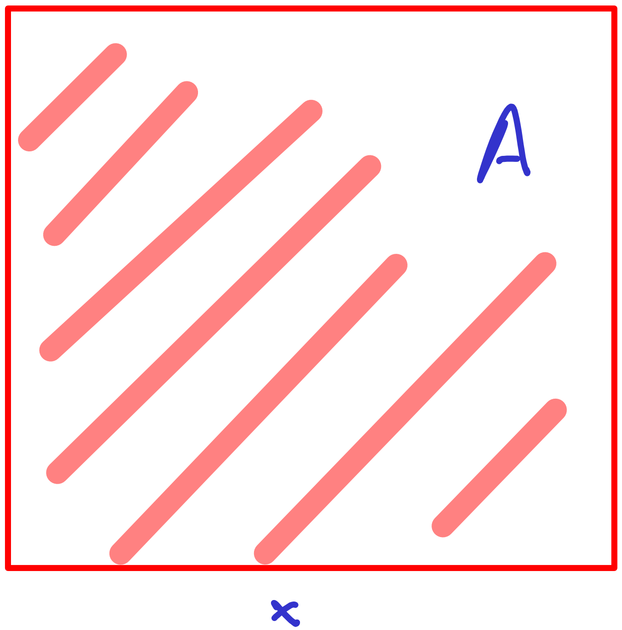
SYSTEMS OF POLYNOMIAL EQUATIONS

MARTÍN SOMBRA (ICREA & U. BARCELONA)
http://atlas.mat.ub.es/personals/sombra

UNLP, 15/12/2016

FROM A BABYLONIAN CLAY TABLET (~1800 BC)

PB: ADD THE AREA AND TWO-THIRD OF A SQUARE TO OBTAIN 0:35.
WHICH IS THE SIDE OF THE SQUARE?



$$x^2 + \frac{2}{3}x = \frac{35}{60}$$

Sol: Take 1. Two-thirds of 1 is 0:40. Half of this, 0:20, you multiply by 0:20 and it 0:6:40, you add to 0:35 and the result 0:41:40, has 0:50 as it square root. The 0:20 which you have multiplied by itself, you substract from 0:50, and 0:30 is the side of the square.

D. Burton. The history of mathematics, 1997.

In other words:

$$x = \sqrt{\left(\frac{0{:}40}{2}\right)^2 + 0{:}35} - \frac{0{:}40}{2} = \cdots = 0{:}30 = \frac{30}{60} = \frac{1}{2}$$

THE FUNDAMENTAL THEOREM OF ALGEBRA

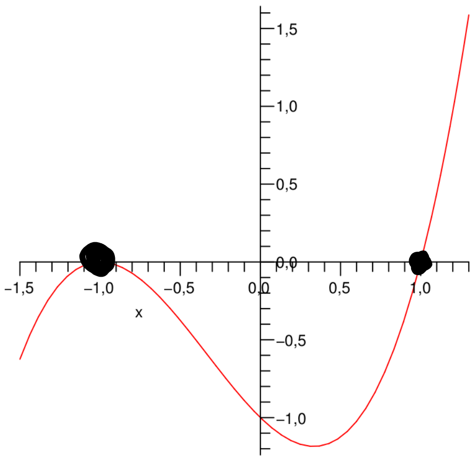
THM (D'ALEMBERT 1746 - GAUSS 1798)

LET $f \in \mathbb{C}[x]$. THEN $f(x) = 0$ HAS $\deg(f)$ SOLUTIONS

i.e. $\mathbb{C}$ IS "ALGEBRAICALLY CLOSED"

Ex: $f = x^3 + x^2 - x - 1$

$$V(f) = \{ x \in \mathbb{C} \mid f(x) = 0 \} = \{ \pm 1 \}$$



BÉZOUT'S THEOREM (1764)

