Dynamics of 4d symplectic maps near a double resonance

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Let $F_\delta$ be a 2-parameter family of analytic symplectic 4d maps, 
$\delta = (\delta_1, \delta_2) \in \mathbb{R}^2$ small enough parameter.

Assume:

1. $F_\delta(0) = 0$ totally elliptic fixed point (for all $\delta$),
2. $\text{Spec}(DF_\delta)(0) = \{\lambda_1, \overline{\lambda}_1, \lambda_2, \overline{\lambda}_2\}$, $\lambda_k = \exp(2\pi i \alpha_k)$, $k = 1, 2$.

We will always assume that the eigenvalues are simple, i.e., $\alpha_1 \neq \alpha_2$.

The local dynamics can be described using Birkhoff NF.

Set of resonances

$$\Gamma = \{(k_1, k_2) : k_1 \alpha_1 + k_2 \alpha_2 = 0 \pmod{1}\} \subset \mathbb{Z}^2.$$

- $r = (k_1, k_2) \in \Gamma$ is a resonance of order $|r| = |k_1| + |k_2|$.
- $(k_1, k_2) \in \Gamma \iff \lambda_1^{k_1} \lambda_2^{k_2} = 1$.
- $(0, 0)$ trivial (or unavoidable) resonance.
- We assume $k_1 \geq 0$ to avoid trivial symmetries in resonances.
The totally elliptic fixed point of $F_0$, at the origin, can be:

1. **Non-resonant** ($\Gamma$ is a trivial group).
   In this case $\{\alpha_1, \alpha_2, 1\}$ are rationally independent.

2. **Simply resonant** ($\Gamma$ is a one-dimensional lattice).
   In this case there are two possibilities:
   a) $\alpha_1 \in \mathbb{Q}$, $\alpha_2 \in \mathbb{R} \setminus \mathbb{Q}$ (or vice versa).
   b) $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \mathbb{Q}$ but $\{\alpha_1, \alpha_2, 1\}$ are rationally dependent.

3. **Doubly resonant** ($\Gamma$ is a two-dimensional lattice).
   In this case $\alpha_1, \alpha_2 \in \mathbb{Q}$

   $$\alpha_1 = \frac{p_1}{q_1} \quad \text{and} \quad \alpha_2 = \frac{p_2}{q_2}, \quad p_1, p_2, q_1, q_2 \in \mathbb{N}.$$
Each resonant relation \((k_1\alpha_1 + k_2\alpha_2 = k_3, k_i \in \mathbb{Z})\) defines a line on the torus

\[
\mathcal{T} = \{ (\lambda_1, \lambda_2) \in \mathbb{C}^2 : |\lambda_1| = |\lambda_2| = 1 \}.
\]

Simply and doubly resonant eigenvalues are dense in \(\mathcal{T}\).

**General idea of this work**

We fix \((\alpha_1, \alpha_2) \in \mathcal{T}\) which we assume close to a double resonant relation \((\alpha_j = p_j/q_j + \delta_j)\):

\[
k_1\alpha_1 + k_2\alpha_2 = k_3 \quad j_1\alpha_1 + j_2\alpha_2 = j_3
\]

and we study the dynamics of \(F_\delta\) at the double resonance \((\delta\text{ small} \rightarrow \text{res. BNF + unfolding})\).

Resonant lines (of order \(\leq 12\)) on the plane \((\alpha_1, \alpha_2)\).
Diffusion along phase space takes place basically along single resonances but multiple resonances play a key role in an explanation of the Arnold diffusion (e.g. Nekhoroshev theory – upper bounds on the rate of diffusion).

\[
(\psi, J) \rightarrow (\bar{\psi}, \bar{J}) = (\psi_1 + \delta(\bar{J}_1 + a_2\bar{J}_2), \psi_2 + \delta(a_2\bar{J}_1 + a_3\bar{J}_2), J_1 + \delta \sin \psi_1, J_2 + \delta \epsilon \sin \psi_2).
\]

\(\rightarrow \delta = 0.5, \epsilon = 0.1, a_2 = 0.5\) and \(a_3 = 1.25\).
List of contents

1. **Takens NF at the double resonance**: interpolating Hamiltonian.

2. **Arithmetic properties of the double resonances.**
   (a) Minimal generators / **primary** resonances.
   (b) Classification by primary resonances & **doubly strong** resonances.

3. **Non-integrability of the NF for generic weak double resonances.**
   (a) ** Normally hyperbolic** invariant cylinder $\Pi_\epsilon$.
   (b) The **splitting** between $W^{u/s}(C_h)$, $C_h$ p.o. (or the separatrices of the $HH$ fixed point) in $\Pi_0$ “at energy” $H_0 = h$.

