# Optimal Sobolev embeddings and Function Spaces


Nadia Clavero

Director: Javier Soria

Tutor: Joan Cerdà


Máster en Matemática Avanzada y Profesional
Especialidad Académica Avanzada
Curso 2010/2011


UNIVERSITAT DE BARCELONA

# Contents

# Chapter 1

# Introduction

We study the optimality of rearrangement invariant Banach spaces in Sobolev embeddings. In other words, we would like to know that the rearrangement invariant Banach range space and the rearrangement invariant Banach domain space are optimal in the Sobolev embedding, in the sense that domain space cannot be replaced by a larger rearrangement invariant Banach space and range space cannot be replaced a smaller one. Before commenting on a brief description of the central part of this work, we will present some facts about rearrangement invariant Banach spaces.

Let $n$ be a positive integer with $n \geq 2$. Let $\Omega \subset \mathbb{R}^n$ be an open subset with $|\Omega| = 1$. Let $f$ be a real-valued measurable function in $\Omega$. The decreasing rearrangement of $f$ is the function $f^*$ on $[0, \infty)$ defined by

$$f^*(t) = \inf \{\lambda > 0 \, ; |x \in \Omega \, ; |f(x)| > \lambda| \leq t\} \, , \quad 0 < t < |\Omega| \, .$$

A set of real-valued measurables functions in $\Omega$ is called a rearrangement invariant Banach space if it is a linear space equipped with a norm $\|\cdot\|_{X(\Omega)}$ satisfying the following properties:

- if $0 \leq f \leq g$ a.e. and $g \in X(\Omega)$, then $f \in X(\Omega)$ and $\|g\|_{X(\Omega)} \leq \|f\|_{X(\Omega)}$ ;
- if $0 \leq f_n \uparrow f$ a.e. and $f \in X(\Omega)$, then $\|f_n\|_{X(\Omega)} \uparrow \|f\|_{X(\Omega)}$ ;
- $\|\chi_E\|_{X(\Omega)} < \infty$ for every $E \subseteq \Omega$ such that $|E| < \infty$; here, $\chi_E$ denotes the chacteristic function of the set $E$.
- for every $E \subseteq \Omega$ with $|E| < \infty$, there exists a constant $C$ such that

$$\int_E f(x) dx \leq C \, \|f\|_{X(\Omega)} \, , \quad \text{for all } f \in X(\Omega);$$

- if $f \in X(\Omega)$ and $f^* = g^*$, then $g \in X(\Omega)$ and $\|g\|_{X(\Omega)} = \|f\|_{X(\Omega)}$ .

According to a fundamental result of Luxemburg [2, Theorem II.4.10] to every rearrangement invariant Banach space $X(\Omega)$ there corresponds a rearrangement invariant Banach space $\overline{X}\,(0, |\Omega|)$ such that

$$\|f\|_{X(\Omega)} = \|f^*\|_{\overline{X}(0,|\Omega|)} \, , \quad \text{for every } f \in X(\Omega).$$

As examples of rearrangement invariant Banach spaces let us mention Lebesgue, Orlicz, Lorentz and Lorentz-Zygmund spaces. For a detailed treatment of the theory of rearrangement invariant Banach spaces, we refer to [2] and [13].

Fix $m \in \mathbb{N}$. Let $X(\Omega)$ be a rearrangement invariant Banach space. The Sobolev space $W^m X(\Omega)$ is defined as

$$W^m X(\Omega) := \left\{ f : \Omega \to \mathbb{R} \,; D^\alpha f \text{ is defined and } \|D^\alpha u\|_{X(\Omega)} < \infty,\ 0 \le |\alpha| \le m \right\},$$

where $D^\alpha f = \frac{\partial^\alpha f}{\partial x^\alpha}$ ($\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$) represents the distributional derivate of $f$. This space is a Banach space with respect to the norm given by

$$\|f\|_{W^m X(\Omega)} = \sum_{0 \le |\alpha| \le m} \|D^\alpha f\|_{X(\Omega)},$$

The notation $W_0^m X(\Omega)$ is employed for the closure of $\mathcal{D}(\Omega)$ in $W^m X(\Omega)$, where $\mathcal{D}(\Omega)$ is the set of $C^\infty(\Omega)$ functions with compact support in $\Omega$. The Sobolev embedding theorem states that:

- if $1 \le p < n$, then
$$W_0^1 L^p(\Omega) \hookrightarrow L^{p^*}(\Omega), \quad \text{where } p^* = (n-p)/pn; \tag{1.1}$$
- if $p = n$, then
$$W_0^1 L^p(\Omega) \hookrightarrow L^q(\Omega), \quad \text{for every } 1 \le q < \infty; \tag{1.2}$$
- if $p > n$, then
$$W_0^1 L^p(\Omega) \hookrightarrow L^\infty(\Omega). \tag{1.3}$$

For more information on Sobolev spaces and Sobolev embeddings theorem, we refer to [1], [19], [21] and [27].

Let $m \in \mathbb{N}$ with $1 \le m \le n-1$. We study the optimality of rearrangement invariant Banach spaces in Sobolev embeddings. In other words, we want to solve the following problem: Given two rearrangement invariant Banach spaces $X(\Omega)$ and $Y(\Omega)$ such that

$$W_0^m X(\Omega) \hookrightarrow Y(\Omega), \tag{1.4}$$

we want to find the optimal pair of rearrangement invariant Banach spaces in the Sobolev embedding (1.4). To solve this problem, we start with the rearrangement invariant Banach space $X(\Omega)$ and then find the smallest rearrangement invariant Banach space $Y_X(\Omega)$ that still renders (1.4) true. Thus, the embedding

$$W_0^m X(\Omega) \hookrightarrow Y_X(\Omega) \hookrightarrow Y(\Omega),$$

has an optimal range, but does not necessarily have an optimal domain; that is $X(\Omega)$ could be replaced by a larger rearrangement invariant Banach space without losing (1.4). We take one more step in order to get the optimal domain partner for $Y_X(\Omega)$, let us call it $X_{Y_X}(\Omega)$. Altogether, we have

$$W_0^m X(\Omega) \hookrightarrow W_0^m X_{Y_X}(\Omega) \hookrightarrow Y_X(\Omega) \hookrightarrow Y(\Omega),$$

and $W_0^m X_{Y_X}(\Omega)$ can be either equivalent to $W_0^m X(\Omega)$ or strictly larger. In any case, after two steps, the pair $(X_{Y_X}(\Omega), Y_X(\Omega))$ forms an optimal pair in the Sobolev embedding and no futher iterations of the process can bring anything new.

Before commenting on our main theorem, let us discuss some refinements of Sobolev embeddings.

The embedding (1.1), which is known as classical Sobolev embedding, cannot be improved in the context of Lebesgue spaces; in other words, if we replace $L^p(\Omega)$ by a larger Lebesgue space $L^q$ with $q < p$, the resulting embedding does not hold. Likewise, if we replace $L^{p^*}(\Omega)$ by smaller Lebesgue space $L^r(\Omega)$ with $r > p^*$, then again the resulting embedding does not hold. However, if we consider Lorentz spaces, we have the following refinement of (1.1):

$$W_0^1 L^p(\Omega) \hookrightarrow L^{p^*,p}(\Omega). \tag{1.5}$$

This embedding was observed by Peetre [24] and O'Neil [23]. Now, when $p = n$ it is known that the Sobolev space $W_0^1 L^n(\Omega)$ can be embedded in every $L^q(\Omega)$ with $1 \le q < \infty$, and it cannot be embedded in $L^\infty(\Omega)$. So, (1.5) is optimal within the Lebesgue spaces, where no improvement is available. However, if we consider Orlicz spaces, $L^\Phi(\Omega)$, we have the following refinement

$$W_0^1 L^n(\Omega) \hookrightarrow L^\Phi(\Omega), \quad \Phi(t) = \exp(t^{n'}). \tag{1.6}$$

This result was shown, independently, by Pokhozhaev [25], Trudinger [29] and Yudovich [17]. It turns out that $L^\Phi(\Omega)$ is the smallest Orlicz space that still renders (1.6). This optimally is due to Hempel, Morris and Trudinger [15]. An improvement of (1.6) is possible, but we need to introduce the Lorentz-Zymund spaces. Equipped with these spaces, we have

$$W_0^1 L^n(\Omega) \hookrightarrow L^{\infty,n;-1}(\Omega). \tag{1.7}$$

This embedding is due to Brézis-Waigner [3] and Hansson [14].

The study of the optimality in Sobolev embeddings can be formulated as follows. We are interesting in determining those rearrangement invariant Banach spaces such that

$$W_0^m X(\Omega) \hookrightarrow Y(\Omega), \quad m \in \mathbb{N}, \text{with } 1 \le m \le n-1. \tag{1.8}$$

We would like to know that $X(\Omega)$ and $Y(\Omega)$ are optimal. Kerman and Pick [18] solved this problem. The central part of their work may be summarized as follows. They developed a method that enables us to reduce the Sobolev embedding (1.8) to the boundedness of certain weighted Hardy operator; and then used it to characterize the largest rearrangement invariant Banach domain space and the smallest rearrangement invariant Banach range space in the Sobolev embedding (1.8). Their method allows us to conclude that:

- if $1 \le p < n$, the couple $\big(L^p(\Omega), L^{p^*,p}(\Omega)\big)$ forms an optimal pair for the Sobolev embedding (1.1);

- if $p = n$, the range space $L^{\infty,n;-1}(\Omega)$ is an optimal rearrangement invariant Banach space in the Sobolev embedding (1.7), but the domain space $L^n(\Omega)$ is not optimal, since it can be replaced by a strictly larger space $X_{Y_X}(\Omega)$ equipped with the norm

$$\|f\|_{X_{Y_X}(\Omega)} = \left\| \int_\Omega^1 s^{1/n-1} u^*(s) ds \right\|_{L^{\infty,n;-1}(\Omega)};$$

- if $p > n$, the couple $(L^{n,1}(\Omega), L^\infty(\Omega))$ forms an optimal pair for the Sobolev embedding (1.3).

The contents of this work are as follows. In Chapter 2, we first provide some definitions and results about rearrangement invariant Banach space. In particular, we define and consider some properties of Lorentz spaces, Lorentz Zygmund spaces and Orlicz spaces. This chapter also contains definitions and results from interpolation theory. For a detailed treatment of interpolation theory, we refer to [2]. This chapter also includes some properties of Hardy operator and its dual, which will be usefull to prove the main theorem of this work. For these properties see [18].

In Chapter 3, we present a brief description of those aspects of distributions that are relevant for our purposes. Of special importance is the notion of weak or distributional derivative of an integrable function. For a detailed treatment of distributions, we refer to [5]. In Chapter 3, we also define Sobolev space and collect its most important properties. We conclude this chapter proving the Sobolev embeddings theorem, that is, we prove the embeddings (1.1), (1.2) and (1.3). These proofs can be found in [19] and [27].

In Chapter 4, we prove (1.5) and (1.7). Concerning the proofs, they can be found in [20]. Moreover, we prove (1.6). This proof can be found in [1].

In Chapter 5, we study the optimality of Sobolev embeddings in the context of rearrangement invariant Banach spaces. The central part of this chapter is the following theorem, which is known as *Reduction Theorem*.

Theorem 5.2.1 *Let $X(\Omega)$ and $Y(\Omega)$ be rearrangement invariant Banach spaces. Then,*

$$W_0^m X(\Omega) \hookrightarrow Y(\Omega), \quad m \in \mathbb{N}, \ 1 \le m \le n-1,$$

*if and only if there is a positive constant $C$ such that*

$$\left\| \int_t^1 f(s) s^{m/n-1} ds \right\|_{\overline{Y}(0,1)} \le C \|f\|_{\overline{X}(0,1)}, \ f \in \overline{X}(0,1).$$

When $m = 1$, this theorem was proved in [10] and the case $m = 2$ was studied in [6]. Finally, Kerman and Pick [18] proved our version of *Reduction Theorem* using results from interpolation theory. In Chapter 5, we also use Theorem 5.2.1 to determine the largest rearrangement invariant Banach domain space and the smallest rearrangement invariant Banach range space. To conclude this chapter, we apply Kerman and Pick's theorem to find the optimal pair of (1.1), (1.2) and (1.3).

In Chapter 6, we focus on the following question: what can we say about the optimal range space with mixed norm in

$$\dot{W}^1L^1(\mathbb{R}^n) \hookrightarrow L^1_{\widehat{x_n}}(\mathbb{R}^{n-1})\left[L^\infty_{x_n}(\mathbb{R})\right]?$$

The innovative part of this chapter are Proposition 6.3 and Proposition 6.5. These results allow to conclude that $L^1(\mathbb{R}^n)\left[L^\infty_{x_n}(\mathbb{R})\right]$ is the partial optimal range space.

Before discussing our results, we will mention our motivation. But, we first need to introduce spaces with mixed norm. We denote by

$$V_k = L^1_{\widehat{x_k}}(\mathbb{R}^{n-1})\left[L^\infty_{x_k}(\mathbb{R})\right], \quad 1 \le k \le n,$$

the spaces with mixed norm

$$\|f\|_{V_k} = \|\Psi_k\|_{L^1(\mathbb{R}^{n-1})}, \quad \text{where } \Psi\left(\widehat{x_k}\right) = \operatorname{ess\,sup}_{x_k\in\mathbb{R}} |f(x)|.$$

We present a brief history of our point of departure. The first proof of (1.1) [26] did not apply to the case $p = 1$, but later Gagliardo [12] and Nirenberg [22] found a method of proof which worked in the exceptional case. Gagliardo's idea was to observe that

$$W^1L^1(\mathbb{R}^n) \hookrightarrow V_k, \quad 1 \le k \le n, \tag{1.9}$$

and to deduce from this that $f \in L^{n'}(\mathbb{R}^n)$.

The embedding (1.9) motivates us to formulate above question: Let $X(\mathbb{R}^{n-1})$ and $Y(\mathbb{R})$ be rearrangement invariant Banach spaces such that

$$\dot{W}^1L^1(\mathbb{R}^n) \hookrightarrow X_{\widehat{x_n}}(\mathbb{R}^{n-1})\left[Y_{x_n}(\mathbb{R})\right].$$

We want to find its optimal range space with mixed norm. In Chapter 6, we begin to solve this problem. First, we take $X(\mathbb{R}^n) = L^1(\mathbb{R}^n)$, and we prove that

$$\dot{W}L^1(\mathbb{R}^n) \hookrightarrow L^1_{\widehat{x_n}}(\mathbb{R}^{n-1})\left[L^\infty_{x_n}(\mathbb{R})\right].$$

has the optimal range space with mixed norm (see Proposition 6.3). Second, we take $Y(\mathbb{R}) = L^\infty(\mathbb{R})$, and we prove that

$$\dot{W}L^1(\mathbb{R}^n) \hookrightarrow L^1_{\widehat{x_n}}(\mathbb{R}^{n-1})\left[L^\infty_{x_n}(\mathbb{R})\right],$$

has the optimal range space with mixed norm (See Proposition 6.5). These results allows us to conclude that $L^1_{\widehat{x_n}}(\mathbb{R}^{n-1})\left[L^\infty_{x_n}(\mathbb{R})\right]$ is the partial optimal range space.

# Chapter 2

# Preliminaries

In this chapter, we first provide some definitions and results about rearrangement invariant Banach space. In particular, we define and consider some properties of Lorentz spaces, Lorentz Zygmund spaces and Orlicz spaces. Next, we present some definitions and results from interpolation theory. This chapter also contains some properties of Hardy operator and its dual, which will appear in Chapter 5.

For any measurable subset $E$ of $\mathbb{R}^n$, we define

$$\mathcal{M}(E) := \{f \ : E \to \mathbb{R} \, ; f \text{ is measurable}\} \, ,$$

and denote by $\mathcal{M}_+(E)$ the class of non-negative functions in $\mathcal{M}(E)$. We recall the notation $X \lessapprox Y$, which means $X$ is no bigger than a constant times $Y$, with the constant independent of all function involved.

## 2.1 The distribution function

**Definition 2.1.1.** *Let $f \in \mathcal{M}(\mathbb{R}^n)$, the distribution function of $f$ is the function $d_f$ defined on $[0\,,\infty)$ as follows:*

$$d_f(\lambda) = |\{x \in \mathbb{R}^n \ : |f(x)| > \lambda\}| \, .$$

The distribution function $d_f$ provides information about the size of $f$ but not about the behavior of $f$ itself near any point. For instance, a function and any of its translates have the same distribution function. It follows from Definition 2.1.1 that $d_f$ is a decreasing function of $\lambda$ (not necessarily strictly).

**Example 2.1.2.** Recall that simple functions are finite linear combinations of characteristic fuctions of sets of finite measure. Now, we compute the distribution function $d_f$ of a non-negative simple function

$$f(x) = \sum_{j=1}^{N} a_j \chi_{E_j}(x), \tag{2.1}$$

where the sets $E_j$ are pairwise disjoint and $a_1 > \ldots > a_N > 0$. If $\lambda \geq a_1$, then $d_f(\lambda) = 0$. However, if $a_2 \leq \lambda < a_1$ then $|f(x)| > \lambda$ when $x \in E_1$. In general, if $a_{j+1} \leq \lambda < a_j$, then $|f(x)| > \lambda$ when $x \in E_1 \cup \ldots \cup E_j$. Setting $B_j = \sum_{k=1}^{j} |E_k|$ we have

$$d_f(\alpha) = \sum_{j=1}^{N} B_j \chi_{[a_{j+1}\,, a_j\,)}(\alpha), \quad \textit{where} \quad a_{N+1} = 0.$$

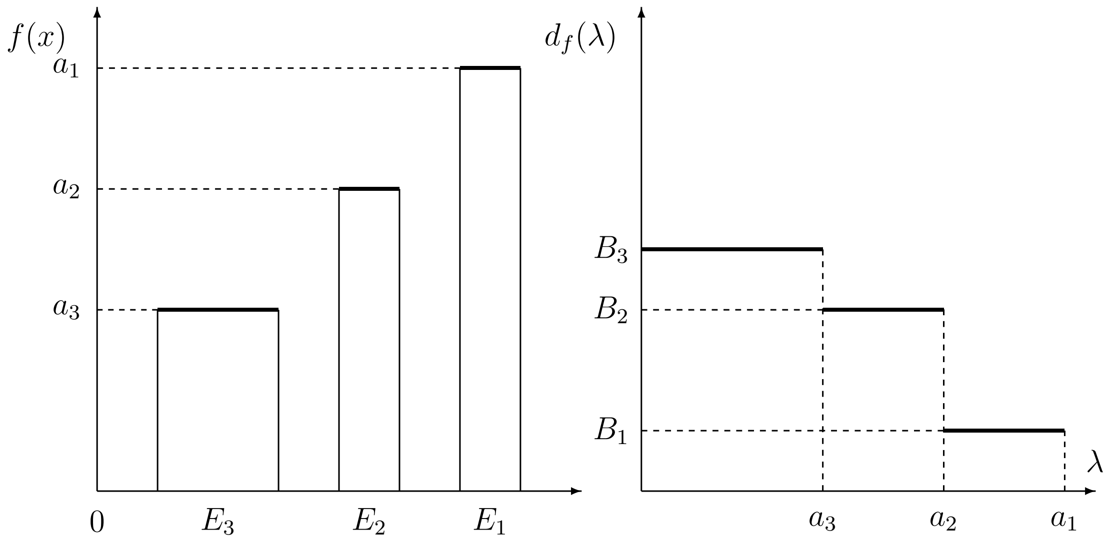


Figure 2.1: The graph of a simple function $f = \sum_{k=1}^{3} a_k \chi_{E_k}$ and its distribution function $d_f(\lambda)$. Here $B_j = \sum_{k=1}^{j} |E_k|$ .

We now state some properties about the distribution function $d_f$.

**Lemma 2.1.3.** *Let $f, g \in \mathcal{M}(R^n)$. Then for all $\alpha\,, \beta > 0$ we have:*

1. *if $|g| \leq |f|$ a.e. then $d_g \leq d_f$;*
2. *$d_{cf}(\alpha) = d_f(\alpha/\,|c|)$, for all $c \in \mathbb{R}^+$;*
3. *$d_{f+g}(\alpha + \beta) \leq d_f(\alpha) + d_g(\beta)$;*

*Proof.* See [2, Theorem II.1.3]. □

## 2.2 Decreasing rearrangements

**Definition 2.2.1.** *Let $f \in \mathcal{M}(\mathbb{R}^n)$. The decreasing rearrangement of $f$ is the function $f^*$ on $[0\,, \infty)$ defined by*

$$f^*(t) = \inf\{s > 0 \ : d_f(s) \leq t\}\,.$$

We adopt the convention $\inf \emptyset = \infty$, thus having $f^*(t) = \infty$ whenever $d_f(\alpha) > t$ for all $\alpha \geq 0$. Note that $f^*$ is decreasing.

**Example 2.2.2.** Consider the simple function of Example 2.1.2,

$$f(x) = \sum_{j=1}^{N} a_j \chi_{E_j}(x),$$

where the sets $E_j$ are pairwise disjoint and $a_1 > \ldots > a_N > 0$. We saw in Example 2.1.2 that

$$d_f(\alpha) = \sum_{j=1}^{N} B_j \chi_{[a_{j+1}, a_j)},$$

where $B_j = \sum_{k=1}^{j} |E_k|$ and $a_{N+1} = 0$. Observe that for $0 \leq t < B_1$ the smallest $s > 0$ with $d_f(s) \leq t$ is $a_1$. Similarly, for $B_1 \leq t < B_2$ the smallest $s > 0$ with $d_f(s) \leq t$ is $a_2$. Arguing this way, we obtain

$$f^*(t) = \sum_{j=1}^{N} a_j \chi_{[B_{j-1}, B_j)}(t), \quad \textit{where} \quad B_0 = 0.$$

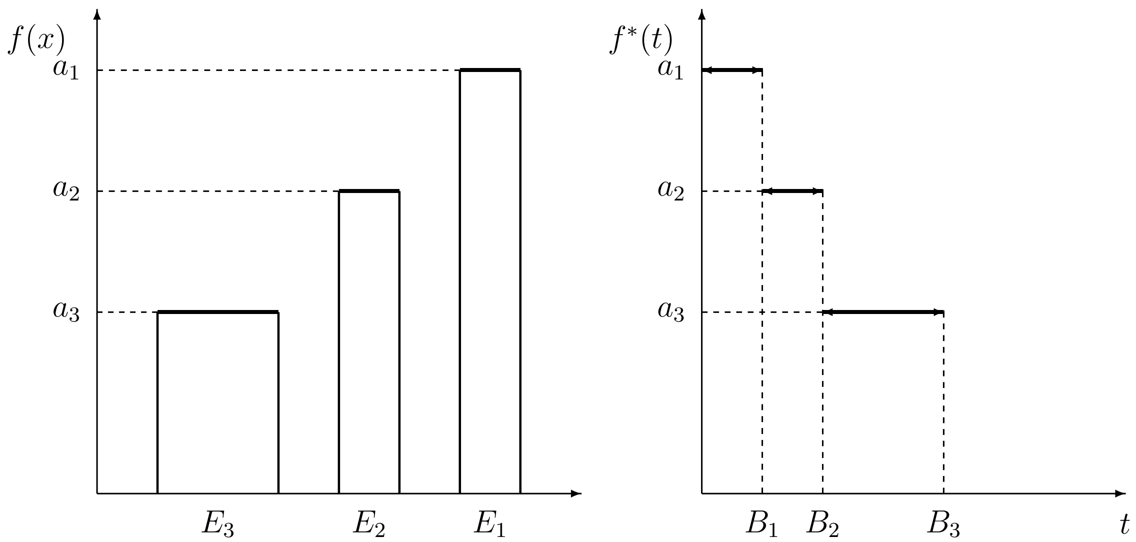


Figure 2.2: The graph of a simple function $f(x)$ and its decreasing rearrangement $f^*(t)$.

**Example 2.2.3.** It is sometimes more usefull to section functions into horrizontal blocks rather than vertical ones. Thus, the simple function in (2.1) may be represented also as follows

$$f(x) = \sum_{k=1}^{n} b_k \chi_{F_k}(x) \tag{2.2}$$

where the coefficients $b_k$ are positive and the sets $F_k$ each have finite measure and form an increasing sequenece $F_1 \subset F_2 \subset \ldots \subset F_n$. Comparison with (2.1) shows that

$$b_k = a_k - a_{k+1}, \quad F_k = \cup_{j=1}^{k} E_j, \quad k = 1\,, 2\,, \ldots\,, n.$$

Thus

$$f^*(t) = \sum_{k=1}^{n} b_k \chi_{[0\,, |F_k|)}(t).$$

**Lemma 2.2.4.** *Suppose $f$ and $g$ belong to $\mathcal{M}(\mathbb{R}^n)$ and let $\alpha$ be any scalar.*

1. *$f^*$ is a right-continuous function on $[0\,, \infty)$;*
2. *if $|g| \le |f|$ a.e. then $g^* \le f^*$;*
3. *$(\alpha f)^* = |\alpha|\, f^*$;*
4. *$(f+g)^*(t_1+t_2) \le f^*(t_1) + g^*(t_2)$; $(t_1\,, t_2 \ge 0)$;*
5. *$d_f(f^*(t)) \le t$ for all $0 \le t < \infty$;*
6. *$f$ and $f^*$ have the same distribution function.*

The following lemma is useful in proving other properties of $f^*$, since it allows us to reduce these proofs to the case when $f$ is a simple funcion.

**Lemma 2.2.5.** *Let $\{f_m\} \in \mathcal{M}(\mathbb{R}^n)$, such that for all $x \in \mathbb{R}^n$, $|f_m(x)| \le |f_{m+1}(x)|$, $m \ge 1$. If $f$ is a measurable function satisfying*

$$|f(x)| = \lim_{m\to\infty} |f_m(x)|\,, \quad x \in \mathbb{R}^n,$$

*then for each $t > 0$, $f_m^*(t) \uparrow f^*(t)$.*

*Proof.* It follows from Lemma 2.2.4 that $f_m^*(t) \le f_{m+1}^*(t) \le f^*(t)$ for $m \ge 1$. Let

$$\ell = \lim_{m\to\infty} f_m^*(t).$$

Since $f_m^*(t) \le \ell$, we have

$$d_{f_m}(\ell) \le d_{f_m}(f^*(t)) \le t, \text{ thus } d_f(\ell) = \lim_{m\to\infty} d_{f_m}(\ell) \le t.$$

Hence, $f^*(t) \le \ell$. But, from the inequality $f_m^*(t) \le f^*(t)$ we obtain $\ell \le f^*(t)$. It therefore follows that $\ell = f^*(t)$ and the lemma is proved. □

**Proposition 2.2.6.** *Let $f \in \mathcal{M}(\Omega)$. If $0 < p < \infty$, then*

$$\int_{\mathbb{R}^n} |f(x)|^p\, dx = \int_0^\infty f^*(t)^p dt.$$

*Moreover, if $p = \infty$, $\|f\|_\infty = f^*(0)$.*

*Proof.* In view of Lemma 2.2.5, we will prove this property for an arbitary non-negative simple function $f$. Using Example 2.2.2 we have

$$\int_{\mathbb{R}^n} |f(x)|^p \, dx = \sum_{j=1}^{N} a_j^p \, |E_j| = \int_0^\infty \sum_{j=1}^{N} a_j^p \chi_{[B_{j-1}, B_j)}(t) dt = \int_0^\infty (f^*(t))^p \, dt.$$

□

We continue with some properties of $f^*$.

**Theorem 2.2.7.** *If $f$ and $g$ belong to $\mathcal{M}_+(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} |f(x)| \, |g(x)| \, dx \le \int_0^\infty f^*(s) g^*(s) ds. \tag{2.3}$$

To prove this theorem we require the following lemma. Its proof can be found in [2].

**Lemma 2.2.8.** *Let $g$ be a non-negative simple function and let $E$ be an arbitrary measurable subset of $\mathbb{R}^n$. Then*

$$\int_E g(x) dx \le \int_0^{|E|} g^*(s) ds.$$

*Proof.* (Theorem 2.2.7) It is enough to establish (2.3) for non-negative functions $f$ and $g$. There is no loss of generality in assuming $f$ and $g$ to be simple. In that case, we may write

$$f(x) = \sum_{j=1}^{m} a_j \chi_{E_j}(x),$$

where $E_1 \subset E_2 \subset \ldots \subset E_m$ and $a_j > 0 \, , j = 1 \, , 2 \, , \ldots \, , m$. Then

$$f^*(t) = \sum_{j=1}^{m} a_j \chi_{[0 \, , |E_j|)}(t).$$

Hence, by Lemma 2.2.8,

$$\begin{aligned} \int_{\mathbb{R}^n} f(x) g(x) dx &= \sum_{j=1}^{m} a_j \int_{E_j} g(x) dx \le \sum_{j=1}^{m} a_j \int_0^{|E_j|} g^*(s) ds \\ &= \int_0^\infty \sum_{j=1}^{m} a_j \chi_{[0 \, , |E_j|)}(s) g^*(s) ds = \int_0^\infty f^*(s) g^*(s) ds \end{aligned}$$

□

**Definition 2.2.9.** *Let $f$ be a Lebesgue-measurable function on $\mathbb{R}^n$. Then $f^{**}$ will denote the maximal function of $f^*$ defined by*

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds \quad (t > 0) \, .$$

Some properties of the maximal function, $f^{**}$, are listed in the following lemma, which is proved in [2].

**Lemma 2.2.10.** *Suppose $f, g$ and $\{f_n\}$ belong to $\mathcal{M}(\mathbb{R}^n)$ and let $\alpha$ be any scalar. Then $f^{**}$ is non-negative, decreasing and continuous on $(0\,,\infty)$. Furthermore, the following properties hold:*

- $f^{**} \equiv 0$ *if and only if* $f \equiv 0$ *a.e;*
- $f^{*} \leq f^{**}$;
- *if* $|g| \leq |f|$ *a.e. then* $g^{**} \leq f^{**}$;
- $(\alpha f)^{**} = |\alpha|\, f^{**}$;
- $(f+g)^{**}(t) \leq f^{**}(t) + g^{**}(t)$
- *if* $|f_n| \uparrow |f|$ *a.e. then* $f_n^{**} \uparrow f^{**}$.