Let $f_1, \dots, f_n \in \mathbb{C}[x_1, \dots, x_n]$ st

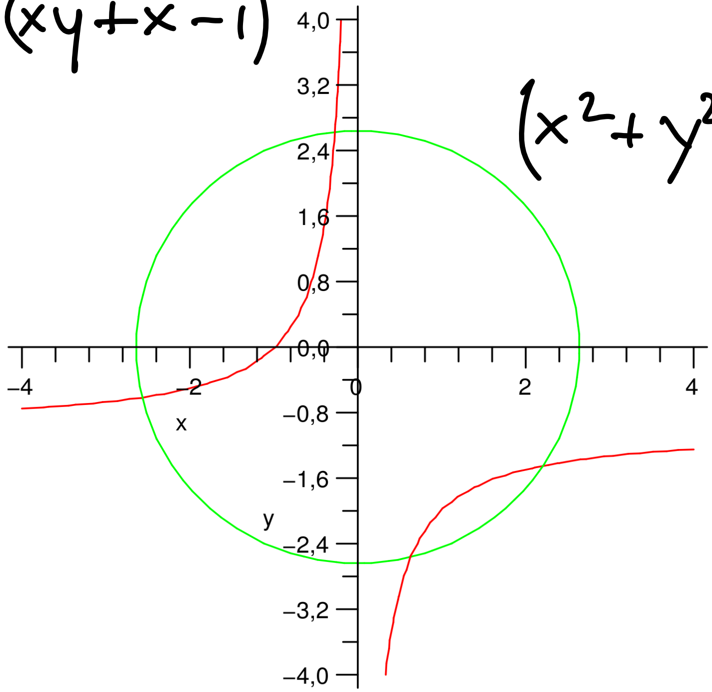
$$f_1(x) = \dots = f_n = 0$$

has finite solutions

Then it has $\leq \prod_{i=1}^{n} \deg(f_i)$ solutions

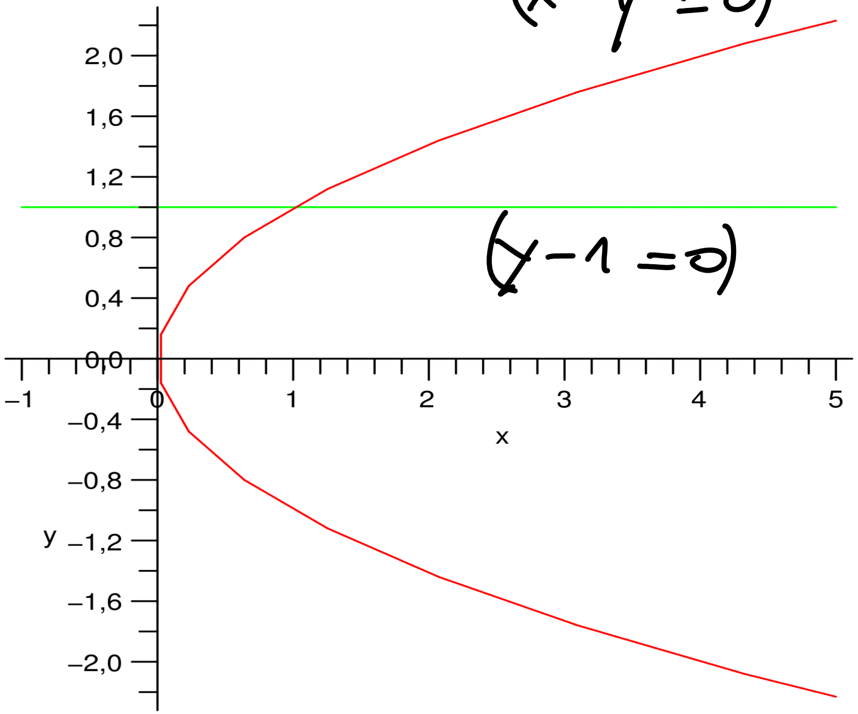
$(xy+x-1)$

$(x^2+y^2-7=0)$



$(x-y^2=0)$

$(y-1=0)$



Bézout Theorem on $\mathbb{P}^n$

Given homogeneous $f_1, \dots, f_n \in \mathbb{C}[x_1, \dots, x_n]$

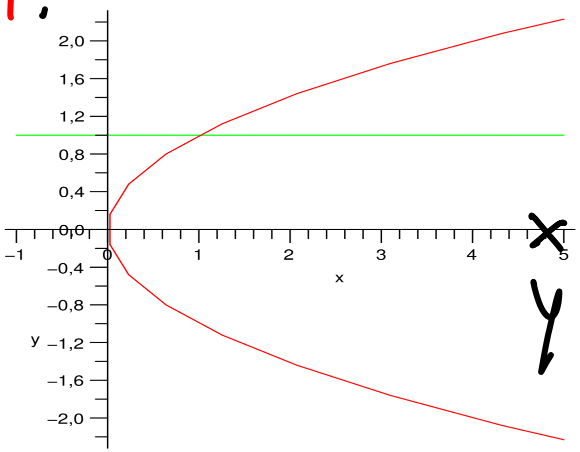
set $V(f_1, \dots, f_n) = \{ \underline{x} \in \mathbb{P}^n \mid f_1(\underline{x}) = \dots = f_n(\underline{x}) = 0 \}$

I: If $\# V(f_1, \dots, f_n) < \infty$ then

$$\# V(f_1, \dots, f_n) = \prod_{i=1}^{n} \deg(f_i)$$

Example (cont.): $(z : x : y) \in \mathbb{P}^2$

In $z = 1$:



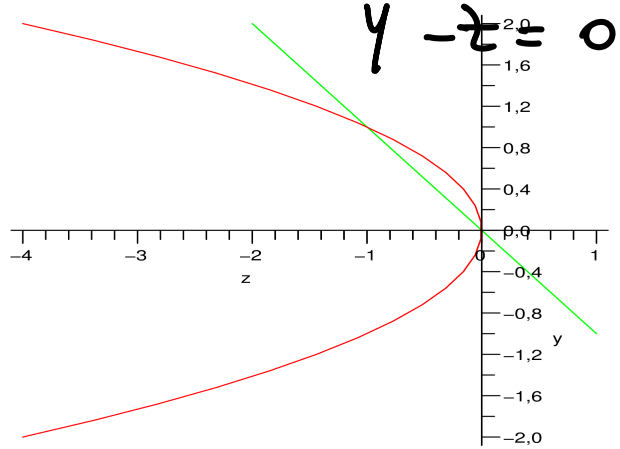
$x - y^2 = 0$

$y - 1 = 0$

In $x = 1$

$z - y^2 = 0$

$y - z = 0$

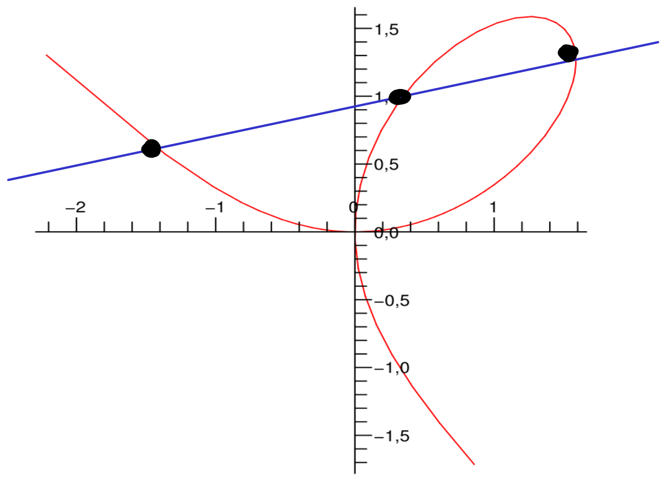


DEGREE OF VARIETIES

Let $X^r \subset \mathbb{P}^n$ irreducible variety

Degree of X: $\deg(X) = \# X \cap E \in \mathbb{N}^*$

for E^{n-r} generic linear space



- $r=0$: $\deg(X) = \# X$
- X linear: $\deg(X) = 1$
- $r = n-1$: $X = V(f)$ with $f \in \mathbb{C}[x_0, \dots, x_n]$ irreducible

$$\deg(X) = \deg(f)$$

HILBERT FUNCTION OF IDEALS

Let $I \subset \mathbb{C}[x_0,\dots,x_n]$ homogeneous ideal

$$\mathbb{C}[x_0,\dots,x_n]/I = \bigoplus_{d\geq 0} \left(\mathbb{C}[\underline{x}]/I\right)_d$$