4. **Non-analyticity of $\Pi_\epsilon$: a numerical experiment.**

5. **Beyond NF theory: a 4D standard-like map.**
   (a) Homoclinic trajectories.
   (b) **Exponentially small** splitting of the 2D inv. manifolds of the $HH$ point.
Takens NF

$F_\delta$ symplectic 4d maps ($\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$), $\delta \in \mathbb{R}^2$ small enough, $F_\delta(0) = 0$, Spec=$\{\lambda_1, \lambda_2, \bar{\lambda}_1, \bar{\lambda}_2\}$, $\lambda_k = \exp(2\pi i \alpha_k)$, $k = 1, 2$.

$$\alpha_1 \neq \alpha_2 \implies DF_0(0) \sim \Lambda_0 = \begin{pmatrix} R_{2\pi \alpha_1} & 0 \\ 0 & R_{2\pi \alpha_2} \end{pmatrix}$$

A canonical change of variables reduces $F_\delta$ to BNF $N_\delta$:

$$N_\delta \circ \Lambda_0 = \Lambda_0 N_\delta.$$

Since $DN_0(0) = \Lambda_0$ the map $\Lambda_0^{-1} N_\delta$ is tangent to the identity $\implies$ it can be formally interpolated (in a compact domain around 0) by a (Hamiltonian) vector field:

$$N_\delta = \Lambda_0 \Phi_{H_\delta}^1 + \exp. \text{ small error}$$
Moreover $H_δ$ is $Λ_0$-invariant ($H_δ = H_δ \circ Λ_0$) $\implies N_δ^j = Λ_0^j Φ_{H_δ}^j$ for all $j \in \mathbb{N}$ $\implies$ study the flow of $H_δ$ instead of iterations of $N_δ$.

To obtain $H_δ$:

→ Complex vbles $(z_k = x_k + iy_k, \bar{z}_k = x_k - iy_k)$, $Λ_0 = \text{diag}(λ_1, λ_2, \bar{λ}_1, \bar{λ}_2)$.

→ $z_1^j \bar{z}_1^k z_2^l \bar{z}_2^m$ resonant $\iff Λ_0$-invariant $\iff (j - k, l - m) \in Γ$.

Then $H_δ$ is a sum of res. monomials: $H_δ = \sum_{(j-k,l-m)\in \Gamma} h_{jklm}(δ) z_1^j \bar{z}_1^k z_2^l \bar{z}_2^m$

In Poincaré vbles ($I_j = \frac{|z_j|^2}{2}$, $φ_j = \text{arg} z_j$):

$$H_δ = \sum_{(k_1,k_2)\in \Gamma} a_{k_1k_2pq}(δ) I_1^{p+k_1/2} I_2^{q+k_2/2} \cos(k_1φ_1 + k_2φ_2 + b_{k_1k_2pq})$$

**Q:** Dominant terms of $H_δ$? Arithmetic properties of $Γ$ depending on $(α_1, α_2)$. 


Minimal generators of $\Gamma$: primary resonances

Recall: $\Gamma = \{ (k_1, k_2) : k_1 \alpha_1 + k_2 \alpha_2 = 0 \pmod{1} \} \subset \mathbb{Z}^2$ lattice.

Consider:

$\rightarrow r_0 = (k_1, k_2) \in \Gamma$ a smallest (maybe non-unique) non-trivial element,

$\rightarrow r_1 = (m_1, m_2) \in \Gamma$ any of the smallest elements independent from $r_0$.

$\implies r_0$ and $r_1$ generate $\Gamma$ (provided $\alpha_1 \neq \alpha_2$).

We call $r_0$, $r_1$ the primary resonances (minimal generators of $\Gamma$).
We denote by $n_j$ the order of $r_j$, $j = 1, 2$ (then, $n_0 \leq n_1$).

Remark: The primary resonances are unique for most of the frequencies. However, there are two situations of non-uniqueness:

- At the leading order. Consider $\alpha_1 = 1/8$, $\alpha_2 = 3/8$. Then $n_0 = n_1 = 4$ and there are 3 resonances of order 4: $(1, -3)$, $(3, -1)$, $(2, 2)$.
- At order $n_1$. Consider $\alpha_1 = 1/11$, $\alpha_2 = 4/11$. Then $n_0 = 4$ and the only resonance of order 4 is $(1, -3)$, and $n_1 = 5$ and there are 2 resonances: $(4, -1)$ and $(3, 2)$. 
Classification by primary resonances

If $n_0 \geq 5$ the fixed point is called weakly resonant and otherwise it is strongly resonant. There are different situations to study:

- $5 \leq n_0 < n_1$: Up to order $n_0 - 1$ the interp. Hamiltonian of BNF looks like the one of a non-resonant point. Moreover, up to order $n_1 - 1$ is also integrable.

