Dynamics in chaotic zones of area preserving maps (APMs): close to separatrix and global instability zones

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Carles Simó and Arturo Vieiro

Introduction

We consider a one-parameter family of maps $F_\delta : \mathcal{U} \longrightarrow \mathbb{R}^2$, $\mathcal{U} \subset \mathbb{R}^2$ domain, such that

1. F_δ real analytic in the (x, y)-coordinates of $\mathcal{U}$,
2. $\det DF_\delta(x, y) = 1$, for all $(x, y) \in \mathbb{R}^2$ and for all $\delta \in \mathbb{R}$, (APMs)
3. F_δ has a fixed point E_0 that will be assumed to be at the origin $\forall \delta \in \mathbb{R}$,
4. spec $DF(E_0) = \{\mu, \bar{\mu}\}$, $\mu = \exp(2\pi i \alpha)$, $\alpha = q/m + \delta$, $q, m \in \mathbb{Z}$.

Through the presentation the Hénon map will be used as a paradigm of APM.

One of the formulations is

$$H_c : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} c(1 - x^2) + 2x + y \\ -x \end{pmatrix}$$

APMs are not large dimensional systems

- The study of APMs becomes relevant for the phase analysis of **larger** dimensional systems.
  $H(\theta, I)$ – n-d.o.f. Hamiltonian system, $\dot{\theta} = \omega(I)$ frecuency vector, assume that for $I = I_*$, $\langle k, \omega(I_*)\rangle = 0$, $k \in \mathbb{Z}^n$ (single resonance).
  Then, after isoenergetic reduction and considering a Poincaré section, the dynamics for $\omega(I) \approx \omega(I_*)$ can be reduced to that of a $\mathbb{R}^2 \times \mathbb{T}^{n-2} \times \mathbb{R}^{n-2}$ map ("APM dynamics $\times$ dense motion on a family of $(n-2)$ invariant tori") .

- The methodology we use consists in **gluing universal models** to describe dynamical properties in different regions of the phase space. Similar approaches can be used in higher dimensional problems although the models become more complicated (and maybe should be obtained/adapted to each problem using (semi)-numerical techniques).
  $\longrightarrow$ following scheme of Chirikov approach...

Goal of our study of APMs

- We focus on a **semi-global** description of the phase space.
- More precisely, we want to describe the dynamics in the **resonant chains** emanating from (but **relatively far** from) the elliptic fixed point E_0.
- Special interest is focused on the **dynamics in chaotic zones** either small regions around the separatrices of the resonant island (with size of the order of the splitting of the separatrices) or the larger regions due to interaction of different resonances (for example, Birkhoff zones of instability if the region is confined between invariant rotational curves).
- We are interested in a topological/qualitative description but **our goal is to obtain quantitative information** of the system: size of the chaotic zones, distance to invariant curves, measure of the stability regions within the chaotic zones, transport properties,...

  sometimes hard (large chaotic regions)... but we should try!!

Some preliminary local/quantitative results

- BNF+Interpolating Hamiltonian
- Dynamics within resonant chains
- Inner/outer splittings of separatrices in a resonant island

BNF + Interpolating Hamiltonian

F_δ one-parameter δ-family of APMs with $F(E_0) = E_0$ elliptic fixed point.
Spec $DF(E_0) = \{\mu, \bar{\mu}\}$, $\mu = e^{2\pi i \alpha}$, $\alpha = q/m + \delta$, δ small enough.
(x, y)-Cartesian coord., $(z, \bar{z})$-complex coord. ($z = x + iy$, $\bar{z} = x - iy$).
(I, φ)-Poincaré variables ($z = \sqrt{2I}\exp(i\varphi)$).
$\longrightarrow$ Consider the Birkhoff NF of $F_\delta(x, y)$ to order m around E_0
(say BNF$_m(F_\delta)(z, \bar{z})$) and let $K(z, \bar{z}) =$ BNF$_m^m(F_\delta)(z, \bar{z})$ (near Id).
$\longrightarrow$ Define
$\mathcal{H}_{nr}(I) = \pi \sum_{n=0}^{s} \frac{b_n}{n+1}(2I)^{n+1}$ and $\mathcal{H}_r(I, \varphi) = \frac{1}{m}(2I)^{\frac{m}{2}} \cos(m\varphi)$.
The time-1 map generated by the flow defined by the Hamiltonian