The Hilbert function of I is

$$H_I : \mathbb{N} \to \mathbb{N} \qquad d \mapsto \dim_{\mathbb{C}} \left(\mathbb{C}[\underline{x}]/I\right)_d$$

$\underline{\text{T}}$ (Hilbert 1893) $\exists\, P_I \in \mathbb{Q}[t]$ and $d_0 \geq 0$ st

$$H_I(d) = P_I(d) \quad \text{for } d \geq d_0$$

Moreover $\deg(P_I) = \dim(V(I))$

If $I = I(X)$: $\quad P_I = \deg(X) \dfrac{t^r}{r!} + O(t^{r-1})$

Bézout Theorem III

I Let $X^r \subset \mathbb{P}^n$ irreducible variety and $H \subset \mathbb{P}^n$ hypersurface st $X \not\subset H$. Then

$$\deg(X \cap H) = \deg(X)\deg(H)$$

Cor (classical Bézout)

Let $H_1, \dots, H_n \subset \mathbb{P}^n$ st $\# H_1 \cap \dots \cap H_n < \infty$. Then

$$\#(H_1 \cap \dots \cap H_n) = \prod_{i=1}^{n} \deg H_i$$

BACK TO SYSTEMS OF EQUATIONS

$$1+x-y+2xy-x^2y+xy^2 = 2-x+y-7xy+x^2y+xy^2=0$$

Bezout predicts $3\cdot 3 \leq 9$ solutions in $\mathbb{C}^2$

There are 5.

Why this discrepancy ?

Possible answer:

There are also two double roots $(0:1:0)$, $(0:0:1)$ at the "line at infinity" $(z=0)$

Alternative Answer

Do not apply classical Bézout but the "intermediate version"

$$\varphi : \mathbb{C}^2 \hookrightarrow \mathbb{P}^5 \qquad (x,y) \mapsto (1 : x : y : xy : x^2y : xy^2)$$

$$X := \overline{\operatorname{im}(\varphi)} \subset \mathbb{P}^5 \qquad \deg(X) = 5$$

$$\Rightarrow V(f,g) = \varphi^{-1}(X \cap V(l_1, l_2))$$

with
$$l_1 = y_0 + y_1 - y_2 + 2y_3 - y_4 + y_5$$
$$l_2 = 2y_0 - y_1 + y_2 - 7y_3 + y_4 + y_5$$

$$\Rightarrow \# V(f,g) \leq \deg(X) = 5$$

no "lost points"!

THE DEGREE OF A TORIC VARIETY

Let $a_0, a_1, \dots, a_N \in \mathbb{Z}^n$ with $a_0 = 0$ and $\sum_j \mathbb{Z} a_j = \mathbb{Z}^n$

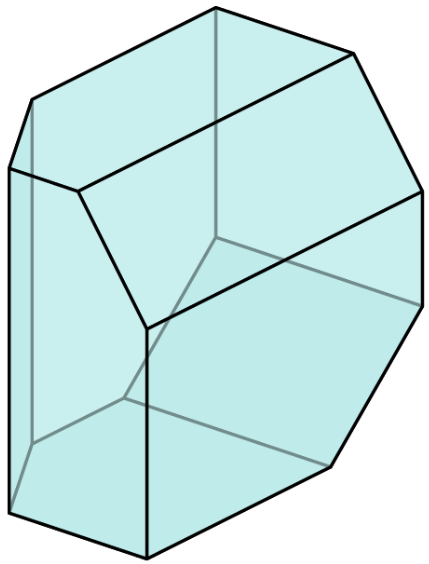
Consider the monomial map

$$(\mathbb{C}^\times)^n \xrightarrow{\varphi} \mathbb{P}^N \qquad \underline{x} \mapsto (\underline{x}^{a_0} : \underline{x}^{a_1} : \dots : x^{a_N})$$

the (projective) toric variety $X = \overline{\mathrm{im}(\varphi)} \subset \mathbb{P}^N$

and the lattice polytope

$$\Delta = \mathrm{conv}(a_0, a_1, \dots, a_N) \subset \mathbb{R}^n$$

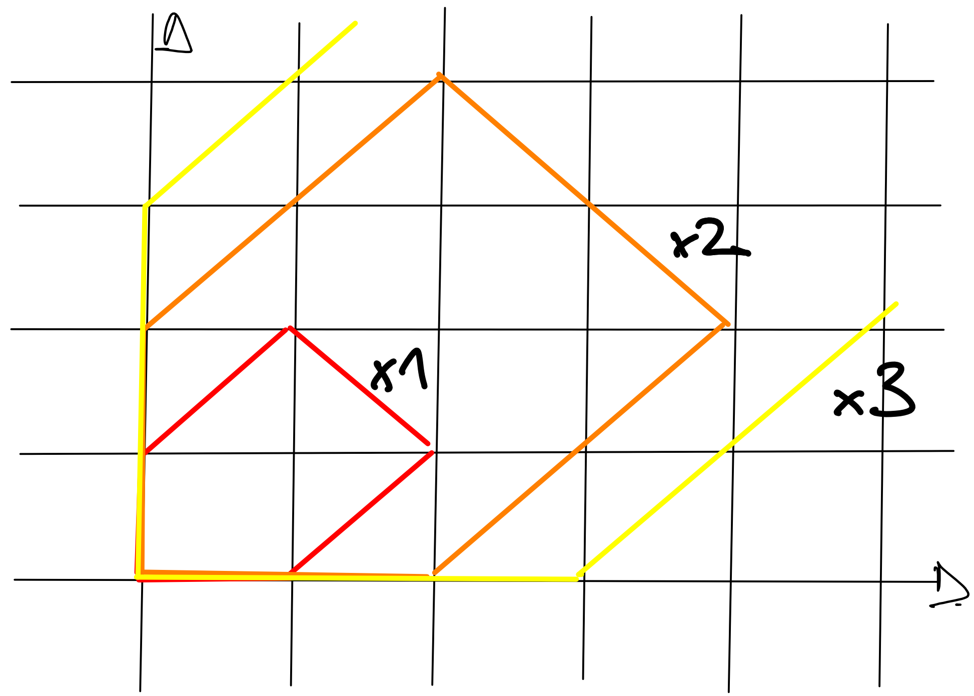


$\underline{\mathrm{T}}$ (Teissier 1979) $\deg(X) = n!\, \mathrm{vol}(\Delta)$

Pf: Consider the morphism of algebras

$$\mathbb{C}[y_0, \dots, y_N]/I(X) \overset{\varphi^*}{\hookrightarrow} \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \qquad y_j \mapsto \underline{x}^{a_j}$$

$$H_{I(X)}(d) = \# \sum_{\substack{\lambda_j \in \mathbb{N} \\ \sum_j \lambda_j = d}} \lambda_j a_j \simeq \mathrm{vol}(\Delta)\, d^n$$



