HEIGHT OF PROJECTIVE VARIETIES

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For $N, D, H \in \mathbb{N}$ consider the Laurent polynomials

$$f_1 := x_1 - H, \quad f_2 := x_2\, x_1^{-D} - H, \quad f_3 := x_3\, x_2^{-D} - H,$$
$$\ldots, \quad f_N := x_N\, x_{N-1}^{-D} - H \quad \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$$

and the associated equation system

$$f_1 = 0, \quad \ldots \quad , f_N = 0$$

The solution set $Z \subset (\mathbb{C}^*)^N$ has only one point, namely

$$\big(H, H^{1+D}, H^{1+D+D^2}, \ldots, H^{1+D+\cdots+D^{N-1}}\big) \quad \in (\mathbb{C}^*)^N$$

For $N := 5$, $D := 3$, $H := 2$

$$Z = \Big\{\big(2; \quad 16; \quad 8{,}192; \quad 1{,}099{,}511{,}627{,}776;$$
$$2{,}658{,}455{,}991{,}569{,}831{,}745{,}807{,}614{,}120{,}560{,}689{,}152\big)\Big\} \quad \subset (\mathbb{C}^*)^5$$

Heights of points of $\mathbb{T}^N$

For $\quad m=(m_1,\dots,m_N)\in\mathbb{Z}^N \quad$ the *height*

$$h(m)=\log\max\{0,m_1,\dots,m_N\}$$

is a measure for the complexity of writing down ξ.

In the example

$$\deg(Z)=1 \qquad , \qquad h(Z)=(1+D+\cdots+D^{N-1})\,\log(H)$$

For $N:=5,\ D:=3,\ H:=2$ we have $\quad h(Z)=83.87$.

Set $\mathbb{T}^N:=(\mathbb{Q}^*)^N$ for the algebraic torus of dimension N. The previous defn is compatible with the group law : for $k\in\mathbb{N}$ we set

$$[k]:\mathbb{T}^N\to\mathbb{T}^N \quad , \quad (t_1,\dots,t_N)\mapsto(t_1^k,\dots,t_N^k)$$

for the multiplication by k over $\mathbb{T}^N$; then

$$h\big([k]\,m\big)=k\,h(m)$$

How does this extends to 0-dimensional varieties ?

Let $X \subset \mathbb{T}^N$ be a 0-dimensional $\mathbb{Q}$-variety. Consider the (primitive) Chow form

$$\mathcal{C}h_X = \gamma \prod_{\xi \in Z} \Big(U_0 + U_1\, \xi_1 + \cdots + U_N\, \xi_N \Big) \quad \in \mathbb{Z}[U_0, \ldots, U_N]$$

which is well-defined up to $\pm$. Set

$$h_{\mathsf{naive}}(X) := h(\mathcal{C}h_X) = \log \max \Big\{ \big| \text{Coeffs of } \mathcal{C}h_X \big| \Big\}$$

This is *linear* up to a bounded function : there exists $c \geq 0$ st

$$h_{\mathsf{naive}}([k]\, X) = c\, k + O(1) \qquad , \qquad k \gg 0$$

then the *height* of X is defined as

$$h(X) := \lim_{k \mapsto \infty} \frac{1}{k}\, h_{\mathsf{naive}}([k]\, X)$$

This is the normalized (or Neron-Tate) height of points of $\mathbb{T}^N$ introduced by [Weil 51] ; this approach is due to [Neron65] for points in Abelian varieties

It is linear :

$$h([k]\, X) = k\, h(X)$$

For $\xi \in (\mathbb{Q}^*)^N$

$$h(\xi) = \log \max\{q, m_1, \ldots, m_N\}$$

where $\xi = \dfrac{1}{q}\, (m_1, \ldots, m_N)$ is an irredundant expression for ξ

In general

$$\big| h(X) - h(\mathcal{C}h_X) \big| \quad \leq \quad \log(N+1)\ \#X$$

Intersection theory on $\mathbb{P}^N$

Let

$$F_1, \dots, F_N \quad \in \mathbb{C}[x_0, \dots, x_N]$$

be homogeneous polynomials, then (Bézout theorem, 1764)

$$\#Z(F_1, \dots, F_N)_0 \quad \leq \quad \prod_{i=1}^{N} \deg(F_i)$$

For an equidimensional variety $\quad X \subset \mathbb{P}^N \quad$ the *degree* is

$$\deg(X) := \#\big(X \cap Z(\ell_1, \dots, \ell_n)\big)$$

for generic linear forms $\ell_1, \dots, \ell_n$ and $n = \dim(X)$.