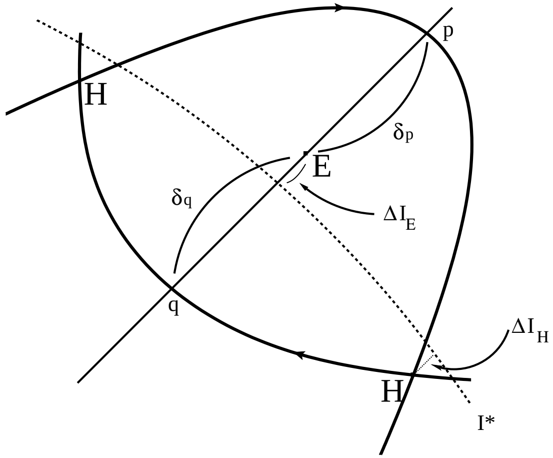
$$\mathcal{H}(I, \varphi) = \mathcal{H}_{nr}(I) + \mathcal{H}_r(I, \varphi)$$

interpolates K with an error of order $m + 1$ with respect to the $(z, \bar{z})$-coordinates, in a suitable annulus containing the resonant m-island.

Description of resonances

Generic case: $\alpha = q/m + \delta$, $m > 5$, δ sufficiently small, $b_1 \neq 0$.

- If $b_1\delta < 0$ then F **has** a resonant island of order m.
- The resonant zone is determined by **two periodic orbits** of period m located near two concentric circumferences (in the BNF variables). The closest orbit to the external circumference is elliptic while the one located close to the inner circumference is hyperbolic.
- The **width** of the resonant island is $\mathcal{O}(I_*^{m/4})$, $I_* = -\delta/2b_1$.



A model around a generic resonance

For a generic APM such that $\delta < 0,\ b_1 > 0,\ b_2 \neq 0$, the dynamics around an island of the m-resonance strip $(m \geq 5)$ can be modeled, after suitable scaling ($J \sim \delta^{-m/4}(I - I_*)$), by the time-$\log(\lambda)$ map of the flow generated by

$$\mathcal{H}(J,\psi) = \frac{1}{2}J^2 + \frac{c}{3}J^3 - (1 + dJ)\cos(\psi),$$

where $c = \mathcal{O}(\delta^{\frac{m}{4}}),\ d = \mathcal{O}(\delta^{\frac{m}{4}-1})$. Bounding the errors it is shown that it gives a “good” enough approximation of the dynamics in an annulus containing the m-islands.

$\longrightarrow$ Then, we have the following... [a]

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[a] The details of the proof (singularities, suitable Hamiltonian,...) can be found in:
Resonant zones, inner and outer splittings in generic and low order resonances of area preserving maps. Nonlinearity 22, 5:1191–1245, 2009.

Main result: comments on the hypothesis

- **A1.** $b_1(\delta)$ is non-zero for $\delta = 0$.
- **A2.** $W^u = G(W^s)$, G periodic (between homo p and $F(p)$),

  s scaled variable s.t. $G(s) = \sum_{k=-\infty}^{\infty} c_k \exp(ik\,2\pi s)$.

  We assume: The maximum of the norms of the functions $c_{\pm 1}\exp(\pm i 2\pi s)$ is bounded away from zero, when δ tends to zero, on suitable lines whose imaginary part tend to $\tau_\pm$ when $\delta \to 0$.
- **A3.** There exists a fixed $\alpha > 0$ s.t.

$$\sigma_\pm = \exp\left(-\frac{2\pi\,\mathrm{Im}\,\tau_\pm - \eta_\pm}{\log(\lambda(\epsilon))}\right)\left(\cos\left(\frac{2\pi\,\mathrm{Re}\,\tau_\pm}{\log(\lambda(\epsilon))} - \phi_\pm\right) + o(1)\right),$$

  where $|\eta_\pm| < \log(\lambda(\epsilon))^{1-\alpha}$ for ϵ sufficiently small.
- **A4.** F maybe meromorphic but the singularity remains at a finite distance when δ goes to 0.

Main result

Theorem. Let F be an APM. Assume that it has an m-order resonance strip, $m > 4$, located at an average distance $I = I_* = \mathcal{O}(\delta)$ from the elliptic fixed point and δ is sufficiently small. Under the assumptions **A1**, **A2**, **A3** and **A4**, the following conclusions hold.

a) The outer splitting is larger than the inner one being the difference between the position of the corresponding nearest singularities $\mathcal{O}(\delta^{m/4-1})$.