$\Rightarrow \quad \deg(X) = n!\, \mathrm{vol}(\Delta)$

Cor (Bernstein-Kushnirenko thm, 1975)

Let $\Delta \subset \mathbb{R}^n$ lattice polytope and $f_1, \dots, f_n \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ st $N(f_i) \subset \Delta$ $\forall i$ and $\# V(f_1, \dots, f_n) < \infty$. Then

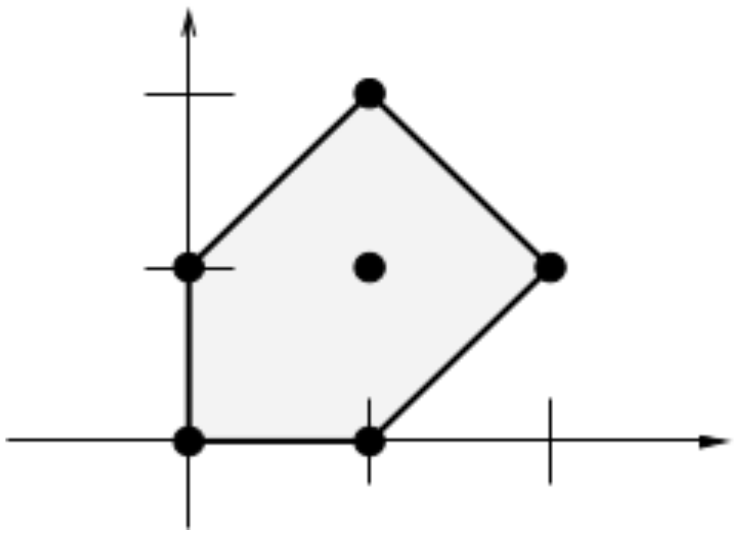
$$\# V(f_1, \dots, f_n) \leq n! \operatorname{vol}(\Delta)$$

Example (cont)

$f = 1 + x - y + 2xy - x^2y + xy^2$

$g = 2 - x + y - 7xy + x^2y + xy^2$

$\Rightarrow \# V(f, g) \leq 5$



A FURTHER EXAMPLE

$f = (s-1) + (s-1)^2 x - 3 s x^2 \qquad g = -7(s-1) + (s-1)^2 x + 3 s x^2$

$f = g = 0$ has two solutions: $(4,1)$ simple; $\left(-\frac{1}{2}, -2\right)$ double

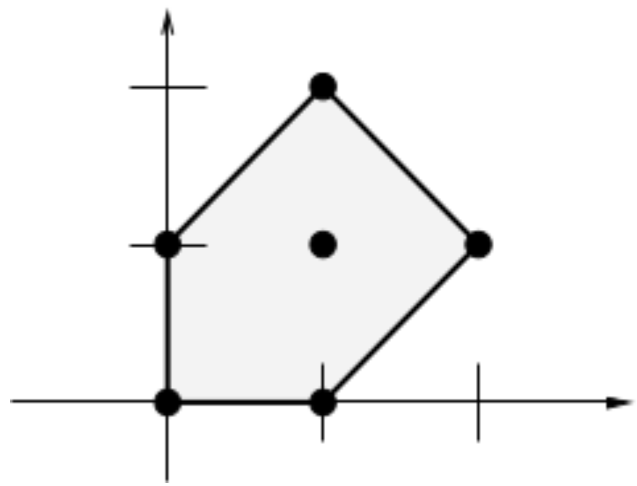
$\Rightarrow \# V(f,g) = 3$

Bounds:

- Bezout: $\# V \leq \deg(f)\deg(g) = 9$
- Bihomogeneous Bezout:

  $\# V \leq \deg_x(f)\deg_y(g) + \deg_y(f)\deg_x(g)$

  $= 2\cdot 2 + 2 \cdot 2 = 8$
- BK: $\# V \leq 2!\,\mathrm{vol}(\Delta) = 5$



v-ADIC NEWTON POLYTOPES

Let $f = \sum_{j=0}^{N} \alpha_j(s)\, \underline{x}^{a_j} \in \mathbb{C}(s)[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$

For $v \in \mathbb{P}^1$, the v-adic Newton polytope is

$$N_v(f) = \operatorname{conv}\left((a_j, -\operatorname{ord}_v(\alpha_j))\right)_j \subset \mathbb{R}^{n+1}$$

with $\operatorname{ord}_v(\alpha_j)$ order of vanishing of α_j at v

Projects onto

$$N(f) = \operatorname{conv}(a_0, \dots, a_N) \subset \mathbb{R}^n$$

The v-adic roof function is

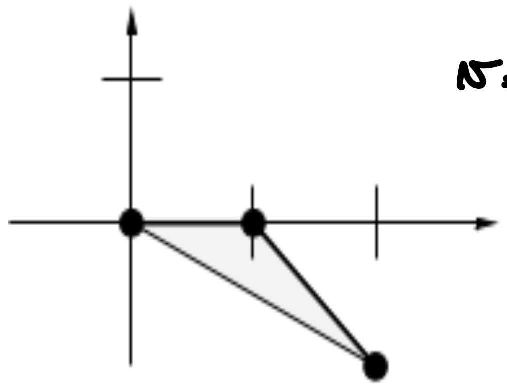
$$\vartheta_v : N(f) \to \mathbb{R}$$

parametrizing the upper envelope of $N_v(f)$

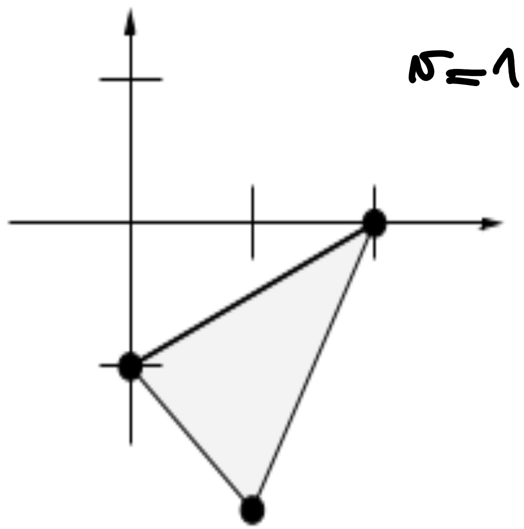
Example (cont.)