- $5 \leq n_0 = n_1$: Up to order $n_0 - 1$ the interp. Hamiltonian of BNF looks like the one of a non-resonant point. Adding order $n_0$-terms becomes generically non-integrable.

- Simply strong resonances: $n_0 \leq 4 < n_1$.

- Doubly strong resonances: $n_0 = 3, n_1 = 4$ and $n_0 = n_1 \leq 4$. 
**Doubly strong resonances** $\alpha_1 \neq \alpha_2$

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Weak double resonances: a truncated model

Recall: $F_\delta, \lambda_k = \exp(2\pi i \alpha_k), \alpha_k = p_k/q_k + \delta_k$ for $k = 1, 2$, $\delta = ||\delta||$ small.

Takens NF: $H_\delta = \sum_{(k_1, k_2) \in \Gamma} a_{k_1 k_2 pq}(\delta) I_1^{p+k_1/2} I_2^{q+k_2/2} \cos(k_1 \varphi_1 + k_2 \varphi_2 + b_{k_1 k_2 pq})$

Assume (most common case!) that

→ $\alpha_1, \alpha_2$ are close to be doubly resonant,
→ $r_0 = (k_1, k_2)$ and $r_1 = (m_1, m_2)$ are the unique minimal generators of $\Gamma$,
→ $5 \leq n_0 < n_1$ (weak double resonance).

Adapt vbles: $\psi_1 = k_1 \varphi_1 + k_2 \varphi_2, \psi_2 = m_1 \varphi_1 + m_2 \varphi_2, I_1 = k_1 J_1 + m_1 J_2, I_2 = k_2 J_1 + m_2 J_2$

\[
H_\delta = H_0(J, \delta) + H_1(J, \psi_1, \delta) + H_2(J_1, J_2, \psi_1, \psi_2, \delta) + O_{n_1+1}(z)
\]

\[
H_0 = A_{00}(J_1, J_2, \delta),
\]

\[
H_1 = \sum_{l_1=1}^{n_1/n_0} I_1^{l_1 |k_1|/2} I_2^{l_1 |k_2|/2} A_{l_1 0}(J_1, J_2, \delta) \cos(l_1 \psi_1 + B_{l_1 0}(J_1, J_2, \delta)),
\]

\[
H_2 = I_1^{m_1/2} I_2^{m_2/2} A_{01}(0, 0, \delta) \cos(\psi_2 + B_{01}(0, 0, \delta)).
\]
Localizing around the double resonance

In a neighbourhood of the origin

$$H_0 = c_1 \delta J_1 + c_2 \delta J_2 + a_1 J_1^2 + a_2 J_1 J_2 + a_3 J_2^2 + \mathcal{O}(\delta^5)$$

$\rightarrow$ inv. $\mathbb{T}^2$ at $J_1 = \delta r_1$, $J_2 = \delta r_2 \Rightarrow$ inv. $\mathbb{T}^2$ for the NF system if $I_1, I_2 > 0$.

Then $J_k = \delta r_k + \delta n_0/4 \tilde{J}_k$ and $H = \delta n_0/2 \tilde{H}$ gives

$$H_0(J_1, J_2, \delta) = a_1 J_1^2 + a_2 J_1 J_2 + a_3 J_2^2 + \mathcal{O}(\delta n_0/4),$$

$$H_1(J_1, J_2, \psi_1, \delta) = \sum_{l_1=1}^{n_1/n_0} \delta^{(l_1-1)n_0/2} \tilde{A}_{l_10}(J_1, J_2, \delta) \cos(l_1 \psi_1 + \tilde{B}_{l_10}(J_1, J_2, \delta)), $$

$$H_2(J_1, J_2, \psi_1, \psi_2, \delta) = \delta^{(n_1-n_0)/2} a_{01} \cos(\psi_2 + b_{01}).$$

Furthermore, if $n_1 < 2n_0$ (different but similar order resonances) then

$$H_1(J_1, J_2, \psi_1, \delta) = (a_{10} + \delta n_0/4^{1-1} \hat{A}_{10}(J_1, J_2, \delta)) \cos \psi_1$$

$\rightarrow$ No other harmonics in $H_1$ appear!
Analysis of the truncated model

\[ H(\psi_1, \psi_2, J_1, J_2) = \frac{J_1^2}{2} + a_2 J_1 J_2 + a_3 \frac{J_2^2}{2} + \cos(\psi_1) + \epsilon \cos(\psi_2) \]

→ For the moment \( \epsilon \sim \delta^{(n_1-n_0)/2} \) will be considered as a small parameter.