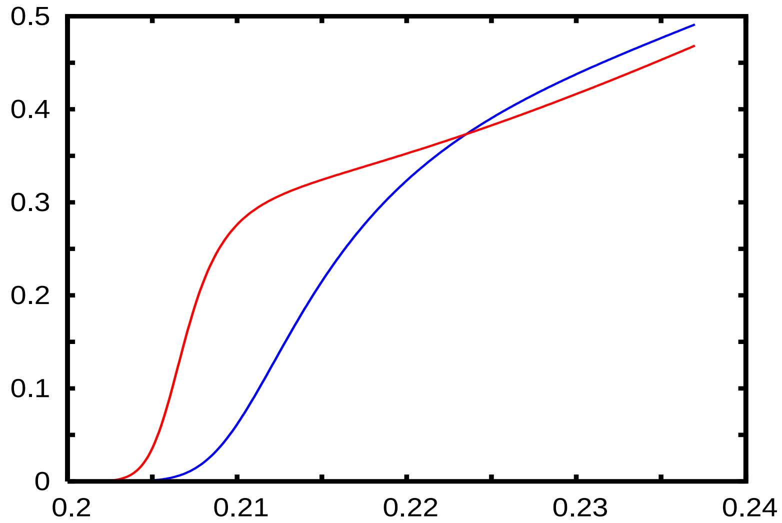
b) Neither the inner nor the outer splittings oscillate.

$\longrightarrow$ **Question:** Consequences in the width of the chaotic zones of this fact?

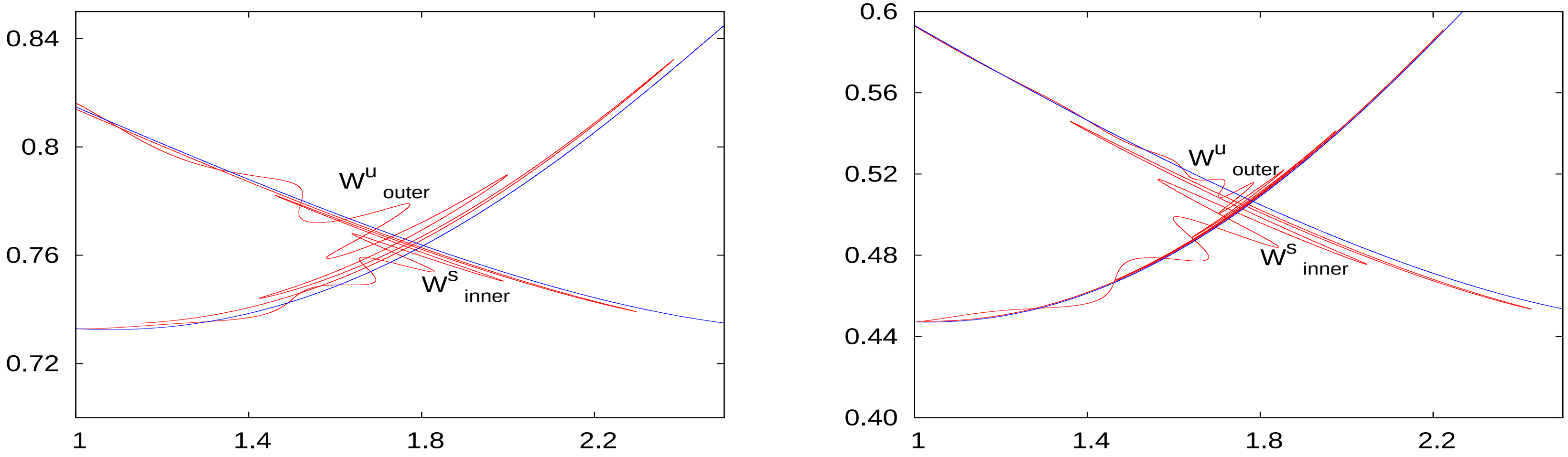
Before some comments...