$$f = (s-1) + (s-1)^2 x - 3 s x^2$$

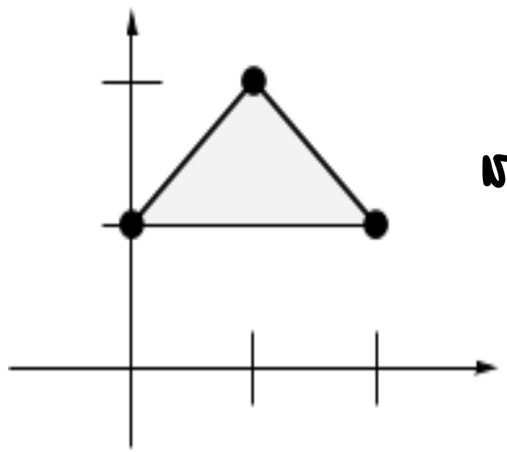
$$g = -7(s-1) + (s-1)^2 x + 3 s x^2$$



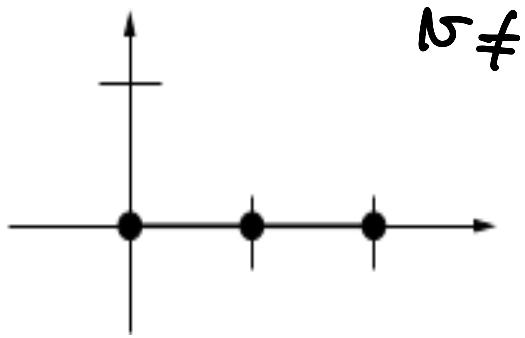
$s=0$



$s=1$



$s=\infty$



$s \neq 0, 1, \infty$

A REFINEMENT OF THE BK THEOREM

I (Philippon - S 2008)

$f_0, \dots, f_n \in \mathbb{C}[s][x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ st $\# V(\underline{f}) \subset \mathbb{C} \times (\mathbb{C}^\times)^n < \infty$

Let $\Delta \subset \mathbb{R}^n$ and $\vartheta_v : \Delta \to \mathbb{R}$ $(v \in \mathbb{P}^1)$