## 2.3 Rearrangement invariant Banach function spaces

**Definition 2.3.1.** *A rearrangement-invariant (r.i.) Banach function norm $\rho$ on $\mathcal{M}_+(\mathbb{R}^n)$ satisfies the following axioms*

- $\rho(f) \geq 0$ *with* $\rho(f) = 0$ *if and only if* $f = 0$ *a.e. on* $\mathbb{R}^n$;
- $\rho(cf) = c\rho(f)$, $c \in \mathbb{R}^+$;
- $\rho(f+g) \leq \rho(f) + \rho(g)$;
- $f_n \uparrow f$ *implies* $\rho(f_n) \uparrow \rho(f)$;
- $E \subset \mathbb{R}^n$ *with* $|E| < \infty$, *then* $\rho(\chi_E) < \infty$;
- $E \subset \mathbb{R}^n$ *with* $|E| < \infty$, *then* $\int_E f(x)dx \lessapprox \rho(f)$;
- $f^* = g^*$, *then* $\rho(f) = \rho(g)$

**Definition 2.3.2.** *Let $\rho$ be a r.i. Banach functin norm. The collection, $X(\mathbb{R}^n)$, of all function $f$ in $\mathcal{M}(\mathbb{R}^n)$ for which $\rho(|f|) < \infty$ is called a rearrangement-invariant Banach function space. For each $f \in X(\Omega)$, define*

$$\|f\|_{X(\mathbb{R}^n)} = \rho\left(|f|\right).$$

**Definition 2.3.3.** *Given a r.i. space $X(\mathbb{R}^n)$, the set*

$$X' = \left\{ f \in \mathcal{M}(\mathbb{R}^n)\, ; \int_\Omega |f(x)g(x)|\, dx < \infty \text{ for every } g \in X(\mathbb{R}^n) \right\},$$

*endowed with the norm*

$$\|f\|_{X'(\mathbb{R}^n)} = \sup_{\|g\|_{X(\mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} |f(x)g(x)|\,,$$

*is called the associate space of $X(\mathbb{R}^n)$.*

**Proposition 2.3.4.** *Let $X(\mathbb{R}^n)$ be an r.i. Banach function space. Then the associate $X'(\Omega)$ is also an r.i. Banach function space. Furthermore,*

$$\|f\|_{X'(\mathbb{R}^n)} = \sup_{\|g\|_{X(\mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} f^*(t)g^*(t)dt, \quad \text{and} \quad \|f\|_{X(\mathbb{R}^n)} = \sup_{\|g\|_{X'(\mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} f^*(t)g^*(t)dt.$$

A basic tool for working with r.i. spaces is the Hardy-Littlewood-Pólya principle:

**Theorem 2.3.5.** *Let $X(\mathbb{R}^n)$ be an r.i. Banach function spaces. If $f^{**}(t) \leq g^{**}(t)$ for all $t > 0$, then $\|f\|_{X(\mathbb{R}^n)} \leq \|g\|_{X(\mathbb{R}^n)}$.*

To prove this theorem we will use the following result. For this result see [2].

**Proposition 2.3.6.** *Let $f_1$ and $f_2$ be non-negative measurable functions on $(0\,,\infty)$ and suppose*

$$\int_0^t f_1(s)ds \leq \int_0^t f_2(s)ds,$$

*for all $t > 0$. Let $h$ be any non-negative decreasing function on $(0\,,\infty)$. Then,*

$$\int_0^\infty f_1(s)h^*(s)ds \leq \int_0^\infty f_2(s)h^*(s)ds.$$

*Proof.* (Theorem 2.3.5) By Proposition 2.3.4, it needs only be shown that

$$\int_0^\infty f^*(s)h^*(s)ds \leq \int_0^\infty g_1^*(s)h^*(s)ds,$$

for every $g$ such that $\|g\|_{X'(\mathbb{R}^n)} \leq 1$. But this is an immediate consequence of Proposition 2.3.6, since $\int_0^t f^*(t)dt \leq \int_0^t g^*(t)dt$ and $h^*$ is non-negative and decreasing. □

**Theorem 2.3.7.** *(Luxemburg representation theorem). Let $\Omega \subset \mathbb{R}^n$ be an open subset. Let $\rho$ be a rearrangement-invariant function norm on $\mathcal{M}_+(\Omega)$. Then, there is a (not necessarily unique) rearrangement-invariant function norm $\overline{\rho}$ on $\mathcal{M}_+(I)$, where $I = (0\,, |\Omega|)$, such that*

$$\rho(f) = \overline{\rho}(f^*), \;\forall f \in \mathcal{M}_+(\Omega).$$

*Furthemore, if $\sigma$ is any rearrangement-invariant function norm on $\mathcal{M}_+(I)$ which represent $\rho$, in the sense that*

$$\rho(f) = \sigma(f^*), \quad f \in \mathcal{M}_+(\Omega),$$

*then the associate norm $\rho'$ of $\rho$ is representaed in the same way by the associate norm $\sigma'$ of $\sigma$, that is*

$$\rho'(g) = \sigma'(g), \quad g \in \mathcal{M}_+(I).$$

## 2.4 Orlicz spaces

Now, we recall the definition of Orlicz spaces. For a detailed treatment of Orlicz spaces, we refer to [2].

**Definition 2.4.1.** *Let $\phi : [0\,,\infty\,) \to [0\,,\infty]$ be increasing and left-continuous function with $\phi(0) = 0$. Suppose on $(0\,,\infty)$ that $\phi$ is neither identically zero nor identically infinite. Then the function $\Phi$ defined by*

$$\Phi(s) = \int_0^s \phi(u)du, \quad (s \geq 0)$$

*is said to be a Young's function.*

**Remark 2.4.2.** Note that a Young's function is convex on the interval where it is finite. Indeed, given $s\,,t \geq 0$ and $\lambda \in (0\,,1)$ we have

$$\begin{aligned} \Phi\left(\lambda s + (1-\lambda)t\right) &= \int_0^{\lambda s+(1-\lambda)t} \phi(r)dr = \int_0^s \phi(r)dr + \int_s^{\lambda s+(1-\lambda)t} \phi(r)dr \\ &= \lambda \int_0^s \phi(r)dr + (1-\lambda)\int_0^s \phi(r)dr + \int_s^{\lambda s+(1-\lambda)t} \phi(r)dr. \end{aligned} \tag{2.4}$$

Since $\phi$ is increasing and left continuous we have

$$\int_s^{\lambda s+(1-\lambda)t} \phi(r)dr \leq (1-\lambda)(t-s)\phi\left(\lambda s + (1-\lambda)t\right)$$

$$\int_{\lambda s+(1-\lambda)t}^t \phi(r)dr \geq \lambda(t-s)\phi\left(\lambda s + (1-\lambda)t\right).$$

Then, comparing the two previous inequalities we obtain

$$\lambda \int_s^{\lambda s+(1-\lambda)t} \phi(r)dr \leq (1-\lambda)\int_{\lambda s+(1-\lambda)t}^t \phi(r)dr,$$

and so,

$$\begin{aligned}\int_s^{\lambda s+(1-\lambda)t}\phi(r)dr &= \lambda\int_s^{\lambda s+(1-\lambda)t}\phi(r)dr+(1-\lambda)\int_s^{\lambda s+(1-\lambda)t}\phi(r)dr\\ &\leq (1-\lambda)\int_s^{\lambda s+(1-\lambda)t}\phi(r)dr+(1-\lambda)\int_{\lambda s+(1-\lambda)t}^{t}\phi(r)dr\\ &= (1-\lambda)\int_s^t\phi(r)dr. \qquad (2.5)\end{aligned}$$

Finally, plugging (2.5) in (2.4), we obtain

$$\Phi\left(\lambda s+(1-\lambda)t\right)\leq\lambda\int_0^s\phi(r)dr+(1-\lambda)\int_0^t\phi(r)dr=\lambda\Phi(s)+(1-\lambda)\Phi(t).$$

Moreover, note that $\lim_{s\to\infty}\Phi(s)/s=\infty$. Indeed, for all $t>0$ we have

$$\frac{\Phi(t)}{t}=\frac{1}{t}\int_0^t\phi(s)ds\geq\frac{1}{t}\int_{t/2}^t\phi(s)ds\geq\frac{1}{2}\phi\left(\frac{t}{2}\right),$$

and then, letting $t\to\infty$, we obtain the desired result.

**Definition 2.4.3.** *Let $\Phi$ be a Young's function. The Luxemburg norm $\rho^\Phi$ is defined by*

$$\rho^\Phi(f)=\inf\left\{k>0\,:\,M^\Phi(kf)\leq 1\right\},\ f\in\mathcal{M}(\mathbb{R}^n), \qquad (2.6)$$

*where*

$$M^\Phi(kf)=\int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{k}\right)dx.$$

**Remark 2.4.4.** If $\rho^\Phi(f)>0$, the infimun in (2.6) is attained. Indeed, we denote

$$A=\left\{k>0\,:\,\int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{k}\right)dx\leq 1\right\}.$$

Let $\{k_n\}\in A$ be a decreasing sequence. We claim that $\lim_n k_n=\rho^\Phi(f)$. In fact, given $\varepsilon>0$, there exists $k_n$ such that $k_n-\rho^\Phi(f)<\epsilon$, since $\rho^\Phi(f)$ is the infimum of $A$. Then, if $n<N$ we have

$$k_N-\rho^\Phi(f)<k_n-\rho^\Phi(f)<\varepsilon,$$

and so $\lim_n k_n=\rho^\Phi(f)$. Thus $\Phi\left(|f(x)|/k_n\right)\uparrow\Phi\left(|f(x)|/\rho^\phi(f)\right)$, beucase $\Phi$ is left-continuous. Finally, we obtain, by monotone convergence,

$$\int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{\rho^\phi(f)}\right)dx\leq 1.$$

Therefore $\rho^\Phi(f)\in A$ and

$$\rho^\Phi(f)=\min\left\{k>0\,:\,\int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{k}\right)dx\leq 1\right\}.$$

In order to show that $\rho^{\Phi}$ is an r.i. norm, we shall need the following preliminary result.

**Lemma 2.4.5.** *If $\Phi$ is a Young's function, then*

$$f = 0 \text{ a.e.} \Leftrightarrow M^{\Phi}(kf) \leq 1, \ \forall\, k > 0.$$

*Proof.* Suppose $f = 0$ a.e. Then $\rho^{\Phi}(f) = 0$ and so $M^{\Phi}(kf) \leq 1$ for all $k > 0$. Conversely, suppose that $M^{\Phi}(kf) \leq 1$ for all $k > 0$, but for some $\varepsilon > 0$ we have $|f| \geq \varepsilon$ on a set $E$ of positive measure. Then

$$M^{\Phi}(kf) \geq \int_E \Phi(k\varepsilon)dx = |E|\,\Phi(k\varepsilon).$$

Since $\Phi(s) \uparrow \infty$ as $s \uparrow \infty$, we therefore obtain the contradiction that $M^{\Phi}(kf) \uparrow \infty$ as $k \uparrow \infty$. □

**Theorem 2.4.6.** *If $\Phi$ is a Young's function, then $\rho^{\Phi}$ is an r.i. norm.*

*Proof.* We need to verify the following properties:

- $\rho^{\Phi}(f) = 0 \ \Leftrightarrow\ f = 0$ a.e. Indeed, it follows from Lemma 2.4.5.
- $\rho^{\Phi}(\alpha f) = \alpha\rho^{\phi}(f) \ \ \forall \alpha > 0$. It is suffices to consider $\rho^{\Phi}(f) > 0$. We have

$$\int_{\Omega} \Phi\left(\frac{\alpha\,|f(x)|}{\alpha\rho^{\Phi}(f)}\right) dx = \int_{\Omega} \Phi\left(\frac{|f(x)|}{\rho^{\Phi}(f)}\right) dx \leq 1,$$

  hence $\rho^{\Phi}(\alpha f) \leq \alpha\rho^{\Phi}(f)$. On the other hand, since $\int_{\Omega} \Phi\left(\frac{\alpha|f(x)|}{\rho^{\Phi}(\alpha f)}\right) dx \leq 1$, we have $\rho^{\Phi}(f) \leq \rho^{\Phi}(\alpha f)/\alpha$.

- $\rho^{\phi}(f+g) \leq \rho^{\Phi}(f) + \rho^{\Phi}(g)$, for all $f, g \in \mathcal{M}(\mathbb{R}^n)$. Indeed, let $\gamma = \rho^{\Phi}(f) + \rho^{\Phi}(g) < \infty$ and let $\alpha = \rho^{\Phi}(f)/\gamma$ and $\beta = \rho^{\Phi}(g)/\gamma$ with $\alpha + \beta = 1$. By (2.6),

$$M^{\Phi}\left(f/\rho^{\Phi}(f)\right) \leq 1 \text{ and } M^{\Phi}\left(g/\rho^{\Phi}(g)\right) \leq 1.$$

  Since $\Phi$ is convex, we have

$$\begin{aligned} M^{\Phi}\left(\frac{f+g}{\gamma}\right) &= M^{\Phi}\left(\frac{\alpha f}{\rho^{\Phi}(f)} + \frac{\beta g}{\rho^{\Phi}(g)}\right) \leq \alpha M^{\Phi}\left(\frac{f}{\rho^{\Phi}(f)}\right) + \beta M^{\Phi}\left(\frac{g}{\rho^{\Phi}(g)}\right) \\ &\leq \alpha + \beta = 1. \end{aligned}$$

  Hence, we conclude that $\rho^{\Phi}(f+g) \leq \gamma = \rho^{\Phi}(f) + \rho^{\Phi}(g)$.

- $0 \leq g \leq f$ a.e. $\Rightarrow\ \rho^{\Phi}(g) \leq \rho^{\Phi}(f)$. Indeed, $0 < \rho^{\Phi}(f) < \infty$. Then

$$M^{\Phi}\left(\frac{g}{\rho^{\Phi}(f)}\right) \leq M^{\Phi}\left(\frac{f}{\rho^{\Phi}(f)}\right),$$

  and $\rho^{\Phi}(g) < \rho^{\Phi}(f)$.

- $0 \le f_n \uparrow f$ a.e. $\Rightarrow$ $\rho^{\Phi}(f_n) \uparrow \rho^{\Phi}(f)$. Indeed, by the above property, the sequence $\rho^{\Phi}(f_n)$ is increasing. Let $\alpha_n = \rho^{\Phi}(f_n)$ and put $\alpha = \sup \alpha_n$. Since $\rho^{\Phi}(f) \ge \alpha_n$ for each $n$, it follows that $\rho^{\Phi}(f) \ge \alpha$. We must show $\rho^{\Phi}(f) \le \alpha$. This is clear for $\alpha = 0$ or $\alpha = \infty$, so we may assume that $0 < \alpha_n < \infty$ for all $n \ge 1$. In this case

$$M^{\Phi}\left(\frac{f_n}{\alpha}\right) \le M^{\Phi}\left(\frac{f_n}{\alpha_n}\right) \le 1,$$

and the monotone convergence theorem shows that the quantity on the left converges to $M^{\Phi}\left(\frac{f}{\alpha}\right)$. Hence $M^{\Phi}\left(\frac{f}{\alpha}\right) \le 1$, and therefore $\rho^{\Phi}(f) \le \alpha$.

- Let $E \subset \mathbb{R}^n$ any measurable subset. Let $b$ denote the measure of $E$ (we may assume $b > 0$). We claim that,

$$\rho^{\Phi}(\chi_E) < \infty.$$

Indeed, the Young's function $\Phi$ is not identically infinite on $(0\,,\infty)$, and it is continuous on the interval where it is finite. Since $\Phi(0) = 0$, it follows that there is a number $k > 0$ for which $\Phi(k) \le 1/b$. Then $M^{\Phi}(k\chi_E) = b\Phi(k) \le 1$ and hence $\rho^{\Phi}(\chi_E) \le 1/k < \infty$.

- Let $E$ be a subset of $\mathbb{R}^n$ of measure $b > 0$. Let $f \in \mathcal{M}(\mathbb{R}^n)$ with $0 < \rho^{\Phi}(f) < \infty$. With $k = 1/\rho^{\Phi}(|f|)$, Jensen's inequality gives

$$\Phi\left(\frac{1}{b}\int_E k\,|f(x)|\,dx\right) \le \frac{1}{b}\int_E \Phi\left(k\,|f(x)|\right)dx \le \frac{1}{b}M^{\Phi}(kf) \le \frac{1}{b}.$$

Hence, since $\Phi$ increases to $\infty$, there is a constant $c$ ,which depends on $\Phi$ and $b$, such that

$$\frac{1}{b}\int_E k\,|f(x)|\,dx \le c \Rightarrow \int_E |f(x)|\,dx \le cb\rho^{\Phi}(f).$$

- The rearrangement-invariance follows from the fact that $M^{\Phi}(f) = M^{\Phi}(g)$ whenever $f$ and $g$ are equimesurables. The latter property needs only be established with $g = f^*$. There is no loss of generality in assuming $f$ to be a simple function. In that case, we may write $f(x) = \sum_{j=1}^{N} a_j \chi_{E_j}(x)$, where the sets $E_j$ are pairwise disjoint and $a_1 > \ldots > a_N$. Then,

$$f^*(t) = \sum_{j=1}^{N} a_j \chi_{[B_{j-1}\,,B_j)}(t), \text{ where } B_j = \sum_{j=1}^{N} |E_j| \text{ and } B_0 = 0.$$

So,

$$M^{\Phi}(f) = \int_{\mathbb{R}^n} \Phi\left(\sum_{j=1}^{N} a_j \chi_{E_j}(x)\right)dx = \sum_{j=1}^{N} |E_j|\,\Phi(a_j),$$

and

$$M^{\Phi}(f^*) = \int_0^{\infty} \Phi\left(\sum_{j=1}^{N} a_j \chi_{[B_{j-1},B_j)}(t)\right) dx = \sum_{j=1}^{N} \int_{B_{j-1}}^{B_j} \Phi(a_j) dx$$
$$= \sum_{j=1}^{N} (B_j - B_{j-1})\Phi(a_j) = \sum_{j=1}^{N} |E_j|\, \Phi(a_j).$$

Hence, $M^{\Phi}(f) = M^{\Phi}(f^*)$.

□

**Definition 2.4.7.** *Let $\Phi$ be a Young's function. The Orlicz space is the rearrangement invariant Banach function space of those $f \in \mathcal{M}(\mathbb{R}^n)$ for which the Luxemburg norm*

$$\|f\|_{L^{\Phi}} = \rho^{\Phi}(f),$$

*is finite.*

**Example 2.4.8.** If we take $\phi(u) = pu^{p-1}$, where $1 \leq p < \infty$, then $\Phi(u) = u^p$ and the Orlicz space $L^{\Phi}(\mathbb{R}^n)$ is the $L^p(\mathbb{R}^n)$.

## 2.5 Lorentz spaces

This section contains definition and some results from Lorentz spaces that will appear later on. For more information on Lorentz spaces see [2] and [13].

**Definition 2.5.1.** *Given $f \in \mathcal{M}(\mathbb{R}^n)$ and $0 < p, q \leq \infty$, define*

$$\|f\|_{L^{p,q}(\mathbb{R}^n)} = \begin{cases} \left(\int_0^{\infty} \left[t^{1/p} f^*(t)\right]^q \frac{dt}{t}\right)^{1/q}, & 0 < q < \infty, \\ \sup_{0<t<\infty} \left(t^{1/p} f^*(t)\right), & q = \infty. \end{cases}$$

*The set of all $f \in \mathcal{M}(\mathbb{R}^n)$ with $\|f\|_{L^{p,q}(\mathbb{R}^n)} < \infty$ is denoted by $L^{p,q}(\mathbb{R}^n)$ and is called Lorentz space.*

**Remark 2.5.2.** Using the notation of Example 2.2.2, when $0 < q < \infty$ we have

$$\|f\|_{L^{p,q}(\mathbb{R}^n)} = \left(\frac{p}{q}\right)^{1/q} \left[a_1^q B_1^{q/p} + a_2^q \left(B_2^{q/p} - B_1^{q/p}\right) + \ldots + a_N^q \left(B_N^{q/p} - B_{N-1}^{q/p}\right)\right]^{1/q}.$$

It follows that the only simple function with finite $\|\cdot\|_{L^{\infty,q}(\mathbb{R}^n)}$ norm is identically equal to zero; for this reason we have that $L^{\infty,q}(\mathbb{R}^n) = \{0\}$, for any $0 < q < \infty$

The next result shows that, for any fixed $p$, the Lorentz spaces increase as the second exponent $q$ increases.

**Proposition 2.5.3.** *Suppose* $0 < p < \infty$ *and* $1 \leq q \leq r \leq \infty$. *Then*

$$L^{p,q}(\mathbb{R}^n) \hookrightarrow L^{p,r}(\mathbb{R}^n).$$

*Proof.* Suppose $r = \infty$. Using the fact that $f^*$ is decreasing, we have

$$t^{1/p} f^*(t) = \left( \frac{p}{q} \int_0^t \left[ s^{1/p} f^*(t) \right]^q \frac{ds}{s} \right)^{1/q} \leq \left( \frac{p}{q} \int_0^t \left[ s^{1/p} f^*(s) \right]^q \frac{ds}{s} \right)^{1/q} \leq \left( \frac{p}{q} \right)^{1/q} \|f\|_{L^{p,q}(\mathbb{R}^n)} .$$

Hence, taking the supremum over all $t > 0$, we obtain

$$\|f\|_{L^{p,\infty}(\mathbb{R}^n)} \leq \left( \frac{p}{q} \right)^{1/q} \|f\|_{L^{p,q}(\mathbb{R}^n)} .$$

Now, suppose $r < \infty$, we have

$$\begin{aligned} \|f\|_{L^{p,r}(\mathbb{R}^n)} &= \left( \int_0^\infty \left[ t^{1/p} f^*(t) \right]^{r-q+q} \frac{dt}{t} \right)^{1/r} \leq \|f\|_{L^{p,\infty}(\mathbb{R}^n)}^{1-q/r} \|f\|_{L^{p,q}(\mathbb{R}^n)}^{q/r} \\ &\leq \left( \frac{p}{q} \right)^{(r-q)/rq} \|f\|_{L^{p,q}(\mathbb{R}^n)} . \end{aligned}$$

□

Note that $\|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$ , does not satisfy the triangle inequality if $p < q \leq \infty$..

**Example 2.5.4.** Consider $f(t) = t$ and $g(t) = 1 - t$ defined on $[0\,,1]$ . Then $f^*(\lambda) = g^*(\lambda) = 1 - \lambda$. A calculation shows that the triangle inequality for these functions with respect to the norm $\|\cdot\|_{L^{p\,,q}(\mathbb{R}^n)}$ would be equivalent to

$$\frac{p}{q} \leq 2\,\frac{\Gamma(q+1)\Gamma(p/q)}{\Gamma(q+1+q/p)}.$$

So, if we take $q = 2$ and $p = 4$, we will obtain $2 \leq 1/12$.

However, there is the following result.

**Theorem 2.5.5.** *Suppose* $1 \leq q \leq p < \infty$ *or* $p = q = \infty$. *Then,* $\left( L^{p,q}(\mathbb{R}^n)\,, \|\cdot\|_{L^{p,q}(\mathbb{R}^n)} \right)$ *is a rearrangement-invariant Banach function space.*

*Proof.* The result is clear when $p = q = 1$ or $p = q = \infty$ since $L^{p\,,q}(\mathbb{R}^n)$ reduces to the Lebesgue spaces $L^1(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$, respectively. Hence, we may assume that $1 < p < \infty$ and $1 \leq q \leq p$. We have

$$\|f+g\|_{L^{p,q}(\mathbb{R}^n) = \sup_{\|h\|_{L^{q'}}=1}} \int_0^\infty (f+g)^*(t) h^*(t) dt,$$

The hypothesis $q \leq p$ implies that $h^*(t)t^{1/p-1/q}$ is decreasing. Hence, since

$$\int_0^t (f+g)^*(t) \leq \int_0^t (f^*+g^*)dt,$$

we may apply Proposition 2.3.6 and then Hölder's inequality to obtain

$$\begin{aligned}\int_0^\infty t^{1/p-1/q}(f+g)^*(t)h^*(t)dt &\leq \int_0^\infty t^{1/p-1/q}(f^*(t)+g^*(t))h^*(t)dt \\ &\leq \left(\int_0^\infty t^{q/p-1}(f^*(t))^q dt\right)^{1/q} \|h\|_{q'} \\ &\quad + \left(\int_0^\infty t^{q/p-1}(g^*(t))^q dt\right)^{1/q} \|h\|_{q'} \\ &= \|f\|_{L^{p,q}(\mathbb{R}^n)} + \|g\|_{L^{p,q}(\mathbb{R}^n)},\end{aligned}$$

since $\|h\|_{q'} = 1$. This establishes the triangle inequality for $\|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$. The remaining properties of a r.i. Banach function norm are easy to verify. □

Although the restriction $q \leq p$ in the previous result is necessary, it can be avoided in the case $p > 1$ by replacing $\|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$ with an equivalent functional which is a norm for all $q \geq 1$.

**Definition 2.5.6.** *Suppose $1 < p \leq \infty$ and $0 < q \leq \infty$. The Lorentz space $L^{(p,q)}(\mathbb{R}^n)$ consist of all Lebesgue-measurables functions on $\mathbb{R}^n$,$f$, for which the quantity*

$$\|f\|_{L^{(p,q)}(\mathbb{R}^n)} = \begin{cases} \left(\int_0^\infty \left[t^{1/p}f^{**}(t)\right]^q \frac{dt}{t}\right)^{1/q}, & 0 < q < \infty, \\ \sup_{0<t<\infty} \left(t^{1/p}f^{**}(t)\right), & q = \infty, \end{cases}$$

*is finite.*

The following inequality is knwon as Hardy's inequality.

**Theorem 2.5.7.** *(Hardy's inequality) Let $\psi$ be a non-negative measurable function on $(0, \infty)$ and suppose $-\infty < \lambda < 1$ and $1 \leq q \leq \infty$. Then,*

$$\left\{\int_0^\infty \left(t^\lambda \frac{1}{t}\int_0^t \psi(s)ds\right)^q \frac{dt}{t}\right\} \leq \frac{1}{1-\lambda}\left\{\int_0^\infty \left(t^\lambda \phi(t)\right)^q \frac{dt}{t}\right\}^{1/q},$$

*with the modification if $q = \infty$.*

*Proof.* Writing $\psi(s) = s^{-\lambda/q'}s^{\lambda/q'}\psi(s)$ and applying Hölder's inequality, we obtain

$$\begin{aligned}\frac{1}{t}\int_0^t \psi(s)ds &\leq \left(\frac{1}{t}\int_0^t s^{-\lambda}ds\right)^{1/q'} \left(\frac{1}{t}\int_0^t s^{\lambda q/q'}\psi(s)^q ds\right)^{1/q} \\ &= (1-\lambda)^{-1/q'}t^{(1-\lambda)(q-1)/q}\left(\int_0^t s^{\lambda(q-1)}\psi(s)^q ds\right)^{1/q}.\end{aligned}$$

Hence,

$$\int_0^\infty \left(t^\lambda \frac{1}{t}\int_0^t \psi(s)ds\right)^q \frac{dt}{t} \leq (1-\lambda)^{1-q}\int_0^\infty t^{\lambda-2}\int_0^t s^{\lambda(q-1)}\psi(s)^q dsdt$$
$$= (1-\lambda)^{1-q}\int_0^\infty s^{\lambda(q-1)}\psi(s)^q \int_s^\infty t^{\lambda-2}dtds.$$

□

**Lemma 2.5.8.** *If $1 < p \leq \infty$ and $0 < q \leq \infty$, then*

$$\|f\|_{L^{p,q}(\mathbb{R}^n)} \leq \|f\|_{L^{(p,q)}(\mathbb{R}^n)} \leq p' \|f\|_{L^{p,q}(\mathbb{R}^n)}, \quad f \in \mathcal{M}(\mathbb{R}^n); \tag{2.7}$$

*if $q = \infty$, the corresponding integral in (2.7) is replaced by the supremum in the usual way.*

*Proof.* The first inequality in (2.7) follows from Lemma 2.2.10. The second follows from Hardy's inequality (Theorem 2.5.7). □

**Theorem 2.5.9.** *If $1 < p < \infty$, $1 \leq q \leq \infty$ or if $p = q = \infty$, then $\left(L^{(p,q)}(\mathbb{R}^n), \|\cdot\|_{(p,q)}\right)$ is a rearrangement invariant Banach function space.*

Let us determine the associate spaces $L^{(p,q)}(\mathbb{R}^n)$.

**Theorem 2.5.10.** *Suppose $1 < p < \infty$, $1 \leq q \leq \infty$ (or $p = q = 1$ or $p = q = \infty$). Then the associate spaces of $L^{(p,q)}(\mathbb{R}^n)$ is, up to equivalence of norms, the Lorentz space $L^{p',q'}(\mathbb{R}^n)$*

*Proof.* See [2]. □

## 2.6 Lorentz Zygmund spaces

In this section, we define the Lorentz-Zygmund spaces. For more details on these space see [2].

**Definition 2.6.1.** *Let $\Omega \subset \mathbb{R}^n$ be a subset with $|\Omega| = 1$. Suppose $0 < p, q \leq \infty$ and $-\infty < \alpha < \infty$. The Lorentz-Zygmund space $L^{p,q;\alpha}(\Omega)$ consist of all Lebesgue measurable functions $f$ on $\Omega$ for which*

$$\|f\|_{p,q;\alpha} = \begin{cases} \left(\int_0^1 \left[t^{1/p}\left(\log \frac{e}{t}\right)^\alpha f^*(t)\right]^q \frac{dt}{t}\right)^{1/q}, & 0 < q < \infty, \\ \sup_{0<t<1} \left[t^{1/p}\left(\log \frac{e}{t}\right)^\alpha f^*(t)\right], & q = \infty, \end{cases}$$

*is finite.*

**Example 2.6.2.** For $\alpha = 0$, $L^{p,q;\alpha}(\Omega)$ coincides with the usual Lorentz space $L^{p,q}(\Omega)$. Moreover, $L^\Phi(\Omega) = L^{\infty,\infty;-1}(\Omega)$ with $\Phi(t) = \exp(t^{n'})$ and $L^{p,p;0}(\Omega) = L^p(\Omega)$.

## 2.7 Interpolation spaces

This section contains definitions and results from interpolation theory that will appear later on. For a detailed treatment of interpolation spaces, we refer to [2].

**Definition 2.7.1.** *A pair $(X_0\,, X_1)$ of Banach spaces $X_0$ and $X_1$ is called a compatible couple if there is some Hausdorff topological vector space in which each of $X_0$ and $X_1$ is continuously embedded.*

Any pair $(X\,, Y)$ of Banach spaces for which $X$ is continuously embedded in $Y$ (or vice versa) is a compatible couple, beucase we may choose for the Hausdorff space the space $Y$ itself.