Some comments: Far from the elliptic point

2:11 Hénon



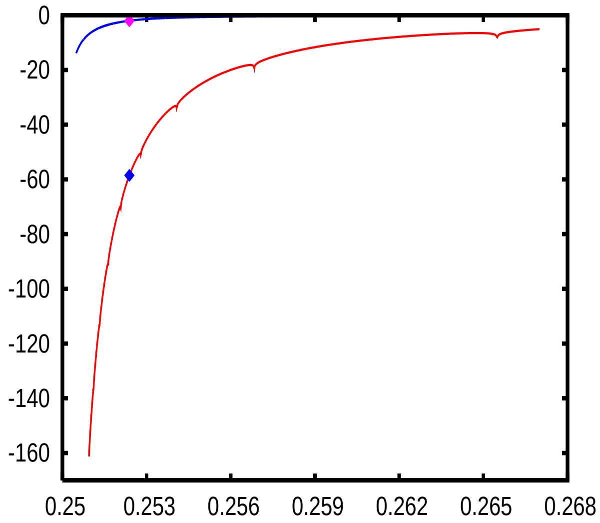
twist: $(I, \theta) \rightarrow (I + 0.14\cos(\theta + \alpha(I)), \theta + \alpha(I)), \alpha(I) = b_1 I + b_2 I^2$.

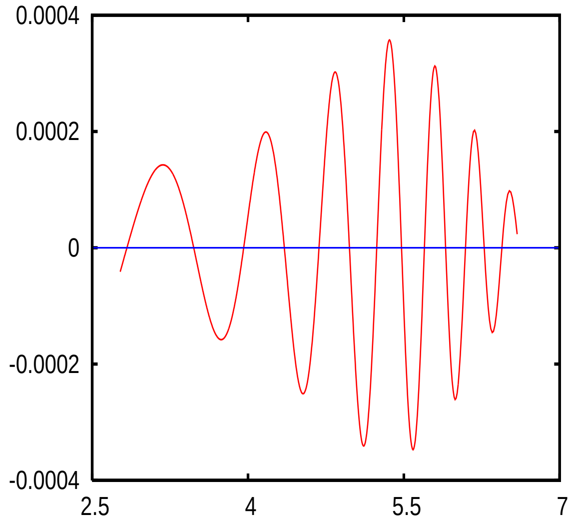


Some comments: Strong resonances

- The description of the resonant structure by means of the interpolating Hamiltonian does not hold if $m \leq 4$.
- We have studied in detail the generic cases for the resonances (1:3) and (1:4), computing the Hamiltonian and the singularities, and also some non-generic cases:

Hénon map 1:4 resonance **Non-generic!!**





Semi-global study of APMs: Chaotic regions

We focus now on the chaotic regions created by the invariant manifolds emanating from a fixed/hyperbolic point h. We do not assume the region of interest to be close to the origin but we require the system to be not too far from integrable in the selected domain to be studied.

$\longrightarrow$ We want:

- To study the resonant islands far from the elliptic point
- To study the dynamics in the chaotic zones

$\longrightarrow$ How?:

Using suitable return maps $\longrightarrow$ **universal models**.

Chaotic regions considered

For each of the following cases we use a concrete return map model to study the dynamics.

- Open case (fish like) — Separatrix map
- Figure eight case (pendulum like) — Double separatrix map
- Large regions of instability (e.g. Birkhoff z.i.) — Biseparatrix map

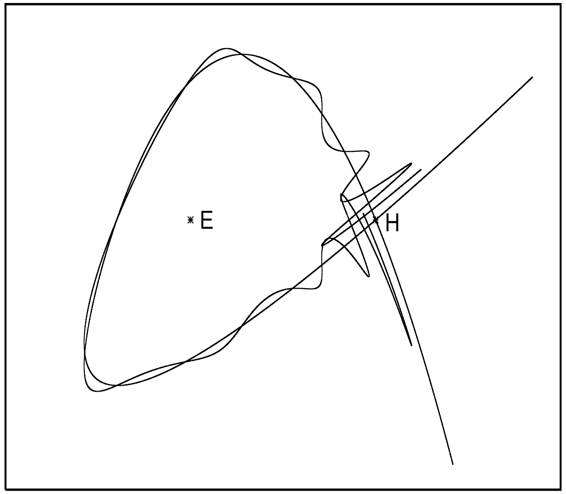
$\longrightarrow$ We look for **quantitative** information on the dynamics within the chaotic zones. However, the biseparatrix model only gives us a topological description of the dynamical behaviour. [a]

[a] The following results can be found in:

Dynamics in chaotic zones of area preserving maps: close to separatrix and global instability zones.