For $\dim(X) = 0$ the degree equals its <u>cardinality</u> :

$$\deg(X) = \#X$$

For a hypersurface $Z(f)$ defined by a squarefree polynomial

$$\deg\big(Z(f)\big) = \deg(f)$$

This notion can be extended to arbitrary varieties. For $Z \subset \mathbb{P}^N$ we set

$$\deg(Z) \quad := \quad \sum_{j=0}^{N} \deg(Z_j)$$

where $Z_j \subset \mathbb{P}^N$ is the jth equidimensional component of Z. Then

$$\deg(X \cap Y) \quad \leq \quad \deg(X)\,\deg(Y)$$

Intersection theory on toric varieties

Let

$$F_1, \ldots, F_N \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$$

be Laurent polynomials, and let $Z_0 \subset (\mathbb{C}^*)^N$ be the set of isolated zeroes of the equation system

$$F_1 = 0, \quad \ldots \quad , F_n = 0$$

Then (Bernstein-Kushnirenko thm 1975)

$$\#Z_0 \quad \leq \quad \sum_{\xi \in Z_0} \ell(\xi) \quad \leq \quad \mathrm{MV}(Q_1, \ldots, Q_N)$$

where $\ell(\xi)$ is the intersection multiplicity of $F_1, \ldots, F_N$ at ξ ;

$$Q_i := \mathrm{NP}(f_i) = \mathrm{Conv}(\mathrm{Supp}(F_i)) \quad \subset \mathbb{R}^N$$

is the Newton polytope of F_i ; and $Q := \mathrm{NP}(f_1, \ldots, f_N) \subset \mathbb{R}^N$

The *mixed volume* $\mathrm{MV}(Q_1, \ldots, Q_N) \in \mathbb{N}$ can be defined as

$$\sum_{J \subset \{1,2,\ldots,N\}} (-1)^{\#J} \, \mathrm{Vol}_{\mathbb{R}^N} \Big(\sum_{j \in J} Q_j \Big)$$

For the unmixed case $Q_1 = \ldots, Q_N = Q$

$$\mathrm{MV}(Q_1, \ldots, Q_N) = N! \, \mathrm{Vol}_{\mathbb{R}}^N(Q)$$

In the example

$$f_1 := x_1 - H, \quad f_2 := x_2\, x_1^{-D} - H, \quad \ldots, \quad f_n := x_N\, x_{N-1}^{-D} - H$$

Let $e_1, \ldots, e_N$ be the *standard basis* of $\mathbb{R}^N$, then

$$Q_1 = \mathrm{Conv}(e_1, 0), \quad Q_2 = \mathrm{Conv}(e_2 - D\, e_1, 0),$$
$$\ldots, \quad Q_n = \mathrm{Conv}(e_N - D\, e_{N-1}, 0) \quad \subset \mathbb{R}^N$$

Then

$$\mathrm{MV}(Q_1, \ldots, Q_N) = 1$$

and so

$$\# Z \quad \leq \quad 1 \quad \ll \quad D^N$$

For $N := 2,\ D := 3$

Heights of subvarieties of $\mathbb{P}^N$

Let's compactify the torus through the standard inclusion

$$i_N : \mathbb{T}^N \hookrightarrow \mathbb{P}^N \quad , \quad (t_1, \dots, t_N) \mapsto (1 : t_1 : \dots : t_N)$$

The multiplication $[k]$ extends to the *k-power map*

$$[k] : \mathbb{P}^N \to \mathbb{P}^N \quad , \quad (x_0 : \dots : x_N) \mapsto (x_0^k : \dots , x_N^k)$$

Let $X \subset \mathbb{P}^N$ be an equidimensional $\mathbb{Q}$-variety of dimension n and let

$$\mathcal{C}h_X \quad \in \mathbb{Z}[U_0, \dots, U_n]$$

be its (primitive) Chow form, where $n := \dim(X)$. This is a homogeneous polynomial in each group of variables $U_i = \{U_{i0}, \dots, U_{in}\}$ of partial degree

$$\deg_{U_i}(\mathcal{C}h_X) \quad = \quad \deg(X)$$

Set

$$h_{\mathsf{naive}}(X) := h(\mathcal{C}h_X) = \log \max \left\{ \left| \text{Coeffs of } \mathcal{C}h_X \right| \right\}$$

for the *naive* height of X (proposed by [Weil 50], reappears in the '80 [Nesterenko 83], [Philippon 86])

