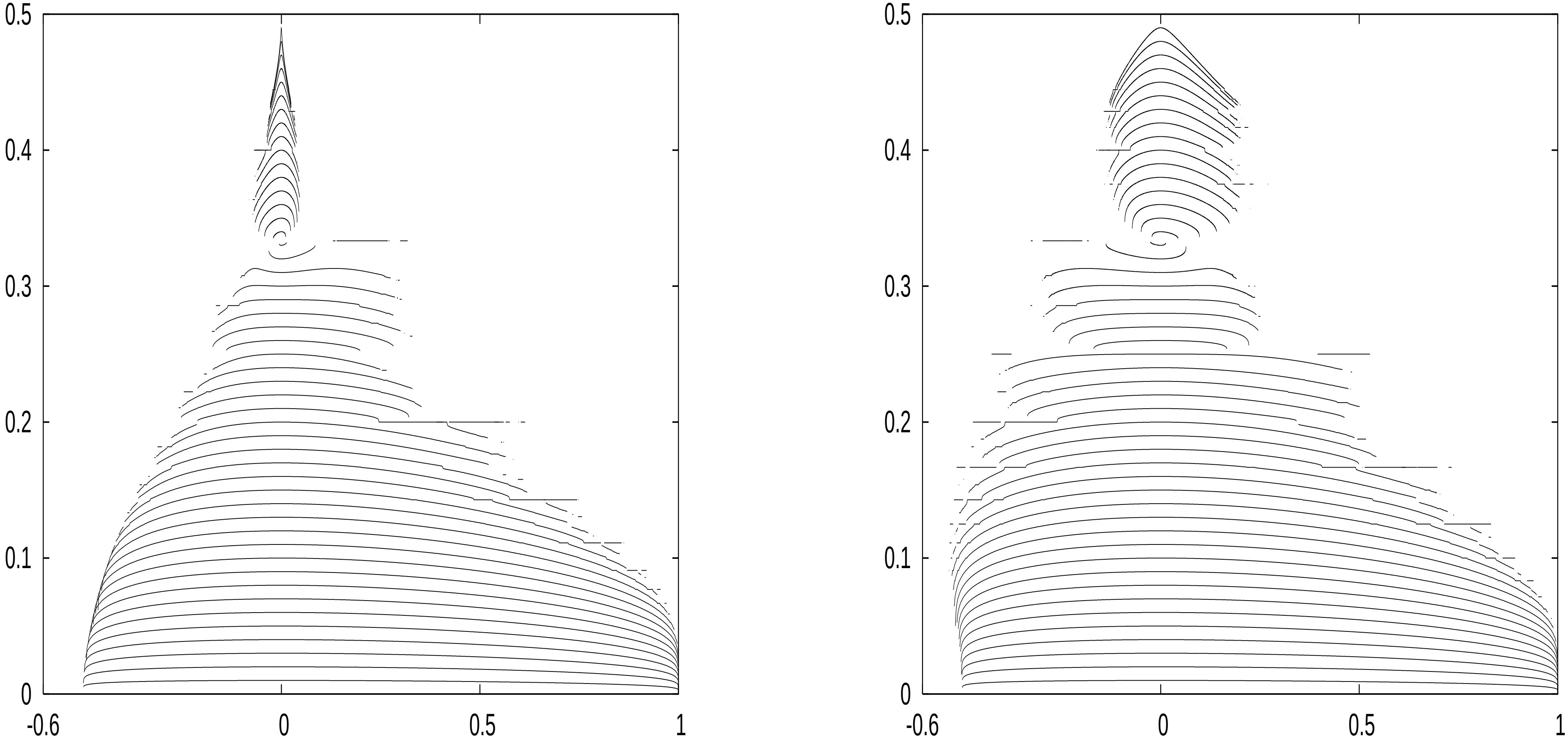
$$\Gamma(F,\epsilon) = \{(x,y) \in \mathcal{A} \text{ (a fixed domain)} \mid \omega(x,y) = E\}\,.$$

$$\lim_{\epsilon\to 0} \mu(\epsilon) = \lim_{\epsilon\to 0} \frac{mes_L\left(\Gamma(F,\epsilon)\right)}{mes_L\left(\mathcal{A}\right)} \stackrel{conj.}{=} A_{islands} + A_{strips}\,,$$

Open question: what happens when approaching to the conservative case and homoclinic points are allowed?

