On the distribution of the solutions of systems of polynomial equations

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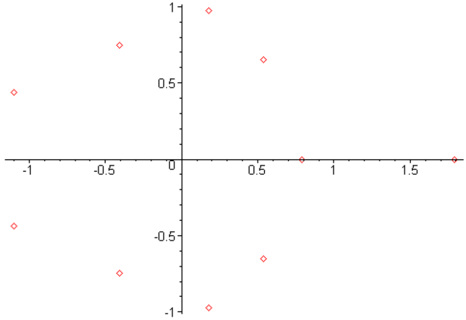
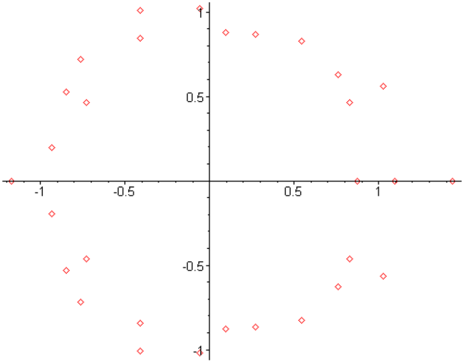
Activity

Let f be a polynomial of degree $d \gg 0$ with coefficients ± 1 or 0. I will plot all complex solutions of $f = 0$, then we will see what it happens. . .

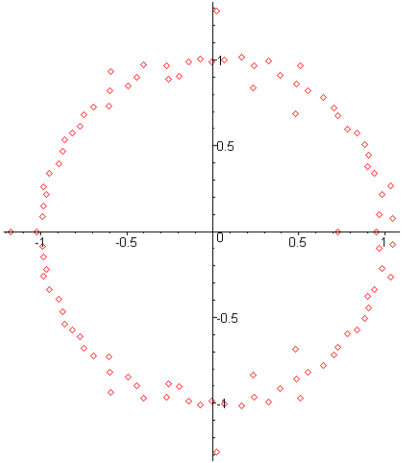
For instance, let $d = 10$ and $f = -x^{10} + x^9 + x^8 + x^6 + x^5 - x^4 + x^3 - x^2 + x - 1$



$d = 30$ and $f = x^{30} - x^{29} - x^{28} + x^{26} + x^{25} - x^{24} - x^{23} - x^{22} + x^{21} - x^{20} + x^{19} + \cdots$



$d = 100$ and $f = -x^{100} - x^{98} + x^{96} + x^{94} - x^{93} + x^{92} - x^{91} - x^{90} + x^{88} - x^{84} + \cdots$



Conclusion???

The Erdös-Turán theorem

Let $f(x) = a_d x^d + \cdots + a_0 = a_d(x - \rho_1 \, \mathrm{e}^{\mathrm{i}\,\theta_1}) \cdots (x - \rho_d \, \mathrm{e}^{\mathrm{i}\,\theta_d})$

Defn

The *angle discrepancy* of f is

$$\Delta_\theta(f) := \sup_{0 \le \alpha < \beta < 2\pi} \left| \frac{\#\{k : \alpha \le \theta_k < \beta\}}{d} - \frac{\beta - \alpha}{2\pi} \right|$$

The ε*-radius discrepancy* of f is

$$\Delta_{\mathrm{r}}(f; \varepsilon) := \frac{1}{d} \, \#\left\{ k : 1 - \varepsilon < \rho_k < \frac{1}{1 - \varepsilon} \right\}$$

Also set $||f|| := \sup_{|z|=1} |f(z)|$

Thm [Erdös-Turán 1948], [Hughes-Nikeghbali 2008]

$$\Delta_\theta(f) \leq c\sqrt{\frac{1}{d}\log\left(\frac{||f||}{\sqrt{|a_0 a_d|}}\right)} \quad , \quad 1-\Delta_{\mathrm{r}}(f;\varepsilon) \leq \frac{2}{\varepsilon d}\log\left(\frac{||f||}{\sqrt{|a_0 a_d|}}\right)$$

Here $\sqrt{2} \leq c \leq 2{,}5619$ [Amoroso-Mignotte 1996]

Cor: the equidistribution

Let $f_d(x)$ of degree d st $\log\left(\frac{||f_d||}{\sqrt{|a_{d,0}a_{d,d}|}}\right) = o(d)$, then

$$\lim_{d\to\infty}\frac{1}{d}\#\left\{k : \alpha \leq \theta_{dk} < \beta\right\} = \frac{\beta-\alpha}{2\pi}$$

$$\lim_{d\to\infty}\frac{1}{d}\#\left\{k : 1-\varepsilon < \rho_{dk} < \frac{1}{1-\varepsilon}\right\} = 1$$

Some consequences

1. The number of real roots of f is $\leq 51\sqrt{d \log\left(\frac{||f||}{\sqrt{|a_0 a_d|}}\right)}$ [Erhardt-Schur-Szego]