**Theorem 2.7.2.** *Let $(X_0\,, X_1)$ be a compatible couple. Then $X_0 + X_1$ and $X_0 \cap X_1$ are Banach spaces under the norms*

$$\|f\|_{X_0+X_1} = \inf_{f=f_0+f_1} \left\{\|f_0\|_{X_0} + \|f_1\|_{X_1}\right\}, \text{ and } \|f\|_{X_0\cap X_1} = \max\left\{\|f\|_{X_0}\,, \|f\|_{X_1}\right\},$$

*respectively.*

*Proof.* See [2, Theorem V.1.3]. □

**Definition 2.7.3.** *Let $(X_0\,, X_1)$ be a compatible couple of Banach spaces. The Peetre $K$-functional is defined for each $f \in X_0 + X_1$ and $t > 0$ by*

$$K(f, t; X_0, X_1) = \inf\left\{\|f_0\|_{X_0} + t\,\|f_1\|_{X_1} : f = f_0 + f_1\right\},$$

*where the infimum extends over all representations $f = f_0 + f_1$ of $f$ with $f_0 \in X_0$ and $f_1 \in X_1$.*

**Example 2.7.4.** Consider the compatiple couple $(L^1(\Omega)\,, L^\infty(\Omega))$. Then

$$K(f, t; L^1(\Omega), L^\infty(\Omega)) = \int_0^t f^*(s)ds.$$

**Theorem 2.7.5.** *Let $T$ be an admisible linear operator with respect to compatible couples $(X_0\,, X_1)$ and $(Y_0\,, Y_1)$. Then*

$$K(Tf, t; Y_0, Y_1) \le M_0 K(f, tM_1/M_0; X_0, X_1)$$

*for all $f$ in $X_0 + X_1$ and all $t > 0$.*

*Proof.* The admisible operator $T$ satisfies

$$\|Tf_i\|_{Y_i} \le M_i\,\|f_i\|_{X_i}\,, \quad f_i \in X_i\,, i = 0, 1.$$

If $f \in X_0 + X_1$ and $f = f_0 + f_1$ is any decomposition of $f$ with $f_i \in X_i$ $(i = 0\,,1)$, then $Tf = Tf_0 + Tf_1$ and $Tf_i \in Y_i$, $(i = 0\,,1)$. Hence,

$$K(Tf,t;Y_0,Y_1) \leq \|Tf_0\|_{X_0} + t\,\|Tf_1\|_{Y_1} \leq M_0\left(\|f_0\|_{X_0} + t\frac{M_0}{M_1}\,\|f_1\|_{X_1}\right).$$

Taking the infimum over all such representations $f = f_0 + f_1$ of $f$, we obtain

$$K(Tf,t;Y_0,Y_1) \leq M_0 K(f, tM_1/M_0; X_0, X_1).$$

□

**Definition 2.7.6.** *Let $(X_0\,,X_1)$ be a compatible couple. The space $(X_0\,,X_1)_{\theta,q}$ consist of all $f$ in $X_0 + X_1$ for which the functional*

$$\|f\|_{\theta,q} = \begin{cases} \left(\int_0^\infty \left[t^{-\theta}K(f,t)\right]^q \frac{dt}{t}\right)^{1/q}, & 0 < \theta < 1\,, 1 \leq q < \infty, \\ \sup_{t>0} t^{-\theta}K(f,t), & 0 \leq \theta \leq 1\,, q = \infty, \end{cases}$$

*is finite where $K(f,t) = K(f,t,X_0,X_1)$.*

**Theorem 2.7.7.** *Let $(X_0\,,X_1)$ be a compatible couple of Banach spaces. Then $(X_0\,,X_1)_{\theta,q}$ endowed with the norm $\|\cdot\|_{\theta,q}$ is a Banach space.*

$$X_0 \cap X_1 \hookrightarrow (X_0\,,X_1)_{\theta,q} \hookrightarrow X_0 + X_1.$$

**Example 2.7.8.** An important example is the case $X_0 = L^1(\Omega)$, $X_1 = L^\infty(\Omega)$ for which the corresponding interpolation spaces are the Lorentz spaces: for $1 < p < \infty$ and $1 \leq q \leq \infty$ one write

$$L^{p,q}(\Omega) = \left(L^1(\Omega)\,, L^\infty(\Omega)\right)_{1/p',q}.$$

**Remark 2.7.9.** Let $(X_0\,,X_1)$ be a compatible couple and consider two interpolation spaces

$$\overline{X}_{\theta_0} = (X_0\,,X_1)_{\theta_0\,,q_0}\,,\quad \overline{X}_{\theta_1} = (X_0\,,X_1)_{\theta_1\,,q_1}\,,$$

where $0 < \theta_0 < \theta_1 < 1$ and $1 \leq q_0\,,q_1 < \infty$. Then $\left(\overline{X}_{\theta_0}\,,\overline{X}_{\theta_1}\right)$ is itself a compatible couple. The following theorem relates the $K$-functional of the underlying couples $(X_0\,,X_1)$ and $\left(\overline{X}_{\theta_0}\,,\overline{X}_{\theta_1}\right)$. Its proof may be found in [16]. We shall write

$$K(f\,,t) = K(f,t;X_0,X_1),\ \text{and}\ \overline{K}(f,t) = K(f,t;\overline{X}_{\theta_0},\overline{X}_{\theta_1}).$$

**Theorem 2.7.10.** *Let $(X_0\,,X_1)$ be a compatible couple and consider two interpolation spaces*

$$\overline{X}_{\theta_0} = (X_0\,,X_1)_{\theta_0\,,q_0}\,,\quad \overline{X}_{\theta_1} = (X_0\,,X_1)_{\theta_1\,,q_1}\,,$$

*where $0 < \theta_0 < \theta_1 < 1$ and $1 \leq q_0\,,q_1 \leq \infty$. Then*

$$\overline{K}(f,t) \approx \left(\int_0^{1/\nu} \left(s^{-\theta_0}K(f,t)\right)^{q_0} \frac{ds}{s}\right)^{1/q_0} + t\left(\int_{1/\nu}^\infty \left(s^{-\theta_1}K(f,t)\right)^{q_1} \frac{ds}{s}\right)^{1/q_1},$$

*where $\nu = \theta_1 - \theta_0$.*

**Remark 2.7.11.** With the same technique we can estimate $\overline{K}(f,t)$ in the two extreme cases $K(f,t;X_0,\overline{X}_{\theta_1})$ and $K(f,t;\overline{X}_{\theta_1},X_1)$. The result in these two cases is

$$K(f,t;X_0,\overline{X}_{\theta_1}) \approx t\left(\int_{t^{1/\theta_1}}^{\infty}\left(s^{-\theta_1}K(f,s)\right)^{q_1}\frac{ds}{s}\right)^{1/q_1}$$

$$K(f,t;\overline{X}_{\theta_0},X_1) \approx \left(\int_0^{t^{1/(1-\theta_0)}}\left(s^{-\theta_0}K(f,s)\right)^{q_0}\frac{ds}{s}\right)^{1/q_0}.$$

## 2.8 Weighted Hardy operators

Let $n \leq 2$ and $1 \leq m \leq n-1$. We establish properties of Weighted Hardy operadors. Throughout this section we present results proved in [18].

**Lemma 2.8.1.** *Let $H_{n/m}$ and $H'_{n/m}$ be the associate weighted Hardy operators defined by*

$$(H_{n/m}f)(t) := \int_t^1 f(s)s^{m/n-1}ds \quad (H'_{n/m}f)(t) := t^{m/n-1}\int_0^t f(s)ds,$$

$f \in \mathcal{M}_+(I)\,, t\in I=(0\,,1)\,.$ *Then,*

$$H_{n/m}\ : L^1(I) \to\ L^{n/(n-m),1}(I), \quad H_{n/m}\ : L^{n/m,1}(I) \to\ L^{\infty}(I), \tag{2.8}$$

*and*

$$H'_{n/m}\ : L^1(I) \to\ L^{n/(n-m),\infty}(I), \quad H'_{n/m}\ : L^{n/m,\infty}(I) \to\ L^{\infty}(I). \tag{2.9}$$

**Remark 2.8.2.** Note that $H_{n/m}f$ is nonincreasing in $t$. Indeed, let $t_1 < t_2 \in I$. Then,

$$H_{n/m}(t_2) = \int_{t_2}^1 f(s)s^{m/n-1}ds < \int_{t_1}^1 f(s)s^{m/n-1}ds = H_{n/m}(t_1).$$

*Proof.* (Lemma 2.8.1). Let us prove (2.8).

$$\begin{aligned}\left\|H_{n/m}f\right\|_{L^{n/(n-m),1}(I)} &= \int_0^1\left(H_{n/m}f(t)\right)^{**}t^{-m/n}dt = \int_0^1 t^{-m/n-1}\left(\int_0^t\left(H_{n/m}f\right)(y)dy\right)\\ &\lessapprox \int_0^1 s^{-m/n}\left(\int_s^1 f(t)t^{m/n-1}dt\right)ds \approx \int_0^1 f(t)dt = C\left\|f\right\|_{L^1(I)}.\end{aligned}$$

Hence, $H_{n/m}\ : L^1(I)\to\ L^{n/(n-m),1}(I)$. Now,

$$\left\|H_{n/m}f\right\|_{L^\infty(I)} = \sup_{0<t<1}\left(H_{n/m}f\right)^*(t) \leq \int_0^1 f(s)s^{m/n-1}ds \leq \int_0^1 f^*(s)s^{m/n-1}ds = \left\|f\right\|_{L^{n/m,1}(I)}.$$

Therefore, $H_{n/m} : L^{n/m,1}(I) \to L^{\infty}(I)$. Let us prove (2.9).

$$\begin{aligned}\left\|H'_{n/m}f\right\|_{L^{n/(n-m),\infty}(I)} &= \sup_{g\neq 0}\frac{\int_0^1 (H'_{n/m}f)(t)g(t)dt}{\|g\|_{L^{n/m,1}(I)}} = \sup_{g\neq 0}\frac{\int_0^1 (H_{n/m}g)(t)f(t)dt}{\|g\|_{L^{n/m,1}(I)}}\\ &\leq \sup_{g\neq 0}\frac{\left\|H_{n/m}g\right\|_{L^{\infty}(I)}\|f\|_{L^1(I)}}{\|g\|_{L^{n/m,1}(I)}} \lessapprox \|f\|_{L^1(I)}\,.\end{aligned}$$

Therefore, $H'_{n/m} : L^1(I) \to L^{n/(n-m),\infty}(I)$. Finally,

$$\begin{aligned}\left\|H'_{n/m}\right\|_{L^{\infty}(I)} &= \sup_{g\neq 0}\frac{\int_0^1 (H'_{n/m}f)(t)g(t)dt}{\|g\|_{L^1(I)}} = \sup_{g\neq 0}\frac{\int_0^1 (H_{n/m}g)(t)f(t)dt}{\|g\|_{L^1(I)}}\\ &\leq \sup_{g\neq 0}\frac{\left\|H_{n/m}g\right\|_{L^{n/(n-m),1}(I)}\|f\|_{L^{n/m,\infty}(I)}}{\|g\|_{L^1(I)}} \lessapprox \|f\|_{L^{n/m,\infty}(I)}\,.\end{aligned}$$

Hence, the proof is complete. □

Additional results involving $H_{n/m}$ and $H'_{n/m}$ require the supremum operator $T_{n/m}$ defined by

$$(T_{n/m}f)(t) := t^{-m/n}\sup_{t\leq s<1} s^{m/n}f^*(s),\ \text{ with } f\in\mathcal{M}(I) \text{ and } t\in I.$$

**Remark 2.8.3.** Note that $(T_{n/m}f)(t)$ is non-increasing in $t$. Indeed, let $t_1 \leq t_2 < 1$. We have

$$t_2^{-m/n}\sup_{t_2\leq s<1} s^{m/n}f^*(s) \leq t_1^{-m/n}\sup_{t_2\leq s<1} s^{m/n}f^*(s) \leq t_1^{-m/n}\sup_{t_1\leq s<1} s^{m/n}f^*(s),$$

that is $(T_{n/m}f)(t_2) \leq (T_{n/m}f)(t_1)$.

**Lemma 2.8.4.** *The operators $T_{n/m}$ have the following endpoint mapping properties:*

$$T_{n/m} : L^{n/m,\infty}(I) \to L^{n/m,\infty}(I), \tag{2.10}$$

*and*

$$T_{n/m} : L^1(I) \to L^1(I). \tag{2.11}$$

*Proof.* Let us prove (2.10)

$$\begin{aligned}\left\|T_{n/m}f\right\|_{L^{n/m,\infty}(I)} &= \sup_{0<t<1} t^{m/n-1}\int_0^t s^{-m/n}\sup_{s\leq y<1} y^{m/n}f^*(y)ds\\ &\leq \sup_{0<t<1} t^{m/n-1}\int_0^t s^{-m/n}\sup_{0\leq y<1} y^{m/n}f^{**}(y)ds\\ &\approx \|f\|_{L^{n/m,\infty}(I)}\,.\end{aligned}$$

Now, let us prove (2.11). It sufficies to verify $\left\|T_{n/m}f\right\|_{L^1} \leq \|f\|_{L^1}$ , for $f \in \mathcal{D}(I)$. Given such an $f \neq 0$, define

$$(Rf)(t) = \sup_{t \leq s<1} s^{m/n} f^*(s),$$

and set $A = \{k \in \mathbb{N} \ : (Rf)(r_k) > (Rf)(r_{k-1})\}$ , where $r_k$ is given by

$$\int_0^{r_k} t^{-m/n} dt = \frac{n}{n-m} 2^{-k};$$

that is, $r_k = 2^{-nk/(n-m)}$, $k = 0, 1, \ldots$ Then, $A$ is non-empty. Take $k \in A$ and define

$$z_k = \begin{cases} 0 & \text{if } (Rf)(t) = (Rf)(r_k),\ t \in (0\,, r_k]; \\ \min\{r_j\ ; (Rf)(r_j) = (Rf)(r_k)\} & \text{otherwise.} \end{cases}$$

Note that,

$$\begin{aligned} Rf(0) &= \sup_{0 \leq s<1} s^{m/n} f^*(s) = \sup_{0<s \leq 1} s^{m/n} f^*(s) = \sup_{0<u<1} \sup_{u<s \leq 1} s^{m/n} f^*(s) \\ &= \sup_{0<u<1} Rf(u) = \sup_{0<u<r_k} Rf(u) = Rf(r_k). \end{aligned}$$

Thus,

$$(Rf)(t) = (Rf)(r_k), \quad k \in A\,, t \in [z_k\,, r_k]\,.$$

Moreover, by the definition of $A$, $\sup_{r_k \leq t<1} t^{m/n} f^*(t)$ is attained in $[r_k\,, r_{k-1})$ when $k \in A$. Therefore, for every $k \in A$ and $t \in [z_k\,, r_{k-1})$ , we have

$$(Rf)(t) \leq (Rf)(r_k) = \sup_{r_k \leq s<r_{k-1}} s^{m/n} f^*(s) \leq r_{k-1}^{m/n} f^*(r_k).$$

So,

$$\begin{aligned} \left\|T_{n/m}f\right\|_{L^1(I)} &\leq \sum_{k \in A} \int_{z_k}^{r_{k-1}} (Rf)(t) t^{-m/n} dt + \int_{r_{k_0}}^1 (Rf)(t) t^{-m/n} dt \\ &\leq \sum_{k \in A} r_{k-1}^{m/n} f^*(r_k) \int_0^{r_{k-1}} t^{-m/n} dt \leq \sum_{k \in A} r_{k-1} f^*(r_k) \\ &\lessapprox \sum_{k \in A} \int_{r_{k+1}}^{r_k} f^*(t) dt \leq \|f\|_{L^1(I)}\,, \end{aligned}$$

where $k_0 = \max\{k \in A\}$ . □

We continue with some further properties of $T_{n/m}$

**Theorem 2.8.5.** *Let $X(I)$ be an r.i. space. Then,*

$$\left\|t^{m/n}(T_{n/m}f^{**})(t)\right\|_{X(I)} \lessapprox \left\|t^{n/m} f^{**}(t)\right\|_{X(I)}.$$

The following two lemmas are essential to the proof of Theorem 2.8.5.

**Lemma 2.8.6.** *For all $f \in \mathcal{M}(I)$ and $t \in I$*

$$\left(T_{n/m} f\right)^{**}(t) \leq \left(T_{n/m} f^{**}\right)(t).$$

*Proof.* Given $f \in \mathcal{M}_+(I)$ and $t \in I$, set

$$f_t(s) = \min\left[f(s), f^*(t)\right], \quad f^t(s) = \max\left[f(s) - f^*(t)\,, 0\right], \; s \in (0\,, 1)\,.$$

Then,

$$(f_t)^*(s) = \min\left[f^*(s), f^*(t)\right], \quad (f^t)^*(s) = \left(f^*(s) - f^*(t)\right)\chi_{(0,t)}(s),$$

and so, $f^*(s) = (f_t)^*(s) + (f^t)^*(s),\; s \in I$. Since, $(T_{n/m} f)(t)$ is noncreasing in $t$,

$$\begin{aligned}
\left(T_{n/m} f\right)^{**}(t) &= \frac{1}{t}\int_0^t s^{-m/n} \sup_{s \leq y < 1} y^{m/n} f^*(y) ds \\
&\leq \frac{1}{t}\int_0^t s^{-m/n} \sup_{s \leq y < 1} y^{m/n} (f_t)^*(y) ds + \frac{1}{t}\int_0^t s^{-m/n} \sup_{s \leq y < 1} y^{m/n} (f^t)^*(y) ds \\
&= I + II.
\end{aligned}$$

Since,

$$\begin{aligned}
I &= \frac{1}{t}\int_0^t s^{-m/n} \sup_{s \leq y < 1} y^{m/n} \min\left[f^*(y), f^*(t)\right] ds \\
&= \frac{1}{t}\int_0^t s^{-m/n} \max\left[\sup_{s \leq y < t} y^{m/n} f^*(t)\,,\, \sup_{t \leq y < 1} y^{m/n} f^*(y)\right] ds \\
&= \frac{1}{t}\int_0^t s^{-m/n} \sup_{t \leq y < 1} y^{m/n} f^*(y) ds \\
&= \frac{n}{n-m} t^{-m/n} \sup_{t \leq y < 1} y^{m/n} f^*(y) = \frac{n}{n-m}\left(T_{n/m} f\right)(t),
\end{aligned}$$

and

$$\begin{aligned}
II &= \frac{1}{t}\int_0^t s^{-m/n} \sup_{s \leq y < 1} y^{m/n} (f^t)^*(y) ds \leq \frac{1}{t}\int_0^1 \left(T_{n/m} f^t\right)(s) ds \\
&\leq \frac{1}{t}\int_0^1 (f^t)^*(s) ds \\
&= \frac{1}{t}\int_0^t \left[f^*(s) - f^*(t)\right] ds \leq f^{**}(t).
\end{aligned}$$

We conclude that, for $f \in \mathcal{M}_+(I)$, $t \in I$,

$$\left(T_{n/m} f\right)^{**}(t) \lessapprox f^{**}(t) + (T_{n/m} f)(t) \lessapprox \left(T_{n/m} f^{**}\right)(t).$$

□

**Lemma 2.8.7.** *Let $X(I)$ be an r.i. spaces. Then,*

$$\left\| \sup_{t\leq s<1} s^{m/n} f^{**}(s) \right\|_{X(I)} \lessapprox \left\| t^{m/n} f^{**}(t) \right\|_{X(I)}, \ t \in I.$$

*Proof.* We only need to prove

$$\int_0^t \sup_{0\leq y<1} y^{m/n} f^{**}(y) ds \leq C \int_0^t \left( y^{m/n} f^{**}(y) \right)^* \left( \frac{s}{2} \right) ds.$$

with $C > 0$ independent of $f \in \mathcal{M}_+(I)$, since in that case by the Hardy-Littlewood-Pólya principle we will obtain the result:

$$\left\| \sup_{t\leq s<1} s^{m/n} f^{**}(s) \right\|_{X(I)} \lessapprox \left\| E_{1/2}(y^{m/n} f^{**}(y)) \right\|_{X(I)} \lessapprox \left\| s^{m/n} f^{**}(s) \right\|_{X(I)}.$$

To this end, fix $f \in \mathcal{M}_+(I)$ and $t \in \left(0\,, \frac{1}{3}\right)$ and take $f_t$ and $f^t$ as in the proof of Lemma 2.8.6. Then

$$\begin{aligned} \sup_{s\leq y<1} y^{m/n} f^{**}(y) &= \sup_{s\leq y<1} y^{m/n} \left[ (f_t)^{**}(y) + (f^t)^{**}(y) \right] \\ &\leq \sup_{s\leq y<1} y^{m/n} (f_t)^{**}(y) + \sup_{s\leq y<1} y^{m/n} (f^t)^{**}(y). \end{aligned}$$

Now,

$$(f^t)^{**}(y) = \begin{cases} f^{**}(y) - f^*(t), & 0 < y < t, \\ \frac{t}{y} \left[ f^{**}(t) - f^*(t) \right], & t \leq y < 1. \end{cases}$$

Hence, for $0 < s \leq t$

$$\sup_{s\leq y<1} y^{m/n} (f^t)^{**}(y) \leq \max \left[ \sup_{s\leq y<t} y^{m/n} f^{**}(y)\,, \sup_{t\leq y<1} y^{m/n-1} t f^{**}(t) \right] \leq \sup_{s\leq y\leq t} y^{m/n} f^{**}(y),$$

and so

$$\begin{aligned} \int_0^t \sup_{s\leq y<1} y^{m/n} (f^t)^{**}(y) ds &\leq \int_0^t \sup_{s\leq y\leq t} y^{m/n-1} \int_0^y f^*(z) dz ds \\ &\leq \left( \int_0^t \sup_{s\leq y\leq t} y^{m/n-1} ds \right) \left( \int_0^t f^*(s) ds \right) \\ &\leq \frac{n}{m} t^{m/n} \int_0^t f^*(s) ds. \end{aligned}$$

But, since $\int_t^{2t} g \leq \int_0^t g^*$ and $g\left(\frac{\cdot}{2}\right)^*(s) = g^*\left(\frac{s}{2}\right)$,

$$\begin{aligned} \int_0^t \left( y^{m/n} f^{**}(y) \right)^* \left( \frac{s}{2} \right) ds &\geq \int_t^{2t} \left( \frac{s}{2} \right)^{m/n} f^{**}\left( \frac{s}{2} \right) ds \geq 2^{-m/n} t^{m/n+1} f^{**}(t) \\ &= 2^{-m/n} t^{m/n} \int_0^t f^*(s) ds, \end{aligned}$$

which yields

$$\int_0^t \sup_{s\le y<1} y^{m/n}\left(f^t\right)^{**}(y)ds \le \frac{n}{m}2^{m/n}\int_0^t \left(y^{m/n}f^{**}(y)\right)^*\left(\frac{s}{2}\right)ds.$$

To prove

$$\int_0^t \sup_{s\le y<1} y^{m/n}\left(f_t\right)^{**}(y)ds \le C\int_0^t \left(y^{m/n}f^{**}(y)\right)^*\left(\frac{s}{2}\right)ds$$

we will show there is a constant $C>0$ so that, for each $f\in\mathcal{M}_+(I)$,

$$\sup_{s\le y<1} y^{m/n}\left(f_t\right)^{**}(y) \le C\sup_{t\le y<1} y^{m/n-1}\int_t^y f^*\left(\frac{z}{2}\right)dz,\ \ 0<s<1\,,0<t<\frac{1}{3}, \tag{2.12}$$

and, moreover

$$s^{m/n}f^{**}\left(\frac{s}{2}\right) = s^{m/n-1}\int_0^s f^*\left(\frac{z}{2}\right)dz \ge C^{-1}\sup_{t\le y<1} y^{m/n-1}\int_t^y f^*\left(\frac{z}{2}\right)dz, \tag{2.13}$$

on a set of measure al least $t$. This will suffice, since the right-hand side of (2.12) does not depend on $s$. Let us prove (2.12):

$$(f_t)^{**}(y) = f^*(t)\chi_{(0,t)}(y) + \left[\frac{t}{y}f^*(t) + \frac{1}{y}\int_t^y f^*(z)dz\right]\chi_{[t,1)}(y),$$

whence

$$\begin{aligned}\sup_{s\le y<1} y^{m/n}(f_t)^{**}(y) &\le t^{m/n}f^*(t) + \sup_{t\le y<1} y^{m/n-1}\int_t^y f^*(z)dz\\ &\le (2^{-m/n+1}+1)\sup_{t\le y<1} y^{m/n-1}y^{m/n-1}\int_t^y f^*\left(\frac{z}{2}\right)dz,\end{aligned}$$

since

$$\sup_{t\le y<1} y^{m/n-1}\int_t^y f^*\left(\frac{z}{2}\right) \ge (2t)^{m/n-1}\int_t^{2t} f^*\left(\frac{z}{2}\right)dz \ge 2^{m/n-1}t^{m/n}f^*(t).$$

Thus, (2.12) holds with $C=(2^{-m/n+1}+1)$. Now, let's prove (2.13). Next, suppose $y_0\in(t\,,1]$ is such that

$$\sup_{t\le y<1} y^{m/n-1}\int_t^y f^*\left(\frac{z}{2}\right)dz = y_0^{m/n-1}\int_t^{y_0} f^*\left(\frac{z}{2}\right)dz.$$

We consider two cases.

- Let $y_0 \leq 2t$. First note that $t < y_0$ implies $(2y_0)^{m/n-1} \leq (y_0+t)^{m/n-1}$. Next, since $y_0/2 \leq t$ and $f^*$ is decreasing,

$$\int_0^{y_0} f^*\left(\frac{z}{2}\right) dz \geq 2\int_t^{y_0} f^*\left(\frac{z}{2}\right) dz.$$

Altogether, for $y_0 < s < y_0 + t$, we have

$$\begin{aligned} s^{m/n-1}\int_0^{s} f^*\left(\frac{z}{2}\right) dz &\geq (y_0+t)^{m/n-1}\int_0^{y_0} f^*\left(\frac{z}{2}\right) dz \geq 2^{m/n}y_0^{m/n-1}\int_t^{y_0} f^*\left(\frac{z}{2}\right) dz \\ &= 2^{m/n}\sup_{t\leq y<1} y^{m/n-1}\int_t^{y} f^*\left(\frac{z}{2}\right) dz. \end{aligned}$$

- Let $y_0 \geq 2t$. For $y_0 - t < s < y_0$, we have

$$\begin{aligned} s^{m/n-1}\int_0^{s} f^*\left(\frac{z}{2}\right) dz &\geq y_0^{m/n-1}\int_0^{y_0-t} f^*\left(\frac{z}{2}\right) dz \geq y_0^{m/n-1}\int_t^{y_0} f^*\left(\frac{z}{2}\right) dz \\ &= \sup_{t\leq y<1} y^{m/n-1}\int_t^{y} f^*\left(\frac{z}{2}\right) dz. \end{aligned}$$

Therefore, the proof is complete. □

*Proof.* (Theorem 2.8.5). By Lemma 2.8.6 we have $(T_{n/m}f)^{**}(t) \leq (T_{n/m}f^{**})(t)$. Then by Lemma 2.8.7 we have

$$\begin{aligned} \left\|t^{m/n}(T_{n/m}f)^{**}(t)\right\|_{X(I)} \leq \left\|t^{m/n}(T_{n/m}f^{**})(t)\right\|_{X(I)} &= \left\|\sup_{t\leq s<1} s^{m/n}f^{**}(s)\right\|_{X(I)} \\ &\lessapprox \left\|s^{m/n}f^{**}(s)\right\|_{X(I)}. \end{aligned}$$

□

# Chapter 3

# Sobolev spaces

## 3.1 Introduction

In this chapter, we present a brief description of those aspects of distributions that are relevant for our purposes. Of special importance is the notion of weak or distributional derivative of an integrable function. We also define the Sobolev spaces and collect their most important properties. We conclude this chapter proving the Sobolev embedding theorem. Such theorem tells us that $W^1L^p(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ for certain values of $q$ depending on $p$ and $n$.

## 3.2 Definitions and basic properties

Consider an open set $\Omega \subset \mathbb{R}^n$ and fix a compact set $K \subset \Omega$. We define

$$\mathcal{D}_k(\Omega) = \{f \in C^\infty(\Omega) \ : \operatorname{supp} f \subset K\}\,.$$

We say that $\phi_k \to \phi$ in $\mathcal{D}_k(\Omega)$ if $D^\alpha \phi_k \to D^\alpha \phi$ uniformly for every $\alpha \in \mathbb{N}^n$.

**Definition 3.2.1.** *Let $\Omega \subset \mathbb{R}^n$ be an open subset. A distribution on $\Omega$ is a linear mapping $u\ : \mathcal{D}(\Omega) \to \mathbb{C}$ such that for every compact set, $K \subset \Omega$, $u_{|\mathcal{D}_k(\Omega)} \in \mathcal{D}_k(\Omega)'$. We denote $\mathcal{D}'(\Omega)$ the complex linear space of all distributions on $\Omega$.*

**Remark 3.2.2.** Let $f \in L^1_{\text{loc}}(\Omega)$ and $\langle \phi\,, f\rangle = \int_\Omega f(x)\phi(x)dx$. Then $u_f = \langle .\,, f\rangle$ belongs to $\mathcal{D}'(\Omega)$. Indeed, let $K \subset \Omega$ be a compact subset and let $\{\phi_k\} \in \mathcal{D}_k(\Omega)$ such that $\phi_k \to 0$ in $\mathcal{D}_k(\Omega)$ (i.e $D^\alpha \phi_k \to 0$ uniformly for every $\alpha \in \mathbb{N}^n$) ; we have

$$|\langle \phi_k\,, f\rangle| = \left| \int_K \phi_k(x) f(x)dx \right| \le \|\phi_k\|_\infty \int_K f(x)dx.$$

Then $u_f(\phi_k) \to 0$; hence $u_f \in \mathcal{D}'(\Omega)$. The linear mapping $f \in L^1_{\text{loc}} \to \langle .\,, f\rangle \in \mathcal{D}(\Omega)$ is one to one, so we may consider $L^1_{\text{loc}}(\Omega) \subset \mathcal{D}'(\Omega)$ and therefore $L^p(\Omega) \subset \mathcal{D}'(\Omega)$ for $1 \le p \le \infty$, since $L^p(\Omega) \subset L^1_{\text{loc}}(\Omega)$ for $1 \le p \le \infty$.

