The mean height of the solution set of a system of polynomial equations

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Systems of polynomial equations

For $n \geqslant 0$ and $0 \leqslant r \leqslant n$ let

$$f_i \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}], \quad i = r+1, \ldots, n$$

Consider its *zero set* in $(\overline{K}^{\times})^n$

$$Z := (f_{r+1} = \cdots = f_n = 0)$$

How **large** is Z?

- K any: *degree*
- K Diophantine: *height*

An example

Let

$$f = 1 + x + y$$

and for $\boldsymbol{\omega} = (\omega_1, \omega_2) \in (\overline{\mathbb{Q}}^\times)^2$ consider the *twist*

$$\boldsymbol{\omega}^* f(x, y) := f(\omega_1 x, \omega_2 y) = 1 + \omega_1 x + \omega_2 y$$

If $\boldsymbol{\omega} \in \mu_\infty^2$ then for all $v \in M_{\mathbb{Q}}$ the coefficients of $\boldsymbol{\omega}^* f$ have the same v-adic absolute value as those of f:

$$|\omega_1|_v = |\omega_2|_v = 1$$

An example (cont.)

However, the Weil height of the corresponding zero set

$$(f = \boldsymbol{\omega}^* f = 0) = (1 + x + y = 1 + \omega_1 x + \omega_2 y = 0)$$

depends on $\boldsymbol{\omega}$

For $\boldsymbol{\xi} \in (\overline{\mathbb{Q}}^\times)^2$

$$\mathrm{h}_{Weil}(\boldsymbol{\xi}) = \frac{1}{\mathrm{Gal}(\boldsymbol{\xi})} \sum_{v \in M_{\mathbb{Q}}} \sum_{\boldsymbol{\eta} \in \mathrm{Gal}(\boldsymbol{\xi})} \log \max(1, |\eta_1|_v, |\eta_2|_v)$$

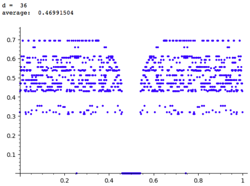
$\approx$ the bit complexity of $\boldsymbol{\xi}$

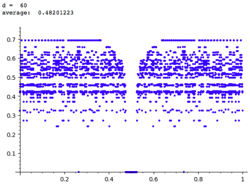
- if $\boldsymbol{\omega} = (\zeta_3, \zeta_3^2)$ then $Z = (\zeta_3, \zeta_3^2)$ and $h = 0$
- if $\boldsymbol{\omega}' = (-1, \mathrm{i})$ then $Z' = (-\mathrm{i}, \mathrm{i} - 1)$ and $h' = \frac{\log(2)}{2}$

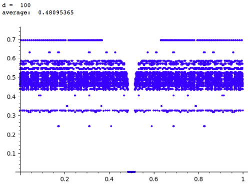
What about the *mean* of these heights?

$$\lim_{d\to+\infty} \frac{1}{\mu_d^2} \sum_{\omega\in\mu_d^2} \mathrm{h}(f = \omega^* f = 0) = \frac{2\,\zeta(3)}{3\,\zeta(2)} = 0.487175\ldots$$

Indeed most of them concentrate around this value







Degree of cycles of toric varieties

- $f_i \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, $i = r+1, \dots, n$

  $\rightsquigarrow \mathrm{NP}(f_i) \subset \mathbb{R}^n$ Newton polytope and $Z(\boldsymbol{f})$ r-cycle of $\mathbb{G}_{\mathrm{m}}^n$

  $\mathbb{G}_{\mathrm{m}}^n$ the n-dimensional *algebraic torus*

- X toric variety with torus $\mathbb{G}_{\mathrm{m}}^n$
- D_i nef toric divisor on X, $i = 1, \dots, r$

  $\rightsquigarrow \Delta_i \subset \mathbb{R}^n$ lattice polytope

Theorem 1 (Bernstein 1975)

If $\boldsymbol{f}$ is *generic* then

$$\deg_{D_1,\dots,D_r}(Z(\boldsymbol{f})) = \mathrm{MV}(\Delta_1, \dots, \Delta_r, \mathrm{NP}(f_{r+1}), \dots, \mathrm{NP}(f_n))$$

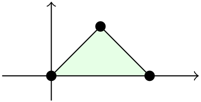
MV the *mixed volume*

Example

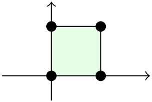
For $n = 2$ and $r = 0$ set

$$f_1 = 1 + x_1 + x_1^2 + x_1 x_2 \quad \text{and} \quad f_2 = 1 + x_1 + x_2 + x_1 x_2$$

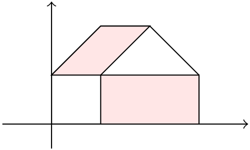
Hence



$\mathrm{NP}(f_1)$



$\mathrm{NP}(f_2)$



$\mathrm{NP}(f_1) + \mathrm{NP}(f_2)$

and so

$$\deg(Z(f_1, f_2)) = \mathrm{MV}(\Delta_1, \Delta_2) = 3$$

"And the reader is likely to discover a new and interesting question by just asking for the arithmetic analogue of her favorite statement in classical algebraic geometry."