2. If $g(z) = 1 + b_1 z + b_2 z^2 + \ldots$ converges on the unit disk, then the zeros of its d-partial sums distribute uniformely on the unit circle for $d \to \infty$ [Jentzsch-Szego]

Systems of equations

For $f_1, \dots, f_n \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ consider its set of zeros

$$V(f_1, \dots, f_n) = \{\xi \in (\mathbb{C}^\times)^n : f_1(\xi) = \cdots = f_n(\xi) = 0\} \subset (\mathbb{C}^\times)^n$$

and V_0 the subset of isolated points Set $Q_i := \mathrm{N}(f_i) \subset \mathbb{R}^n$ the

Newton polytope, then

$$\# V_0 \leq \mathrm{MV}_n(Q_1, \dots, Q_n) =: D \quad \text{[BKK]}$$

From now on, we will assume $\# V_0 = D$, in particular $V(\boldsymbol{f}) = V_0$.

Pb: estimate $\Delta_\theta(\boldsymbol{f})$ and $\Delta_{\mathrm{r}}(\boldsymbol{f}, \varepsilon)$

For $v \in \mathbb{R}^n \setminus \{0\}$ let $\pi_v : \mathbb{R}^n \to v^\perp$ the orthogonal projection and

$$\gamma(\boldsymbol{f}) := \frac{1}{D} \sup_{v \in \mathbb{R}^n \setminus \{\mathbf{0}\}} \sum_{j=1}^{n} \mathrm{MV}_{n-1}\left(\pi_v(Q_k) : k \neq j\right) \log ||f_j||$$

For f_i dense of degree d_i we have $\gamma(\boldsymbol{f}) = \sqrt{n} \sum_j \frac{\log ||f_j||}{d_j}$

Thm (D'Andrea-Galligo-S)

Let $f_1, \ldots, f_n \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, then

$$\Delta_\theta(f) \leq c(n)\gamma(\boldsymbol{f})^{\frac{1}{2(n+1)}} \quad , \quad 1 - \Delta_{\mathrm{r}}(f; \varepsilon) \leq \frac{2}{\varepsilon d}\gamma(\boldsymbol{f})$$

with $c(n) \leq 2^{3n} n^{\frac{n+1}{2}}$

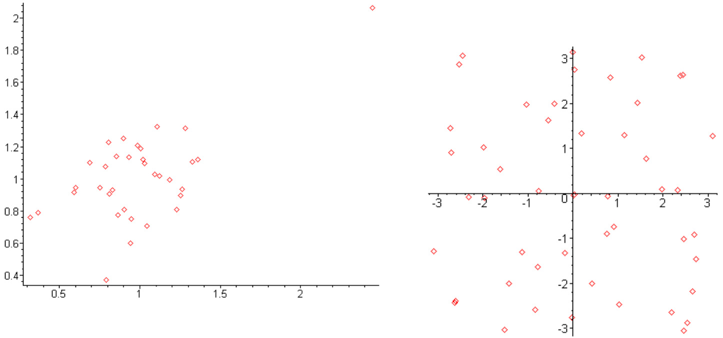
Cor. Let $\boldsymbol{f}_d$ such that $\gamma(\boldsymbol{f}_d) = o(d)$, then $V(\boldsymbol{f}_d)$ converges to equidistribution on $(S^1)^n$ for $d \to \infty$

For instance, let $d = 7$ and

$$f{=}x^7{+}x^6y{+}x^5y^2{-}x^4y^3{+}x^3y^4{+}xy^6{-}y^7{-}x^6{+}x^4y^2{-}x^3{*}y^3{+}x^2{*}y^4{+}x{*}y^5{+}y^6{+}\cdots$$

$$g{=}{-}x^7{-}x^5y^2{+}x^4y^3{+}x^3y^4{-}x^2y^5{-}y^7{+}x^5y{-}xy^5{-}y^6{+}x^5{+}x^4y{-}x^2y^3{-}xy^4{+}x^2y^2{+}\cdots$$

The joint modulus and arguments of $f = g = 0$ plot as



The size of an eliminant form

Let $a \in \mathbb{Z}^n \setminus \{0\}$ and consider the monomial projection $\chi_a : (\mathbb{C}^\times)^n \to \mathbb{C}^\times$, $\xi \mapsto \xi^a = \xi_1^{a_1} \cdots \xi_n^{a_n}$. The associated *eliminant polynomial* is

$$E(\boldsymbol{f}, a)(z) := c \prod_{\xi \in V} (z - \chi_a(\xi))^{\mathrm{mult}(\xi)} \in \mathbb{Z}[z]$$

Thm (a variant of the arithmetic Bezout theorem)

$$\log ||E(\boldsymbol{f}, a)|| \leq ||a|| \sum_{j=0}^{n} \mathrm{MV}_{n-1}\left(\pi_a(Q_k) : k \neq j\right) \log ||f_j||$$

For the estimate of Δ_θ: apply E-T to $E(\boldsymbol{f}, a)$ to estimate the exponential sums on its roots, then recover V by tomography *via* Fourier analysis

Thank you!