→ Change: \( \tilde{\psi}_1 = \psi_1, \tilde{\psi}_2 = \psi_2 - a_2 \psi_1, \tilde{J}_1 = J_1 + a_2 J_2, \tilde{J}_2 = J_2 \)

\[ H(\psi_1, \psi_2, J_1, J_2) = \frac{J_1^2}{2} + dJ_2^2/2 + \cos(\psi_1) + \epsilon \cos(\psi_2 + a_2 \psi_1), \]

where \( d = a_3 - a_2^2 \). We assume \( d \neq 0 \).

→ 4 fixed points: If \( \nu = \epsilon d > 0 \) and \( |\epsilon| \) small enough

\[ p_1 = (0, 0, 0, 0) - HH, \quad p_2 = (0, \pi, 0, 0) - HE \]
\[ p_3 = (\pi, -a_2 \pi, 0, 0) - EH, \quad p_4 = (\pi, (1-a_2) \pi, 0, 0) - EE \]

→ Reversiblities:

\[ R_0(\psi_1, \psi_2, J_1, J_2) = (2\pi - \psi_1, -2\pi a_2 - \psi_2, J_1, J_2), \]
\[ R_1(\psi_1, \psi_2, J_1, J_2) = (2\pi - \psi_1, 2\pi (1-a_2) - \psi_2, J_1, J_2). \]
\[ H(\psi_1, \psi_2, J_1, J_2) = \frac{J_1^2}{2} + dJ_2^2/2 + \cos(\psi_1) + \epsilon \cos(\psi_2 + a_2 \psi_1) \]

\[ \epsilon = 0: \]

→ \( \psi_2 \) is cyclic (\( J_2 \) is a first integral) \( \Rightarrow \) Pendulum (fast) dynamics in \( \psi_1, J_1 \)-coord. given by \( H_1^0 = \frac{J_1^2}{2} + \cos(\psi_1) \).

→ The cylinder \( \Pi_0^0 = \Pi_0^{2\pi} = \{ \psi_1 = 0 \pmod{2\pi}, J_1 = 0 \} \) is a 2D NHIM.

→ \( \Pi_0^0 \) is foliated by p.o. \( C_h^0 = \Pi_0^0 \cap \{ H = h \} \).

→ \( W^u(\Pi_0^0) \) is given by \( J_1 = 2 \sin(\psi_1/2) \).

→ Non-transversal: \( W^u(\Pi_0^0) = W^s(\Pi_0^0) \).
The perturbed NHIM

\[ H(\psi_1, \psi_2, J_1, J_2) = J_1^2/2 + dJ_2^2/2 + \cos(\psi_1) + \epsilon \cos(\psi_2 + a_2 \psi_1) \]

\( \epsilon \neq 0 \): Normal hyperbolicity theory (Fenichel)

\[ \exists \Pi_\epsilon^0 \text{ (resp. } \Pi_\epsilon^{2\pi}) \mathcal{O}(\epsilon)-\text{close to } \Pi_0^0 \text{ (resp. } \Pi_0^{2\pi}). \]

(the perturbed system is not \( \psi_1 \)-periodic: \( R_k(\Pi_\epsilon^0) \neq \Pi_\epsilon^0 \))

\[ \exists \ W^{u/s}(\Pi_\epsilon^0) \mathcal{O}(\epsilon)-\text{close to } W^{u/s}(\Pi_0^0) \]

\[ W^u(\Pi_\epsilon^0) \text{ given by a graph } J_1 = 2 \sin \frac{\psi_1}{2} + \epsilon f(\psi_1, \psi_2, J_2; \epsilon), \]

with \( f_0 = \frac{1}{2 \sin \frac{\psi_1}{2}} \int_0^{\psi_1} g(s, \psi_2 + dJ_2 \log[\tan(\frac{s}{4})/\tan(\frac{\psi_1}{4})]) \, ds \)

and \( g(\psi_1, \psi_2) = a_2 \sin(\psi_2 + a_2 \psi_1) \).