st $N(f_i) \subset \Delta$ and $\vartheta_{i,v} \leq \vartheta_v$

with $\vartheta_{i,v} : N(f_i) \to \mathbb{R}$ v-adic roof of f

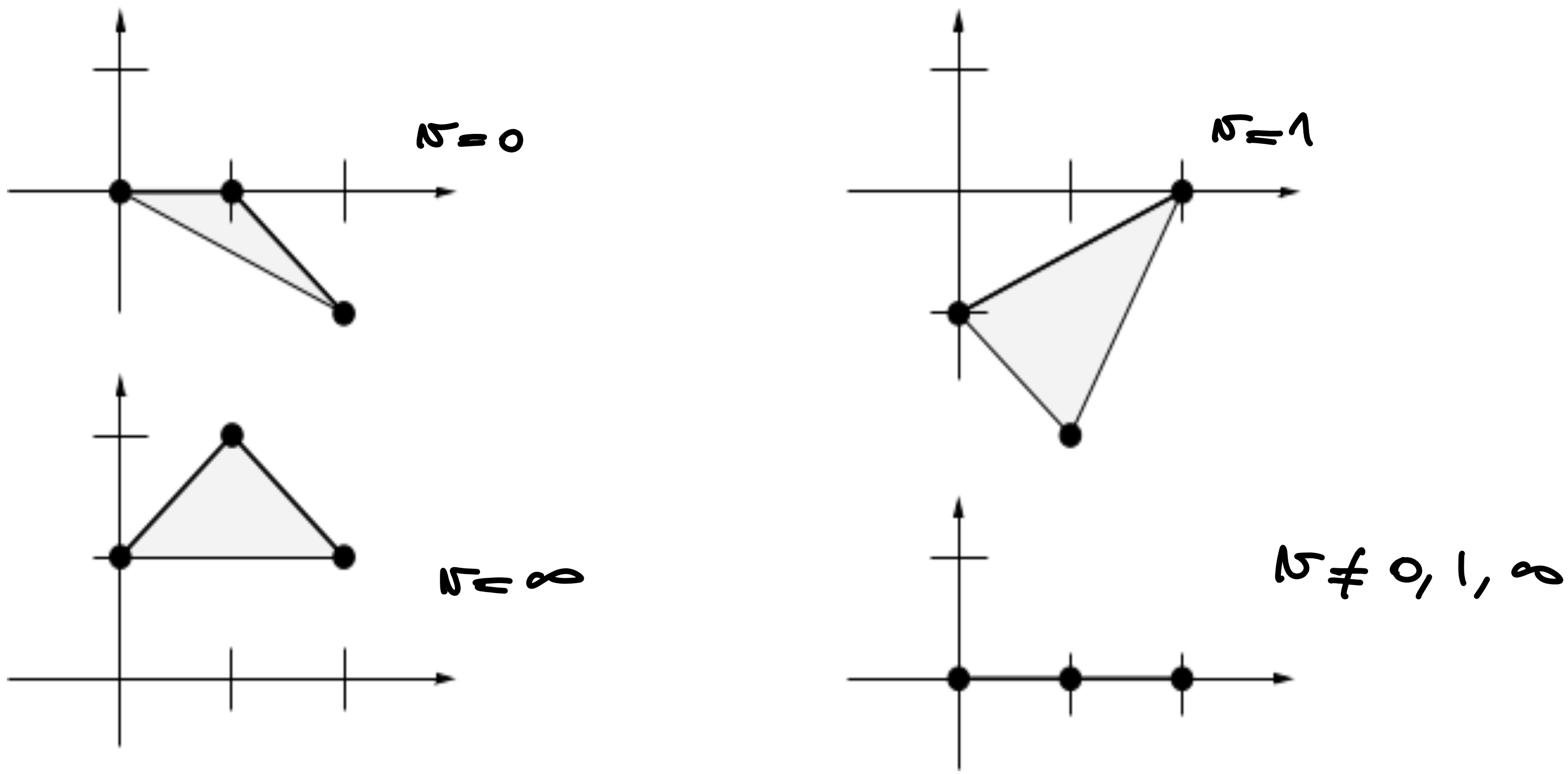
Then

$$\# V(f_0, \dots, f_n) \leq (n+1)! \sum_{v \in \mathbb{P}^1} \int_\Delta \vartheta_v(u)\, du$$

Example (cont.)

$f = (s-1) + (s-1)^2 x - 3 s x^2 \qquad g = -7(s-1) + (s-1)^2 x + 3 s x^2$

$N(f) = N(g) = [0,2]$



$$\# V(f,g) \leqslant 2! \left(\int_0^2 \vartheta_0 \, du + \int_0^2 \vartheta_1 \, du + \int_0^2 \vartheta_\infty \, du \right) = 2\left(-\frac{1}{2} - 1 + 3\right) = 3$$

PROOF

$f = (s-1) + (s-1)^2 x - 3 s x^2 \qquad g = -7(s-1) + (s-1)^2 x + 3 s x^2$

Let

$\varphi : \mathbb{C} \times \mathbb{C}^\times \to \mathbb{P}^1 \times \mathbb{P}^2 \quad (s,t) \mapsto \left((1:s), (s-1 : (s-1)^2 t : s t^2)\right)$

$X := \overline{\mathrm{im}(\varphi)} \subset \mathbb{P}^1 \times \mathbb{P}^2$

The system $f = g = 0$ on $\mathbb{C} \times \mathbb{C}^\times$ is equiv to

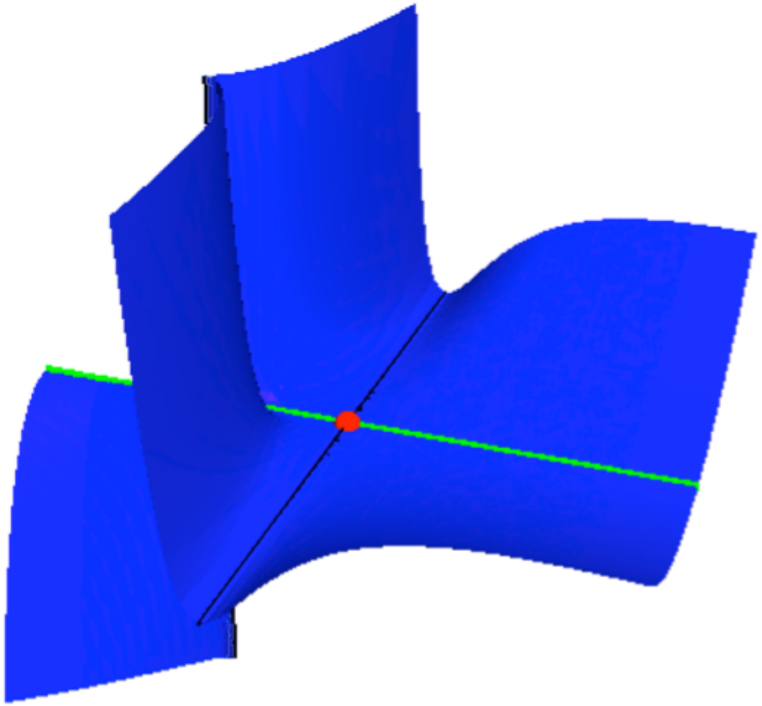
$y_0 + y_1 - 3 y_2 = -7 y_0 + y_1 + 3 y_2 = 0$ on X