**Definition 3.2.3.** *Let $\alpha = (\alpha_1, \dots \alpha_n) \in \mathbb{N}^n$ and $f \in L^1_{loc}(\Omega)$, we define the distributional derivatives of $f$, $D^\alpha$, as follows*

$$\int_\Omega D^\alpha f(x)\phi(x)dx = (-1)^{|\alpha|} \int_\Omega f(x) D^\alpha \phi(x)dx \quad \forall \phi \in \mathcal{D}(\Omega),$$

*where $|\alpha| = \alpha_1 + \ldots + \alpha_n$.*

**Definition 3.2.4.** *Let $\Omega \subset \mathbb{R}^n$ be an open set, $1 \le p \le \infty$ and $m \in \mathbb{N}$. The Sobolev space of order $m \in \mathbb{N}$ is defined by*

$$W^m L^p(\Omega) := \{u \in L^p(\Omega)\,; D^\alpha u \in L^p(\Omega)\,, |\alpha| \le m\}\,,$$

*where $D^\alpha u$ represent the distributional derivatives of $u$.*

**Theorem 3.2.5.** *Let $\Omega \subset \mathbb{R}^n$ an open set, $1 \le p \le \infty$ and $m \in \mathbb{N}$. The space $W^m L^p(\Omega)$ is a Banach space with the norm*

$$\|u\|_{W^m L^p(\Omega)} = \begin{cases} \sum_{0\le|\alpha|\le m} \|D^\alpha u\|_{L^p(\Omega)}\,, & \text{if } 1 \le p < \infty, \\ \max_{0\le|\alpha|\le m} \|D^\alpha u\|_{L^\infty(\Omega)}\,, & \text{if } p = \infty. \end{cases}$$

*Proof.* Let $\{u_n\}$ be a Cauchy sequence in $W^m L^p(\Omega)$. Then $\{D^\alpha u_n\}$ is a Cauchy sequence in $L^p(\Omega)$ for $0 \le |\alpha| \le m$. Since $L^p(\Omega)$ is complete, there exit $u\,, u^\alpha \in L^p(\Omega)$, such that $u_n \to u$ and $D^\alpha u_n \to u^\alpha$ in $L^p(\Omega)$ norm as $n \to \infty$. We claim that $u_n \to u$ in $\mathcal{D}'(\Omega)$. Indeed, for any $\phi \in \mathcal{D}(\Omega)$ we have

$$\int_\Omega |u_n(x) - u(x)|\,|\phi(x)|\,dx \le \|\phi\|_{p'} \|u_n - u\|_p \to 0$$

as $n \to \infty$. Similary $D^\alpha u_n \to u^a$ in $\mathcal{D}'(\Omega)$. It follows that

$$\begin{aligned}\lim_{n\to\infty} \int_\Omega D^\alpha u_n(x)\phi(x)dx &= \lim_{n\to\infty} (-1)^\alpha \int_\Omega u_n(x) D^\alpha \phi(x)dx = (-1)^\alpha \int_\Omega u(x) D^\alpha\phi(x)dx \\ &= \int_\Omega D^\alpha u(x)\phi(x)dx.\end{aligned}$$

Thus $u_\alpha = D^\alpha u$ in the distributional sense on $\Omega$ for $0 \le |\alpha| \le m$, whence $u$ belongs to $W^m L^p(\Omega)$. Moreover, we have $\lim_{n\to\infty} \|u_n - u\|_{W^m L^p(\Omega)} = 0$, and so the space $W^m L^p(\Omega)$ is complete. □

**Proposition 3.2.6.** *Let $1 \le p < \infty$. Then $\mathcal{D}(\mathbb{R}^n)$ is dense in $W^m L^p(\mathbb{R}^n)$.*

*Proof.* Let be $\phi \in \mathcal{D}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \phi(x)dx = 1$. For every $\varepsilon > 0$, we consider $\phi_\varepsilon(x) := \varepsilon^{-n}\phi(x/\varepsilon)$. Let $f \in W^m L^p(\mathbb{R}^n)$, and set $f_\varepsilon := f * \phi_\varepsilon$. We have $f_\varepsilon \in C^\infty(\mathbb{R}^n)$; moreover

$f * \phi_\varepsilon \to f$ in $L^p$-norm as $\varepsilon \to 0$. Indeed,

$$\begin{aligned}\int_{\mathbb{R}^n} |f * \phi_\varepsilon(x) - f(x)|^p \, dx &\leq \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |f(x-y) - f(x)| \, \phi_\varepsilon(y) dy \right)^p dx \\ &\leq \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |f(x-y) - f(x)|^p \, \phi_\varepsilon(y) dy \right) dx \\ &\times \left( \int_{\mathbb{R}^n} \phi_\varepsilon(y) dy \right)^{1/p'} \\ &= \int_{\mathbb{R}^n} \|\tau_y f - f\|_{L^p(\mathbb{R}^n)} \, \phi_\varepsilon(y) dy \\ &= \int_{\mathbb{R}^n} \|\tau_{\varepsilon y} f - f\|_{L^p(\mathbb{R}^n)} \, \phi(y) dy\end{aligned}$$

then by the Lebesque dominated convergence and the fact that $\|\tau_{\varepsilon y} f - f\|_p \to 0$ as $\varepsilon \to 0$, we obtain the desire result. Moreover, $D^\alpha f_\varepsilon \to D^\alpha f$ in $L^p$-norm, as $\varepsilon \to 0$, $\forall \alpha \in \mathbb{N}^n$ such that $1 \leq |a| \leq m$. Note that $\forall \psi \in \mathcal{D}(\mathbb{R}^n)$

$$\begin{aligned}\int_{\mathbb{R}^n} D^\alpha f_\varepsilon(x) \psi(x) dx &= (-1)^{|\alpha|} \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} f(x-y) \phi_\varepsilon(y) dy \right) D^\alpha \psi(x) dx \\ &= (-1)^{|\alpha|} \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} f(x-y) D^\alpha \psi(x) dx \right) \phi_\varepsilon(y) dy \\ &= \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} D_x^\alpha f(x-y) \psi(x) dx \right) \phi_\varepsilon(y) dy \\ &= \int_{\mathbb{R}^n} D^\alpha f * \phi_\varepsilon(x) \psi(x) dx.\end{aligned}$$

Thus, we have $D^\alpha f_\varepsilon \to D^\alpha f$, since $D^\alpha f \in L^p$. Note that the functions $\{f_\varepsilon\}$ give the required aproximation, but they not have compact support. So we need to introduce the two-paremetrer family

$$\{\tau_\delta(\eta) f_\varepsilon\} \in \mathcal{D}(\mathbb{R}^n) \, , \delta \, , \varepsilon \in \mathbb{R}^+,$$

with $\eta \in \mathcal{D}(\mathbb{R}^n)$ such that $\eta(0) = 1$. Fix $\varepsilon > 0$, if we prove,

$$D^\alpha(\tau_\delta(\eta) f_\varepsilon) \to D^\alpha(f_\varepsilon)$$

in $L^p$-norm as $\delta \to 0$ for all $0 \leq |\alpha| \leq m$, where $\tau_\delta(\eta)(x)$ is the dilation operador defined by

$$\tau_\delta(\eta)(x) := \eta(\delta x), \;\; \delta \in \mathbb{R}^+;$$

we will complete the proof. Indeed, if $|\alpha| = 0$ since

$$\lim_{\delta \to 0} \eta(\delta x) f_\varepsilon(x) = f_\varepsilon(x) \;\; \text{and} \;\; |\eta(\delta x) f_\varepsilon(x)| \leq |f_\epsilon(x)| \;\; \text{a.e.},$$

using the Lebesgue dominated convergence we obtain the desire result. Next, if $1 \leq |a| \leq m$,

$$\lim_{\delta \to 0} D^\alpha(\eta(\delta x) f_\varepsilon(x)) = D^\alpha f_\varepsilon(x) \;\; \text{and} \;\; |D^\alpha(\eta(\delta x) f_\varepsilon(x))| \leq |f_\varepsilon(x)| + |D^\alpha f_\varepsilon(x)| \;\; \text{a.e.}$$

Then, by the Lebesgue dominated convergence, we obtain the desire result. Therefore, the proof is complete. □

The following example shows that the proposition is not true for an arbitrary domain $\Omega \subset \mathbb{R}^n$.

**Example 3.2.7.** Let $\Omega = \{(x\,,y) \in \mathbb{R}^2 \ : 0 < |x| < 1\,, 0 < y < 1\}$. Let $u$ be a function defined on $\Omega$.

$$u(x\,,y) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Denote $K = \overline{\Omega}$. Suppose that there exists $\phi \in C^1(K)$ such that $\|u - \phi\|_{W^{1,p}(\Omega)} < \varepsilon$. Let

$$L = \{(x\,,y) \ : -1 \leq x \leq 0\,, 0 \leq y \leq 1\}\,,\ R = \{(x\,,y) \ : 0 \leq x \leq 1\,, 0 \leq y \leq 1\}\,.$$

We have $\|\phi\|_{L^1(L)} \leq \|\phi\|_{L^p(L)} < \varepsilon$ and similarly $\|1 - \phi\|_{L^1(R)} < \epsilon$, from which we obtain $\|\phi\|_{L^1(R)} > 1 - \varepsilon$. Let $\Phi(x) = \int_0^1 \phi(x\,,y)dy$, by the Integral Mean Value Theorem, we know that there exists $a$ and $b$ with $-1 < a < 0$ and $0 < b < 1$ such that $\Phi(a) < \varepsilon$ and $\Phi(b) > 1 - \varepsilon$. If $0 < \epsilon < 1/2$

$$1 - 2\varepsilon < \Phi(b) - \Phi(a) = \int_a^b \Phi'(x)dx \leq \int_{\bar{\Omega}} |D_x\phi(x\,,y)|\, dxdy \leq 2^{1/p'} \|D_x\phi\|_{L^p(\Omega)}\,.$$

Hence, $1 < \varepsilon(2 + 2^{1/p'})$, which is impossible for small $\varepsilon$. The problem with this domain is that lie on both sides of part of its boundary. The condition which is called **the segment condition** prevents this from happening and guarantees that $\mathcal{D}(\mathbb{R}^n)$ is dense in $W^mL^p(\Omega)$ for $1 \leq p < \infty$.

## 3.3 Riesz potencials

**Definition 3.3.1.** *Let $0 < \alpha < n$ and $f \in \mathcal{D}(\mathbb{R}^n)$. We define the Riesz potencials by*

$$I_\alpha(f)(x) = \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} |x - y|^{-n+\alpha} f(y)dy, \quad \textit{with } \gamma(\alpha) = \frac{\pi^{n/2} 2^\alpha \Gamma(\alpha/2)}{\Gamma(n/2 - \alpha/2)}. \tag{3.1}$$

*with* $\gamma(\alpha) = \frac{\pi^{n/2}2^\alpha\Gamma(\alpha/2)}{\Gamma(n/2-\alpha/2)}$.

**Remark 3.3.2.** Since the Riesz potencials are integral operators it is natural to inquire about their actions on the spaces $L^p(\mathbb{R}^n)$. We formulate the following problem: given $\alpha\,, 0 < \alpha < n$ for what pairs $p$ and $q$, is the operator $f \to I_\alpha(f)$ bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Suppose that we had an estimate

$$\|I_\alpha(f)\|_{L^q(\mathbb{R}^n)} \leq C\, \|f\|_{L^q(\mathbb{R}^n)}\,,$$

for some positive indices $p\,,q$ and all $f \in L^p(\mathbb{R}^n)$. Then $p$ and $q$ must be related by

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}.$$

In fact, let $f \in \mathcal{D}(\mathbb{R}^n)$ non-negative. Consider the dilation operador defined by

$$\tau_\delta(f)(x) = f(\delta x), \quad \delta \in \mathbb{R}^+.$$

Then

$$\begin{aligned} I_\alpha(\tau_\delta f)(x) &= \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} |x-y|^{-n+\alpha} f(\delta y)dy = \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} |x - z/\delta|^{-n+\alpha} f(z)\delta^{-n}dy \\ &= \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} |\delta x - z|^{-n+\alpha} f(z)\delta^{-\alpha}dy = \delta^\alpha \tau_\delta(I_\alpha f)(x); \end{aligned}$$

thus

$$\|I_\alpha(\tau_\delta(f))\|_q = \delta^{-\alpha} \|\tau_\delta I_\alpha f\|_q = \delta^{-\alpha-n/q} \|I_\alpha f\|_q \le C \|\tau_\delta f\|_p = C\delta^{-n/p} \|f\|_p\,.$$

Hence

$$\|I_\alpha f\|_q \le C\delta^{-1/p+1/q+\alpha/n} \|f\|_p\,.$$

Suppose now that $1/p > 1/q + \alpha/n$, letting $\delta \to \infty$ obtain that $I_\alpha(f) = 0$.Similary, if $1/p < 1/q + \alpha/n$ letting $\delta \to 0$, we obtain that $\|f\|_p = \infty$. Thus, this inequality is possible only if $1/q = 1/p - \alpha/n$.

**Theorem 3.3.3.** *Let* $0 < \alpha < n\,, 1 \le p < q < \infty$, $1/q = 1/p - \alpha/n$. *If* $f \in L^1(\mathbb{R}^n)$, *then*

$$|\{x\,:|I_\alpha f(x) > \lambda|\}| \le \left(\frac{A\|f\|_1}{\lambda}\right)^q.$$

*Proof.* Let be $K(x) = |x|^{-n+\alpha}$, we consider the transformation $f \to K * f$ (which differs from $f \to I_\alpha f$ by a contant multiple). We decompose $K$ as $K_1 + K_\infty$ where

$$K_1(x) = \begin{cases} K(x), & \text{if } |x| \le \mu \\ 0 & \text{if } |x| > \mu \end{cases}$$

$$K_\infty(x) = \begin{cases} K(x), & \text{if } |x| > \mu \\ 0 & \text{if } |x| \le \mu \end{cases}$$

with $\mu$ a fixed positive constant which need not to be specified. Note that $K_1 \in L^1(\mathbb{R}^n)$ and $K_\infty \in L^{p'}$. Suppose that $\|f\|_p^p = 1$. Since $K * f = K_1 * f + K_\infty * f$, we have

$$|\{x\,:|K * f(x)| > 2\lambda\}| \le |\{x\,:|K_1 * f(x)| > \lambda\}| + |\{x\,:|K_\infty * f|\,(x) > \lambda\}|$$

However

$$|\{x\,:\,|K_1 * f(x)| > \lambda\}| \le \frac{\|K_1 * f\|_p^p}{\lambda^p} \le \frac{\|K_1\|_1^p\,\|f\|_p^p}{\lambda^p} = \frac{\|K_1\|_1^p}{\lambda^p} = c\frac{\mu^\alpha}{\lambda^p},$$

since

$$\|K_1\|_1 = \int_{|x|\le\mu} |x|^{-n+\alpha}\,dx = c\mu^\alpha.$$

Next

$$\|K_\infty * f\|_\infty \le \|K_\infty\|_{p'}\,\|f\|_p = \|K_\infty\|_{p'} = c\mu^{-n/q},$$

since

$$\|K_\infty\|_{p'} = \left(\int_{|x|>\mu} |x|^{(-n+\alpha)p'}\,dx\right)^{1/p'} = c\mu^{-n/q}.$$

Now, if $c\mu^{-n/q} = \lambda$, we obtain $\|K_\infty\|_{p'} = \lambda$. Take $\mu = c\lambda^{-q/m}$ to have this value; then $\|K_\infty * f\|_\infty \le \lambda$ and so $|\{x\,:\,|K_\infty * f(x)| > \lambda\}| = 0$. Finally

$$|\{x\,:\,|K * f(x)| > 2\lambda\}| \le \frac{c\mu^{\alpha\,p}}{\lambda} = c\lambda^{-q} = c\left(\frac{\|f\|_p}{\lambda}\right)^q$$

Hence, the mapping $f \to K * f$ is of weak type $(p\,,q)$, in particular when $p = 1$. □

**Theorem 3.3.4.** *Let $0 < \alpha < n$ and $1 < p < q < \infty$ with $1/q = 1/p - \alpha/n$. Then,*

$$\|I_\alpha f\|_q \le A_{p,q}\,\|f\|_p\,.$$

*Proof.* It follows from Theorem 3.3.3 and the Marcinkiewicz interpolation theorem. □

## 3.4 Sobolev embedding theorem

In this section, we will study the Sobolev embedding theorem. It asserts that:

- if $1 \le p < n$, then $W^1L^p(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ where $p^* = np/(n-p)$;
- if $p = n$, then $W^1L^p(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$, for every $p \le q < \infty$;
- if $p > n$, then $W^1L^p(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$.

### Embeddings: $1 \le p < n$

**Theorem 3.4.1.** *Let $n \ge 2$. If $1 < p < n$, then*

$$W^1L^p(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n), \;\; \textit{where } p^* = \frac{pn}{n-p}.$$

To prove this theorem, we need the following lemma. It gives an appropriate way of expressing a function in terms of its partial derivates. Its proof can be found in [27].

**Lemma 3.4.2.** *Let $f \in \mathcal{D}(\mathbb{R}^n)$ then*

$$f(x) = \frac{1}{\omega_{n-1}} \sum_{j=1}^{n} \int_{\mathbb{R}^n} \frac{D_j(x-y)y_j}{|y|^n} dy,$$

*where $\omega_{n-1}$ is the area of the sphere $S^{n-1}$.*

*Proof.* (Theorem 3.4.1) Assume that $f \in \mathcal{D}(\mathbb{R}^n)$. By Lemma 3.4.2 we have

$$|f(x)| \lessapprox \sum_{j=1}^{n} \int_{\mathbb{R}^n} |D_j f(x-y)| \, |y|^{-n+1} \, dy = \sum_{j=1}^{n} I_1 \left(D_j f\right)(x).$$

Hence, we get

$$\|f\|_{L^q(\mathbb{R}^n)}^q \lessapprox \int_{\mathbb{R}^n} \left| \sum_{j=1}^{n} I_1 \left(D_j f\right)(x) \right|^q dx \lessapprox \sum_{j=1}^{n} \int_{\mathbb{R}^n} \left| I_1 \left(D_j f\right)(x) \right|^q dx \lessapprox \sum_{j=1}^{n} \|D_j f\|_{L^p(\mathbb{R}^n)}^p .$$

Now, let $f \in W^1 L^p(\mathbb{R}^n)$. Since $\mathcal{D}(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$, there exists $\{f_k\} \in \mathcal{D}(\mathbb{R}^n)$ such that $\|f_k - f\|_{W^{1,p}(\mathbb{R}^n)} \to 0$. Hence, we get

$$\|f_k - f_{k'}\|_{L^q(\mathbb{R}^n)} \lessapprox \sum_{j=1}^{n} \|D_j f_k - D_j f_{k'}\|_{L^p(\mathbb{R}^n)} ,$$

and so the sequence $\{f_k\}$ also converges in $L^q(\mathbb{R}^n)$ norm and this limit is equal $f$. Thus $f \in L^q(\mathbb{R}^n)$ and

$$\|f\|_{L^q(\mathbb{R}^n)} \lessapprox \sum_{j=1}^{n} \|D_j f\|_{L^p(\mathbb{R}^n)} \lessapprox \|f\|_{W^1 L^p(\mathbb{R}^n)} .$$

This shows that $f \in L^q(\mathbb{R}^n)$ and the inclusion mapping of $W^1 L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ is continuous. □

At the end of fifties, Gagliardo [12] and Nirenberg [22] extended Theorem 3.4.1 to the case $p = 1$. Note that the argument used in Theorem 3.4.1 does not work in case $p = 1$ beucase Theorem 3.3.4 fails for $p = 1$. A different idea is neeeded and it is contained in the next lemma, which is proved in [19]. We use the notation $\widehat{x}_k$ for the vector in $\mathbb{R}^{n-1}$ obtained from a given $x \in \mathbb{R}^n$ by removing its $k$th coordenate, that is

$$\widehat{x}_k = (x_1 , \dots , , x_{k-1} , x_{k+1} , \dots , x_n) \in \mathbb{R}^{n-1}.$$

**Lemma 3.4.3.** *Let $n \geq 2$. Assume that the functions $g_k \in L^1(\mathbb{R}^{n-1})$ $k = 1, \dots , n$ are non-negative. Then*

$$\int_{\mathbb{R}^n} \prod_{k=1}^{n} g_k(\widehat{x}_k)^{1/(n-1)} dx \leq \left( \prod_{k=1}^{n} \int_{\mathbb{R}^{n-1}} g_k(\widehat{x}_k) d\widehat{x}_k \right)^{1/(n-1)} .$$

*Proof.* The proof is by induction on $n$. If $n = 2$, let

$$g(x) := g_1(x_2)g_2(x_1) \text{ with } x = (x_1\,, x_2) \in \mathbb{R}^2.$$

Using Tonelli's theorem we get

$$\int_{\mathbb{R}^2} g(x)dx = \left(\int_{\mathbb{R}} g_1(x_2)dx_2\right)\left(\int_{\mathbb{R}} g_2(x_1)dx_1\right).$$

Assume next that the result is true for $n$ and let us prove it for $n+1$. Let

$$g(x) = \prod_{k=1}^{n+1} g_i(\widehat{x_i})^{1/n}, \text{ with } g_i \in L^1(\mathbb{R}^n).$$

Fix $x_{n+1} \in \mathbb{R}$. Integrating both sides of the previous identity with respect to $x_1\,, \dots\,, x_n$ and using Hölder's inequality, we get

$$\int_{\mathbb{R}^n} g(x)d\widehat{x_{n+1}} \leq \left(\int_{\mathbb{R}^n} g_{n+1}(\widehat{x_{n+1}})d\widehat{x_{n+1}}\right)^{1/n}\left(\int_{\mathbb{R}^n}\prod_{k=1}^{n} g_k(\widehat{y_k}\,, x_{n+1})^{1/(n-1)}dy\right)^{1-1/n}$$

where $y = (x_1\,, \dots\,, x_n)$. Since $x_{n+1}$ is fixed, by induction hypothesis we have

$$\left(\int_{\mathbb{R}^n}\prod_{k=1}^{n} g_k(\widehat{y_k}\,, x_{n+1})^{1/(n-1)}dy\right)^{1-1/n} \leq \left(\prod_{k=1}^{n}\int_{\mathbb{R}^{n-1}} g_k(\widehat{y_k}\,, x_{n+1})d\widehat{y_k}\right)^{1/n}.$$

Thus

$$\int_{\mathbb{R}^n} g(x)d\widehat{x_{n+1}} \leq \left(\int_{\mathbb{R}^n} g_{n+1}(\widehat{x_{n+1}})d\widehat{x_{n+1}}\right)^{1/n}\left(\prod_{k=1}^{n}\int_{\mathbb{R}^{n-1}} g_k(\widehat{y_k}\,, x_{n+1})d\widehat{y_k}\right)^{1/n}.$$

Integrating both sides of the previous identity with respect $x_{n+1}$ and using Hölder's inequality we get

$$\int_{\mathbb{R}^n} g(x)dx \leq \left(\prod_{k=1}^{n+1}\int_{\mathbb{R}^n} g_k(\widehat{x_k})d\widehat{x_k}\right)^{1/n}.$$

□

**Theorem 3.4.4.** *Let $n \geq 2$ and $p = 1$. Then,*

$$W^1L^1(\mathbb{R}^n) \hookrightarrow L^{n'}(\mathbb{R}^n).$$

*Proof.* Let $f \in \mathcal{D}(\mathbb{R}^n)$, we have

$$|f(x)| \leq \frac{1}{2}\int_{\mathbb{R}} |D_k f(x)|\, dx_k \equiv g_k(\widehat{x_k}) \quad k = 1\,, \dots\,, n.$$

Applying Lemma 3.4.3 we obtain

$$\int_{\mathbb{R}^n} |f(x)|^{n/(n-1)}\, dx \leq \int_{\mathbb{R}^n} \prod_{k=1}^{n} g_k(\widehat{x_k})^{1/(n-1)} dx \leq \left( \prod_{k=1}^{n} \int_{\mathbb{R}^{n-1}} g_k(\widehat{x}_k) d\widehat{x}_k \right)^{1/(n-1)}$$
$$\leq \frac{1}{2^{n/(n-1)}} \prod_{k=1}^{n} \|D_k f\|_{L^1(\mathbb{R}^n)}^{1/(n-1)};$$

hence

$$\|f\|_{L^{n'}(\mathbb{R}^n)} \leq \frac{1}{2} \left( \prod_{k=1}^{n} \|D_k f\|_{L^1(\mathbb{R}^n)} \right)^{1/n}.$$

If we use the fact that $\left(\prod_{j=1}^n a_j\right)^{1/n} \leq \frac{1}{n}\sum_{j=1}^n a_j$ if $a_j \geq 0$; then as consequence we have

$$\|f\|_{L^{n'}(\mathbb{R}^n)} \leq \frac{1}{2n} \sum_{k=1}^{n} \|D_k f\|_{L^1(\mathbb{R}^n)}. \tag{3.2}$$

Now, let $f \in W^1L^1(\mathbb{R}^n)$. Since $\mathcal{D}(\mathbb{R}^n)$ is dense in $W^{1,1}(\mathbb{R}^n)$, there exists $\{f_m\} \in \mathcal{D}(\mathbb{R}^n)$ such that $\|f - f_m\|_{W^1L^1(\mathbb{R}^n)} \to 0$, as $m \to \infty$. Hence, by (3.2) we have

$$\|f_m - f_{m'}\|_{L^{n'}} \leq \frac{1}{2n} \sum_{k=1}^{n} \|D_k f_m - D_k f_{m'}\|_{L^1(\mathbb{R}^n)},$$

hence $f \in L^{n'}(\mathbb{R}^n)$, and

$$\|f\|_{L^{n'}(\mathbb{R}^n)} \leq \frac{1}{2n} \sum_{k=1}^{n} \|D_k f\|_{L^1(\mathbb{R}^n)} \leq \frac{1}{2n} \|f\|_{W^1L^1(\mathbb{R}^n)}.$$

□

**Corollary 3.4.5.** *Let $n \geq 2$. Let $k$ be a positive integer such that $1 \leq k \leq n-1$. Suppose $1 \leq p < n/k$. Then,*

$$W^1L^p(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n), \quad \textit{with } p^* = \frac{np}{n-kp}.$$

*Proof.* The proof proceeds by induction on $k$. Note that Theorem 3.4.1 and Theorem 3.4.4 establish the case $k = 1$. Now, assume that

$$\|v\|_{L^{q_{k-1}}(\mathbb{R}^n)} \lessapprox \|v\|_{W^{k-1}L^p(\mathbb{R}^n)}, \ v \in W^{k-1}L^p(\mathbb{R}^n),$$

where $q_{k-1} = \frac{np}{n-kp+p}$. Let $u \in W^kL^p(\mathbb{R}^n)$; we take $v = D_j u$ $1 \leq j \leq n$ we obtain

$$\|D_j u\|_{L^{q_{k-1}}(\mathbb{R}^n)} \lessapprox \|D_j u\|_{W^{k-1}L^p(\mathbb{R}^n)}.$$

Therefore,

$$\|u\|_{W^1L^{q_{k-1}}(\mathbb{R}^n)} = \sum_{j=1}^n \|D_j u\|_{L^{q_{k-1}}(\mathbb{R}^n)} \lessapprox \sum_{j=1}^n \sum_{0\le|\alpha|\le k-1} \|D^\alpha D_j u\|_{L^p(\mathbb{R}^n)} \lessapprox \|u\|_{W^kL^p(\mathbb{R}^n)} .$$

Now, since $kp < n$, we have $q_{k-1} < n$ and so

$$\|u\|_{L^q(\mathbb{R}^n)} \lessapprox \|u\|_{W^1L^{q_{k-1}}(\mathbb{R}^n)} \lessapprox \|u\|_{W^kL^q(\mathbb{R}^n)} ,$$

where $q = \frac{nq_{k-1}}{n-q_{k-1}} = \frac{np}{n-kp}$. □

## Embeddings: $p = n$

We have seen that for a given function $u \in W^1L^p(\mathbb{R}^n)$ and $1 \le p < n$, then

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \lessapprox \|u\|_{W^1L^p(\mathbb{R}^n)} ,$$

where $p^* = np/(n-p)$. Note that when $p$ tends to $n$, $p^*$ tends to $\infty$ and so one would be tempted to say that if $u \in W^1L^n(\mathbb{R}^n)$ then $u \in L^\infty(\mathbb{R}^n)$. The following example shows that this is false if $n > 1$.

**Example 3.4.6.** Put $n = 2$ and define $u(x) := \log\log\left(1 + \frac{1}{|x|}\right)$. Let us prove that $u \in W^1L^2\left(B(0\,,1)\right)$. Indeed, it suffices to prove that for all $k = 0\,,\ldots,n$ there is a positive constant, $c > 0$, such that

$$\left|\int_{B(0\,,1)} u(x)D_k\phi(x)dx\right| \le c\,\|D_k\phi\|_2 \quad \forall\phi \in \mathcal{D}\left(B(0\,,1)\right). \tag{3.3}$$

Indeed, if (3.3) holds the lineal form

$$\phi \in \mathcal{D}\left(B(0\,,1)\right) \to (-1)^k \int_{B(0\,,1)} u(x)D_k\phi(x)dx$$

defined in a dense subspace of $L^2\left(B(0\,,1)\right)$ is continuous for the $L^2$ norm; therefore by Hahn-Banach's Theorem it extends to a continuous lineal form $F$ in $L^2\left(B(0\,,1)\right)$. Then, by Riesz's Theorem there exits $g \in L^2\left(B(0\,,1)\right)$ such that

$$\langle F\,,\phi\rangle = \int g(x)\phi(x)dx \quad \forall\phi \in L^2\left(B(0\,,1)\right),$$

in particular

$$(-1)^k \int u(x)D_k\phi(x)dx = \int g(x)\phi(x)dx \;\; \forall\phi \in \mathcal{D}\left(B(0\,,1)\right)$$

and so $u \in W^1L^2\left(B(0\,,1)\right)$.

Now we prove that $u \in W^1L^2(B(0,1))$, let $\phi \in \mathcal{D}(B(0,1))$

$$\begin{aligned}\left|\int_{B(0,1)} u(x)D_k\phi(x)dx\right| &\le \int_{B(0,1)} |u(x)|\,|D_k\phi(x)|\,dx \\ &\le \left(\int_{B(0,1)} |u(x)|^2\,dx\right)^{1/2} \|D_k\phi\|_{L^2(B(0,1))}.\end{aligned}$$

Since

$$\begin{aligned}\int_{B(0,1)} |u(x)|^2\,dx &= 2\pi\int_0^1 r\left(\log\log\left(1+\frac{1}{r}\right)\right)^2 dr \\ &= 2\pi\int_1^\infty \frac{\log^2\log(1+t)}{t^3}dt \le 2\pi\int_1^\infty \frac{\log(1+t)}{t^3}dt \\ &\le 2\pi\int_1^\infty \frac{(1+t)}{t^3}dt = 2\pi\lim_{b\to\infty}\int_1^b \frac{(1+t)}{t^3}dt = C,\end{aligned}$$

we have

$$\left|\int_{B(0,1)} u(x)D_k\phi(x)dx\right| \le c\,\|D_k\phi\|_{L^2(B(0,1))}.$$

Hence, $u \in W^1L^2((B(0,1))$ but $u \notin L^\infty((B(0,1))$ beucase $u(x)\to\infty$ when $|x|\to 0$.