Then there exists $c \geq 0$ st for $k \gg 0$

$$\frac{h_{\mathsf{naive}}([k]\,X)}{\deg([k]\,X)} = c\,k + O(1)$$

The *normalized* (or Neron-Tate) height of X is defined as

$$h(X) := \deg(X) \lim_{k\to\infty} \frac{1}{k} \frac{h_{\mathsf{naive}}\big([k]\,X\big)}{\deg\big([k]\,X\big)} \quad \in \mathbb{R}_+$$

[Zhang 95], [David-Philippon 98]

Then

$$\frac{h\big([k]\,X\big)}{\deg\big([k]\,X\big)} = k\,\frac{h\big(X\big)}{\deg\big(X\big)}$$

This can be compared with the naive height as

$$\big|h(X) - h(\mathcal{C}h_X)\big| \quad \leq \quad 2\,(n+1)\,\log(N+1)$$

Vanishing

$h(X) = 0$ iff $X = \bigcup_{i=1}^{M} X_i$ where each $X_i = \omega_i\,H$ is the translated of an algebraic group H by a torsion point ω

The "$\Rightarrow$" implication is equivalent to the Bogomolov conjecture, solved by [Zhang95]

Examples

- $\dim(X) = 0$

This height was first introduced by A. Weil (1951) as

$$h(X) := \sum_{\xi \in X} h(\xi)$$

with

$$h(\xi) := \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_{\mathbb{Q}}} \sum_{\sigma : K \hookrightarrow \mathbb{C}_v} \log \max \left\{ |\sigma(\xi_0)|_v, \ldots, |\sigma(\xi_N)|_v \right\}$$

where

K is a number field such that $\xi \in (K^*)^N$;

$[K:\mathbb{Q}]$ is the extension degree of K;

$M_{\mathbb{Q}} := \{\infty\} \cup \{p\,;\, p \text{ prime}\}$ is the canonical set of absolute values of $\mathbb{Q}$;

$|\cdot|_\infty$ is the ordinary absolute value;

$|\cdot|_p$ is the p-adic absolute value defined by

$$|\alpha|_p := p^{-\mathrm{ord}_p(\alpha)};$$

$\mathbb{C}_v$ is the completion of the algebraic closure of $\mathbb{Q}_v$;

σ runs over all inclusions of K into $\mathbb{C}_v$.

Examples (cont.)

- $\dim(X) > 0$

There is no general algorithm for computing $h(X)$. Moreover we don't know which is its arithmetic nature in the general case (is it a period à la Kontsevich-Zagier ?)

For $X = Z(f)$ the height equals the *Mahler measure* of f

$$h(X) = m(f) = \int_{S^1 \times \cdots \times S^1} \log|f(z)| \quad dz_1 \cdots dz_N$$

the integral being w.r. to the unitary Haar measure over the compact torus

For plenty of $f \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ this is related to special values of Dirichlet L functions and of L functions of elliptic curves

E.g. [Smyth81]

$$m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2)$$

with $L(\chi_{-3}, 2) = 1 - \frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{5^2} + \cdots$

This is a very active area of research : work of D. Boyd, C. Deninger, F. Rodríguez Villegas, V. Maillot, . . .

More examples : monomials varieties

Joint work with P. Philippon

Let $\mathcal{A} := \big\{(a_0, \alpha_0), \dots, (a_N, \alpha_N)\big\} \subset \mathbb{Z}^n \times \mathbb{Q}^*$ st

$$\big(a_1 - a_0, \dots, a_N - a_0\big)_{\mathbb{Z}} = \mathbb{Z}^n$$

Let

$$\varphi_{\mathcal{A}} : \mathbb{T}^n \to \mathbb{P}^N \quad , \quad s \mapsto \big(\alpha_0\, s^{a_0} : \cdots : \alpha_N\, s^{a_N}\big)$$

and set

$$X_{\mathcal{A}} := \overline{\varphi_{\mathcal{A}}(\mathbb{T}^n)} \quad \subset \mathbb{P}^N$$

for the associated *monomial variety* When $\alpha_i = 1$ for all i this is a projective toric variety