THE END...

Rotation number



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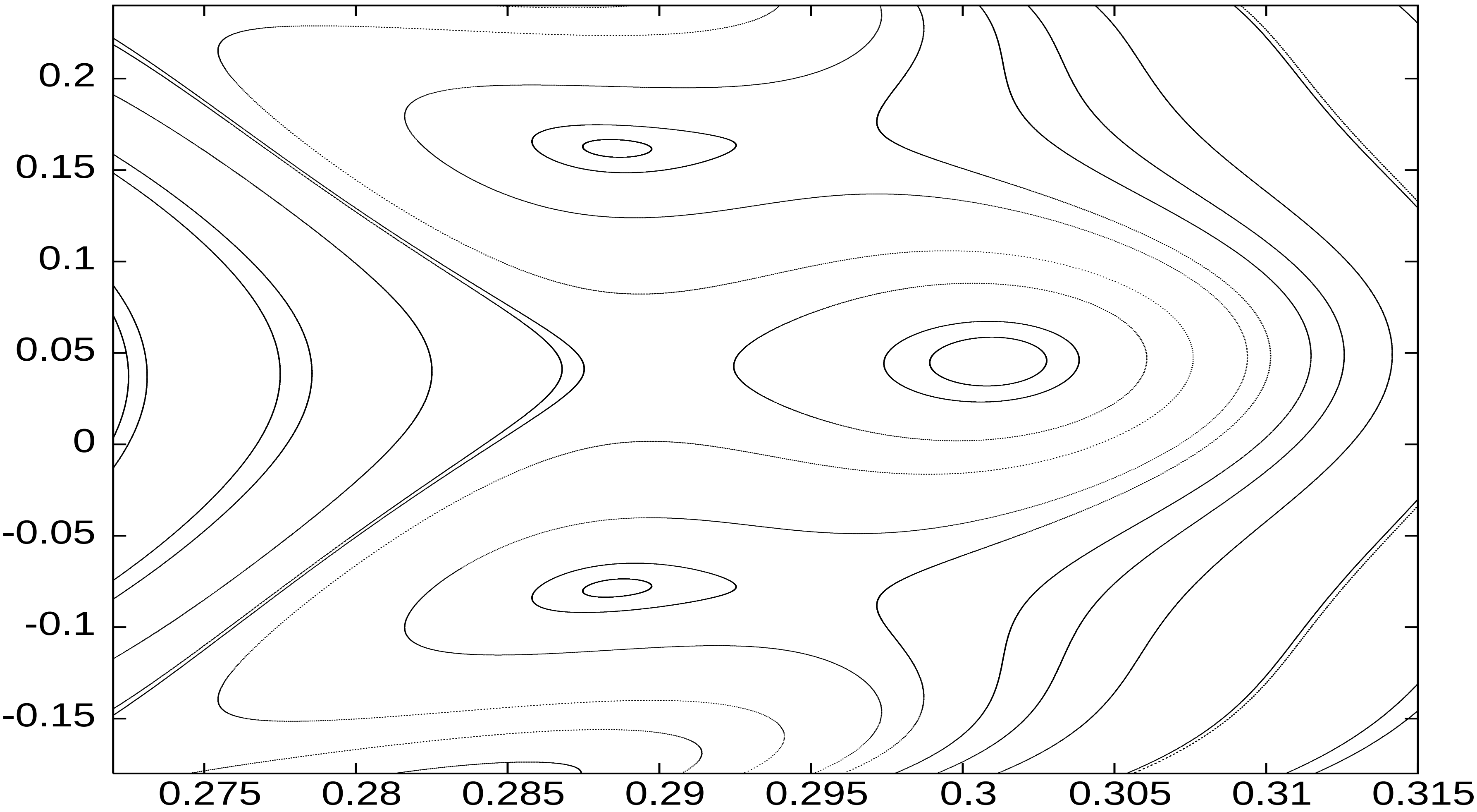
Meandering curve



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