Transversality? The distance between \( W^{u/s}(\Pi_\epsilon^{0/2\pi}) \) on \( \psi_1 = \pi \) is

\[ J_1^s - J_1^u = \epsilon \left( f_0(\pi, -2\pi a_2 - \psi_2, J_2) - f_0(\pi, \psi_2, J_2) \right) + \mathcal{O}(\epsilon^2) = \]

\[ = \epsilon A(J_2) \sin(\psi_2 + \pi a_2) + \mathcal{O}(\epsilon^2), \]

where

\[ A(J_2) = a_2 \int_0^\pi \cos(a_2(s - \pi) + dJ_2 \log[\tan(\frac{s}{4})]) \, ds. \]
Transversality properties

\[ J^s_1 - J^u_1 \] vanishes for \( \psi_2 = -\pi a_2 \pmod{\pi} \) \( \Rightarrow \) 2 lines of homoclinics

\[ \ell_0 = \{(\psi_1, \psi_2, J_1, J_2) : J_1 = 2 + \epsilon f(\psi_1, \psi_2, J_2; \epsilon), \psi_1 = \pi, \psi_2 = -\pi a_2 \}, \]

\[ \ell_1 = \{(\psi_1, \psi_2, J_1, J_2) : J_1 = 2 + \epsilon f(\psi_1, \psi_2, J_2; \epsilon), \psi_1 = \pi, \psi_2 = \pi(1 - a_2) \}. \]

\( W^u(\Pi_0^0) \) intersects \( W^s(\Pi_2^{2\pi}) \) transversally provided \( A(J_2) \neq 0. \)

\[ \rightarrow \] For \( J_2 = 0, A(J_2) = \sin(\pi a_2) \Rightarrow \text{transversality if } a_2 \notin \mathbb{Z}. \]

\[ \rightarrow \] For \( J_2 \) arbitrary, \( A(J_2) \) vanishes on the lines

Fix \( a_2 \) and \( d \), then:

* For \( J_2 \to \infty \), the lines accumulate to \( k/2, k \in \mathbb{Z} \).

* Finite number of zeros of \( A(J_2) \) \( \Rightarrow \)

Given \( a_2 \) except for some concrete values of \( J_2 \) the intersections are transversal.
Dynamics on the NHIM and transversality

The dynamics on $\Pi^0_\epsilon$ is given by

$$H^0_{2,\epsilon}(\psi_2, J_2) = d\frac{J_2^2}{2} + \epsilon \cos(\psi_2) + 1 + O(\epsilon^2),$$

where $(\psi_2, J_2)$ are used as coordinates on the cylinder.

$\Pi^0_\epsilon$ contains the fixed points $p_1$ (HH) and $p_2$ (HE).

**Q:** Assume that $W^u(\Pi^0_\epsilon)$ intersects $W^s(\Pi^{2\pi}_\epsilon)$ transversally. Does it imply that the separatrices of $C^e_h$ intersect transversally inside $\{ H = h \}$? **NO.**

Any $p \in \ell_k$ is homoclinic to $C^e_h$ with $h = H(p)$. Then,

transversality inside $\{ H = h \} \iff T_p\ell_k \not\in T_p\{ H = h \} \iff h \neq 1 + \epsilon(-1)^k \cos \pi a_2 + O(\epsilon^2), k = 0, 1.$

**Remark.** If $a_2 \not\in \mathbb{Z} \Rightarrow H(p_2) < E_k < H(p_1), k = 1, 2$ (inside the pendulum within $\Pi^0_\epsilon$).
Dynamics on the NHIM and transversality (II)

Assume $a_2 \notin \mathbb{Z}$:

- Consider $h = H(p_1) = 1 + \epsilon$ then $C^\epsilon_h$ are the separatrices.
  - $C^h_0 = \{ J_2 = 0 \}$ (line of fixed points) $\Rightarrow W^u(C^\epsilon_h)$ and $W^s(R_k(C^\epsilon_h))$
    intersect transversally because $A(0) \neq 0$ and $H(p_1) > E_k$.
  - Melnikov (Kovacic) or slow-fast analysis (Haller) $\Rightarrow$ angle is $\mathcal{O}(\sqrt{\epsilon})$.

- Consider $h < H(p_1)$: initially $\ell_k \cap W^u(C^\epsilon_h)$ consists of 2 homoclinic points which collide when $h = E_k$ and disappear $\Rightarrow$ there are no primary homoclinic orbits to $p_2$ (EH) because $E_k > H(p_2)$. 

$\epsilon = 0.1$, $a_2 = 0.25$, $d = 0.5$ and $h = 1 + \epsilon$
Non-analyticity of $\Pi_\epsilon$: a numerical experiment

Normal hyperbolicity theory $\Rightarrow$ $\Pi_\epsilon$ is (at most) $C^r$ with $r = \tilde{\lambda}_1 / \tilde{\lambda}_2$ the quotient of the normal and tangent maximal Lyapunov exponents.

Q: For concrete parameters, can we easily detect the lack of analyticity?