— Christophe Soulé

Metrics on toric varieties and roof functions

Set $K = \mathbb{Q}$ and let X be a toric variety with torus $\mathbb{G}_{\mathrm{m}}^{n}$, and let

$$\overline{D} = (D, (\|\cdot\|_v)_{v \in M_{\mathbb{Q}}})$$

a *semipositive toric metrized divisor* on X

D nef toric divisor on X

$\|\cdot\|_v$ semipositive and rotation invariant metric on the analytic line bundle $\mathcal{O}(D)_v^{\mathrm{an}}$

Recall that $D \rightsquigarrow \Delta$

Now we can also construct an adelic family of continuous and concave functions on the polytope

$$\overline{D} \rightsquigarrow (\vartheta_v : \Delta \to \mathbb{R})_{v \in M_{\mathbb{Q}}}$$

ϑ_v is the v-adic *roof function* of $\overline{D}$ Burgos, Philippon and S. 2014

Height of toric varieties

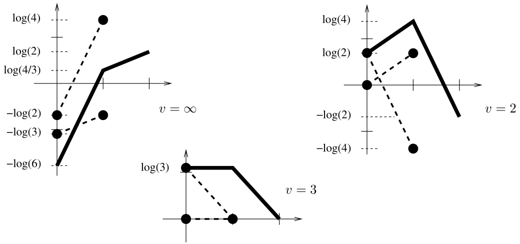
Theorem 2 (Burgos, Philippon and S. 2014)

Let $\overline{D}_i$ SP toric metrized divisor on X, $i = 0, \dots, n$. Then

$$h_{\overline{D}_0,\dots,\overline{D}_n}(X) = \sum_{v \in M_{\mathbb{Q}}} \mathrm{MI}(\vartheta_{0,v}, \dots, \vartheta_{n,v})$$

MI the *mixed integral*

$h_{\overline{D}_0,\dots,\overline{D}_n}(X)$ the *height* of X



and so

$$h_{\overline{D}_0,\overline{D}_1}(\mathbb{P}^1) = (2\ \log 2 - \log 3) + 2\ \log 2 + \log 3 = 4\ \log 2$$

Ronkin functions

Let

$$f = \sum_{\boldsymbol{m} \in \mathbb{Z}^n} \alpha_{\boldsymbol{m}}\, x_1^{m_1} \cdots x_n^{m_n} \in \mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \setminus \{0\}$$

For each v consider the *v-adic Ronkin function* $\rho_v : \mathbb{R}^n \to \mathbb{R}$ defined as

$$\rho_v(\boldsymbol{u}) = \text{mean of } \log |f|_v \text{ on the fiber at } \boldsymbol{u}$$
$$\text{of the } v\text{-adic tropicalization map } (\mathbb{C}_v^\times)^n \to \mathbb{R}^n$$

$$= \begin{cases} \displaystyle\int_{(S^1)^n} \log |(e^{-\boldsymbol{u}})^* f| \, d\text{Haar} & \text{if } v = \infty \\ \min_{\boldsymbol{m}} \langle \boldsymbol{m}, \boldsymbol{u} \rangle - \log |\alpha_{\boldsymbol{m}}|_v & \text{if } v \neq \infty \end{cases}$$

Passare and Rullgard 2004, Gualdi 2017

Height of hypersurfaces

Then consider its *Legendre-Fenchel dual* $\rho_v^\vee : \Delta \to \mathbb{R}$ defined as

$$\rho_v^\vee(\boldsymbol{t}) = \inf_{\boldsymbol{u}\in\mathbb{R}^n} \langle \boldsymbol{t}, \boldsymbol{u}\rangle - \rho_v(\boldsymbol{u})$$

The adelic family of continuous concave functions on the polytope

$$(\rho_v^\vee : \Delta \to \mathbb{R})_{v\in M_{\mathbb{Q}}}$$

is the arithmetic analogue of $\mathrm{NP}(f)$

Theorem 3 (Gualdi 2017)

Let $\overline{D}_i$ SP toric metrized divisor on X, $i = 0, \ldots, n-1$. Then

$$\mathrm{h}_{\overline{D}_0,\ldots,\overline{D}_{n-1}}(Z(f)) = \sum_{v\in M_{\mathbb{Q}}} \mathrm{MI}(\vartheta_{0,v}, \ldots, \vartheta_{n-1,v}, \rho_v^\vee)$$

Limit heights

Conjecture (Gualdi and S.)

- $f_i \in \mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \setminus \{0\}$, $i = r+1, \dots, n$
- $\overline{D}_i$ SP toric metrized divisor on X, $i = 0, \dots, r$
- $\boldsymbol{\omega}_\ell \in (\boldsymbol{\mu}_\infty^n)^{n-r}$, $\ell \geqslant 1$, a strict sequence

strict sequence = eventually escapes any proper algebraic subgroup

Then

$$\lim_{\ell \to +\infty} \mathrm{h}_{\overline{D}_0, \dots, \overline{D}_r}(Z(\boldsymbol{\omega}_\ell^* \boldsymbol{f})) = \sum_{v \in M_{\mathbb{Q}}} \mathrm{MI}(\vartheta_{0,v}, \dots, \vartheta_{r,v}, \rho_{r+1,v}^\vee, \dots, \rho_{n,v}^\vee)$$

$\boldsymbol{\omega}_\ell^* \boldsymbol{f} = (\omega_{\ell,r+1}^* f_{r+1}, \dots, \omega_{\ell,n}^* f_n)$

Limit heights (cont.)