Submitted to Physica D.

Open case



$$SM : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + a + b \log |y'| \\ y + \sin(2\pi x) \end{pmatrix}$$

where $b = 1/\log(\lambda)$, λ the dominant eigenvalue of $DF(h)$ and a is a “shift”.

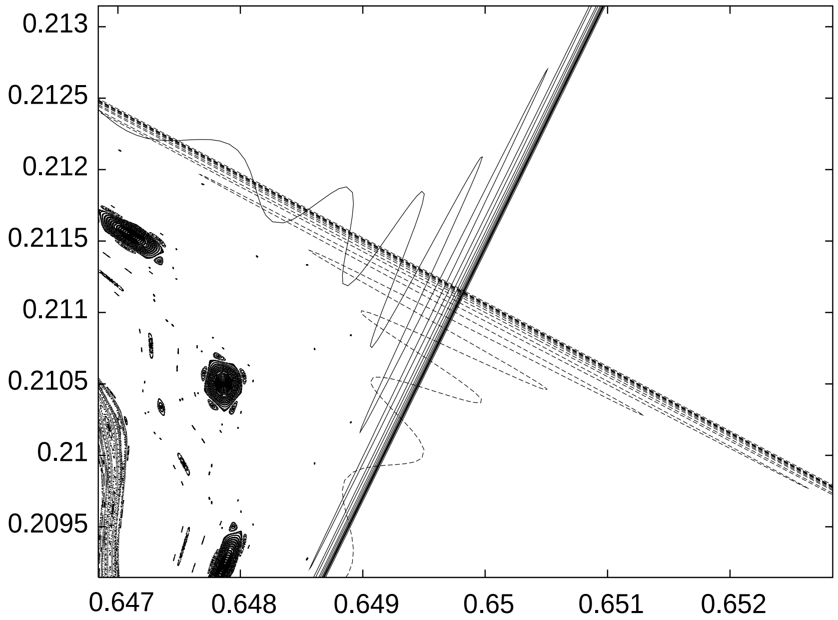
The y-vble. is scaled by the amplitude of the splitting.

We deal with an **a priori stable** case: $\log(\lambda) = \mathcal{O}(\epsilon)$ and $a = \mathcal{O}(1/\epsilon) \Rightarrow A = \mathcal{O}(\exp(-ctant/\epsilon^r)$. Here ϵ is a “distance-to-integrable” parameter.

Open case: results

- Distance to invariant curves from the separatrix: $d_c \sim |b|/k^*$ (SM is approximated by STM, $k^* \approx 0.97/(2\pi)$ Greene value).
  - When coming back to the original variables: $D_c \sim \sigma\ell/(2\pi k^* \log(\lambda))$,
  - If measured from the hyperbolic point, assuming the map close to the time-ϵ flow of $H(x,y) = y^2/2 - \alpha x^3 - \beta x^2$, one has: $D_c^h \approx (3LD_c/2)^{1/2}$, where L is the distance between the hyperbolic and the elliptic point inside the "fish". This result can be improved using higher order interpolating Hamiltonians.
- Distance to islands from the separatrix: $d_i \sim |b|/\tilde{k}$, $\tilde{k} = 2/\pi$.
- Expected number of "central" islands before the r.i.c. $\#\{islands\} \approx 1.415 \times b$.