The dimension and degree are

$$\dim(X_{\mathcal{A}}) = n \quad , \quad \deg(X_{\mathcal{A}}) = n!\,\mathrm{Vol}_{\mathbb{R}^n}(Q)$$

where $Q := \mathrm{Conv}(a_0, \dots, a_N) \subset \mathbb{R}^n$

E.g. Let $S \subset \mathbb{P}^4$ be the surface associated to the monomial map

$$(s,t) \mapsto (1 : s : t : s^2\, t : s\, t^2)$$

its degree is

$$\deg(S) = 2!\,\mathrm{Vol}_{\mathbb{R}^n}(Q) = 5$$

Thm. (Philippon-S. 03)
Let $v \in M_{\mathbb{Q}}$ and set

$$Q_v := \mathrm{Conv}\big((a_0, \log|\alpha_0|_v), \ldots, (a_N, \log|\alpha_N|_v)\big) \quad \subset \mathbb{R}^{n+1}$$

for the v-adic polytope of $\mathcal{A}$, and set

$$E_v : Q \to \mathbb{R}$$

for the parameterization of its upper convex envelope w.r. to $Q := \mathrm{Conv}(a_0, \ldots, a_N) \subset \mathbb{R}^n$; then set

$$E \quad := \quad \sum_v E_v$$

Then

$$h(X_{\mathcal{A}}) \quad = \quad (n+1)! \int_Q E \quad dx_1 \cdots dx_N$$

E.g. Let $\mathcal{A} := \big\{(0,1), (1,5), (2,7), (3,1)\big\} \subset \mathbb{Z} \times \mathbb{Q}^*$; then $X_{\mathcal{A}} \subset \mathbb{P}^3$ is (the closure of) the image of the map

$$s \mapsto (1 : 5\,s : 7\,s^2 : s^3)$$

Set

$$Q_\infty \quad := \quad \mathrm{Conv}\big((0,0), (1, \log(5)), (2, \log(7)), (3,0)\big) \quad \subset \mathbb{R}^2$$

then

$$h(X_{\mathcal{A}}) = 2!\,\mathrm{Vol}_{\mathbb{R}^{n+1}}(Q_\infty) = 2\,\big(\log(5) + \log(7)\big)$$

Cor.

$$h(X_{\mathcal{A}}) \quad \in \quad \big(\log(\overline{\mathbb{Q}}^*)\big)_{\mathbb{Q}}$$

$\Rightarrow$ either $h(X_{\mathcal{A}}) = 0$ or $h(X_{\mathcal{A}}) \notin \overline{\mathbb{Q}}$ (by Baker's theorem)

For $\mathcal{A}$ *symmetric* that is when

$$[-1]\, X_{\mathcal{A}} = X_{\mathcal{A}}$$

then

$$h(X_{\mathcal{A}}) \quad = \quad \frac{(n+1)!}{2} \sum_{v \in M_{\mathbb{Q}}} \mathrm{Vol}_{\mathbb{R}^{n+1}}(Q_v)$$

Idea of the proof. Set $\mathcal{H}_{\text{géom}}(X_{\mathcal{A}}; D)$ for the Hilbert function of $X_{\mathcal{A}}$; then

$$\mathcal{H}_{\text{géom}}(X_{\mathcal{A}}; D) = \#\Big(D\, Q \cap \mathbb{Z}^n\Big) = \mathrm{Vol}_{\mathbb{R}^n}(Q)\, D^n + O(D^{n-1})$$

which implies that $\deg(X_{\mathcal{A}}) = \dfrac{\mathrm{Vol}_{\mathbb{R}^n}(Q)}{n!}$. For the height : set

$$I_D^{\mathbb{Z}} := I(X_{\mathcal{A}}) \cap \mathbb{Z}[x_0, \ldots, x_N]_D$$

which is a lattice of $I_D^{\mathbb{R}} := I_D^{\mathbb{Z}} \otimes \mathbb{R}$. We can compute the *arithmetic Hilbert function* of $X_{\mathcal{A}}$, which is defined as

$$\mathcal{H}_{\text{arith}}(X_{\mathcal{A}}; D) := \mathrm{Vol}\big(I_D^{\mathbb{R}} / I_D^{\mathbb{Z}}\big)$$