The particular case

$$X = \mathbb{P}^n, \quad \overline{D}_0 = \overline{H}_\infty^{can} \quad \text{and} \quad r = 0$$

H hyperplane at infinity of $\mathbb{P}^n$

would imply that

$$\lim_{\ell \to +\infty} \mathrm{h}_{\mathrm{Weil}}(Z(\boldsymbol{\omega}_\ell^* \boldsymbol{f})) = \sum_{v \in M_{\mathbb{Q}}} \mathrm{MI}(0_\Delta, \rho_{1,v}^\vee, \dots, \rho_{n,v}^\vee)$$

0_Δ the zero function on the standard simplex of $\mathbb{R}^n$

Limit heights (cont.)

Theorem 4 (Gualdi and S.)

Conjecture 1 holds when $n = 2$, $r = 0$ and f_1, f_2 are affine.

Proof (sketch):

From the definition of heights we construct an adelic family of functions

$$F_v : (\mathbb{C}_v^\times)^2 \to \mathbb{R} \cup \{-\infty\}, \quad v \in M_\mathbb{Q}$$

such that

$$\mathrm{h}_{\overline{D}_0}(Z(f_1, \omega_\ell^* f_2)) = \mathrm{h}_{\overline{D}_0, \overline{D}_1}(Z(f_1)) + \sum_v \left(\frac{1}{\# \mathrm{Gal}(\omega_\ell)} \sum_{\eta \in \mathrm{Gal}(\omega_\ell)} F_v(\eta) \right)$$

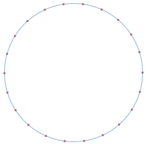
with $\overline{D}_1$ the *Ronkin metrized divisor* of f_2:

$\rightsquigarrow$ the first term coincides with the RHS (Theorem 2)

Logarithmic adelic equidistribution of torsion points

The second term tends to 0 for $\ell \to +\infty$:

$$\lim_{\ell \to +\infty} \sum_v \left(\frac{1}{\# \operatorname{Gal}(\boldsymbol{\omega}_\ell)} \sum_{\boldsymbol{\eta} \in \operatorname{Gal}(\boldsymbol{\omega}_\ell)} F_v(\boldsymbol{\eta}) \right) = \sum_v \int F_v \, d\nu_v = 0$$



Proven using

- v-adic distribution of torsion points
- lower bounds for linear forms in logarithms (Baker)
- lower bounds for p-adic linear forms in roots of unity (Tate-Voloch)

A mixed integral computation

Corollary

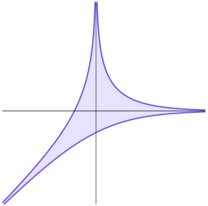
Let $\boldsymbol{\omega}_\ell \in \mu_\infty^2$, $\ell \geqslant 1$, be a strict sequence. Then

$$\lim_{\ell \to +\infty} \mathrm{h}_{\mathrm{Weil}}(Z(1+x+x, 1+\omega_{\ell,1}\, x+\omega_{\ell,2}\, x)) = \mathrm{MI}(0_\Delta, \rho_\infty^\vee, \rho_\infty^\vee) = \frac{2\,\zeta(3)}{3\,\zeta(2)}$$

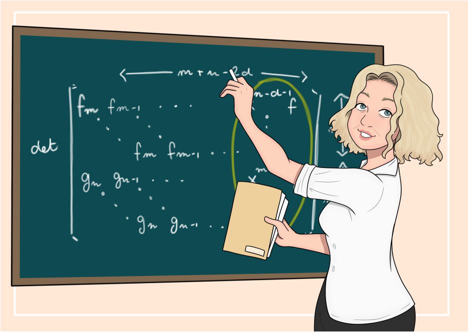
ρ_∞ the Archimedean Ronkin function of $1+x+y$

Indeed, if $v \neq \infty$ then $\mathrm{MI}(0_\Delta, \rho_v^\vee, \rho_v^\vee) = \mathrm{MI}(0_\Delta, 0_\Delta, 0_\Delta) = 0$. Else

$$\mathrm{MI}(0_\Delta, \rho_\infty^\vee, \rho_\infty^\vee) = \frac{-2}{\pi^2} \int_{\mathcal{A}_{1+x+y}} \min(0, u_1, u_2) du_1 du_2 = \frac{4\,\zeta(3)}{\pi^2} = \frac{2\,\zeta(3)}{3\,\zeta(2)}$$



$\mathcal{A}_{1+x+y}$ the *Archimedean amoeba* of $1+x+y$



Happy birthday Teresa!