Notations:

- Consider $p_1$ (HH fixed point). We define

\[ W^{u,\Pi_0}_\text{slow}(p_1) = W^u(p_1) \cap \Pi_0 \]

(the separatrices of the pendulum inside $\Pi_\epsilon$)

- We say that an 1D invariant submanifold $W^{u}_\text{slow}(p_1) \subset W^u(p_1)$ is a slow unstable invariant manifold if it is tangent at $p_1$ to the eigenvector related to the slow eigenvalue $\tilde{\lambda}_2$.

There are infinitely many slow unstable invariant manifolds (in particular, $W^{u,\Pi_0}_\text{slow}(p_1)$).
Basic idea

Denote by $X_H$ the vector field related to

$$H(\psi_1, \psi_2, J_1, J_2) = \frac{J_1^2}{2} + a_2 J_1 J_2 + a_3 \frac{J_2^2}{2} + \cos(\psi_1) + \epsilon \cos(\psi_2).$$

→ In these coord. $\Pi^0_0$ is given by $\psi_1 = 0, J_1 + a_2 J_2 = 0$.

→ Denote by $\tilde{\Lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2)$ the unstable eigenvalues of $DX_H(p_1)$.

Basic observations:

1. If $\Pi^0_{\epsilon}$ is an analytic manifold then $W^u_{\epsilon, \Pi^0_{\epsilon}}(p_1)$ is analytic.

2. If $\tilde{\Lambda}$ is non-resonant, then there is a unique analytic slow unstable invariant submanifold $W^u_{\text{slow}}(p_1)$ which will be denoted by $W^u_{\text{slow}}(p_1)$. On the other hand, if $\tilde{\Lambda}$ is resonant then generically all slow unstable submanifolds are non-analytic.
Proof of the basic observations

1. Assume $\Pi_0^0$ analytic $\implies H|_{\Pi_0^0}$ is an analytic Hamiltonian which has a non-degenerated saddle fixed point with eigenvalues $\pm \tilde{\lambda}_2$
$\implies W^{u,\Pi_0^0}(p_1)$ is a 1D analytic inv. manifold.

2. The restriction $X$ of the vector field (v.f.) to $W^u(p_1)$ is a 2D analytic v.f. with $(0, 0)$ as a repulsive fixed point with eigenvalues $\tilde{\Lambda}$.
   - If $\tilde{\Lambda}$ non-resonant $X$ is conjugated to $\dot{s}_1 = \tilde{\lambda}_1 s_1$, $\dot{s}_2 = \tilde{\lambda}_2 s_2$.
     Solutions: $s_1 = C s_2^{\tilde{\lambda}_1/\tilde{\lambda}_2}$ (or $s_2 = \hat{C} s_1^{\tilde{\lambda}_2/\tilde{\lambda}_1}$) non-analytic except $s_1 = 0$ and $s_2 = 0$.
   - If $\tilde{\Lambda}$ resonant (i.e. $\tilde{\lambda}_1 = k\tilde{\lambda}_2$, $k \in \mathbb{N}$) $X$ is conjugated to
     \[\begin{align*}
     \dot{s}_1 &= \tilde{\lambda}_1 s_1 + \nu s_2^k, \\
     \dot{s}_2 &= \tilde{\lambda}_2 s_2.
     \end{align*}\]
     Solutions: $s_1 = \nu s_2^k\left(\log(s_2) + C\right)/\tilde{\lambda}_2$ non-analytic except $s_2 = 0$. 
How to proceed

Conclusion:

- If $\tilde{\Lambda}$ is resonant then $\Pi_\varepsilon^0$ is generically non-analytic.
- If $\tilde{\Lambda}$ is non-resonant then the analyticity of $\Pi_\varepsilon^0$ implies
  \[ W_{\text{slow}}^{u,\Pi_\varepsilon^0}(p_1) = W_{\text{slow}}^{u,\omega}(p_1) \]
  by uniqueness of the analytic invariant unstable manifold.

To check non-analyticity of $\Pi_\varepsilon^0$:

1. Consider the generic non-resonant case.
2. Numerically check that the analytic $W_{\text{slow}}^{u,\omega}(p_1)$ leaves the cylinder $\Pi_\varepsilon^0$.
3. It follows that $W_{\text{slow}}^{u,\Pi_\varepsilon^0}(p_1) \neq W_{\text{slow}}^{u,\omega}(p_1)$ and, consequently, that $\Pi_\varepsilon^0$ is non-analytic.