However, we have the following result which is proved in [19].

**Theorem 3.4.7.** *If $n \le q < \infty$, then*

$$W^1L^n(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n).$$

*Proof.* Let $u \in \mathcal{D}(\mathbb{R}^n)$ and $v := |u|^t$ where $t > 1$, by Theorem 3.4.4 we have

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |u(x)|^{tn'}\,dx\right)^{1/n'} &= \left(\int_{\mathbb{R}^n} |v(x)|^{n'}\,dx\right)^{1/n'} \\ &\lessapprox \sum_{k=1}^n \int_{\mathbb{R}^n} |D_kv(x)|\,dx \approx \sum_{k=1}^n \int_{\mathbb{R}^n} |u(x)|^{t-1}\,|D_ku(x)|\,dx \\ &\lessapprox \left(\int_{\mathbb{R}^n} |u(x)|^{(t-1)n'}\,dx\right)^{1/n'} \left(\sum_{k=1}^n \|D_ku\|_{L^n(\mathbb{R}^n)}\right),\end{aligned}$$

where in the last inequality we have used Hölder's inequality. Hence

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |u(x)|^{tn'}\,dx\right)^{1/(n't)} &\lessapprox \left(\int_{\mathbb{R}^n} |u(x)|^{(t-1)n'}\,dx\right)^{1/(n't)} \left(\sum_k^n \|D_ku\|_{L^n(\mathbb{R}^n)}\right)^{1/t} \\ &\lessapprox \left(\|u\|_{L^{(t-1)n'}(\mathbb{R}^n)} + \sum_{k=1}^n \|D_ku\|_{L^n(\mathbb{R}^n)}\right),\end{aligned}$$

where we have used Young's inequality with exponent $t$ and $t/(t-1)$. Taking $t = n$ yields

$$\|u\|_{L^{n^2/(n-1)}(\mathbb{R}^n)} \lessapprox \|u\|_{W^1L^n(\mathbb{R}^n)}\,.$$

Now, assume $u \in W^1L^{n^2/(n-1)}(\mathbb{R}^n)$ and let $\{u_i\} \in \mathcal{D}(\mathbb{R}^n)$ such that $u_i \to u$ in $W^1L^{n^2/(n-1)}(\mathbb{R}^n)$; then we get

$$\|u_i - u_j\|_{L^{n^2/(n-1)}(\mathbb{R}^n)} \lessapprox \|u_i - u_j\|_{W^1L^n(\mathbb{R}^n)}\,;$$

thus $u_i \to u$ in $L^{n^2/(n-1)}(\mathbb{R}^n)$ and so the embedding $W^1L^n(\mathbb{R}^n) \hookrightarrow L^{n^2/(n-1)}(\mathbb{R}^n)$ is continuous. Now, we claim that

$$W^1L^n(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$$

is continuous for all $n \le q \le n^2/(n-1)$. We denote $n^2/(n-1) = q_1$. Indeed, assume that $n < q < q_1$ and write $1/q = \lambda/n + (1-\lambda)/q_1$ for some $0 < \lambda < 1$. Then

$$\|u\|_{L^q(\mathbb{R}^n)} \le \Big(\|u\|_{L^n(\mathbb{R}^n)}\Big)^{\lambda} \Big(\|u\|_{L^{q_1}(\mathbb{R}^n)}\Big)^{(1-\lambda)} \le \|u\|_{L^n(\mathbb{R}^n)} + \|u\|_{L^{q_1}(\mathbb{R}^n)}\,,$$

where we have used Young's inequality with exponents $1/\lambda$ and $(1/\lambda)'$. Therefore

$$\|u\|_{L^q(\mathbb{R}^n)} \le \|u\|_{L^n(\mathbb{R}^n)} + \|u\|_{L^{q_1}(\mathbb{R}^n)} \lessapprox \|u\|_{L^n(\mathbb{R}^n)} + \|u\|_{W^1L^n(\mathbb{R}^n)} \lessapprox \|u\|_{W^1L^n(\mathbb{R}^n)}\,,$$

which shows our assertion. Taking $t = n+1$ and using what we just proved gives that the emdedding

$$W^1L^n(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$$

is continuous for all $n \le q \le n(n+1)/(n-1)$. We proceed in this fashion taking $t = n+2$, $n+3$, etc. □

**Corollary 3.4.8.** *Let $n \ge 2$. Let $k$ be non-negative integer such that $1 \le k \le n-1$. If $kp = n$, then*

$$W^kL^p(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n), \quad n/k \le q < \infty.$$

*Proof.* Assume that $u \in W^kL^{n/k}(\mathbb{R}^n)$, and so $u \in W^{k-1}L^{n/k}(\mathbb{R}^n)$ and $D_ju \in W^{k-1}L^{n/k}(\mathbb{R}^n)$ for all $j = 1, \dots, n$. Since $n/k < n/(k-1)$ by Corollary 3.4.5, we obtain $u \in L^n(\mathbb{R}^n)$ and $D_ju \in L^n(\mathbb{R}^n)$ for all $j = 1, \dots, n$. Therefore $u \in W^1L^n(\mathbb{R}^n)$, and so by the Theorem 3.4.7 $u \in L^q(\mathbb{R}^n)$ for all $n \le q < \infty$. In other words, we have the following inequality for all $n \le q < \infty$

$$\begin{aligned}
\|u\|_{L^q(\mathbb{R}^n)} \lessapprox \|u\|_{W^1L^n(\mathbb{R}^n)} &= \|u\|_{L^n(\mathbb{R}^n)} + \sum_{j=1}^{n} \|D_ju\|_{L^n(\mathbb{R}^n)} \\
&\lessapprox \|u\|_{W^{k-1}L^{n/p}(\mathbb{R}^n)} + \sum_{j=1}^{n} \left\|D^ju\right\|_{W^{k-1}L^{n/k}(\mathbb{R}^n)} \\
&\le \|u\|_{W^kL^{n/k}(\mathbb{R}^n)} + \sum_{0\le|\alpha|\le k} \|D^{a}u\|_{L^{n/k}(\mathbb{R}^n)} \\
&\le \|u\|_{W^kL^{n/k}(\mathbb{R}^n)}\,.
\end{aligned}$$

Therefore the proof is complete. □

## Embeddings: $n < p < \infty$

**Theorem 3.4.9.** *(Morrey's Theorem) Let $n < p < \infty$. Then,*

$$W^1L^p(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n). \tag{3.4}$$

*Moreover*

$$\sup_{x\,,y\in\mathbb{R}^n\,,x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \leq C\,\|u\|_{W^1L^p(\mathbb{R}^n)}\,, \text{ where } \alpha := 1-\frac{n}{p}. \tag{3.5}$$

*Proof.* We start by proving (3.5). Let $Q$ be an open cube containing 0 whose edges have length $r$ and are parallel to coordinate axis. Given $x\in Q$, we have

$$u(x)-u(0) = \int_0^1 \frac{d}{dt}u(tx)dt,$$

and so

$$|u(x)-u(0)| \leq \int_0^1 \sum_{k=1}^n |x_k|\,|D_k u(tx)|\,dt \leq r\int_0^1 \sum_{k=1}^n |D_k u(tx)|\,dt.$$

Let $u_Q$ be the measure of $u$ on $Q$, i.e.

$$u_Q = \frac{1}{|Q|}\int_Q u(x)dx,$$

we have

$$|u_Q - u(0)| \leq \int_Q \frac{r}{|Q|}\left(\sum_{k=1}^n \int_0^1 |D_k u(tx)|\,dt\right)dx = \frac{1}{r^{n-1}}\int_0^1 t^{-n}\left(\sum_{k=1}^n \int_{tQ} |D_k u(y)|\,dy\right)dt.$$

Since $tQ\subset Q$ for $0<t<1$, using Hölder's inequality we have

$$\int_{tQ} |D_k u(y)|\,dy \leq \|D_k u\|_{L^p(\mathbb{R}^n)}\, t^{n/p'} r^{n/p'}.$$

Thus,

$$|u_Q - u(0)| \leq \frac{r^{1-n/p}}{1-\frac{n}{p}}\sum_{k=1}^n \|D_k u\|_{L^p(\mathbb{R}^n)}\,. \tag{3.6}$$

By translation, (3.6) is valid for all cube ,$Q$, whose sides have length and its edges are parallel to coordinate axes; that is for all $x\in Q$

$$|u_Q - u(x)| \leq \frac{r^{1-n/p}}{1-\frac{n}{p}}\sum_{k=1}^n \|D_k u\|_{L^p(\mathbb{R}^n)}\,. \tag{3.7}$$

Then, we obtain

$$|u(y)-u(x)| \leq \frac{2r^{1-n/p}}{1-\frac{n}{p}}\sum_{k=1}^n \|D_k u\|_{L^p(\mathbb{R}^n)}\,.$$

Next, for any two points, $x$, $y \in \mathbb{R}^n$, there exist a cube of side $r = 2|x-y|$ containing $x$ and $y$, hence

$$\sup_{x,y\in\mathbb{R}^n, x\neq y} \frac{|u(y)-u(x)|}{|x-y|^{1-n/p}} \leq c\sum_{k=1}^{n} \|D_k u\|_{L^p(\mathbb{R}^n)} \leq \|u\|_{W^1L^p(\mathbb{R}^n)}\,.$$

Thus, it follows (3.5) for all $u \in \mathcal{D}(\mathbb{R}^n)$. Now, we show (3.4). Given $u \in \mathcal{D}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and cube of edge $r = 1$ which contains $x$ by (3.7) we have

$$|u(x)| \lessapprox |u_Q| + \sum_{k=1}^{n} \|D_k u\|_{L^p(\mathbb{R}^n)} \lessapprox \|u\|_{W^1L^p(\mathbb{R}^n)}\,.$$

Hence $\|u\|_{L^\infty(\mathbb{R}^n)} \lessapprox \|u\|_{W^1L^p(\mathbb{R}^n)}\,.$ □

**Corollary 3.4.10.** *Let $k$ be integer such that $1 \leq k \leq n-1$. If $n/k < p < \infty$, we then*

$$W^kL^p(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n).$$

**Remark 3.4.11.** The previous results can be formulated in term of functions in $W^kL^p(\Omega)$, where $\Omega$ is a domain which satifies certain properties. For more details see [1] and [21].

# Chapter 4

# Orlicz spaces and Lorentz spaces

## 4.1 Introduction

Let $n$ be positive integer with $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be an open subset. We denote as $W_0^1 L^p(\Omega)$ the clousure of $\mathcal{D}(\Omega)$ in $W^1 L^p(\Omega)$. Throughout this chapter, we assume that $|\Omega| < \infty$.

In this chapter, present some refinements of Sobolev embeddings theorem. We have seen in Theorem 3.4.1 that

$$W_0^1 L^p(\Omega) \hookrightarrow L^{p^*}(\Omega), \quad \text{with } p^* = np/(n-p), \quad 1 \leq p < n. \tag{4.1}$$

Although (4.1) cannot be improved within Lebesgue space, if we consider Lorentz spaces we have the following improvement

$$W_0^1 L^p(\Omega) \hookrightarrow L^{p^*,p}(\Omega), \ 1 \leq p < n. \tag{4.2}$$

Those embeddings were observed by Peetre [24] and O'Neil [23]. Now, when $p = n$ it is known that $W_0^1 L^n(\Omega)$ can be embedded in $L^q(\Omega)$ for every $n \leq q < \infty$ (Theorem 3.4.7), and that $W_0^1 L^n(\Omega)$ cannot be embedded in $L^\infty(\Omega)$ (Example 3.4.6). Thus

$$W_0^1 L^n(\Omega) \hookrightarrow L^q(\Omega), \quad n \leq q < \infty. \tag{4.3}$$

cannot improved within the Lebesgue spaces. However, if we consider Orlicz spaces we have the following refinement

$$W_0^1 L^n(\Omega) \hookrightarrow L^\Phi(\Omega), \quad \Phi(t) = \exp(t^{n'}). \tag{4.4}$$

This result was shown, independently, by Pokhozhaev [25], Trudinger [29] and Yudovich [17]. It turns out that $L^\Phi(\Omega)$ is the smallest Orlicz space that still renders (4.4) true. This optimality result is due to Hempel, Morris, and Trudinger [15]. It turns out that an improvement of (4.4) is still possible. If we consider Lorentz Zygmund spaces, we have the following refinement of (4.4)

$$W_0^1 L^n(\Omega) \hookrightarrow L^{\infty,n\,;-1}(\Omega). \tag{4.5}$$

This embedding is due to Brézis-Waigner [3] and independently to Hansson [14]. It can be also derived from capacity estimates of Maz'ya [21].

## 4.2 Sobolev embeddings into Orlicz spaces

In this section we will prove the Sobolev embedding

$$W_0^1 L^n(\Omega) \hookrightarrow L^\Phi(\Omega), \quad \Phi(t) = \exp(t^{n'}).$$

First we prove the following result which will be useful later.

**Lemma 4.2.1.** *Let $x \in \mathbb{R}^n$. If $0 \le s < n$, then*

$$\int_\Omega |x-y|^{-s}\, dy \le \frac{\alpha(n)^{s/n}}{n-s} |\Omega|^{1-s/n},$$

*where $\alpha(n)$ is the volume of the unit $n$-ball.*

*Proof.* Let $B(x\,,r)$ be the ball such that $|B(x\,,r)| = |\Omega|$. Observe that for each $y \in \Omega \setminus B(x\,,r)$ and $z \in B(x\,,r) \setminus \Omega$, we have $|x-y|^{-s} \le |x-z|^{-s}$, and beucase

$$|\Omega \setminus B(x\,,r)| = |B(x\,,r) \setminus \Omega|,$$

it therefore follows that

$$\int_{\Omega \setminus B(x\,,r)} |x-y|^{-s}\, dy \le \int_{B(x\,,r)\setminus\Omega} |x-z|^{-s}\, dz.$$

Consequently,

$$\int_\Omega |x-y|^{-s}\, dy \le \int_{B(x\,,r)} |x-z|^{-s}\, dz = \frac{\alpha(n) r^{-s+n}}{-s+n},$$

where $\alpha(n)$ is the measure of the unit $n$-ball. But $\alpha(n)r^n = |\Omega|$ and hence

$$\int_\Omega |x-y|^{-s}\, dy \le \frac{\alpha(n)^{s/n} |\Omega|^{1-s/n}}{-s+n}.$$

□

**Theorem 4.2.2.** *Let $n > 1$. Then,*

$$W_0^1 L^n(\Omega) \hookrightarrow L^\Phi(\Omega), \quad \Phi(t) = \exp(t^{n'}).$$

*Proof.* It is sufficient to prove the theorem for functions $u \in \mathcal{D}(\Omega)$. By Lemma 3.4.2 we know that

$$|u(x)| \lessapprox \sum_{k=1}^n \int_\Omega \frac{|D_k u(y)|}{|x-y|^{n-1}} dy.$$

Suppose $s > 1$ and $v \in L^{s'}(\Omega)$, then

$$\begin{aligned}
\int_\Omega |u(x)|\,|v(x)|\,dx &\lessapprox \sum_{k=1}^n \int_\Omega\int_\Omega \frac{|v(x)|\,|D_k u(y)|}{|x-y|^{n-1}}dxdy \\
&\lessapprox \sum_{k=1}^n \int_\Omega |D_k u(y)| \left(\int_\Omega \frac{|v(x)|}{|x-y|^{\frac{(n-1)}{s}}}dx\right)^{1/n} dy \\
&\qquad \times \left(\int_\Omega \frac{|v(x)|}{|x-y|^{n-(1/s)}}dx\right)^{1/n'} \\
&\lessapprox \sum_{k=1}^n \left(\int_\Omega\int_\Omega \frac{|D_k u(x)|^n\,|v(x)|}{|x-y|^{(n-1)/s}}dxdy\right)^{1/n} \\
&\qquad \times \left(\int_\Omega\int_\Omega \frac{|v(x)|}{|x-y|^{n-(1/s)}}dxdy\right)^{1/n'}
\end{aligned}$$

By Lemma 4.2.1 we obtain

$$\int_\Omega\int_\Omega \frac{|v(x)|}{|x-y|^{n-(1/s)}}dydx \le s\,|\Omega|^{1/sn}\,K_1 \int_\Omega |v(x)|\,dx \le K_1 s\,|\Omega|^{1/(sn)+1/s}\,\|v\|_{L^{s'}(\Omega)}\,.$$

with $K_1 = \alpha(n)^{(ns-1)/sn}$. Also

$$\begin{aligned}
\int_\Omega\int_\Omega \frac{|D_k u(y)|^n\,|v(x)|}{|x-y|^{(n-1)/s}}dydx &\le \|v\|_{L^{s'}(\Omega)} \int_\Omega |D_k u(y)|^n \left(\int_\Omega \frac{1}{|x-y|^{n-1}}dx\right)^{1/s} \\
&\le K_2\,|\Omega|^{1/(ns)}\,\|v\|_{s'} \int_\Omega |D_k u(y)|^n\,dy \\
&= K_2\,\|D_k u\|^n_{L^n(\Omega)}\,\|v\|_{L^{s'}(\Omega)}\,|\Omega|^{1/(ns)}\,,
\end{aligned}$$

with $K_2 = \alpha(n)^{(ns-1)/(sn)}$. It follows from these estimates that

$$\int_\Omega |u(x)|\,|v(x)|\,dx \le K_3\,\|v\|_{L^{s'}(\Omega)} \sum_{k=1}^n s^{1/n'}\,\|D_k u\|_{L^n(\Omega)}\,|\Omega|^{1/s}\,,$$

with $K_3 = K_1^{1/n'}K_2^{1/n}$. We have

$$\|u\|_{L^s(\Omega)} = \sup_{v\neq 0} \frac{\int_\Omega |u(x)|\,|v(x)|\,dx}{\|v\|_{L^{s'}(\Omega)}} \le K_5 s^{1/n'}\,\|u\|_{W_0^1L^n(\Omega)}\,|\Omega|^{1/s}\,.$$

Setting $s = nj/(n-1)$, we obtain

$$\begin{aligned}
\int_\Omega |u(x)|^{nj/(n-1)}\,dx &\le |\Omega| \left(\frac{nj}{n-1}\right)^j \left(K_5\,\|u\|_{W_0^1L^n(\Omega)}\right)^{nj/(n-1)} \\
&= |\Omega| \left(\frac{j}{e^{n/(n-1)}}\right)^k \left(K_5 e\left[\frac{n}{n-1}\right]^{(n-1)/n} \|u\|_{W^1L_0^n(\Omega)}\right)^{nj/(n-1).}
\end{aligned}$$

Note that $\sum_{j=1}^{\infty} \frac{1}{j!}\left(\frac{j}{e^{n/(n-1)}}\right)^j$ converges. Now, let

$$\tilde{A} = \max\left\{1\,, |\Omega| \sum_{j=1}^{\infty} \frac{1}{j!}\left(\frac{j}{e^{n/(n-1)}}\right)^j\right\}$$

and

$$K = eK_5\tilde{A}\left(\frac{n}{n-1}\right)^{(n-1)/n} \|u\|_{W_0^1 L^n(\Omega)}\,.$$

Then

$$\int_\Omega \left(\frac{|u(x)|}{K}\right)^{nj/(n-1)} dx \leq \frac{|\Omega|}{\tilde{A}^{nj/(n-1)}}\left(\frac{j}{e^{n/(n-1)}}\right)^j < \frac{|\Omega|}{\tilde{A}}\left(\frac{j}{e^{n/(n-1)}}\right)^j$$

since $\tilde{A} \geq 1$ and $nk/(n-1) > 1$. Expanding $\Phi(t) = \exp(t^{n'})$ in power series, i.e.

$$\Phi(t) = \sum_{j=1}^{\infty} \frac{1}{j!} t^{nj/(n-1)},$$

we obtain

$$\int_\Omega \Phi\left(\frac{|u(x)|}{K}\right) dx = \sum_{j=1}^{\infty} \frac{1}{k!}\int_\Omega \left(\frac{|u(x)|}{K}\right)^{nj/(n-1)} dx < \frac{|\Omega|}{\tilde{A}} \sum_{j=1}^{\infty} \frac{1}{j!}\left(\frac{j}{e^{n/(n-1)}}\right)^j \leq 1.$$

Hence $u \in L^\Phi(\Omega)$ and $\|u\|_{L^\Phi(\Omega)} \leq K$. □

**Corollary 4.2.3.** *Let $n > 1$ and $\Phi(t) = \exp(t^{n'})$ then*

$$W^1 L^p(\mathbb{R}^n) \hookrightarrow L^\Phi(\mathbb{R}^n).$$

*Proof.* It follows from Theorem 4.2.2, taking $|\operatorname{supp} u| = |\Omega|$ where $u \in \mathcal{D}(\mathbb{R}^n)$. □

## 4.3 Sobolev embeddings into Lorentz spaces

Now, we will prove

$$W_0^1 L^p(\Omega) \hookrightarrow L^{p^*,p}(\Omega), \quad 1 \leq p < n\,,\; p^* = \frac{np}{n-p}.$$

Throughout this section, we present results proved in [20]. We first establish the weak version of the Sobolev-Gagliardo-Niremberg inequality.

**Lemma 4.3.1.** *For every $u \in W_0^1 L^1(\Omega)$ and $\lambda > 0$, the estimate*

$$\lambda \left( |\{x \in \Omega \, : |u(x)| \geq \lambda\}| \right)^{1/n'} \lessapprox \sum_{k=1}^{n} \int_{\Omega} |D_k u(x)| \, dx,$$

*holds, i.e.* $W_0^1 L^1(\Omega) \hookrightarrow L^{n',\infty}(\Omega)$.

*Proof.* Fix $u \in \mathcal{D}(\Omega)$. We denote by $G$ the set $\{x \in \Omega \, : |u(x)| \geq \lambda\}$. Let $K \subset G$ be a compact set. Then, using Lemma 3.4.2, Lemma 4.2.1 and Fubini's theorem we have

$$|K| \leq \frac{1}{\lambda} \int_K |u(x)| \, dx \lessapprox \frac{1}{\lambda} \int_K \left( \sum_{k=1}^{n} \int_{\Omega} \frac{D_k u(y)}{|x-y|^{n-1}} dy \right) dx \lessapprox \frac{|K|^{1/n}}{\lambda} \sum_{k=1}^{n} \int_{\Omega} |D_k u(y)| \, dy.$$

Thus, since $|K| < \infty$,

$$\lambda \, |K|^{1/n'} \lessapprox \sum_{k=1}^{n} \int_{\Omega} |D_k u(y)| \, dy.$$

If one takes the supremum over all such that $K \subset G$, the lemma is proved for $u \in \mathcal{D}(\Omega)$. The general case follows by standard approximation arguments. □

**Corollary 4.3.2.** *For every $u \in W^1 L^1(\mathbb{R}^n)$ and $\lambda > 0$, the estimate*

$$\lambda \left( |\{x \in \mathbb{R}^n \, : |u(x)| \geq \lambda\}| \right)^{1/n'} \lessapprox \sum_{k=1}^{n} \int_{\mathbb{R}^n} |D_k u(x)| \, dx,$$

*holds.*

**Lemma 4.3.3.** *Let $1 \leq p \leq n$ and $u \in W_0^1 L^p(\Omega)$. We denote $t_k = |\Omega| \, 2^{1-k}$ and $a_k = u^*(t_k)$, $k \in \mathbb{N}$. Then*

$$\sum_{k=1}^{\infty} t_k^{p/p^*} \left( a_{k+1} - a_k \right)^p \lessapprox \sum_{j=1}^{n} \int_{\Omega} |D_j u(x)|^p \, dx.$$

*Proof.* If $p > 1$, using Hölder's inequality we obtain

$$\lambda \left( |\{x \in \Omega \, : |u(x)| \geq \lambda\}| \right)^{1/n'} \lessapprox \left( \sum_{j=1}^{n} \int_{\Omega} |D_j(u)(x)|^p \, dx \right)^{1/p} \left( |\{x \in \Omega \, : |u(x)| > 0\}| \right)^{1/p'} . \tag{4.6}$$

With the convention $\frac{1}{1'} = 0$ this also holds for $p = 1$. Now, given $0 < a < b < \infty$, we use a smooth function $\varphi_a^b$ on $\mathbb{R}$ such that

$$\begin{cases} \varphi_a^b(s) = 0 & \text{for } s \in (-\infty \, , a] \, , \\ 0 < \left(\varphi_a^b\right)'(s) < 2 & \text{for } s \in (a \, , b) \, , \\ \varphi_a^b(s) = b - a & \text{for } s \in [b \, , \infty) \, . \end{cases}$$

Applying (4.6) to the function $\varphi_a^b(|u(x)|)$ and $\lambda = b - a$, we arrive at

$$(b-a)\left(|\{x \in \Omega : |u(x)| \geq b\}|\right)^{1/n'} \lessapprox \left(\sum_{j=1}^{n} \int_{a<|u(x)|<b} |D_j(u)(x)|^p \, dx\right)^{1/p} \times \left(|\{x \in \Omega : |u(x)| > a\}|\right)^{1/p'}.$$

Then, if we take $a = a_k$ and $b = a_{k+1}$ we obtain

$$(a_{k+1}-a_k)\left(|\{x \in \Omega : |u(x)| \geq a_{k+1}\}|\right)^{1/n'} \lessapprox \left(\sum_{j=1}^{n} \int_{a_k<|u(x)|<a_{k+1}} |D_j(u)(x)|^p \, dx\right)^{1/p} \times \left(|\{x \in \Omega : |u(x)| > a_k\}|\right)^{1/p'},$$

but, since $|\{x \in \Omega : |u(x)| > a_k\}| \leq t_k \leq |\{x \in \Omega : |u(x)| \geq a_k\}|$, we obtain

$$t_{k+1}^{1/n'}(a_{k+1}-a_k) \lessapprox \left(\sum_{j=1}^{n} \int_{a_k<|u(x)|<a_{k+1}} |D_j(u)(x)|^p \, dx\right)^{1/p} t_k^{1/p'},$$

that is (recall that $2t_{k+1} = t_k$),

$$t_{k+1}^{1/p^*}(a_{k+1}-a_k) \lessapprox \left(\sum_{j=1}^{n} \int_{a_k<|u(x)|<a_{k+1}} |D_j(u)(x)|^p \, dx\right)^{1/p},$$

where the convention $\frac{1}{n^*} = 0$ is used. We raise this estimate to the power $p$ and sum over $k$. We obtain

$$\sum_{k=1}^{\infty} t^{p/p^*}(a_{k+1}-a_k)^p \leq \sum_{j=1}^{n}\left(\sum_{k=1}^{\infty} \int_{a_k<|u(x)|<a_{k+1}} |D_j(u)(x)|^p \, dx\right) \leq \sum_{j=1}^{n} \int_{\Omega} |D_j(u)(x)|^p \, dx,$$

finishing the proof. □

**Corollary 4.3.4.** *Let $1 \leq p \leq n$ and $u \in \mathcal{D}(\mathbb{R}^n)$. Denote $t_k = 2^{1-k}|\operatorname{supp} u|$ and $a_k = u^*(t_k)$ with $k \in \mathbb{N}$. Then, $\sum_{k=1}^{\infty} t_k^{p/p^*}(a_{k+1}-a_k)^p \lessapprox \sum_{j=1}^{n} \int_{\Omega} |D_j u(x)|^p \, dx$.*

**Lemma 4.3.5.** *The inequality*

$$(a+b)^p \leq (1+\varepsilon)^{p-1}a^p + (1+1/\varepsilon)^{p-1}b^p, \quad 1 \leq p < \infty,$$

*holds for arbitrary $a, b \in \mathbb{R}^+$ and $\varepsilon > 0$.*

*Proof.* It follows from the fact that $t \to t^p$ is a convex function and therefore

$$(a+b)^p = \left[\lambda\frac{a}{\lambda} + (1-\lambda)\left(\frac{b}{1-\lambda}\right)\right]^p \leq \lambda\left(\frac{a}{\lambda}\right)^p + (1-\lambda)\left(\frac{b}{1-\lambda}\right)^p,$$

whenever $0 < \lambda < 1$. Taking $\lambda = \frac{1}{(1+\varepsilon)}$ establishes the result. □

**Theorem 4.3.6.** *Assume that* $1 \leq p < n$. *Then*

$$\int_0^{|\Omega|} t^{p/p^*} (u^*(t))^p \, dt \lessapprox \sum_{k=1}^n \int_\Omega |D_k u(x)|^p \, dx,$$

*for all* $u \in W_0^1 L^p(\Omega)$, *i.e.* $W_0^1 L^p(\Omega) \hookrightarrow L^{p^*,p}(\Omega)$

*Proof.* Fix $u \in \mathcal{D}(\Omega)$. Let $t_k$ and $a_k$ have the same meaning as in Lemma 4.3.3. Given $\varepsilon > 0$, Lemma 4.3.7 yields

$$a_{k+1}^p \leq \left(1 + \frac{1}{\varepsilon}\right)^{p-1} (a_{k+1} - a_k)^p + (1+\varepsilon)^{p-1} a_k^p.$$

Hence, taking into account that $a_1 = u^*(t_1) = 0$,

$$2^{p/p^*} \sum_{k=1}^\infty t_{k+1}^{p/p^*} a_k^p = \sum_{k=1}^\infty t_k^{p/p^*} a_k^p = \sum_{k=1}^\infty t_{k+1}^{p/p^*} a_{k+1}^p \leq (1+\varepsilon)^{p-1} \sum_{k=1}^\infty t_{k+1}^{p/p^*} a_k^p$$
$$+ \left(1 + \frac{1}{\varepsilon}\right)^{p-1} \sum_{k=1}^\infty t_{k+1}^{p/p^*} (a_{k+1} - a_k)^p \,.$$

Choosing $\varepsilon > 0$ so small that $(1+\varepsilon)^{p-1} < 2^{p/p^*}$, and by Lemma 4.3.3 we obtain

$$\sum_{k=1}^\infty t_{k+1}^{p/p^*} (u^*(t_k))^p \lessapprox \sum_{j=1}^n \int_\Omega |D_j u(x)|^p \, dx.$$

The sum on the right is an infinite Riemann sum, i.e.

$$2^{-p/p^*} \int_0^{|\Omega|} t^{p/p^*} u^*(t) \frac{dt}{t} = 2^{-p/p^*} \sum_{k=1}^\infty t_k^{p/p^*} (u^*(t_k))^p \frac{1}{t_k} (t_{k-1} - t_k) \,,$$

therefore

$$\int_0^{|\Omega|} t^{p/p^*} (u^*(t)) \frac{dt}{t} \lessapprox \sum_{j=1}^n \int_{\mathbb{R}^n} |D_j u(x)|^p \, dx.$$

Finally, by standard approximation arguments we extend the result to all functions $u \in W_0^1 L^p(\Omega)$. □

**Corollary 4.3.7.** *Assume that* $1 \leq p < n$. *Then*

$$\int_0^\infty t^{p/p^*} (u^*(t))^p \, dt \leq A \sum_{k=1}^n \int_{\mathbb{R}^n} |D_k u(x)|^p \, dx,$$

*for all* $u \in W^1 L^p(\mathbb{R}^n)$, *i.e.* $W^1 L^p(\mathbb{R}^n) \hookrightarrow L^{p^*,p}(\mathbb{R}^n)$.