Can show $\deg_{\pi_2^* \mathcal{O}(1)}(X) = 3$

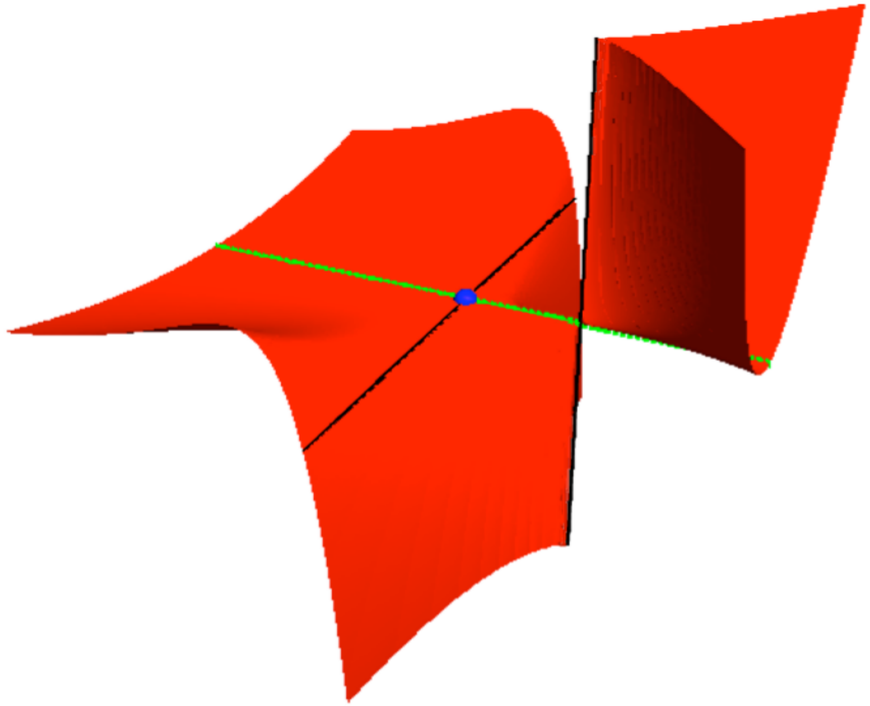
hence $\# V(f,g) \leq 3$

Pics of X

In $(s_1 \neq 0,\ x_0 \neq 0)$



In $(s_0 \neq 0,\ x_0 \neq 0)$



X is a toric curve over $\mathbb{P}^1$

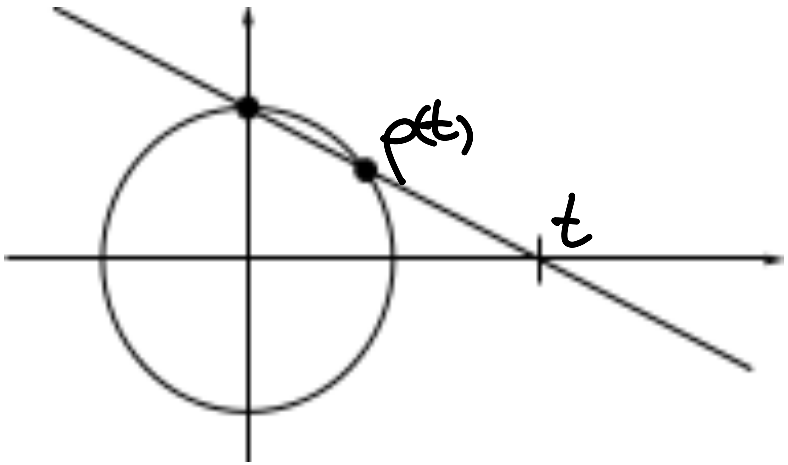
In particular, it is a T-variety of complexity 1

THE NEWTON POLYGON OF A RATIONAL CURVE

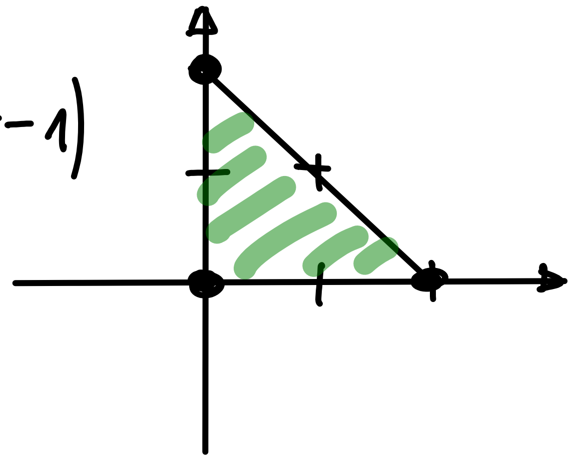
Let $\rho : \mathbb{C} \dashrightarrow \mathbb{C}^2$ rtl map $\Rightarrow C = \overline{\mathrm{im}(\rho)}$ plane curve

Pb: Compute the Newton polygone of an equation of C

Ex $\rho(t) = \left(\dfrac{2t}{t^2+1}, \dfrac{t^2-1}{t^2+1} \right)$ $\qquad C = V(x^2+y^2-1)$



$N(x^2+y^2-1)$



For $v \in \mathbb{P}^1$ set

$$\mathrm{ord}_v(\rho) = (\mathrm{ord}_v(\rho_1), \mathrm{ord}_v(\rho_2)) \in \mathbb{Z}^2$$

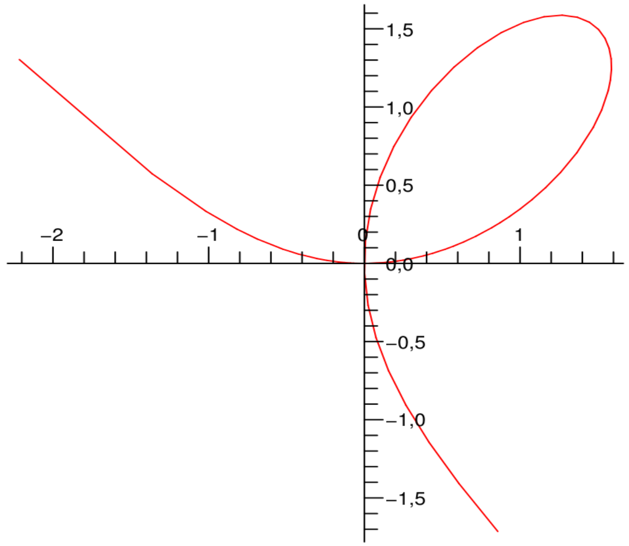
order of zero / pole of ρ at v

- $\mathrm{ord}_v(\rho) = (0,0)$ $\forall v$
- $\sum_{v \in \mathbb{P}^1} \mathrm{ord}_v(\rho) = (0,0)$

Th (Dickenstein-Feichtner-Sturmfels-Tevelev 2007)
$N(E_c)$ constructed by rotating $-90°$ the $\mathrm{ord}_v(\rho)$ and concatenating counterclockwise.

$\exists$ alternative pf using the PS theorem (D'Andrea, S. 2008)

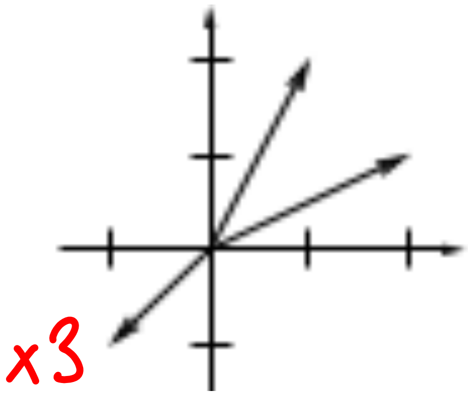
The Descartes' Folium



$$\rho : t \mapsto \left(\frac{3t^2}{t^3+1} \, , \, \frac{3t}{t^3+1} \right)$$

- $\mathrm{ord}_0(\rho) = (2,1)$
- $\mathrm{ord}_\infty(\rho) = (1,2)$
- $\mathrm{ord}_\omega = (-1,-1)$ for $\omega = 1, \; \frac{-1}{2} \pm \frac{\sqrt{3}}{2}$