Hénon map $H_\alpha(x, y) = R_{2\pi\alpha}(x, y - x^2)$



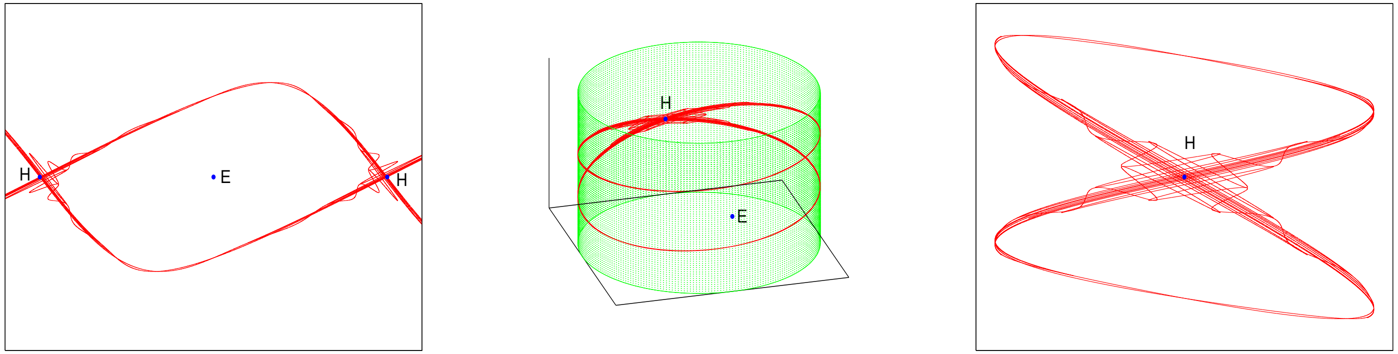
$\alpha = 0.1$

Experimental values: $(D_c^H)_e \approx 2.94 \times 10^{-3}$, $(D_i^H)_e \approx 2.08 \times 10^{-3}$

“Fish” interpolating Hamiltonian: $D_c^H \approx 2.47 \times 10^{-3}$, $D_i^H \approx 1.85 \times 10^{-3}$

5-order interp. Hamiltonian: $D_c^H \approx 2.731 \times 10^{-3}$, $D_i^H \approx 2.050 \times 10^{-3}$

Figure eight case



$$DSM : \begin{pmatrix} x \\ y \\ s \end{pmatrix} \longmapsto \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{s} \end{pmatrix} = \begin{pmatrix} x + a_{\bar{s}} + b \log|\bar{y}| \ (\text{mod } 1) \\ y + \nu_{\bar{s}} \sin 2\pi x \\ \text{sign}(y)\, s \end{pmatrix},$$

where ν is such that $\nu_1 = 1$ and $\nu_{-1} = A_{-1}/A_1$, being A_1 and A_{-1} the amplitudes of the outer and inner splittings, respectively, of the resonant island.

Comments:

- It is defined on a domain $\mathcal{W} = \mathcal{U} \cup \mathcal{D}$ (upper and lower domains around the outer and inner separatrices of the resonance).
- $y > 0$ means we are outside the stable manifold (either in $\mathcal{U}$ or $\mathcal{D}$).

Figure eight: results

Generic resonances close to the origin. Assume $b_1\delta < 0$ and that the hypothesis of the theorem concerning the difference of the inner and outer splittings hold. Then,

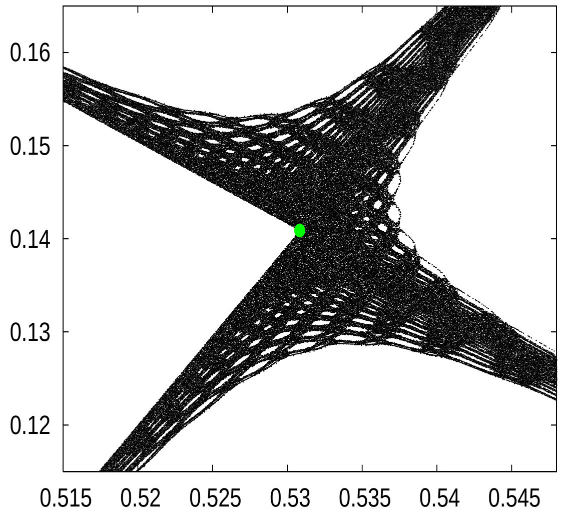
- The width of the outer chaotic zone is larger than the width of the inner chaotic one if, and only if, sign $b_1 \cdot$ sign $b_2 < 0$.
- Both amplitudes of the stochastic layer are of the order of magnitude of the outer splitting (the largest one).

Pendulum-like islands: comments

The idea is to construct an interpolating Hamiltonian of the map (in a domain containing the resonance) and to use preservation of energy to see how the distance to the rotational invariant curves changes when measuring from the upper $\mathcal{U}$ and the lower $\mathcal{D}$ domains. This can be done computing the ratio