By the "theorem of arithmetic amplitude" of [Gillet-Soulé 93] and

[Randriam 01] we can read the height from the asymptotics of this function

$$\mathcal{H}_{\text{arith}}(X_{\mathcal{A}}; D) = \frac{h(X_{\mathcal{A}})}{(n+1)!}\, D^{n+1} + o(D^{n+1})$$

$\square$

The "arithmetic" Bézout theorem

Let $X \subset \mathbb{P}^N$ be equidimensional and $f = \sum_a f_a \, x^a \in \mathbb{Z}[x_0, \ldots, x_N]$ a homogeneous polynomial ; then

$$h\big(X \cap Z(f)\big) \quad \leq \quad h(X) \, \deg(f) + \deg(X) \, h_1(f)$$

where $h_1(f) := \log \big(\sum_a |f_a| \big)$ is the height associated with the ℓ^1-norm [Philippon 86]

For homogeneous polynomials $F_1, \ldots, F_N \in \mathbb{Z}[x_0, \ldots, x_N]$ of degree D, this implies that

$$h\big(Z(F_1, \ldots, F_N)\big) \quad \leq \quad D^{N-1} \sum_{i=1}^{N} h_1(F_i)$$

Thm. ([Bost-Gillet-Soulé 94], [Philippon 95])
Let $X, Y \subset \mathbb{P}^N$ be (any) varieties ; then

$$h(X \cap Y) \quad \leq \quad h(X) \, \deg(Y) + \deg(X) \, h(Y) + (N+1) \, \log(N+1)$$

The "arithmetic" Bernstein-Kushnirenko theorem

Thm. ([S. 02] based on work of [Maillot 97])
Let

$$F_1, \dots, F_N \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_N^{\pm 1}]$$

be Laurent polynomials, and let $Z_0 \subset \mathbb{T}^N$ be the set of isolated points of the equation system

$$F_1 = 0, \quad \dots, \quad F_N = 0$$

Let $\quad Q_0 \subset \mathbb{R}^N \quad$ be an arbitrary convex polytope ; then

$$h\big(\varphi_{Q_0}(Z)\big) \quad \leq \quad \sum_{i=1}^{N} \mathrm{MV}(Q_0, \dots, Q_{i-1}, Q_{i+1}, \dots, Q_N)\, h_1(F_i)$$

The inclusion $i_N : \mathbb{T}^N \hookrightarrow \mathbb{P}^N$ corresponds to the standard polytope $S := \mathrm{Conv}(0, e_1, \dots, e_N)$ Hence in the example this gives

$$\begin{aligned} h\big(\varphi_S(Z)\big) \; &\leq \; \sum_{i=1}^{N} \mathrm{MV}(S, Q_1, \dots, Q_{i-1}, Q_{i+1}, \dots, Q_N)\, h_1(f_i) \\ &= \; (1 + D + \cdots + D^{n-1})\, \log(H+1) \end{aligned}$$

while in fact

$$h(Z) = (1 + D + \cdots + D^{n-1})\, \log(H)$$

On the other hand, set

$$Q_0 := \mathrm{Conv}\big(0, e_1, e_2 - D\, e_1, e_3 - D\, e_2, \ldots, e_N - D\, e_{N-1}\big) \quad \subset \mathbb{R}^N$$

then $\quad \mathrm{MV}(Q_0, Q_1, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_N) \; = \; 1 \quad$ for all i
and so the previous thm gives

$$\begin{aligned} h\big(\varphi_{Q_0}(Z)\big) \; &\leq \; \sum_{i=1}^{N} \mathrm{MV}(Q_0, Q_1, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_N)\, h_1(f_i) \\ &= \; (N+1)\, \log(H+1) \end{aligned}$$

In fact

$$\varphi_{Q_0}(Z_0) = (1 : H : \cdots : H)$$

which shows that $\quad h_{Q_0}(Z_0) = \log(H)$

Some applications

- This gives an <u>a priori estimate</u> for the size of the output

  $\Rightarrow$ certificates and precises the application of modular methods in polynomial equation solving (as e.g. in the Magma package `Kronecker` [Lecerf 99])

- Sometimes this allows to <u>compress the output</u> (by an appropriate choice of Q_0)

- Estimates for the height of the polynomials in <u>Hilbert's Nullstellensatz</u> [Berenstein-Yger96], [Krick-Pardo-S.01]