Remark. The tangency order between the analytic solution $s_1 = 0$ and the other solutions of $\dot{s}_1 = \tilde{\lambda}_1 s_1$, $\dot{s}_2 = \tilde{\lambda}_2 s_2$ is $r_* = \tilde{\lambda}_1/\tilde{\lambda}_2 \implies$ necessary to approximate $W_{\text{slow}}^{u,\omega}(p_1)$ up to order $k > r_*$. 
Example

• Parameters: $a_2 = 0.25$, $a_3 = 0.5625$ and $\epsilon = 0.1$.

• $\tilde{\Lambda}$ non-resonant ($\tilde{\lambda}_1 \approx 1.003282954$ and $\tilde{\lambda}_2 \approx 0.222875109$, then $r_* \approx 4.501547788$).

• Compute the parametric representation $\psi_i(s)$, $J_i(s)$ of $W_u^{\text{slow}}(p_1)$.

  $$
  \psi_i(s) = \sum_{k \geq 1} \psi_k^{(i)} s^k, \quad J_i(s) = \sum_{k \geq 1} J_k^{(i)} s^k, \quad i = 1, 2
  $$

• Truncating at order $k = 120$ the approximation of $W_u^{\text{slow}}(p_1)$ has an error below $10^{-15}$ for $|s| < s_* \approx 0.3635$. 
A posteriori check: $\Pi^0_\epsilon$ is $C^4$ but not $C^5$

We compute $W^u(p_1)$. Parametrization:

$$g(s_1, s_2) = (\psi_1(s_1, s_2), \psi_2(s_1, s_2), J_1(s_1, s_2), J_2(s_1, s_2))$$ where

$$\psi_i(s_1, s_2) = \sum_{k,l \geq 1} \psi_{k,l}^{(i)} s_1^k s_2^l, \quad J_i(s_1, s_2) = \sum_{k,l \geq 1} J_{k,l}^{(i)} s_1^k s_2^l, \quad i = 1, 2.$$

At order 20: error below $10^{-15}$ for $s = (s_1, s_2)$ s.t. $\|s\| < s_* \approx 0.282$.

* $s_1 = 0$ corresponds to $W_{\text{slow}}^u(p_1)$ (analytic)

* We compute the non-analytic $1D$ submanifold of $W^u(p_1)$ within $\Pi^0_\epsilon$ by imposing that the homoclinic has $\psi_1 = 0$.

\[
f(x) = r_* x - 5.55007880852\]
Beyond NF: a 4D map

Truncated NF: \( H(\psi_1, \psi_2, J_1, J_2) = \frac{J_1^2}{2} + a_2 J_1 J_2 + a_3 \frac{J_2^2}{2} + \cos(\psi_1) + \epsilon \cos(\psi_2) \)

Generating function: \( S(\psi_1, \psi_2, J_1, J_2) = \psi_1 \bar{J}_1 + \psi_2 \bar{J}_2 + \delta \mathcal{H}(\psi_1, \psi_2, J_1, J_2). \)

\[
T_{\delta} : \begin{pmatrix} \psi_1 \\ \psi_2 \\ J_1 \\ J_2 \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \\ \bar{J}_1 \\ \bar{J}_2 \end{pmatrix} = \begin{pmatrix} \psi_1 + \delta(\bar{J}_1 + a_2 \bar{J}_2) \\ \psi_2 + \delta(a_2 \bar{J}_1 + a_3 \bar{J}_2) \\ J_1 + \delta \sin(\psi_1) \\ J_2 + \delta \epsilon \sin(\psi_2) \end{pmatrix}
\]

Phase space structure similar to \( H \) (but the homoclinic trajectories split!):

- 4 fixed points: \( p_1 \) HH, \( p_2 \) HE, \( p_3 \) EH, \( p_4 \) EE.
- NHIM \( \Pi_{\epsilon, \delta}^0 \).
- Reversible: \( R_1 = (-\psi_1, 2\pi - \psi_2, \bar{J}_1, \bar{J}_2) \), \( R_2 = (2\pi - \psi_1, 2\pi - \psi_2, \bar{J}_1, \bar{J}_2) \) and \( R_3 = (2\pi - \psi_1, -\psi_2, \bar{J}_1, \bar{J}_2) \).
The homoclinic trajectories

Reversibilities $\Rightarrow T$ has 6 primary homoclinic trajectories.

"Pendulum" separatrix in $\Pi^0_{\epsilon,\delta}$

$\delta = 0.1$
$\epsilon = 0.1$
$a_2 = \frac{1}{4}$
$a_3 = \frac{9}{16}$
Splitting of the invariant manifolds

To measure the splitting of the manifolds $W^{u,s}(p_1)$, at the homoclinic point $p_h$ on $\Sigma_{R_k}$, $k = 1, 2, 3$, we compute the volume $V$ defined as follows:

- Let 
  
  $$(\psi_1, \psi_2, J_1, J_2) = (G_1(s_1, s_2), G_2(s_1, s_2), G_3(s_1, s_2), G_4(s_1, s_2)),$$
  
  $s_1, s_2 \in \mathbb{R}$, be the parametrisation of the 2D local $W^u(p_1)$.