$$f = \nabla\mathcal{H}(J_M)/\nabla\mathcal{H}(J_m)$$

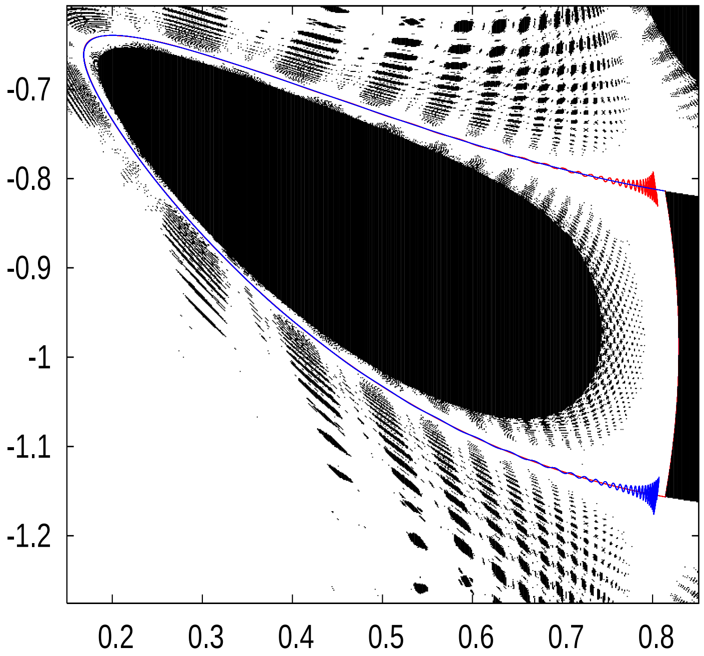
where J_M and J_m are the maximum (minimum) of the outer (inner) separatrix of the Hamiltonian. For close to the origin resonances $f = 1 + \mathcal{O}(\delta^{m/4})$.



$$\alpha = 0.21,$$
$$\sigma_- = \mathcal{O}(10^{-12}),$$
$$\sigma_+ = \mathcal{O}(10^{-3}).$$

Pendulum-like islands: comments II

The same idea applies to resonances far from the origin as well as for strong resonances but, for each case, a suitable interpolating Hamiltonian must be considered. In these cases the chaotic zone width measured in both domains can be of different order of magnitude:



$c = 1.015$,

$\sigma_+ = \mathcal{O}(10^{-54})$, $\sigma_- = \mathcal{O}(10^{-1})$.

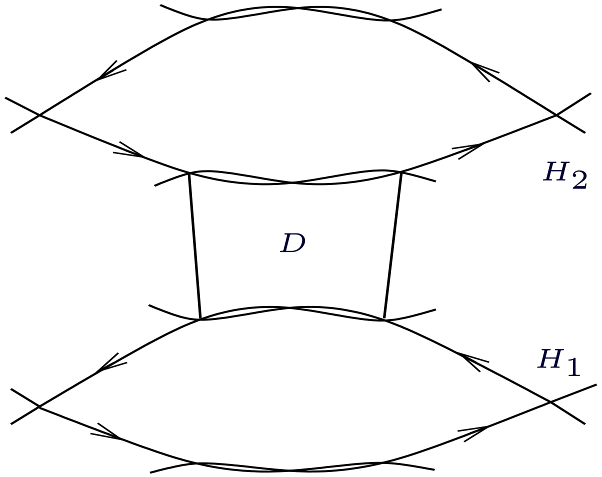
Experimentally, $f \approx -5$. Using interp. Ham. up to order $\delta \approx c - 1$ we obtain $f \approx -5.64$.

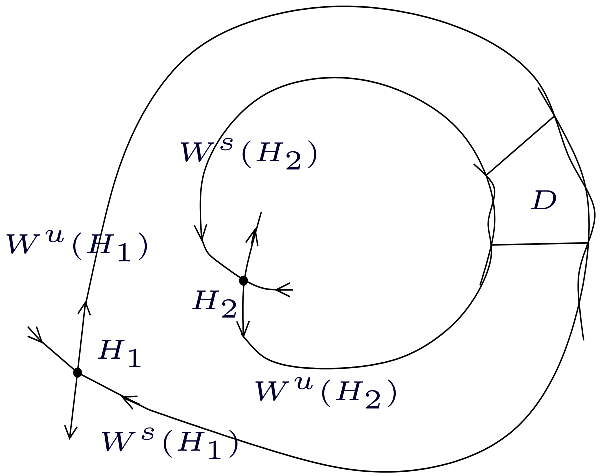
But $\delta = 0.015$ is too large. For δ small we obtain better results (even we can predict # tiny islands).

Large regions of stability

Due to the interaction of resonances large chaotic zones of instability appear. These are regions without rotational invariant curves (e.g. Birkhoff zones of instability). We have considered the **biseparatrix model** and we have studied different situations (twist and non-twist case). On the other hand, it helps to study the phenomena taking place at the border of the stability domain.