*Proof.* To prove this corollary apply Corollary 4.3.4 and the reasoning of the previous theorem. □

Note that, thanks to Proposition 2.5.3 and the inequality $p < p^*$, Theorem 4.3.6 give us a non-trivial improvement of the range space in $W_0^1 L^p(\Omega) \hookrightarrow L^{p^*}(\Omega)$.

## 4.4 Sobolev embeddings into Lorentz Zygmund spaces

In this section, we will prove Now, we are going to show

$$W_0^1 L^n(\Omega) \hookrightarrow L^{\infty,n;-1}(\Omega).$$

Throughout this section, we present results proved in [20].

**Remark 4.4.1.** Note that

$$\int_0^1 \left( \frac{u^*(t)}{\log\left(\frac{1}{t}\right)} \right)^n \frac{dt}{t} \approx \int_0^1 \left( \frac{u^*(t)}{\log\left(\frac{e}{t}\right)} \right)^n \frac{dt}{t},$$

it follows from [4, Theorem 3.1].

**Theorem 4.4.2.** *We have*

$$\int_0^1 \left( \frac{u^*(t)}{\log\left(\frac{1}{t}\right)} \right)^n \frac{dt}{t} \lessapprox \sum_{j=1}^n \int_\Omega |D_j u(x)|^n \, dx, \tag{4.7}$$

*for all* $u \in W_0^1 L^n(\Omega)$*, i.e.* $W_0^1 L^n(\Omega) \hookrightarrow L^{\infty,n\,;-1}(\Omega)$.

*Proof.* Let $u \in \mathcal{D}(\Omega)$. We fix such $u$ and assume that $t_k$ and $a_k$ have the same meaning as in Lemma 4.3.3. Given $m \in \mathbb{N}$ (using $a_1 = 0$)

$$0 \le \frac{a_m^n}{m^{n-1}} = \sum_{k=1}^{m-1} \left( \frac{a_{k+1}^n}{(k+1)^{n-1}} - \frac{a_k^n}{k^{n-1}} \right) = \sum_{k=1}^{m-1} \frac{a_{k+1}^n - a_k^n}{(k+1)^{n-1}} - \sum_{k=1}^{m-1} a_k^n \left( \frac{1}{k^{n-1}} - \frac{1}{(k+1)^{n-1}} \right).$$

Hence, passing to limit for $m \to \infty$

$$\begin{aligned}\sum_{k=1}^\infty \frac{a_k^n}{k^n} \approx \sum_{k=1}^\infty a_k^n \left( \frac{1}{k^{n-1}} - \frac{1}{(k+1)^{n-1}} \right) \lessapprox \sum_{k=1}^\infty \frac{a_{k+1}^n - a_k^n}{(k+1)^{n-1}} \approx \sum_{k=1}^\infty \frac{a_{k+1}^{n-1}(a_{k+1} - a_k)}{(k+1)^{n-1}} \\ \lessapprox \left( \sum_{k=1}^\infty (a_{k+1} - a_k)^n \right)^{1/n} \left( \sum_{k=1}^\infty \frac{a_{k+1}^n}{(k+1)^n} \right)^{1/n'}.\end{aligned}$$

Recalling that $a_1 = 0$, we have

$$\sum_{k=1}^\infty \frac{u^*(t_k)^n}{k^n} = \sum_{k=1}^\infty \frac{a_n^k}{k^n} \lessapprox \sum_{k=1}^\infty (a_{k+1} - a_k)^n \lessapprox \sum_{j=1}^n \int_\Omega |D_j u(x)|^n \, dx,$$

which is a discrete version of (4.7). Finally, by standard truncation and approximation argument, we extend the result to all $u \in W_0^1 L^n(\Omega)$. ☐

A further improvement of $W_0^1 L^n(\Omega) \hookrightarrow L^{\infty,n;-1}(\Omega)$ is possible.

**Definition 4.4.3.** *We define $W_p(\Omega)$ for $1 \leq p \leq \infty$ as the family of all measurable functions on $\Omega$ for which*

$$\|u\|_{W_p(\Omega)} = \begin{cases} \left(\int_0^1 \left(u^*\left(\frac{t}{2}\right) - u^*(t)\right)^p \frac{dt}{t}\right)^{1/p} < \infty, & \textit{when } p < \infty, \\ \sup_{0<t<1} \left(u^*\left(\frac{t}{2}\right) - u^*(t)\right), & \textit{when } p = \infty. \end{cases}$$

**Theorem 4.4.4.** *Let $n > 1$, then*

$$W_0^1 L^n(\Omega) \hookrightarrow W_n(\Omega). \tag{4.4.4}$$

*Proof.* Note that, for $p = n$ for Lemma 4.3.3 reads as

$$\sum_{k=1}^{\infty} (a_{k+1} - a_k)^n \leq A \sum_{j=1}^{n} \int_\Omega |D_j u(x)|^n \, dx,$$

which is just a discrete version of (4.4.4) . □

Since $W_n \subsetneq L^{\infty,n;-1}(\Omega)$ (see Remark 4.4.5 bellow), Theorem 4.4.4 improves Remark 4.4.2; but $W_n(\Omega)$ is not an r.i. space beucase it is not a linear set (see Remark 4.4.5 bellow).

**Remark 4.4.5.** The $W_p(\Omega)$ space has the follwing properties:

1. $\|\chi_E\|_{W_p(\Omega)} = (\log 2)^{1/p}$ for every measurable $E \subset \Omega$ and $1 \leq p < \infty$.
2. $L^\infty(\Omega) \subsetneq W_n(\Omega)$.
3. For $1 \leq p < \infty$ each interger-valued $u \in W_p(\Omega)$ is bounded.
4. For $1 < p < \infty$, $W_p(\Omega)$ is not a linear set.
5. For $1 < p < \infty$ $W_p(\Omega) \subsetneq L^{\infty,p;-1}(\Omega)$.

# Chapter 5

# Optimal Sobolev embeddings on rearrangement invariant spaces

## 5.1 Introduction

Let $n$ be an integer, $n \geq 2$, and let $m$ be an integer satisfying $1 \leq m \leq n-1$. Let $\Omega$ be a bounded subset of $\mathbb{R}^n$ such that $|\Omega| = 1$ (if its measure is different from 1, everything can be modified by the change of variables $t \to |\Omega|\, t$). Denote by $|D^m u|$ the eucledian length of, $D^m u = \left(\frac{\partial^\alpha u}{\partial x^\alpha}\right)_{0 \leq |\alpha| \leq m}$, the vector of all derivates of order $m$ or less, whenever such derivates exits on $\Omega$ in the weak sense.

In this chapter we study of optimality in Sobolev embeddings on rearrangement invariant spaces. This problem can be formulated as follows. We are interesting in determining those rearrangement invariant Banach spaces such that

$$W_0^m X(\Omega) \hookrightarrow Y(\Omega). \tag{5.1}$$

We would like to know that $X(\Omega)$ and $Y(\Omega)$ in (5.1) are optimal; in the sense that $X(\Omega)$ cannot be replaced by an larger r.i. space and $Y(\Omega)$ cannot be replaced by an smaller one. Kerman and Pick [18] solved this problem. The central part of their work may be summarized as follows. They developed a method that enable us to reduce the Sobolev embedding (5.1) to boundedness of certain weighted Hardy operator; and then used it to characterize the largest rearrangement invariant Banach domain space and the smallest rearrangement invariant Banach range space in the Sobolev embedding (5.1).

## 5.2 Reduction Theorem

In this section, we will prove our main theorem (Theorem 5.2.1) which is known as *Reduction Theorem.* Its proof can be found in [18].

**Theorem 5.2.1.** *Let $X(\Omega)$ and $Y(\Omega)$ be an r.i. spaces. Then,*

$$W_0^m X(\Omega) \hookrightarrow Y(\Omega),$$

*if and only if there is a positive constant $C_2$ such that*

$$\left\| \int_t^1 f(s) s^{m/n-1} ds \right\|_{\overline{Y}(I)} \leq C_2 \|f\|_{\overline{X}(I)}, \quad f \in \overline{X}(I)$$

*i.e., $H_{n/m} : \overline{X}(I) \to \overline{Y}(I)$.*

When $m = 1$, Theorem 5.2.1 was proved in [10] and the case $m = 2$ was studied in [6]. Finally, Kerman and Pick [18] proved our version of *Reduction Theorem* for all $m$ using results from interpolation theory.

## The necessity part of Theorem 5.2.1

**Theorem 5.2.2.** *Let $X(\Omega)$ and $Y(\Omega)$ be an r.i. spaces. Suppose that $W_0^m X(\Omega) \hookrightarrow Y(\Omega)$. Then, there is a positive constant $C_2$ such that*

$$\left\| \int_t^1 f(s) s^{m/n-1} ds \right\|_{\overline{Y}(I)} \leq C_2 \|f\|_{\overline{X}(I)}, \quad f \in \overline{X}(I).$$

*Proof.* We may suppose, without loss of generality that $0 \in \Omega$. Let $\sigma$ be a positive number not exceeding $1/2$, and so small that the ball centered at 0 and having measure $\sigma$ is contained in $\Omega$. Given any non-negative function $f \in \overline{X}(I)$ with $\operatorname{supp} f \subset [0, \sigma]$, define

$$u(x) := \int_{C_n|x|^n}^\infty \int_{t_1}^\infty \int_{t_2}^\infty \ldots \int_{t_{m-1}}^\infty f(t_m) t_m^{-m+m/n} dt_m \ldots d_{t_1} = g\left(C_n |x|^n\right),$$

for all $x \in \mathbb{R}^n$. Note that $u$ has compact support in $\Omega$. Moreover induction in $\ell \in \mathbb{N}$, shows that any $\ell^{\text{th}}$ order derivative of $u$ is a linear combination of terms of the form

$$x_{\alpha_1} \ldots x_{\alpha_i} g^{(j)}\left(|x|^n C_n\right) |x|^k, \;\; k + i = jn - \ell\,, 0 < i\,, j\,, \leq \ell.$$

From this we conclude that any $\ell^{\text{th}}$ order derivative of $u$ is in absolute value, dominated by a constant multiple of

$$\sum_{j=1}^{\ell} \left|g^{(j)}\left(|x|^n C_n\right)\right| |x|^{jn-\ell},$$

and hence,

$$\left|\left(D^m u\right)(x)\right| \leq \sum_{\ell=1}^{m} \left|g^{(\ell)}\left(|x|^n C_n\right)\right| |x|^{jn-\ell}.$$

Now, if $1 \leq \ell \leq m-1$

$$\left|g^{(\ell)}\left(C_n |x|^n\right)\right| = \int_{C_n|x|^n}^\infty \int_{t_{\ell+1}}^\infty \ldots \int_{t_{m-1}}^\infty f(t_m) t_m^{-m+m/n} dt_m \ldots d_{\ell+1},$$

and

$$\left|g^{(m)}\left(C_n\left|x\right|^n\right)\right| \approx f\left(C_n\left|x\right|^n\right)\left|x\right|^{-nm+m}.$$

Since we are assuming that supp$f \subset [0,\sigma] \subset [0,1]$, we have

$$u(x) := \int_{C_n|x|^n}^1 \int_{t_1}^1 \int_{t_2}^1 \dots \int_{t_{m-1}}^1 f(t_m) t_m^{-m+m/n} dt_m \dots d_{t_1} = g\left(C_n\left|x\right|^n\right).$$

Moreover, when $j = 1, \dots, m-1$ we have on applying Fubini's theorem $m-j-1$ times

$$g^{(\ell)}\left(C_n\left|x\right|^n\right) \lessapprox \int_{C_n|x|^n}^1 f(s) s^{-j+m/n-1} ds.$$

Hence,

$$\left|D^m u(x)\right| \lessapprox \sum_{\ell=1}^m \left|g^{(\ell)}\left(|x|\right)\right| |x|^{\ell n-m} \lessapprox f\left(C_n\left|x\right|^n\right) + \sum_{\ell=1}^{m-1} |x|^{\ell n-m} \int_{C_n|x|^n}^1 f(s) s^{-\ell+m/n-1} ds.$$

Thus,

$$\left\| |D^m u| \right\|_{X(\Omega)} \lessapprox \|f\|_{\overline{X}(I)} + \sum_{j=1}^{m-1} \left\| t^{j-m/n} \int_t^1 f(s) s^{-j+m/n-1} ds \right\|_{\overline{X}(I)}. \tag{5.2}$$

Considere the linear operator $T$ defined as

$$Tf(t) = t^{j-m/n} \int_t^1 f(s) s^{-j+m/n-1} ds, \quad f \in \mathcal{M}_+(I).$$

The operator $T$ is bounded in $L^1(I)$, since

$$\begin{aligned} \left\| t^{j-m/n} \int_t^1 f(s) s^{-j+m/n-1} ds \right\|_{L^1(I)} &= \int_0^1 t^{j-m/n} \int_t^1 f(s) s^{-j+m/n-1} ds \\ &\approx \int_0^1 f(s) ds = \|f\|_{L^1(I)}. \end{aligned}$$

Moreover, $T$ is bounded on $L^\infty(I)$, since

$$\begin{aligned} \left\| t^{j-m/n} \int_t^1 f(s) s^{-j+m/n-1} ds \right\|_{L^\infty(I)} &= \sup_{0<t<1} t^{j-m/n} \int_t^1 f(s) s^{-j+m/n-1} ds \\ &\le \|f\|_{L^\infty(I)} \sup_{0<t<1} t^{j-m/n} \int_t^1 s^{-j+m/n-1} ds \\ &\lessapprox \|f\|_{L^\infty(I)}. \end{aligned}$$

A theorem by Calderón (see [2, Theorem III.2.12]) then ensures that $T$ is bounded on $\overline{X}(I)$, i.e.

$$\left\| t^{j-m/n} \int_t^1 f(s) s^{-j+m/n-1} ds \right\|_{\overline{X}(I)} \lessapprox \|f\|_{\overline{X}(I)} .$$

Therefore, from (5.2) it follows that

$$\| |D^m u| \|_{X(\Omega)} \lessapprox \|f\|_{\overline{X}(I)} ,$$

whence $u \in W_0^m X(\Omega)$. Next, by hypothesis

$$\begin{aligned} \| |D^m u|^* \|_{\overline{X}(I)} &\gtrapprox \|u^*\|_{\overline{Y}(I)} \\ &= \left\| \int_t^1 \int_{t_1}^1 \dots \int_{t_{m-1}}^1 f(t_m) t^{-m+m/n} dt_m \dots dt_1 \right\|_{\overline{Y}(I)} \\ &= \left\| \int_t^1 f(s) s^{-m+m/n} \frac{(s-t)^{m-1}}{(m-1)!} ds \right\|_{\overline{Y}(I)} \\ &\gtrapprox \left\| \int_{2t}^1 f(s) s^{-1+m/n} (1-t/s)^{m-1} ds \right\|_{\overline{Y}(I)} \\ &\gtrapprox \left\| \int_t^1 f(s) s^{-1+m/n} ds \right\|_{\overline{Y}(I)} \end{aligned}$$

Thus,

$$\left\| \int_t^1 f(s) s^{m/n-1} ds \right\|_{\overline{Y}(I)} \lessapprox \|f\|_{\overline{X}(I)} , \tag{5.3}$$

for every non-negative $f \in \overline{X}(I)$ with $\operatorname{supp} f \subset [0, \sigma]$ . Now, let $f$ be any function from $\overline{X}(I)$. Then

$$\begin{aligned} \left\| \int_t^1 f(s) s^{m/n-1} ds \right\|_{\overline{Y}(I)} \leq & \left\| \int_t^1 \chi_{[0,\sigma]}(s) f(s) s^{m/n-1} ds \right\|_{\overline{Y}(I)} \\ &+ \left\| \int_t^1 \chi_{[\sigma,1]}(s) f(s) s^{m/n-1} ds \right\|_{\overline{Y}(I)} . \end{aligned} \tag{5.4}$$

We have

$$\begin{aligned} \left\| \int_t^1 \chi_{[\sigma,1]}(s) f(s) s^{m/n-1} ds \right\|_{\overline{Y}(I)} &\leq \sigma^{m/n-1} \left\| \int_t^1 f(s) s^{m/n-1} ds \right\|_{\overline{Y}(I)} \\ &\leq \sigma^{m/n-1} \|1\|_{\overline{Y}(I)} \int_0^1 f(s) ds \\ &\leq \sigma^{m/n-1} \|1\|_{\overline{Y}(I)} \|1\|_{\overline{X}'(I)} \|f\|_{\overline{X}(I)} . \end{aligned} \tag{5.5}$$

On estimating the first term on the right-hand side of (5.4) by (5.3) (with $f$ replaced by $\chi_{[0,\sigma]} f$) and the second term by (5.5), the proof is complete. □

## The sufficiency part of Theorem 5.2.1

The following lemma is a generalization of the Pólya-Szegö principle. Its proof may be found in [8].

**Lemma 5.2.3.** *Let $u \in \mathcal{D}(\mathbb{R}^n)$. Then,*

$$\int_0^t \left[ y^{1/n'} \left( -\frac{du^*}{dy} \right) \right]^* (s) \leq n^{-1} K_n^{-1/n} \int_0^t |Du|^* (s) ds, \quad t \in \mathbb{R}^+,$$

*where $K_n = \pi^{n/2} \Gamma \left( n/2 + 1 \right)^{-1}$.*

Now, we recall the definition of the $m$-dimensional Hausdorff measure and some important result such as coarea formula and isoperimetric theorem. Next, we will prove Lemma 5.2.3.

**Definition 5.2.4.** *For each $m \geq 0$, $\varepsilon > 0$ and $E \subset \mathbb{R}^n$, let*

$$\mathcal{H}_\varepsilon^m = \inf \left\{ \sum_{i=1}^\infty \alpha(m) 2^{-m} diam(A_i)^m \ : E \subset \cup_{i=1}^\infty A_i \, , diam \, A_i < \varepsilon \right\}.$$

*Since $\mathcal{H}_\varepsilon^m$ is decreasing in $\varepsilon$, we may define the $m$-dimensional Hausdorff measure as*

$$\mathcal{H}^m(E) = \lim_{\varepsilon \to 0} \mathcal{H}_\varepsilon^m(E).$$

The following theorem is known as coarea formula. A proof of this formula appears in [11].

**Theorem 5.2.5.** *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz continuous function and let $f : \mathbb{R}^n \to \mathbb{R}$ be an integrable function. Then,*

$$\int_{\mathbb{R}^n} f(x) \left| Du(x) \right| dx = \int_0^\infty \left( \int_{\{x \in \mathbb{R}^n : |u(x)| = t\}} f(x) \mathcal{H}^{n-1}(dx) \right) dt.$$

The following theorem is known as isoperimetric theorem in $\mathbb{R}^n$.

**Theorem 5.2.6.** *Let $E \subset \mathbb{R}^n$ be a measurable subset with finite measure. Then,*

$$\mathcal{H}^{n-1}(\partial E) \geq n K_n \left| E \right|^{1-1/n},$$

*where $K_n$ is the measure of the unit $n$-dimensional ball.*

*Proof.* (Lemma 5.2.3). Let $u \in \mathcal{D}(\mathbb{R}^n)$. The following inequality

$$\int_{\{x \in \mathbb{R}^n : u^*(b) < |u(x)| < u^*(a)\}} \left| Du(x) \right| dx = \geq n K_n^{1/n} a^{1/n'} (u^*(a) - u^*(b)), \tag{5.6}$$

holds if $0 \le a < b < \operatorname{supp} u$. In fact,

$$\begin{aligned}
\int_{\{x\in\mathbb{R}^n\,:u^*(b)<|u(x)|<u^*(a)\}} |Du(x)|dx \\
&= \int_{u^*(b)}^{u^*(a)} \mathcal{H}_{n-1}\left(\{x\in\mathbb{R}^n\ :|u(x)|=t\}\right)dt \\
&\ge nK_n^{1/n}\int_{u^*(b)}^{u^*(a)} \left|\{x\in\mathbb{R}^n\ :|u(x)|\ge t\}\right|^{1/n'}dt \\
&\ge nK_n^{1/n}\left|\{x\in\mathbb{R}^n\ :|u(x)|\ge u^*(a)\}\right|^{1/n'}(u^*(a)-u^*(b)) \\
&\ge nK_n^{1/n}a^{1/n'}(u^*(a)-u^*(b)).
\end{aligned}$$

Moreover, we have

$$\left|\{x\in\mathbb{R}^n\,;u^*(b)<|u(x)|<u^*(a)\}\right|\le b-a. \tag{5.7}$$

Now, we claim that the following inequality

$$\frac{d}{ds}\int_{\{x\in\mathbb{R}^n\,:|u(x)|>u^*(s)\}} |Du(x)|\,dx \ge -nK_n^{1/n}s^{1/n'}\frac{du^*}{ds}(s), \tag{5.8}$$

holds for almost every $s\in\mathbb{R}_+$. In fact, the right hand side of (5.8) is zero if $s \ge |\operatorname{supp} u|$. If $0\le s<|\operatorname{supp} u|$ the left hand side of (5.8) is equal to

$$\begin{aligned}
\lim_{h\to 0}\frac{1}{h}\int_{\{x\in\mathbb{R}^n\,:u^*(s+h)<|u(x)|\le u^*(s)\}} |Du(x)|\,dx \\
&\ge \lim_{h\to 0} nK_n^{1/n}s^{1/n'}\frac{1}{h}(u^*(s)-u^*(s+h)) \\
&= -nK_n^{1/n}s^{1/n'}\frac{du^*}{ds}(s).
\end{aligned}$$

Thus,

$$\int_a^b -nK_n^{1/n}s^{1/n'}\frac{du^*}{ds}(s)ds \le \int_{\{x\in\mathbb{R}^n\,:u^*(b)<|u(x)|<u^*(a)\}} |Du(x)|\,dx.$$

Then, by (5.7) and Hardy-Littlewood inequality, we obtain for every countable family $\{(a_j\,,b_j)\}$ of disjoint intervals in $(0\,,|\operatorname{supp} u|)$,

$$\int_{\cup(a_i\,,b_i)} -nK_n^{1/n}s^{1/n'}\frac{du^*}{ds}(s)ds \le \int_0^{\sum(b_i-a_i)} |Du|^*(r)dr.$$

The last estimate yields,

$$\int_0^s\left[-nK_n^{1/n}s^{1/n'}\frac{du^*}{ds}\right]^*(s)ds = \sup_{|E|=s}\int_E -nK_n^{1/n}s^{1/n'}\frac{du^*}{ds}(s)ds \le \int_0^s |Du|^*(r)dr,$$

since every measurable set $E\subset(0\,,|\operatorname{supp} u|)$ can be approximated from outside by sets of the form $\cup(a_i\,,b_i)$. □

Now, we are going to link the Sobolev embedding to the Hardy operator $H_{n/m}$; for this we require the following theorem which is a description of the $K$ functional (see Section 2.7) for the couple $(W_0^m L^1(\Omega)\,, W_0^m L^\infty(\Omega))$ . Its proof may be found in [9].

**Theorem 5.2.7.** *Let $\Omega \subset \mathbb{R}^n$ be an open subset. Then,*

$$K\left(t\,, u\,, W_0^m L^1(\Omega)\,, W_0^m L^\infty(\Omega)\right) \approx \int_0^t |D^m u|^*(s)ds,\ t>0,$$

*with $u \in W_0^m L^1(\Omega) + W^m L_0^\infty(\Omega)$.*

We can identify the Sobolev spaces $W_0^m L^{p,q}(\Omega)$ as interpolation spaces between $W_0^m L^1(\Omega)$ and $W_0^m L^\infty(\Omega)$.

**Corollary 5.2.8.** *Let $1 < p < \infty$ and $1 \le q \le \infty$. Then, up to equivalence of norms, $W_0^m L^{p,q}(\Omega)$ coincides with the interpolation spaces $(W_0^m L^1(\Omega)\,, W_0^m L^\infty(\Omega))_{1/p',q}$.*

**Remark 5.2.9.** Let $1 < p_0 < p_1 < \infty$ and $1 \le q_0\,, q_1 < \infty$. Theorem 2.7.10 yields,

$$\begin{aligned}&K\left(t\,, u\,, W_0^m L^{p_0,q_0}(\Omega)\,, W_0^m L^{p_1,q_1}(\Omega)\right)\\ &\approx \left(\int_0^{t^\alpha} \left(s^{1/p_0-1/q_0}\,|D^m u|^*(s)\right)^{q_0} ds\right)^{1/q_0} + \left(\int_{t^\alpha}^1 \left(s^{1/p_1-1/q_1}\,|D^m u|^*(s)\right)^{q_1} ds\right)^{1/q_1},\end{aligned}$$

where $\frac{1}{\alpha} = \frac{1}{p_0} - \frac{1}{p_1}$. Moreover, if $1 < p_1 < \infty$ and $1 \le q_1 < \infty$

$$\begin{aligned}K\left(t\,, u\,, W_0^m L^1(\Omega)\,, W_0^m L^{p_1,q_1}(\Omega)\right) \approx &\int_0^{t^\alpha} |D^m u|^*(s)ds\\ &+ t\left(\int_{t^\alpha}^1 \left(s^{1/p_1-1/q_1}\,|D^m u|^*(s)\right)^{q_1} ds\right)^{1/q_1},\end{aligned}$$

where $\frac{1}{\alpha} = 1 - \frac{1}{p_1}$. Indeed, it suffices to verify

$$\begin{aligned}K\left(t\,, u\,, W_0^m L^1(\Omega)\,, W_0^m L^{p_1,q_1}(\Omega)\right) \gtrsim &\int_0^{t^\alpha} |D^m u|^*(s)ds\\ &+ t\left(\int_{t^\alpha}^1 \left(s^{1/p_1-1/q_1}\,|D^m u|^*(s)\right)^{q_1} ds\right)^{1/q_1}.\end{aligned}$$

Fix $t > 0$. Let $u = g+h$ be any representation of $u$ with $g \in W_0^m L^1(\Omega)$ and $h \in W_0^m L^{p_1,q_1}(\Omega)$. Then,

$$\begin{aligned}I_1 = \int_0^{t^\alpha} |D^m u|^*(s)ds &\le \int_0^{t^\alpha} |D^m g|^*(s)ds + \int_0^{t^\alpha} |D^m h|^*(s)ds\\ &\lesssim \||D^m g|\|_{L^1} + t\,\||D^m h|\|_{L^{p_1,q_1}}\,,\end{aligned}$$

and

$$\begin{aligned}I_2 &= \left(\int_{t^\alpha}^1 \left(s^{1/p_1-1/q_1}\,|D^m u|^*(s)\right)^{q_1} ds\right)^{1/q_1}\\ &\le \left(\int_{t^\alpha}^1 \left(s^{1/p_1-1/q_1}\,|D^m g|^{**}(s)\right)^{q_1} ds\right)^{1/q_1} + \left(\int_{t^\alpha}^1 \left(s^{1/p_1-1/q_1}\,|D^m h|^{**}(s)\right)^{q_1} ds\right)^{1/q_1}\\ &\lessapprox t^{-1}\,\||D^m g|\|_{L^1} + \||D^m h|\|_{L^{p_1,q_1}}\,.\end{aligned}$$

Therefore,

$$I_1 + tI_2 \lessapprox \||D^m g|^*\|_{L^1} + t\,\||D^m h|\|_{L^{p_1,q_1}}\,,$$

and, taking the infimum over all such representations $u = g + h$, we conclude that

$$I_1 + tI_2 \lessapprox K\left(t\,,u\,,W_0^m L^1(\Omega)\,,W_0^m L^{p_1,q_1}(\Omega)\right).$$

**Theorem 5.2.10.** *For any $u \in W_0^m L^1(\Omega)$ and $t \in I$, we have*

$$\int_0^t s^{-m/n} u^*(s)ds \lessapprox \int_0^t s^{-m/n}\left(\int_{s/2}^1 |D^m u|^*(y) y^{m/n-1} dy\right) ds.$$

To prove this theorem we require some results.

