Hamiltonian-Hopf bifurcation under a periodic forcing
Quasi-periodicity in splitting separatrices

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Abstract
We consider the effect of a non-autonomous periodic perturbation on a 2-dof autonomous system obtained as a truncation of the Hamiltonian-Hopf normal form. We study the splitting of the invariant 2-dimensional stable manifolds of a focal point. Due to the interaction of the resonant angle and the periodic perturbation the splitting behaves quasi-periodically on two angles. Different frequencies are considered: quadratic irrational, frequencies having common fraction expansions with bounded and unbounded quotients, and “typical” frequencies in measure-theoretical sense.

The model
We consider the system
\[\dot{x}(y, t) = H(x, y, t) + \epsilon \tilde{H}(x, y, t),\]
where
\[H(x, y, t) = \begin{pmatrix} x_1 \cdot y_1 \cdot \gamma_1(t) + \epsilon \tilde{H}(x, y, t) \\ \end{pmatrix},\]
and
\[\tilde{H}(x, y, t) = \begin{pmatrix} \tilde{H}_1(x, y, t) \\ \tilde{H}_2(x, y, t) \\ \end{pmatrix} = \begin{pmatrix} \phi(x, y, t) \\ \phi(y, x, t) \\ \end{pmatrix},\]
where \(\gamma(t) = \cos((\pi/2) \cdot t).\)

We will discuss the case \(\gamma(t) = \cos((\pi/2) \cdot t).\)

<table>
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<th>(x)</th>
<th>(y)</th>
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<tr>
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<td>0.2</td>
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<tr>
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<td>0.4</td>
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<tr>
<td>2</td>
<td>0.5</td>
<td>0.6</td>
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The unperturbed system \(\tilde{H}.\) The functions \(G_1 = \Gamma_1 + y_2 + \Gamma_2 - \Gamma_1 + y_1 + G_2\) are independent first integrals. In polar coordinates \(r = \sqrt{x_1^2 + y_1^2}, \quad \theta = \tan^{-1}(y_1/x_1), \quad \psi = y_2 + \Gamma_2 - \Gamma_1 + y_1 + G_2, \quad \psi = y_2 + \Gamma_2 - \Gamma_1 + y_1 + G_2\) the Hamiltonian is \(H = \frac{1}{2} r^2 + \psi.\)

The invariant manifolds \(W^{s/u}(\psi)\) for different \(\psi\) values
The angles \((x_0, y_0)\) are initial conditions on a fundamental domain (torus) \(T.\) We have considered \(\epsilon = 0.1, \delta = 0.01, \gamma = 0.01, \text{ and } \tau = \sqrt{2}.\)

The Poincaré-Melnikov integral
For simplicity, we discuss on the \(G_1\)-splitting (similar to the \(G_2\)-splitting). Recall that \(H_1 = g(y, \phi(t))\), where \(g(y, \phi) = \gamma(y - \phi)^{-1} + \epsilon \tilde{H}(x, y, t)\).

The contribution to the leftmost change is
\[\int_{T} \frac{\partial L}{\partial t} d\tilde{L}, \quad \text{where} \quad s = \frac{1}{m} \frac{\partial L}{\partial t} d\tilde{L}, \quad \text{and} \quad \tilde{L} = \int_{T} \frac{\partial L}{\partial t} d\tilde{L}.

Main result
Assume that \(\epsilon > 0, \delta > 1, \phi \in \mathbb{R}, \varphi \in \mathbb{R}, \text{ and } \epsilon < \frac{\varphi}{\varphi^2} \leq \frac{1}{2}.\)

Theorem. There exists a “universal” (almost independent of \(\gamma\)) function \(\psi(\phi, t)\), which the contribution of the harmonic approximated by \(\psi\) in the splitting satisfies
\[\psi(\xi, \Delta, \gamma, \phi, t) = \sqrt{\mathbb{E}[\tilde{H}(\xi, \Delta, \gamma, \phi, t)]}, \quad \text{where} \quad \psi(\phi, \Delta, \gamma, \phi, t) = \frac{1}{\Delta \gamma \phi} \cdot \mathbb{E}[\tilde{H}(\xi, \Delta, \gamma, \phi, t)].

We are interested in the function \(\psi(\phi, t)\). We denote by \(\psi(\phi, t)\) the function of \(\phi\) periodic in \(t\) with frequency \(\gamma\). Hence, \(\phi(t) \in [\psi(\phi, t) \leq \psi(\phi, t) + 2\pi]\) leads to quasi-periodic phenomena.

Other frequencies: hidden/not hidden best approximants
Assume that (our system satisfies these assumptions):\n\[\psi(\phi, t) \text{ is periodic in } t, \quad \text{and} \quad \psi(\phi, t) = \psi(\phi + 2\pi, t).

The role of CFE appears as \(\psi(\phi, t) \approx 2\pi \text{ for } \phi \approx \psi(\phi, t).\)

Hidden BA (HBA) and “typical” measure-theoretical properties
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References

Figure 1: Splitting of the invariant manifolds: \(G_1\) (left) and \(G_2\) (right) for \(n = 2\) (top) and \(n = 2\) (bottom).

Figure 2: For \(\gamma = \sqrt{\frac{5}{2}} - \frac{1}{2}\), we represent \(\text{P}_{\text{C}}(\text{C}_{\text{H}})(\psi, \phi, t)\) as a function of \(\psi, \phi\).

Figure 3: We display \(\psi(\phi, t)\) as a function of \(\phi\). The right plot shows the function \(\psi(\phi, t)\) in the right plot.

Figure 4: We display the results for \(\gamma = \sqrt{\frac{5}{2}} - \frac{1}{2}\).