Geometrical situation:





The biseparatrix model

Between two concentric chains of islands, the simplest *qualitative* model on the domain $0 < v < d$ is given by

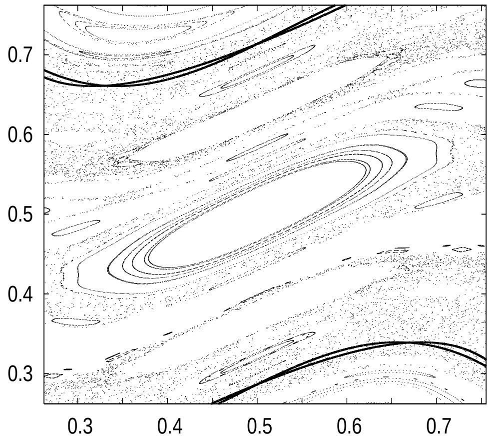
$$\begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} u + \alpha + \beta_1 \log(v') - \beta_2 \log(d - v') \\ v + \sin(2\pi u) \end{pmatrix}.$$

where $\beta_1 = 1/\log(\lambda)$, $\beta_2 = 1/\log(\mu)$, λ being the eigenvalue of modulus greater than one of the hyperbolic point of the bottom separatrix and μ the corresponding one of the top separatrix.

- For this model it is theoretically expected to have rotational invariant curves provided $d > (\sqrt{b_1} + \sqrt{b_2})^2/k^*$ ($k^* =$ Greene's value).
- Changing - for + in the 1st. row it is a model for non-twist Birkhoff zones.
- It remains to generalise it to different order for the top/bottom resonances and to make it quantitative.

BSM figures

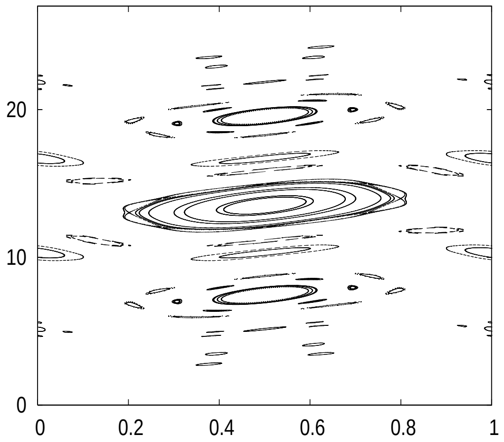
Chirikov standard map with $k = 0.16$.

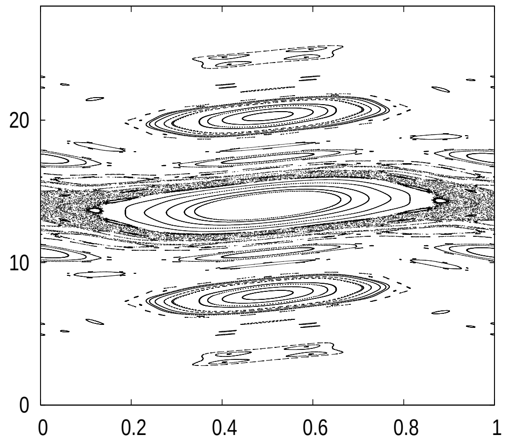


For the corresponding BSM model:

$\beta_1 = \beta_2 \approx 1.0365$

$(\lambda = \mu \approx 2.624248)$





$$\left.\begin{array}{r}\text{Amplitude inner/outer splitting: } A \approx 6 \times 10^{-3} \\ \text{Islands distance: } \hat{d} \approx 0.426987 \\ \text{Adjacent homoclinics distance: } a \approx 0.017796\end{array}\right\} \Rightarrow d \approx 24 \; (d = 27, 28)$$

The end...

We are working in 4D symplectic maps trying to generalise these studies (in collaboration with V. Gelfreich). [a]

The main goal is to get a quantitative description of the dynamics in **double resonances**.

Thank you for your attention!!

[a] Some (numerical) preliminary results can be found in *Some properties of the global behaviour of conservative low dimensional systems*, in Foundations of Computational Mathematics: Hong Kong 2008, F. Cucker et al., editors, London Math. Soc. Lecture Notes Series **363**, Cambridge Univ. Press, 2009.