**Lemma 5.2.11.** *We have*

$$W_0^m L^1(\Omega) \hookrightarrow L^{n/(n-m),1}(\Omega). \tag{5.9}$$

*Moreover,*

$$W_0^m L^{n/m,1}(\Omega) \hookrightarrow L^\infty(\Omega). \tag{5.10}$$

*Proof.* Let us prove (5.9). The proof proceeds by induction on $m$. Note that Theorem 4.3.6 establish the case $m = 1$. Suppose it has been proved for $m-1$, that is

$$\|v\|_{L^{n/(n-m+1),1}} \le C\left\||D^{m-1}v|\right\|_{L^1}\,,\ \ v \in \mathcal{D}(\Omega).$$

Set $v = D_j u$, $j = 1\,,2\,,\ldots\,,n$. Then

$$\||Du|\|_{L^{n/(n-m+1),1}} \le C\,\||D^m u|\|_{L^1}\,.$$

Now, let $0 < q < n$

$$\begin{aligned}\left\|\int_t^1 f(s)s^{1/n-1}ds\right\|_{L^{nq/(n-q),1}} &\approx \int_0^1 f(s)s^{1/q-1}ds \lessapprox \int_0^1 f^*(s)s^{1/q-1}ds\\ &\le \|f\|_{L^{q,1}}\,,\quad f \in \mathcal{M}_+(I).\end{aligned}$$

So, for all $u \in \mathcal{D}(\Omega)$

$$\|u\|_{L^{nq/(n-q),1}} = \left\|\int_t^1\left(-s^{1/n'}\frac{du^*}{ds}(s)\right)s^{-1/n'}ds\right\|_{L^{nq/(n-q),1}} \le C\left\|s^{1/n'}\frac{-du^*(s)}{ds}\right\|_{L^{q,1}}.$$

By Lemma 5.2.3, we get $\|u\|_{L^{nq/(n-q),1}} \lessapprox \||Du|\|_{L^{q,1}}$. Now, taking $q = n/(n-m+1)$, we get

$$\|u\|_{L^{n/(n-m),1}} \le C\,\||Du|\|_{L^{n/(n-m+1)}} \le C\,\||D^m u|\|_{L^1}\,.$$

and the result follows. □

*Proof.* (Theorem 5.2.10) Let

$$X_0(\Omega)=W_0^mL^1(\Omega)\,,\ Y_0(\Omega)=L^{n/(n-m),1}(\Omega)\,,\ X_1(\Omega)=W_0^mL^{n/m,1}(\Omega)\,,\ Y_1(\Omega)=L^{\infty}(\Omega).$$

By Lemma 5.2.11, we have

$$W_0^mL^1(\Omega)\hookrightarrow L^{n/(n-m),1}(\Omega)\ \text{ and }\ W_0^mL^{n/m,1}(\Omega)\hookrightarrow L^{\infty}(\Omega),$$

i.e., the operador Sobolev embedding is an admisible operador with respect to the compatible couple $\left(W_0^mL^1(\Omega)\,,W_0^mL^{n/m,1}(\Omega)\right)$ and $\left(L^{n/(n-m),1}(\Omega)\,,L^{\infty}(\Omega)\right)$ and Theorem 2.7.5

$$K\left(t\,,u\,,L^{n/(n-m),1}(\Omega)\,,L^{\infty}(\Omega)\right)\le CK\left(Ct\,,u\,,W_0^mL^1(\Omega)\,,W_0^mL^{n/m,1}(\Omega)\right),$$

with $u\in W_0^mL^1(\Omega)$. Then, by Remark 2.7.11 we have

$$K\left(t\,,u\,,L^{n/(n-m),1}(\Omega)\,,L^{\infty}(\Omega)\right)\approx\int_0^{t^{n/(n-m)}}s^{-m/n}u^*(s)ds,$$

and by Remark 5.2.9

$$\begin{aligned}K&\left(Ct\,,u\,,W_0^mL^1(\Omega)\,,W_0^mL^{n/m,1}(\Omega)\right)\\&\approx\int_0^{(Ct)^{n/(n-m)}}\left|D^mu\right|^*(s)ds+Ct\int_{(Ct)^{n/(n-m)}}^1\left|D^mu\right|^*(s)s^{m/n-1}ds\end{aligned}$$

Then replacing $(Ct)^{n/(n-m)}$ by $t$ , we get

$$\begin{aligned}&\int_0^t s^{-m/n}u^*(s)ds\\&\le C\left(\int_0^t\left|D^mu)\right|^*(s)ds+Ct^{1-m/n}\int_t^1\left|D^mu\right|^*(s)s^{m/n-1}ds\right)\\&\le C\left[\int_0^t s^{-m/n}\int_{s/2}^s y^{m/n-1}dy\left|D^mu\right|^*(s)ds+\int_t^1\left|D^mu\right|^*(y)y^{m/n-1}dy\int_0^t s^{-m/n}ds\right]\\&\le C\left[\int_0^t s^{-m/n}\int_{s/2}^s y^{m/n-1}\left|D^mu\right|^*(y)dyds+\int_0^t s^{-m/n}\int_s^1\left|D^mu\right|^*(y)y^{m/n-1}dyds\right]\\&\le C\int_0^t s^{-m/n}\int_{s/2}^1\left|D^mu\right|^*(y)y^{m/n-1}dyds,\end{aligned}$$

thus, the proof is complete. □

**Theorem 5.2.12.** *Let $X(\Omega)$ be an r.i. space. Then, the functional*

$$\|f\|_{Z(\Omega)}:=\left\|H'_{n/m}f^*\right\|_{\overline{X}'(I)}=\left\|t^{m/n}f^{**}(t)\right\|_{\overline{X}'(I)},\ f\in\mathcal{M}(\Omega),$$

*is an r.i. norm, being, in fact, the smallest r.i. norm satisfying*

$$\left\|H'_{n/m}f\right\|_{\overline{X}'(I)}\lesssim\|f\|_{\overline{Z}(I)},\quad f\in\mathcal{M}_+(I).\tag{5.11}$$

*Moreover $\|\cdot\|_{Z'(\Omega)}$, the associate norm of $\|\cdot\|_{Z(\Omega)}$, is the largest r.i. norm satisfying*

$$\left\|H_{n/m}f\right\|_{\overline{Z}'(I)}\lesssim\|f\|_{\overline{X}(I)},\ f\in\mathcal{M}_+(I).\tag{5.12}$$

*Proof.* We claim that $\|\cdot\|_{Z(\Omega)}$ is an r.i. norm. Indeed,

- the positivity and homogeneity are clear;
- $\|f+g\|_{Z(\Omega)} = \left\|t^{m/n}(f+g)^{**}(t)\right\|_{\overline{X}'(I)} \le \|f\|_{Z(\Omega)} + \|g\|_{Z(\Omega)}\,, \forall\, f\,, g \in \mathcal{M}(\Omega)$;
- let $\{f_j\}_{j\in\mathbb{N}}$ such that $f_j \uparrow f$. So $(f_j)^{**} \uparrow f^{**}$ and hence

$$\left\|t^{m/n}(f_j)^{**}(t)\right\|_{\overline{X}'(I)} \uparrow \left\|t^{m/n}f^{**}(t)\right\|_{\overline{X}'(I)}\,,$$

  that is, $\|f_j\|_{Z(\Omega)} \uparrow \|f\|_{Z(\Omega)}$;
- $\int_0^1 g(t)(H'_{n/m}\chi_I)(t)dt \le \int_0^1 g(t)dt \le \|g\|_{\overline{X}'(I)}$, for all $g \in \mathcal{M}_+(\Omega)$. Hence,

$$\|\chi_\Omega\|_{Z(\Omega)} = \left\|H'_{n/m}\chi_I\right\|_{\overline{X}'(I)} \le 1;$$

- let $f \in \mathcal{M}(\Omega)$,

$$\begin{aligned}\left\|t^{m/n}f^{**}(t)\right\|_{\overline{X}'(I)} &\ge \frac{1}{\|\chi_I\|_{\overline{X}(I)}}\int_0^1 \chi_I(t)t^{m/n}f^{**}(t)dt \gtrapprox Cf^{**}(1)\\ &= \int_0^1 f^*(t)dt \ge \int_\Omega f(t)dt,\end{aligned}$$

  and so $\int_\Omega f(t)dt \lessapprox \|f\|_{Z(\Omega)}$;
- $\|f^*\|_{Z(\Omega)} = \left\|H'_{n/m}f^*\right\|_{\overline{X}'(I)} = \|f\|_{Z(\Omega)}\,,\ f \in \mathcal{M}(\Omega)$.

Let us prove (5.11),

$$\int_0^t f(s)ds \le \int_0^t f^*(s)ds \;\Rightarrow\; (H'_{n/m}f)(t) \le (H'_{n/m}f^*)(t)\,, \forall\, t \in I$$

and so,

$$\left\|H'_{n/m}f\right\|_{\overline{X}'(I)} \le \|f\|_{Z(\Omega)}\,.$$

Finally, let us prove (5.12). Suppose that $g \in \mathcal{M}(I)$, we have

$$\begin{aligned}\left|\int_0^1 g(t)H_{n/m}f(t)dt\right| &= \left|\int_0^1 f(s)s^{m/n-1}\left(\int_0^s g(t)dt\right)ds\right| = \left|\int_0^1 f(s)H'_{n/m}g(s)ds\right|\\ &\le \left\|H'_{n/m}g\right\|_{\overline{X}'(I)}\|f\|_{\overline{X}(I)} \le \|g\|_{\overline{Z}(I)}\|f\|_{\overline{X}(I)}\,.\end{aligned}$$

Therefore,

$$\|H_{n/m}f\|_{\overline{Z}'(I)} = \sup_{\|g\|_{\overline{Z}(I)}\le 1}\left|\int_0^1 g(t)H_{n/m}f(t)dt\right| \le \sup_{\|g\|_{\overline{Z}(I)}\le 1}\|g\|_{\overline{Z}(I)}\|f\|_{\overline{X}(I)} \le \|f\|_{\overline{X}(I)}\,.$$

$\square$

**Theorem 5.2.13.** *Let $X(\Omega)$ and $Y(\Omega)$ be an r.i. spaces. Suppose that,*

$$H_{n/m} : \overline{X}(I) \to \overline{Y}(I). \tag{5.13}$$

*Then, $W_0^m X(\Omega) \hookrightarrow Y(\Omega)$.*

*Proof.* By Theorem 5.2.12, the functional

$$\|f\|_{Z(\Omega)} := \left\|H'_{n/m} f^*\right\|_{\overline{X}'(I)},$$

is an r.i. norm. Moreover, when (5.13) holds, Theorem 5.2.12 ensures

$$\|u\|_{Y(\Omega)} \lessapprox \|u\|_{Z'(\Omega)}.$$

Note that Hardy-Littlewood-Pólya Principle and Theorem 5.2.10 imply

$$\int_0^1 t^{-m/n} u^*(t) h^*(t) dt \leq \int_0^1 t^{-m/n} \int_{t/2}^1 |D^m u|^* s^{m/n-1} ds h^*(t) dt,$$

Taking $h^*(t) = \sup_{t \leq s < 1} s^{m/n} g^*(s)$, we get

$$\int_0^1 u^*(t)(T_{n/m} g)(t) dt \leq \int_0^1 \left(\int_{t/2}^1 |D^m u|^* s^{m/n-1} ds\right) (T_{n/m} g)(t) dt.$$

Now,

$$\begin{aligned}
\|u^*\|_{\overline{Z}'(I)} &= \sup_{g \neq 0} \frac{\int_0^1 u^*(t) g^*(t) dt}{\|g\|_{\overline{Z}(I)}} \leq \sup_{g \neq 0} \frac{\int_0^1 u^*(t)(T_{n/m} g)(t) dt}{\|g\|_{\overline{Z}(I)}} \\
&\leq \sup_{g \neq 0} \frac{\int_0^1 \left(\int_{t/2}^1 |D^m u|^*(s) s^{m/n-1} ds\right) (T_{n/m} g)(t) dt}{\|g\|_{\overline{Z}(I)}} \\
&\sup_{g \neq 0} \frac{\left\|\int_{t/2}^1 |D^m u|^*(s) s^{m/n-1} ds\right\|_{\overline{Z}'(I)} \left\|T_{n/m} g\right\|_{\overline{Z}(I)}}{\|g\|_{\overline{Z}(I)}} \\
&\lessapprox \left\|\int_{t/2}^1 |D^m u|^*(s) s^{m/n-1} ds\right\|_{\overline{Z}'(I)}.
\end{aligned}$$

in the last inequality we used $T_{n/m}$ is bounded on $\overline{Z}(I)$ (Theorem 2.8.5). Now, since

$$\left\|\int_{t/2}^1 |D^m u|^*(s) s^{m/n-1} ds\right\|_{\overline{Z}'(I)} \leq \left\|\int_t^1 |D^m u|^*(s) s^{m/n-1} ds\right\|_{\overline{Z}'(I)},$$

we get

$$\|u\|_{Y(\Omega)} \lessapprox \|u\|_{Z'(\Omega)} \lessapprox \left\|\int_t^1 |D^m u|^*(s) s^{m/n-1} ds\right\|_{\overline{Z}'(I)} \lessapprox \||D^m u|\|_{X(\Omega)}.$$

□

## 5.3 Optimal range and optimal domain of r.i. norms

Now, we show how Theorem 5.2.1 can be used to characterize the smallest r.i. domain norm and the largest r.i. range norm in the Sobolev embedding

$$W_0^m X(\Omega) \hookrightarrow Y(\Omega).$$

Note that Theorem 5.2.1 implies the following chain of equivalent statements

$$\begin{aligned} W_0^m X(\Omega) \hookrightarrow Y(\Omega) &\Leftrightarrow \left\| H_{n/m} f \right\|_{\overline{Y}(I)} \lessapprox \|f\|_{\overline{X}(I)} \\ &\Leftrightarrow \left\| H'_{n/m} f \right\|_{\overline{X}'(I)} \lessapprox \|f\|_{\overline{Y}'(I)} \\ &\Leftrightarrow \left\| H'_{n/m} f^* \right\|_{\overline{X}'(I)} \lessapprox \|f\|_{\overline{Y}'(I)} . \end{aligned}$$

The first equivalence is Theorem 5.2.1. We claim

$$\left\| H_{n/m} f \right\|_{\overline{Y}(I)} \lessapprox \|f\|_{\overline{X}(I)} \Leftrightarrow \left\| H'_{n/m} f \right\|_{\overline{X}'(I)} \lessapprox \|f\|_{\overline{Y}'(I)} .$$

Indeed,

$$\begin{aligned} \left\| H_{n/m} f \right\|_{\overline{Y}(I)} &= \sup_{g \neq 0} \frac{\int_0^1 (H_{n/m} f)(s) g(s) ds}{\|g\|_{\overline{Y}'(I)}} = \sup_{g \neq 0} \frac{\int_0^1 (H'_{n/m} g)(s) f(s) ds}{\|g\|_{\overline{Y}'(I)}} \\ &\leq \sup_{g \neq 0} \frac{\left\| H'_{n/m} g \right\|_{\overline{X}'(I)} \|f\|_{\overline{X}(I)}}{\|g\|_{\overline{Y}'(I)}} \lessapprox \|f\|_{\overline{X}(I)} . \end{aligned}$$

Moreover

$$\begin{aligned} \left\| H'_{n/m} f \right\|_{\overline{X}'(I)} &= \sup_{g \neq 0} \frac{\int_0^1 (H'_{n/m} f)(s) g(s) ds}{\|g\|_{\overline{X}(I)}} \leq \sup_{g \neq 0} \frac{\left\| H_{n/m} g \right\|_{\overline{Y}(I)} \|f\|_{\overline{Y}'(I)}}{\|g\|_{\overline{X}(I)}} \\ &\lessapprox \|f\|_{\overline{Y}'(I)} . \end{aligned}$$

The last equivalence; the implication $\Rightarrow$) is restriction to monotone functions, while the converse one follows from the estimate

$$\int_0^t g(s) ds \leq \int_0^t g^*(s) ds.$$

In practice, one starts with a Sobolev space $W_0^m X(\Omega)$, and then finds its optimal range space $Y_X(\Omega)$. The description of $Y_X(\Omega)$ is given by the following theorem.

**Theorem 5.3.1.** *Let $X(\Omega)$ be an r.i. space. Let $Y_X(\Omega)$ be the r.i. space whose associate space $Y'_X(\Omega)$ has norm*

$$\|f\|_{Y'_X(\Omega)} = \left\| H'_{n/m} f^* \right\|_{\overline{X}'(I)} , \quad f \in \mathcal{M}(\Omega).$$

*Then the Sobolev embedding $W_0^m X(\Omega) \hookrightarrow Y_X(\Omega)$ holds, and $Y_X(\Omega)$ is the optimal (i.e., the largest possible) such an r.i. space.*

Now, it is natural to ask if there is an optimal domain space, $X_Y(\Omega)$, whose Sobolev space, $W_0^m X_Y(\Omega)$, (possibly bigger that $W_0^m X(\Omega)$) is the largest that still imbeds into $Y(\Omega)$. We will prove the existence of such $X_Y(\Omega)$. The fact that $d_h = d_f$ will be denoted by $f \sim h$

**Theorem 5.3.2.** *Let $Y(\Omega)$ be an r.i. space. Assume that $Y(\Omega) \hookrightarrow L^{n/(n-m),1}(\Omega)$. Then the function space $X_Y(\Omega)$ generated by the norm*

$$\|f\|_{\overline{X}_Y(I)} = \sup_{f\sim h} \left\|H_{n/m}h\right\|_{\overline{Y}(I)}, \; f \in \mathcal{M}(I)\,, h \in \mathcal{M}_+(I),$$

*is an r.i. space such that*

$$H_{n/m} \; : \overline{X}_Y(I) \to \overline{Y}(I).$$

*Moreover, $\|\cdot\|_{X_Y(\Omega)}$ is the smallest such r.i. norm.*

*Proof.* Let's prove $\|\cdot\|_{X_Y(\Omega)}$ is an r.i. norm. The positivity and homogeneity of $\|\cdot\|_{X_Y(\Omega)}$ are clear. Next, when $h \sim f + g$, there exists $h_f \sim f$ and $h_g \sim g$ so that $h = h_f + h_g$. From this observation we get the subadditivity of $\|\cdot\|_{X_Y(\Omega)}$. Suppose that $f_j \uparrow f$. Now, when $h \sim f_j$ there exists a measure preserving transformation $T$ such that $h = f_j \circ T$, and hence $h \le f_{j+1} \circ T = k \sim f_{j+1}$, so $\|f_j\|_{X_Y(\Omega)} \le \|f_{j+1}\|_{X_Y(\Omega)}$. Further, $h \sim f$ once more means $h = f \circ T$ for some measure-preserving transformation $T$. We then have $h_j = f_j \circ T \uparrow f \circ T$, so $\|f_j\|_{X_Y(\Omega)} \uparrow \|f\|_{X_Y(\Omega)}$. Again,

$$\|\chi_I\|_{\overline{X}_Y(I)} = \left\|\frac{n}{m}(1 - t^{m/n})\right\|_{\overline{Y}(I)} < \infty,$$

while

$$\|f\|_{\overline{X}_Y(I)} \ge \left\|\int_t^1 f(s)s^{m/n-1}ds\right\|_{\overline{Y}(I)} \gtrapprox \left\|\int_t^1 f(s)s^{m/n-1}ds\right\|_{L^{n/(n-m),1}(I)}$$
$$\approx \int_\Omega f(s)ds \;\; f \in \mathcal{M}_+(I).$$

□

The following proposition proves that the formula for $X_Y(\Omega)$ can be improved if $Y(\Omega)$ satisfies some properties.

**Proposition 5.3.3.** *Let $Y(\Omega)$ be an r.i space such that $Y(\Omega) \hookrightarrow L^{n/(n-m),1}(\Omega)$. Suppose $T_{n/m} : \overline{Y}'(I) \to \overline{Y}'(I)$. Then*

$$\|f\|_{\overline{X}_Y(I)} \approx \left\|\int_t^1 f^*(s)s^{m/n-1}ds\right\|_{\overline{Y}(I)}.$$

*Proof.* Since $Y(\Omega) \subset L^{n/(n-m),1}(\Omega)$, by Theorem 5.2.12 we have

$$\|f\|_{\overline{X}_Y(I)} = \sup_{h\sim f} \left\| \int_t^1 h(s)s^{m/n-1}ds \right\|_{\overline{Y}(I)}, \quad f \in \mathcal{M}(I)\,, h \in \mathcal{M}_+(I).$$

Now,

$$\left\| \int_t^1 f^*(s)s^{m/n-1} \right\|_{\overline{Y}(I)} \leq \sup_{h\sim f} \left\| \int_t^1 h(s)s^{m/n-1}ds \right\|_{\overline{Y}(I)},$$

since $f^* \sim f$. Moreover, $h \sim f$ means $h^* = f^*$, so will be done if we can prove

$$\left\| \int_t^1 h(s)s^{m/n-1}ds \right\|_{\overline{Y}(I)} \lesssim \left\| \int_t^1 h^*(s)s^{m/n-1}ds \right\|_{\overline{Y}(I)}, \quad h \in \mathcal{M}_+(I).$$

We have

$$\begin{aligned}
\left\| \int_t^1 h(s)s^{m/n-1}ds \right\|_{\overline{Y}(I)} &= \sup_{g>0} \frac{\int_0^1 g^*(t) \int_t^1 h(s)s^{m/n-1}dsdt}{\|g\|_{\overline{Y}'(I)}} \\
&= \sup_{g>0} \frac{\int_0^1 g^{**}(t)h(t)t^{m/n}dt}{\|g\|_{\overline{Y}'(I)}} \\
&\leq \sup_{g>0} \frac{\int_0^1 (T_{n/m}g)^{**}(t)h(t)t^{m/n}dt}{\|g\|_{\overline{Y}'(I)}} \\
&\leq \sup_{g>0} \frac{\int_0^1 (T_{n/m}g)^{**}(t)h^*(t)t^{m/n}dt}{\|g\|_{\overline{Y}'(I)}} \quad (t^{m/n}(T_{n/m}g)^{**}(t)\downarrow) \\
&\lesssim \sup_{g>0} \frac{\int_0^1 (T_{n/m}g)^{**}(t)h^*(t)t^{m/n}dt}{\|T_{n/m}g\|_{\overline{Y}'(I)}} \\
&\lesssim \left\| \int_t^1 h^*(s)s^{m/n-1}ds \right\|_{\overline{Y}(I)}.
\end{aligned}$$

□

Combining Theorem 5.3.1 and Theorem 5.2.12.

**Theorem 5.3.4.** *Let $X(\Omega)$ be an r.i. space. Set*

$$\|f\|_{Y_X'(\Omega)} := \left\|H_{n/m}'f^*\right\|_{\overline{X}'(I)}, \quad \text{and} \quad \|f\|_{X_{Y_X}(\Omega)} := \sup_{h\sim f^*} \left\|H_{n/m}h\right\|_{\overline{Y}_X(I)},$$

*with $h \in \mathcal{M}_+(I)$ and $f \in \mathcal{M}(\Omega)$. Then, both $\|\cdot\|_{Y_X(\Omega)}$ and $\|\cdot\|_{X_{Y_X}(\Omega)}$ are optimal in*

$$W_0^m X_{Y_X}(\Omega) \to Y_X(\Omega).$$

**Remark 5.3.5.** Observe that

$$\|f\|_{Y'_X(\Omega)} = \left\|H'_{n/m}f^*\right\|_{\overline{X}'(I)},$$

and so, since $\|\cdot\|_{L^\infty}$ is the largest r.i. norm

$$\|f\|_{Y'_X(\Omega)} \lessapprox \left\|t^{m/n}f^{**}\right\|_{L^\infty(I)} \approx \|f\|_{L^{n/m,\infty}(\Omega)}$$

hence $Y_X(\Omega) \subset L^{n/(n-m),1}(\Omega)$. Moreover, by Theorem 5.3.2

$$\|f\|_{X_{Y_X}(\Omega)} \approx \left\|\int_t^1 h^*(s)s^{m/n-1}ds\right\|_{\overline{Y}_X(I)}$$

## 5.4 Examples

Given $m \in \mathbb{N}$, $1 \le m \le n-1$, $n \ge 2$ and two r.i spaces $X(\Omega)$, $Y(\Omega)$ such that

$$W_0^m X(\Omega) \hookrightarrow Y(\Omega),$$

we want to find the optimal pair in the Sobolev embedding. To solve this problem, we start with r.i. space $X(\Omega)$ and then find its optimal range partner $Y_X(\Omega)$. Thus, the embedding

$$W_0^m X(\Omega) \hookrightarrow Y_X(\Omega) \hookrightarrow Y(\Omega),$$

has an optimal range, but it does not necessarily have an optimal domain. We take one more step in order to get the optimal domain r.i. partner for $Y_X(\Omega)$, let us call it $X_{Y_X}(\Omega)$. Altogether, we have

$$W_0^m X(\Omega) \hookrightarrow W_0^m X_{Y_X}(\Omega) \hookrightarrow Y_X(\Omega) \hookrightarrow Y(\Omega),$$

and $W_0^m X_{Y_X}(\Omega)$ now can be either equivalent to $W_0^m X(\Omega)$ or strictly larger. In any case, after these two steps, the couple $(X_{Y_X}(\Omega), Y_X(\Omega))$ forms an optimal pair in the Sobolev embedding and no further iterations of the process can bring anything new.

**Example 5.4.1.** Let $1 < p < n/m$. Let $X(\Omega) = L^p(\Omega)$ and $Y(\Omega) = L^{p^*}(\Omega)$. We know, by Theorem 3.4.1

$$W_0^m L^p(\Omega) \hookrightarrow L^{p^*}(\Omega), \quad p^* = \frac{p}{n-mp}$$

Now, we want the optimal pair in the Sobolev embedding. First, we construct the optimal range space; by Theorem 5.3.4 we have

$$\begin{aligned}\|f\|_{Y'_X(\Omega)} &= \left\|t^{m/n}f^{**}(t)\right\|_{L^{p'}(I)} \\ &= \left(\int_0^1 \left(t^{m/n}f^{**}(t)\right)^{p'} dt\right)^{1/p'} = \|f\|_{L^{np/(np-n+mp),p'}(\Omega)}\end{aligned}$$

and so $Y_X(\Omega) = L^{p^*,p}(\Omega)$. Thus,

$$W_0^m L^p(\Omega) \hookrightarrow L^{p^*,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$$

Note that $p < p^*$, so by Proposition 2.5.3 we have

$$L^{p^*,p}(\Omega) \hookrightarrow L^{p^*}(\Omega).$$

Given $Y_X(\Omega) = L^{p^*,p}(\Omega)$, we find its optimal domain r.i. partner. By Remark 5.3.5 we know

$$\|f\|_{X_{Y_X}(\Omega)} \approx \left\| \int_t^1 f^*(s) s^{m/n-1} ds \right\|_{L^{p^*,p}(I)}.$$

Note that $\|f\|_{X_{Y_X}(\Omega)} \approx \|f\|_{L^p(\Omega)}$. Indeed,

$$\begin{aligned}
\left\| \int_t^1 f^*(s) s^{m/n-1} ds \right\|_{L^{p^*,p}(I)} &= \left( \int_0^1 \left( t^{-m/n} \int_t^1 f^*(s) s^{m/n-1} ds \right)^p dt \right)^{1/p} \\
&\lesssim \int_0^1 t^{-mp/n-1} \left( \int_t^1 (f^*(s))^p s^{mp/n} ds \right) dt \\
&= \int_0^1 f^*(s)^p s^{mp/n} \int_0^s t^{-mp/n-1} dt ds \\
&\approx \int_0^1 f^*(s)^p ds,
\end{aligned}$$

hence, $\|f\|_{X_{Y_X}(\Omega)} \lesssim \|f\|_{L^p(\Omega)}$. Let us prove that $\|f\|_{L^p(\Omega)} \lesssim \|f\|_{X_{Y_X}(\Omega)}$. Note that,

$$\int_0^t \left( \int_s^1 f^*(u) u^{m/n-1} du \right) ds \geq \int_0^t f^*(u) u^{m/n} du \gtrsim f^*(t) t^{m/n+1};$$

thus,

$$\left\| \int_t^1 f^*(s) s^{m/n-1} ds \right\|_{L^{np/(n-mp),p}(I)} \gtrsim \left( \int_0^1 f^*(t)^p dt \right)^{1/p}.$$

Therefore, $\|f\|_{X_{Y_X}(\Omega)} \approx \|f\|_{L^p(\Omega)}$. We conclude that

$$W_0^m L^p(\Omega) \hookrightarrow L^{p^*,p}(\Omega) \hookrightarrow L^{p^*}(\Omega).$$

**Example 5.4.2.** Let $X(\Omega) = L^1(\Omega)$ and $Y(\Omega) = L^{n/(n-m)}(\Omega)$. By Theorem 3.4.4

$$W_0^m L^1(\Omega) \hookrightarrow L^{n/(n-m)}(\Omega),$$

We construct the optimal range space

$$\|f\|_{Y_X'(\Omega)} = \left\| t^{m/n} f^{**}(t) \right\|_{L^\infty(I)} = \|f\|_{L^{n/m,\infty}(\Omega)}$$

and so $Y_X(\Omega) = L^{n/(n-m),1}(\Omega)$. Thus,

$$W_0^m L^1(\Omega) \hookrightarrow L^{n/(n-m),1}(\Omega) \hookrightarrow L^{n/(n-m)}(\Omega)$$

By Proposition 2.5.3 we have

$$L^{n/(n-m),1}(\Omega) \hookrightarrow L^{n/(n-m)}(\Omega).$$

Let $Y_X(\Omega) = L^{n/(n-m),1}(\Omega)$, we find its optimal domain r.i. partner. We know

$$\|f\|_{X_{Y_X}(\Omega)} \approx \left\| \int_t^1 f^*(s) s^{m/n-1} ds \right\|_{L^{n/(n-m),1}(I)}$$
$$\int_0^1 t^{-m/n} \left( \int_t^1 f^*(s) s^{m/n-1} ds \right) dt \approx \|f\|_{L^1(\Omega)}$$

Therefore,

$$W_0^m L^1(\Omega) \hookrightarrow L^{n/(n-m),1}(\Omega) \hookrightarrow L^{n/(n-m)}(\Omega)$$

**Example 5.4.3.** Let $L^{n/m}(\Omega) = X(\Omega)$ and $Y(\Omega) = L^q(\Omega)$ $n/m \leq q < \infty$. To find $\left\| t^{m/n} f^{**}(t) \right\|_{\overline{X}'(I)}$, we use the following result which is proved in [10]

**Theorem 5.4.4.** *Let $1 < p < \infty$ and suppose the weight $\phi$ on $(0,1)$ satisfies the following properties*

1. $\int_0^1 \phi(t)^p dt < \infty$,
2. $\int_0^1 (\phi(t)^p / t^p) dt = \infty$,
3. $\int_0^r \phi(t)^p dt \leq C r^p \left( 1 + \int_r^1 \frac{\phi(t)^p}{t^p} dt \right)$, $0 < r < 1$.