Polygon of C and inner normals



$$E_c = x^3 + y^3 - 3xy$$

THE BIT SIZE OF THE SOLUTION SET

Let $f_1, \dots, f_n \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ st $\# V(f_1, \dots, f_n) < \infty$

Pb: Bound the bit size (or height) of $V(f_1, \dots, f_n)$

HEIGHT OF VARIETIES

$\frac{1}{10^{64}}$ is **BIG**

$h(\alpha/\beta) := \log \max |\alpha|, |\beta|$ height ($\alpha, \beta \in \mathbb{Z}$ coprime)

Extends to subvarieties $X^r \subset \mathbb{P}^n$

- $r = 0$ Weil height
- $r = n-1$ $X = V(f)$ $h(X) = \int_{(S^1)^n} \log |f| \, d\mathrm{Haar}$

Mahler measure

- Arithmetic Bézout:

$$h(X \cap V(f)) \leq \deg(f)\, h(X) + \ell(f) \deg(X)$$

$$\ell\left(\sum \alpha_a x^a\right) = \log\left(\sum_a |\alpha_a|\right) \quad \text{length}$$

AN ARITHMETIC BK THEOREM

T (Martínez – S, 2016)

$f_1, \dots, f_n \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ st $\# V(f_1, \dots, f_n) < \infty$

Let $\Delta_i = N(f_i) \subset \mathbb{R}^n$ Newton polytope

$S \subset \mathbb{R}^n$ standard simplex

$$h(V(f_1, \dots, f_n)) \leqslant \sum_{i=1}^{n} MV(S, \Delta_1, \dots, \Delta_n) \, \ell(f_i)$$

"mixed volume" function

Ex: $d, \alpha \geqslant 1$ $\quad f_1 = x_1 - \alpha, \; f_2 = x_2 - \alpha x_1^d, \dots, \; f_n = x_n - \alpha x_{n-1}^d$

$$V(\underline{f}) = \{(\alpha, \alpha^{d+1}, \dots, \alpha^{d^{n-1} + d^{n-2} + \dots + 1})\}$$

$$h(V(\underline{f})) = (d^{n-1} + \dots + 1) \log(\alpha)$$

$$\leqslant (d^{n-1} + \dots + 1) \log(\alpha + 1)$$

upper bound from the MS theorem

"Proof"

An application of the toric dictionary in Arakelov geometry (Burgos – Philippon – S, 2014)

$X^n/\mathbb{Q}$ toric variety

$\overline{D}$ semipositive toric metrized Cartier divisor

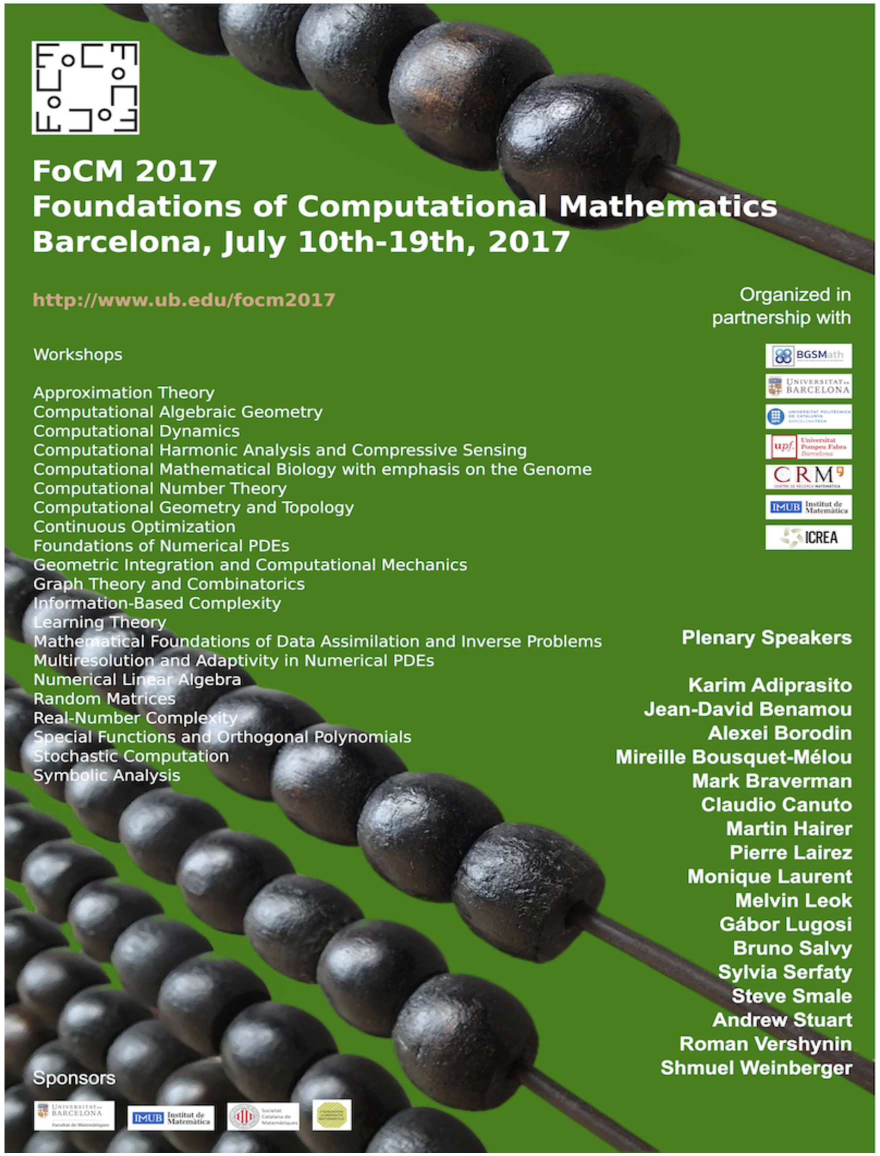
$$h_{\overline{D}}(X) = (n+1)! \int_{\Delta_D} \vartheta_{\overline{D}} \, dx_1 \cdots dx_n$$

polytope

"roof" function

together with the arithmetic Bézout theorem

THANK YOU!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