- Consider $(s_1^h, s_2^h)$ such that the point with coordinates
  
  $$(G_1(s_1^h, s_2^h), G_2(s_1^h, s_2^h), G_3(s_1^h, s_2^h), G_4(s_1^h, s_2^h))$$
  
  belongs to the homoclinic trajectory defined by $p_h$.

- $V$ is the volume defined by the unitary tangent vectors $v_j = \tilde{v}_j / \|\tilde{v}_j\|$

  where

  $\tilde{v}_1(s_1^h, s_2^h) = (\partial G_i/\partial s_1)(s_1^h, s_2^h)$,
  $\tilde{v}_2(s_1^h, s_2^h) = (\partial G_i/\partial s_2)(s_1^h, s_2^h)$,
  $\tilde{v}_3(s_1^h, s_2^h) = R_1(\tilde{v}_1(s_1^h, s_2^h))$ and
  $\tilde{v}_4(s_1^h, s_2^h) = R_1(\tilde{v}_2(s_1^h, s_2^h))$. 
Background from 2D maps

Consider $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ analytic APM. Assume:

* $F(0) = 0$ hyperbolic fixed point, Spec=$\{\lambda, 1/\lambda\}$, $\lambda \approx 1 + \epsilon$, $\epsilon << 1$.
* $F \sim \varphi_{t=\log \lambda}^H$, $H$ is the so-called limit Hamiltonian.

Let $\Gamma(t)$ the separatrix of $H \Rightarrow \Gamma(t)$ has singularities for $t \in \mathbb{C} \setminus \mathbb{R} \Rightarrow$

Let $\tau$ the closest singularity to the real axis.

Then (generically):

$$\sigma \sim A(\log \lambda)^B e^{-2\pi \text{Im} \tau / \log \lambda}$$
Asymptotic behaviour of $V$ in $\Sigma_{R_1}$

For a fixed $\epsilon$, $a_2$ and $a_3$ parameters we study the behaviour as $\delta \to 0$.

At $p_h$ in $\Sigma_{R_1}$ (homoclinic trajectory on $\Pi_{\epsilon,\delta}^0$):

$$V \sim A \mu_2^B e^{-2\pi \text{Im } \hat{\tau}_2/\mu_2}$$

where $\mu_2 = \log \tilde{\lambda}_2$, $A, B \in \mathbb{R}$ and $\hat{\tau}_2 = i\pi/2 + \mathcal{O}(\sqrt{\epsilon})$ is related to the “closest” singularity $\tau_2$ of the homoclinic trajectory of the limit vector field.

$\epsilon = 0.1$, $a_2 = 0.25$, $a_3 = 0.5625$. Left: $\log V$ vs. $\delta$. Right: $\text{Im } \hat{\tau}_2$ vs. $\epsilon$.

Different values of $\epsilon$, $a_2$ and $a_3$ have been considered.
Asymptotic behaviour of $V$ in $\Sigma_{R_2}$ and $\Sigma_{R_3}$

In both cases, the volume $V$ behaves like

$$V(\delta) \sim A\mu_1^B e^{-2\pi\text{Im}\hat{\tau}_1/\mu_1},$$

with $\mu_1 = \log \tilde{\lambda}_1$ and where $\hat{\tau}_1$ is related to the closest singularity $\tau_1$ of the homoclinic trajectory of the limit Hamiltonian flow which has a homoclinic point on the plane $(\psi_1, \psi_2) = (\pi, \pi)$ (or $(\psi_1, \psi_2) = (\pi, 0)$).

→ Note that the limit homoclinic trajectory is not explicitly known.
→ There is no developed theory available (4D!) supporting the previously given asymptotic formulas.

Remark. Our numerics support the fact that $B = -3$ in all the cases and independently of $\epsilon, a_2$ and $a_3$. Further investigations are needed.
Future directions

- Consider $|\epsilon|$ in a non-perturbative regime (e.g. two resonances of equal order).

  In particular, for $|\epsilon|$ large the EE fixed point can suffer a Hamiltonian-Hopf bifurcation (complex instability).

- Clarify the situations where a different truncated model is obtained, and study the strong doubly resonant cases.

- Analyse the diffusion properties (obtain quantitative data from massive numerical simulations, and relate it with the geometry at the simple/double resonances).

→ Work in progress with E.Fontich, V.Gelfreich and C.Simó.
Thanks for your attention!!