*Then, the r.i. norm $\varrho = \varrho_p(\phi(t) f^{**}(t))$ has dual norm*

$$\varrho'(g) \approx \varrho_{p'}(\psi g^*), \quad g \in \mathcal{M}_+(I),$$

*where*

$$\psi(t)^{p'} = (p'-1) \left( 1 + \int_t^1 \frac{\phi(s)^p}{s^p} ds \right)^{-p'} \frac{\phi(t)^p}{t^p}, \; 0 < t < 1.$$

Set $p = n/(n-m)$ and put $\phi(t) = t^{m/n}$; note that $\phi$ satisfies the properties of Theorem 5.4.4 and so $\|f\|_{Y_X(\Omega)} \approx \|\psi f^*\|_{L^{n/m}(I)}$ where

$$\psi(t) \approx (1 - \log(t))^{-1} t^{-m/n} = \log(e/t)^{-1} t^{-m/n};$$

that is $Y_X(\Omega) = L^{\infty,n/m;-1}(\Omega)$. Thus,

$$W_0^m L^{n/m}(\Omega) \hookrightarrow L^{\infty,n/m;-1}(\Omega) \hookrightarrow L^q(\Omega), \quad n/m \le q < \infty.$$

Moreover, by Theorem 5.3.3

$$\|f\|_{X_{Y_X}(\Omega)} \approx \left\| \int_t^1 f^*(s)s^{m/n-1}ds \right\|_{L^{\infty,n/m;-1}(\Omega)}.$$

Therefore, we get

$$W_0^m L^{n/m}(\Omega) \hookrightarrow W_0^m X_{Y_X}(\Omega) \hookrightarrow L^{\infty,n/m;-1}(\Omega) \hookrightarrow L^q(\Omega), \quad n/m \le q < \infty.$$

Note that $L^{n/m}(\Omega) \hookrightarrow X_{Y_X}(\Omega)$. To prove it, we require the following result, which can be found in [4]

**Theorem 5.4.5.** *Let $v, w$ be non-negative mesurables functions on $(0,1)$. Let $0 < p \le q < \infty$. Then the inequality*

$$\left(\int_0^1 (f^*(t))^q\, w(t)dt\right)^{1/q} \lessapprox \left(\int_0^1 (f^*(t))^p\, v(t)dt\right)^{1/p},$$

*holds if and only if*

$$\sup_{0<t<1} (W(t))^{1/q}\,(V(t))^{-1/p} < \infty.$$

So,

$$\begin{aligned}
\|f\|_{X_{Y_X}(\Omega)} &\approx \left\| \int_t^1 f^*(s)s^{m/n-1}ds \right\|_{L^{\infty,n/m;-1}(\Omega)} \\
&= \left(\int_0^1 t^{-1}(1-\log(t))^{-n/m}\left(\int_t^1 f^*(s)s^{m/n-1}ds\right)^{n/m} dt\right)^{m/n} \\
&\lessapprox \int_0^1 t^{-m/n}\left(\int_t^1 f^*(s)s^{m/n-1}ds\right) dt \quad \text{(by Theorem 5.4.5)} \\
&\approx \int_0^1 f^*(t)dt = \|f\|_{L^1(\Omega)} \lessapprox \|f\|_{L^{n/m}(\Omega)}.
\end{aligned}$$

**Example 5.4.6.** Given $Y(\Omega)$, we find its optimal domain r.i. partner. We know

$$\begin{aligned}
\|f\|_{X_Y(\Omega)} &\approx \left\| \int_t^1 f^*(s)s^{m/n-1}ds \right\|_{L^\infty(\Omega)} = \sup_{0<t<1} \int_t^1 f^*(s)s^{m/n-1}ds \\
&= \int_0^1 f^*(s)s^{m/n-1}ds = \|f\|_{L^{n/m,1}}(\Omega).
\end{aligned}$$

By Theorem 3.4.9, we know

$$W_0^p L^p(\Omega) \hookrightarrow L^\infty(\Omega), \quad \text{with } p > n/m;$$

and so, $W_0^m L^p(\Omega) \hookrightarrow W_0^m L^{n/m,1}(\Omega) \hookrightarrow L^\infty(\Omega)$. Now, given $X_Y(\Omega) = L^{n/m,1}(\Omega)$, we find its optimal range r.i. partner:

$$\|f\|_{Y'_{X_Y}(\Omega)} = \left\|H'_{n/m} f^*\right\|_{L^{(n-m)/n,\infty}(I)} \leq \|f\|_{L^1(\Omega)};$$

and so, $Y_{X_Y}(\Omega) \hookrightarrow L^\infty(\Omega)$. Hence, $Y_{X_Y}(\Omega) \approx L^\infty(\Omega)$, since $L^\infty(\Omega)$ is the smallest r.i. space. Therefore,

$$W_0^m L^p(\Omega) \hookrightarrow W_0^m L^{n/m,1}(\Omega) \hookrightarrow L^\infty(\Omega), \quad \text{with } p > n/m.$$

**Example 5.4.7.** Cianchi in [7], proved that $L^\Phi(\Omega)$ with $\Phi(t) = \exp t^{n'}$ is the optimal (that is, the smallest possible) Orlicz range space in

$$W_0^1 L^n(\Omega) \hookrightarrow L^\Phi(\Omega).$$

However, it turns out that $L^n(\Omega)$ is not optimal as an Orlicz domain space, but such an optimal Orlicz domain space does not exist at all. This should be understood as follows: for every Orlicz space $L^A(\Omega)$ such that

$$W_0^1 L^A(\Omega) \hookrightarrow L^\Phi(\Omega),$$

there exists another stricly larger Orlicz space $L^B(\Omega)$ such that

$$W_0^1 L^B(\Omega) \hookrightarrow L^\Phi(\Omega).$$

Since $L^\Phi(\Omega) \hookrightarrow L^\infty(\Omega)$, it suffices to prove:

**Theorem 5.4.8.** *There does not exist any largest space $L^A(\Omega)$ such that*

$$W_0^1 L^A \hookrightarrow L^\infty(\Omega).$$

To prove this result, we require the following theorem, which is proved in [8].

**Theorem 5.4.9.** *The embedding*

$$W_0^1 L^A \hookrightarrow L^\infty(\Omega),$$

*holds if and only if*

$$\int_1^\infty \frac{\widetilde{A}(s)}{s^{n'+1}} ds < \infty. \tag{5.14}$$

*where $\widetilde{A}$ is the complementary function of $A$, which is a Young function defined by $\widetilde{A}(t) = \sup_{s\geq 0}\{st - A(s)\}$ for $t \geq 0$.*

*Proof.* (Theorem 5.4.8) The proof of this theorem can be found in [8]. Let $A$ be a Young function such that (5.14) holds. We claim that there is another Young function, $B$, such that $\widetilde{B}(t) \geq \widetilde{A}(t)$ for all $t > 0$, $\limsup_{t\to\infty} \widetilde{B}(t)/\widetilde{A}(\beta t) = \infty$ for every $\beta > 1$, and

$$\int_1^\infty \frac{\widetilde{B}(s)}{s^{n'+1}} ds < \infty. \tag{5.15}$$

For such $B$ we would have $L^A(\Omega) \subsetneqq L^B(\Omega)$ and $W_0^1 L^B(\Omega) \hookrightarrow L^\infty(\Omega)$, as required. To prove our claim, let us set $a_k = (k \log^2 k)^{-1}$, $k \in \mathbb{N}$. For $t \in [k!\,,(k+1)!\,)$, we define $\tau$ by the identity

$$\frac{\widetilde{A}(\tau)}{\tau} = a_k t^{n'-1}, \quad t \in [k!\,,(k+1)!\,)\,. \tag{5.16}$$

We note that $\tau$ is uniquely defined, since the function $\widetilde{A}(t)/t$ strictly increases from 0 to $\infty$ as $t$ goes from 0 to $\infty$. We claim that for every $\beta > 1$

$$\limsup_{t\to\infty} \frac{\widetilde{A}(\tau)}{\tau} \frac{t}{\widetilde{A}(\beta t)} = \infty. \tag{5.17}$$

Indeed, assume the contrary. Then, for some $\beta > 1$ and $K > 0$,

$$K^{-1} a_k t^{n'} \leq \widetilde{A}(\beta t), \quad t \in [k!\,,(k+1)!\,)\,.$$

But,

$$\int_1^\infty \frac{\widetilde{A}(s)}{s^{n'+1}} ds = \sum_{k=1}^\infty \int_{\beta k!}^{\beta(k+1)!} \frac{\widetilde{A}(s)}{s^{n'+1}} ds = \beta^{-n} \sum_{k=1}^\infty \int_{k!}^{(k+1)!} \frac{\widetilde{A}(\beta y)}{y^{n'+1}} dy$$
$$\geq \frac{1}{K\beta^{n'}} \sum_{k=1}^\infty a_k \log(k+1) = \infty,$$

which contradicts (5.14). This proves (5.17). Now, let $\beta_j \nearrow \infty$ be a fixed sequence. Then, by (5.17), there exits a sequence $t_j \nearrow \infty$ such that $t_j \geq j!$, $t_{j+1} > \tau_j$ (where $\tau_j$ correspond to $t_j$ in the sence of (5.16)), and

$$\lim_{j\to\infty} \frac{\widetilde{A}(\tau_j)}{\tau_j} \frac{t_j}{\widetilde{A}(\beta_j t_j)} = \infty. \tag{5.18}$$

We define,

$$\widetilde{B}(t) = \begin{cases} \widetilde{A}(t_j) + \frac{\widetilde{A}(\tau_j) - \widetilde{A}(t_j)}{\tau_j - t_j}(t - t_j) & t \in (t_j, \tau_j)\,, \\ \widetilde{A}(t) & \text{otherwise.} \end{cases}$$

Then, $\widetilde{B}$ is a Young function and $\widetilde{B}(t) \geq \widetilde{A}(t)$ for $t \in (0\,,\infty)\,$. It follows from (5.18) that, for every $j \in \mathbb{N}$, $\tau_j > 2t_j$, and therefore also $\widetilde{A}(\tau_j) > 2\widetilde{A}(t_j)$. Hence, using (5.18), we get

$$\frac{\widetilde{B}(2t_j)}{\widetilde{A}(\beta_j t_j)} = \frac{\widetilde{A}(t_j) + \frac{\widetilde{A}(\tau_j) - \widetilde{A}(t_j)}{\tau_j - t_j} t_j}{\widetilde{A}(\beta_j t_j)} \geq \frac{1}{2} \frac{\widetilde{A}(\tau_j) t_j}{\tau_j \widetilde{A}(\beta_j t_j)} \nearrow \infty.$$

It remains to show (5.15). We have

$$\int_1^\infty \frac{\widetilde{B}(s)}{s^{n'+1}}ds \leq \int_1^\infty \frac{\widetilde{A}(s)}{s^{n'+1}}ds + \sum_{j=1}^\infty \frac{\widetilde{A}(\tau_j) - \widetilde{A}(t_j)}{\tau_j - t_j} \int_{t_j}^{\tau_j} \frac{s - t_j}{s^{n'+1}}ds.$$

Further, using (5.16), $t_j \geq j!$, and the monotonicity of $\{a_j\}$, we obtain

$$\sum_{j=1}^\infty \frac{\widetilde{A}(\tau_j) - \widetilde{A}(t_j)}{\tau_j - t_j} \int_{t_j}^{\tau_j} \frac{s - t_j}{s^{n'+1}}ds \leq C \sum_{j=1}^\infty \frac{\widetilde{A}(t_j)}{\tau_j} t_j^{1-n'} \leq C \sum_{j=1}^\infty a_j < \infty.$$

Therefore, we get (5.15) on recalling (5.14). The proof is complete. □

# Chapter 6

# Mixed norms

In this chapter, we focus on the following question: what can we say about the optimal range space with mixed norm in

$$\dot{W}^1 L^1(\mathbb{R}^n) \hookrightarrow L^1_{\widehat{x_n}}(\mathbb{R}^{n-1}) \left[ L^\infty_{x_n}(\mathbb{R}) \right]?$$

The innovative part of this chapter are Proposition 6.3 and Proposition 6.5. These results allow to conclude the partial optimality of $L^1_{\widehat{x_n}}(\mathbb{R}^{n-1}) \left[ L^\infty_{x_n}(\mathbb{R}) \right]$ in the above mentioned embedding.

Before discussing our results, we will mention our motivation. But, we first need to introduce spaces with mixed norm. We denote by

$$V_k = L^1_{\widehat{x_k}}(\mathbb{R}^{n-1}) \left[ L^\infty_{x_k}(\mathbb{R}) \right], \quad 1 \leq k \leq n,$$

the spaces with mixed norm

$$\|f\|_{V_k} = \|\Psi_k\|_{L^1(\mathbb{R}^{n-1})}, \quad \text{where } \Psi\left(\widehat{x_k}\right) = \operatorname{ess\,sup}_{x_k \in \mathbb{R}} |f(x)|.$$

We recall that we use the notation $\widehat{x}_k$ for the vector in $\mathbb{R}^{n-1}$ obtained from a given $x \in \mathbb{R}^n$ by removing its $k$th coordenate, that is

$$\widehat{x_k} = (x_1, \ldots, , x_{k-1}, x_{k+1}, \ldots, x_n) \in \mathbb{R}^{n-1}.$$

Now, we present a brief history about our point of departure. In Chapter 3, we have proved that if $1 \leq p < n$, then

$$W^1 L^p(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n), \quad p^* = \frac{np}{n-p}. \tag{6.1}$$

The first proof of (6.1) [26] did not apply to the case $p = 1$, but later Gagliardo [12] and Nirenberg [22] found a method of proof which worked in the exceptional case. Gagliardo's idea was to observe that

$$W^1 L^1(\mathbb{R}^n) \hookrightarrow V_k, \quad 1 \leq k \leq n, \tag{6.2}$$

and to deduce from this that $f \in L^{n'}(\mathbb{R}^n)$.

Now, let us prove (6.2).

**Lemma 6.1.** *Let $n \geq 2$. Then,*

$$W^1L^1(\mathbb{R}^n) \hookrightarrow V_k, \quad 1 \leq k \leq n. \tag{6.3}$$

**Remark 6.2.** Note that

$$\|f\|_{L^\infty(\mathbb{R})} \leq \frac{1}{2} \left\| \frac{df}{dx} \right\|_{L^1(\mathbb{R})}, \; f \in \mathcal{D}(\mathbb{R}).$$

Indeed, let $f \in \mathcal{D}(\mathbb{R})$. From

$$f(x) = \int_{-\infty}^{x} \frac{df}{dx}(y)dy = -\int_{x}^{\infty} \frac{df}{dx}(y)dy,$$

one deduces $|f(x)| \leq \int_{-\infty}^{x} \left|\frac{df}{dx}(y)\right| dy$ and $|f(x)| \leq \int_{x}^{\infty} \left|\frac{df}{dx}(y)\right| dy$, and adding gives

$$2\,|f(x)| \leq \int_{\mathbb{R}} \left|\frac{df}{dx}(y)\right| dy.$$

Hence,

$$\|f\|_{L^\infty(\mathbb{R})} \leq \frac{1}{2} \left\| \frac{df}{dx} \right\|_{L^1(\mathbb{R})}.$$

*Proof.* (Lemma 6.1) Fix $1 \leq k \leq n$. Let $f \in \mathcal{D}(\mathbb{R}^n)$. Then, by Remark 6.2, we have

$$\sup_{x_k} |f(x)| \leq \frac{1}{2} \int_{\mathbb{R}} |D_k f(x)|\, dx_k.$$

Hence,

$$\|f\|_{V_k} = \int_{\mathbb{R}^{n-1}} \sup_{x_k} |f(x)|\, dx_{\widehat{x_k}} \leq \int_{\mathbb{R}^n} |D_k f(x)|\, dx = \|D_k f\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{W^1L^1(\mathbb{R}^n)}.$$

Therefore, the proof is complete. □

Lemma 6.2 motivates us to formulate the following problem. Let $X(\mathbb{R}^{n-1})$ and $Y(\mathbb{R})$ be r.i. spaces. We want to find the optimal range space with mixed norm in the the Sobolev embedding

$$\dot{W}L^1(\mathbb{R}^n) \hookrightarrow X_{\widehat{x_n}}(\mathbb{R}^{n-1})\left[Y_{x_n}(\mathbb{R})\right].$$

We begin to solve this problem. First, we take $X(\mathbb{R}^n) = L^1(\mathbb{R}^n)$ and we prove that

$$\dot{W}^1L^1(\mathbb{R}^n) \hookrightarrow L^1_{\widehat{x_n}}(\mathbb{R}^{n-1})\left[L^\infty(\mathbb{R})\right],$$

has the optimal range space with mixed norm; that is, we prove the following proposition.

**Proposition 6.3.** *Let $n \geq 2$. Let $Y(\mathbb{R})$ be an r.i. space. Assume that*

$$\dot{W}^1L^1(\mathbb{R}^n) \hookrightarrow L^1_{\widehat{x_n}}(\mathbb{R}^{n-1})\,[Y_{x_n}(\mathbb{R})]\,. \tag{6.4}$$

*Then, $L^\infty(\mathbb{R}) \hookrightarrow Y(\mathbb{R})$.*

*Proof.* Let us prove that $1 \in Y(\mathbb{R})$. Let $k \in \mathbb{N}$, $k \geq 1$. Let $\varphi \in \mathcal{D}(\mathbb{R})$ be non-negative function with support in the ball centered at 0 with radius 1, and $\int_{\mathbb{R}} \varphi(x)dx = 1$. Let $\theta \in \mathcal{D}(\mathbb{R})$ be a function defined as follows

$$\theta(x) = \int_{\mathbb{R}} \frac{1}{4}\varphi\left(\frac{y}{4}\right)\chi_{B(0,3/4)}(x-y)dy.$$

Note that $\theta(x) = 1$ for all $x \in B(0,1/2)$, $\theta(x) = 0$ for all $x \notin \overline{B(0,1)}$ and $0 \leq \theta(x) \leq 1$ for all $x \in \overline{B(0,1)}$. Let $\phi \in \mathcal{D}(\mathbb{R}^{n-1})$ be non-negative function with support in the ball centered at 0 with radius 1, and $\int_{\mathbb{R}^{n-1}} \phi(x)dx = 1$. Define

$$h_k(x_n) = \theta\left(\frac{x}{k}\right) \quad \text{and} \quad g_k(\widehat{x}_n) = \phi\left(\frac{\widehat{x}_n}{k}\right).$$

We have

$$\|h_k\|_{L^1(\mathbb{R})} \leq k \quad \text{and} \quad \|h_k'\|_{L^1(\mathbb{R})} \leq A.$$

Moreover

$$\|g_k\|_{L^1(\mathbb{R}^{n-1})} = k^{n-1} \quad \text{and} \quad \sum_{i=1}^{n-1} \|D_i\phi_k\|_{L^1(\mathbb{R}^{n-1})} \leq A(n-1)k^{n-2}.$$

Now, consider $f_k(x) = h_k(x_n)g_k\left(\widehat{x}_n\right)$, $x \in \mathbb{R}^n$. Let us prove that $f_k \in \dot{W}^1L^1(\mathbb{R}^n)$. Indeed,

$$\begin{aligned}\|f_k\|_{\dot{W}^1L^1(\mathbb{R}^n)} &= \|h_k'\|_{L^1(\mathbb{R})}\,\|g_k\|_{L^1(\mathbb{R}^{n-1})} + \|h_k\|_{L^1(\mathbb{R})}\sum_{i=1}^{n} \|D_ig_k\|_{L^1(\mathbb{R}^{n-1})}\\ &\leq Ak^{n-1} + (n-1)k^{n-1} = nAk^{n-1}.\end{aligned}$$

Then, by (6.4), we obtain

$$\|h_k\|_{Y(\mathbb{R})}\,\|g_k\|_{L^1(\mathbb{R}^{n-1})} \leq nAk^{n-1},$$

and so $\|h_k\|_{Y(\mathbb{R})} \leq n$. Since $h_k$ converges almost everywhere to 1, by Fatou's Lemma we get

$$\|1\|_{Y(\mathbb{R})} \leq \liminf_{k\to\infty} \|h_k\|_{Y(\mathbb{R})} < \infty.$$

Therefore, the proof is complete. □

**Remark 6.4.** Note that, if we replace $\dot{W}^1L^1(\mathbb{R}^n)$ in (6.4) by $W^1L^1(\mathbb{R}^n)$, Proposition 6.3 does not hold. Indeed, take $Y(\mathbb{R}) = L^1(\mathbb{R})$ then

$$\|f\|_{L^1(\mathbb{R}^n)} \le \|Df\|_{L^1(\mathbb{R}^n)} + \|f\|_{L^1(\mathbb{R}^n)},$$

but $L^\infty \not\subset L^1$.

Second, we take $L^\infty(\mathbb{R}) = Y(\mathbb{R})$ and we prove that

$$\dot{W}^1L^1(\mathbb{R}^n) \hookrightarrow L^1_{\widehat{x_n}}(\mathbb{R}^{n-1})\left[L^\infty_{x_n}(\mathbb{R})\right],$$

has the optimal range space with mixed norm; that is, we prove the following proposition.

**Proposition 6.5.** *Let $n \ge 2$. Let $X(\mathbb{R}^{n-1})$ be an r.i. space. Assume that*

$$\dot{W}^1L^1(\mathbb{R}^n) \hookrightarrow X_{\widehat{x}_n}(\mathbb{R}^{n-1})\left[L^\infty_{x_n}(\mathbb{R})\right]. \tag{6.5}$$

*Then, $L^1(\mathbb{R}^{n-1}) \hookrightarrow X(\mathbb{R}^{n-1})$.*

*Proof.* Let $k \in \mathbb{N}$, $k \ge 1$. Let $\varphi \in \mathcal{D}(\mathbb{R}^{n-1})$ be non-negative function with support in the ball centered at 0 with radius 1, and $\int_{\mathbb{R}^{n-1}} \varphi(x)dx = 1$. Let $\theta \in \mathcal{D}(\mathbb{R}^{n-1})$ be the function defined as follows

$$\theta(x) = \int_{\mathbb{R}^{n-1}} \frac{1}{4^{n-1}}\varphi\left(\frac{y}{4}\right)\chi_{B(0,3/4)}(x-y)dy.$$

Note that $\theta(x) = 1$ for all $x \in B(0,1/2)$, $\theta(x) = 0$ for all $x \notin \overline{B(0,1)}$ and $0 \le \theta(x) \le 1$ for all $x \in B(0,1)$. Now, let $\psi \in \mathcal{D}(\mathbb{R})$ be non-negative function such that $\operatorname{supp}\psi \subset B(0,1)$ and consider

$$\eta(x) = \int_{\mathbb{R}} \frac{1}{4}\varphi\left(\frac{y}{4}\right)\chi_{B(0,3/4)}(x-y)dy.$$

Define

$$g_k(x_n) = \eta(k\,x_n), \text{ and } u_k\left(\widehat{x_n}\right) = h\left(\widehat{x_n}\right)\theta\left(\frac{\widehat{x_n}}{k}\right),$$

where $h \in \mathcal{D}(\mathbb{R}^{n-1})$. We have

$$\|g_k\|_{L^1(\mathbb{R})} \le \frac{1}{k}, \quad \|g_k'\|_{L^1(\mathbb{R})} \le A \quad \text{and} \quad \|g_k\|_{L^\infty(\mathbb{R})} = 1.$$

Moreover,

$$\|u_k\|_{L^1(\mathbb{R}^{n-1})} \le \|h\|_{L^1(\mathbb{R}^{n-1})}\|\theta(\cdot/k)\|_{L^\infty_{\widehat{x_n}}(\mathbb{R}^{n-1})} = \|h\|_{L^1(\mathbb{R}^{n-1})},$$

and

$$\sum_{j=1}^{n}\|D_j u_k\|_{L^1(\mathbb{R}^{n-1})} \leq \sum_{j=1}^{n}\left(\|D_j h\|_{L^1(\mathbb{R}^{n-1})}\|\theta(\cdot/k)\|_{L^1(\mathbb{R}^{n-1})} + \|h\|_{L^1(\mathbb{R}^{n-1})}\|D_j\theta(\cdot/k)\|_{L^1(\mathbb{R}^{n-1})}\right)$$
$$\leq \sum_{j=1}^{n}\left(\|D_j h\|_{L^1(\mathbb{R}^{n-1})} + \frac{A}{k^{n-1}}\|h\|_{L^1(\mathbb{R}^{n-1})}\right).$$

We define $f_k(x) = u_k(x_{\widehat{x_n}})\, g_k(x_n)$, $x \in \mathbb{R}^n$. Let us prove that $f_k \in \dot{W}^1L^1(\mathbb{R}^n)$. In fact,

$$\|f_k\|_{\dot{W}^1L^1(\mathbb{R}^n)} = \|u_k\|_{L^1(\mathbb{R}^{n-1})}\|g_k'\|_{L^1(\mathbb{R})} + \sum_{j=1}^{n}\|D_j u_k\|_{L^1(\mathbb{R}^{n-1})}\|g_k\|_{L^1(\mathbb{R})}$$
$$\leq A\|h\|_{L^1(\mathbb{R}^{n-1})} + \sum_{j=1}^{n}\left(\frac{A}{k^{n-1}}\|D_j h\|_{L^1(\mathbb{R}^{n-1})} + \|h\|_{L^1(\mathbb{R}^{n-1})}\right).$$

Then, by (6.5)

$$\|u_k\|_{X(\mathbb{R}^{n-1})}\|g_k\|_{L^\infty(\mathbb{R})} \leq A\|h\|_{L^1(\mathbb{R}^{n-1})} + \frac{A}{k}\sum_{j=1}^{n}\left(\frac{A}{k^{n-1}}\|D_j h\|_{L^1(\mathbb{R}^{n-1})} + \|h\|_{L^1(\mathbb{R}^{n-1})}\right).$$

Since $u_k$ converges almost everywhere to $h$, by Fatou's Lemma

$$\|h\|_{X(\mathbb{R}^{n-1})} \leq \liminf_{k\to\infty}\|u_k\|_{X(\mathbb{R}^{n-1})} \leq A\|h\|_{L^1(\mathbb{R}^{n-1})}.$$

Hence,

$$\|h\|_{X(\mathbb{R}^{n-1})} \leq A\|h\|_{L^1(\mathbb{R}^{n-1})}, \ h \in \mathcal{D}(\mathbb{R}^{n-1}).$$

Finally, by standard approximation argument, we extend the result to all $h \in L^1(\mathbb{R}^{n-1})$. Hence, the proof is complete. □

Proposition 6.3 and Proposition 6.5 allow us to conclude that $L^1_{\widehat{x_n}}(\mathbb{R}^{n-1})\left[L^\infty_{x_n}(\mathbb{R})\right]$ is the partial optimal range space. Now, our aim for a future research is to prove that it is the optimal range space with mixed norm. In other words, we would like to prove that

$$\dot{W}L^1(\mathbb{R}^n) \hookrightarrow L^1_{\widehat{x_n}}(\mathbb{R}^{n-1})\left[L^\infty_{x_n}(\mathbb{R})\right] \hookrightarrow X_{\widehat{x_n}}(\mathbb{R}^{n-1})\left[Y_{x_n}(\mathbb{R})\right],$$

for any rearrangement invariant Banach space with mixed norm such that

$$\dot{W}L^1(\mathbb{R}^n) \hookrightarrow X_{\widehat{x_n}}(\mathbb{R}^{n-1})\left[Y_{x_n}(\mathbb{R})\right].$$

# Bibliography


[1] R. A. Adams and J. J. F. Fournier, *Sobolev Spaces*, Pure and Applied Mathematics 140, Academic Press, Amsterdam, 2003.

[2] C. Bennett and R. Sharpley, *Interpolation of Operators*, Pure and Applied Mathematics 129, Academic Press, Princeton, 1988.

[3] H. Brézis and S. Wainger, *A note on limiting cases of Sobolev embeddings and convolution inequalities*, Comm. Partial Differential Equations **5** (1980), 773-789.

[4] M. Carro, L. Pick, J. Soria and V. D. Stepanov, *On embeddings between classical Lorentz spaces*, Math. Inequal. Appl. **4** (2001), 397-428.

[5] J. Cerdà, *Linear Functional Analysis*, Graduate Studies in Mathematics 116, American Mathematical Society, Providence, 2010.

[6] A. Cianchi, *Symmetrization and second-order Sobolev inequalities*, Ann. Mat. Pura Appl. **183** (2004), 45-77.

[7] A. Cianchi, *Optimal Orlicz-Sobolev embeddings*, Rev. Mat. Iberoamericana **20** (2004), 427-474.

[8] A. Cianchi and L. Pick, *Sobolev embeddings into BMO, VMO and $L^\infty$*, Ark. Mat. **36** (1998), 317-340.

[9] R. DeVore and K. Scherer, *Interpolation of linear operators on Sobolev spaces*, Ann. of Math. **109** (1979), 583-599.

[10] D. E. Edmunds, R. Kerman and L. Pick, *Optimal Sobolev imbeddings involving rearrangement-invariant quasinorms*, J. Funct. Anal. **170** (2000), 307-355.

[11] H. Federer, *Geometric Measure Theory*, Grundlehren der Mathematischen Wissenchaften 153, Springer-Verlag, New York, 1969.

[12] E. Gagliardo, *Proprietà di alcune classi di funzioni in più variabili*, Ricerche Mat. **7** (1958), 102-137.

[13] L. Grafakos, *Classical Fourier Analysis*, Graduate Texts in Mathematics 249, Springer, New York, 2008.

[14] K. Hansson, *Imbedding theorems of Sobolev type in potential theory*, Math. Scand. **45** (1979), 77-102.

[15] J. A. Hempel, G. R. Morris and N. S. Trudinger, *On the sharpness of a limiting case of the Sobolev imbedding theorem*, Bull. Austral. Math. Soc. **3** (1970), 369-373.

[16] T. Holmstedt, *Interpolation of quasi-normed spaces*, Math. Scand. **26** (1970), 177-199.

[17] V. I. Judovič, *Some estimates connected with integral operators and with solutions of elliptic equations*, Dokl. Akad. Nauk SSSR **138** (1961), 805-808.

[18] R. Kerman and L. Pick, *Optimal Sobolev imbeddings*, Forum Math. **18** (2006), 535-570.

[19] G. Leoni, *A First Course in Sobolev Spaces*, Graduate Studies in Mathematics 105, American Mathematical Society, Providence, 2009.

[20] J. Malý and L. Pick, *An elementary proof of sharp Sobolev embeddings*, Proc. Amer. Math. Soc. **130** (2002), 555-563.

[21] V. G. Maz'ja, *Sobolev Spaces*, Springer-Verlag, Berlin, 1985.

[22] L. Nirenberg, *On elliptic partial differential equations*, Ann. Sc. Pisa **13** (1959), 116-162.

[23] R. O'Neil, *Convolution operators and $L(p, q)$ spaces*, Duke Math. J. **30** (1963), 129-142.

[24] G. Peetre, *Espaces d'interpolation et théorème de Soboleff*, Ann. Inst. Fourier **16** (1966), 279-317.

[25] S. I. Pohožaev, *On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$*, Dokl. Akad. Nauk SSSR **165** (1965), 36-39.

[26] S.L. Sobolev, *On a theorem of functional analysis*, Mat. Sb **46** (1938), 471-497.

[27] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, 1970.

[28] G. Talenti, *Inequalities in rearrangement invariant function spaces*, Nonlinear analysis, function spaces and applications **5** (1994), 177-230.

[29] N. S. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473-483.