THE DISTRIBUTION OF GALOIS ORBITS OF SMALL POINTS IN TORIC VARIETIES

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THE WEIL HEIGHT

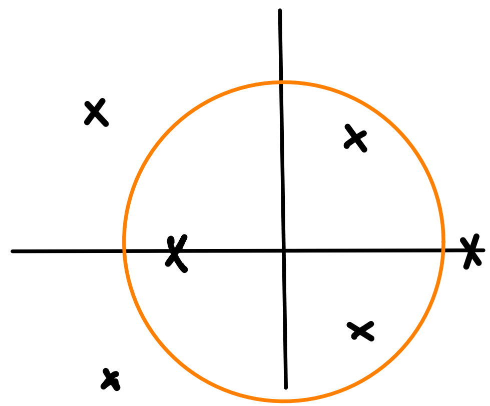
Let $\xi \in \overline{\mathbb{Q}}^\times$ of degree $d \geq 1$

$$P_\xi = \alpha_d x^d + \dots + \alpha_0 = \alpha_d \prod_{\eta \in G_\xi} (x - \eta) \quad \in \mathbb{Z}[x]$$

minimal poly of ξ ← Galois orbit of ξ

The Weil height of ξ is

$$h(\xi) = \frac{1}{d}\left(\sum_{\eta \in G_\xi} \log \max(1, |\eta|) + \log|\alpha_d| \right)$$



- If $\xi = \frac{a}{b} \in \mathbb{Q}^\times$ then $h(\xi) = \log \max(|a|, |b|)$
- $h(\xi) = 0 \iff \xi$ root of 1 (Kronecker)

BILU'S EQUIDISTRIBUTION THM (1997)

THM Let $p_k \in \overline{\mathbb{Q}}^\times$, $k \geq 1$, st

- $\forall p \in \overline{\mathbb{Q}}^\times$, $\#\{k \mid p_k = p\} < \infty$
- $h(p_k) \xrightarrow[k\to\infty]{} 0$

Then $G p_k \to S^1$

I.e. $\forall f \in \mathcal{C}^0(\mathbb{C}^\times)$ with compact support

$$\frac{1}{\# G p_k} \sum_{g \in G p_k} f(g) \xrightarrow[k\to\infty]{} \int f \, d\mu_{S^1}$$

Toric version of Szpiro, Ullmo & Zhang equidistribution (1996)

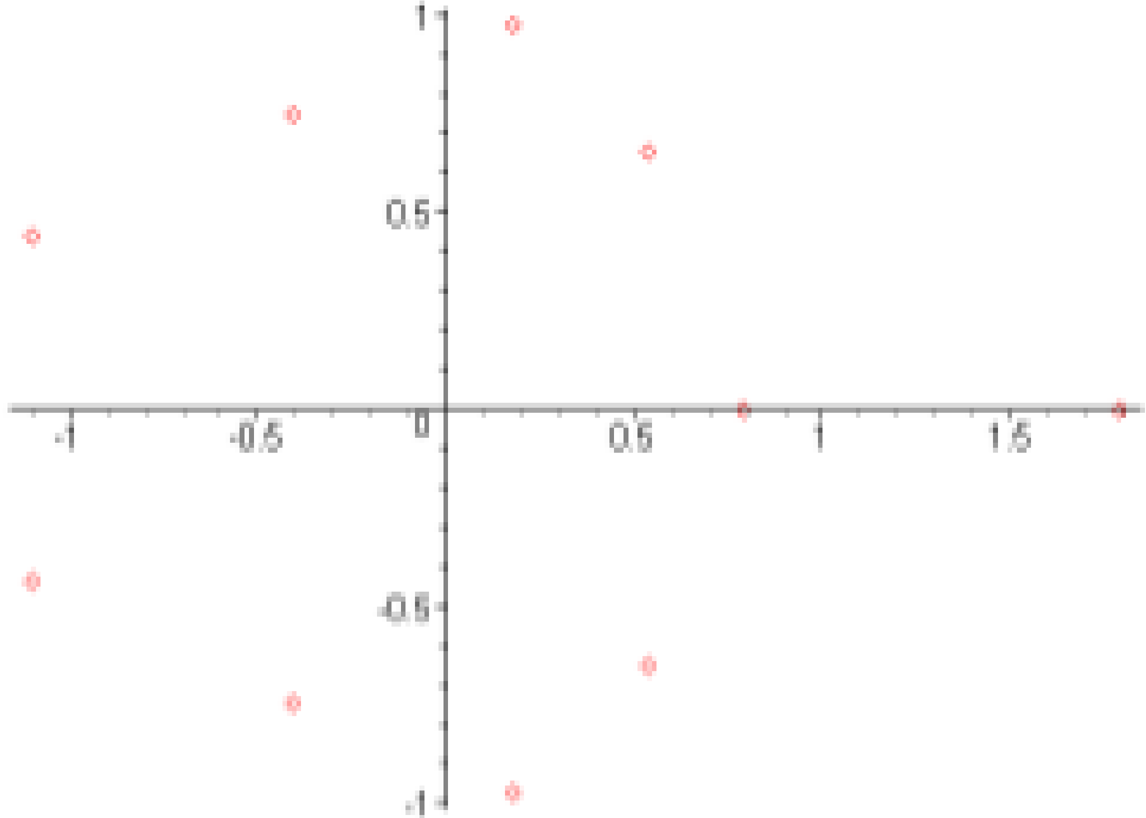
Some experiments

Take $P_d \in \mathbb{Z}[x]$ irreducible with

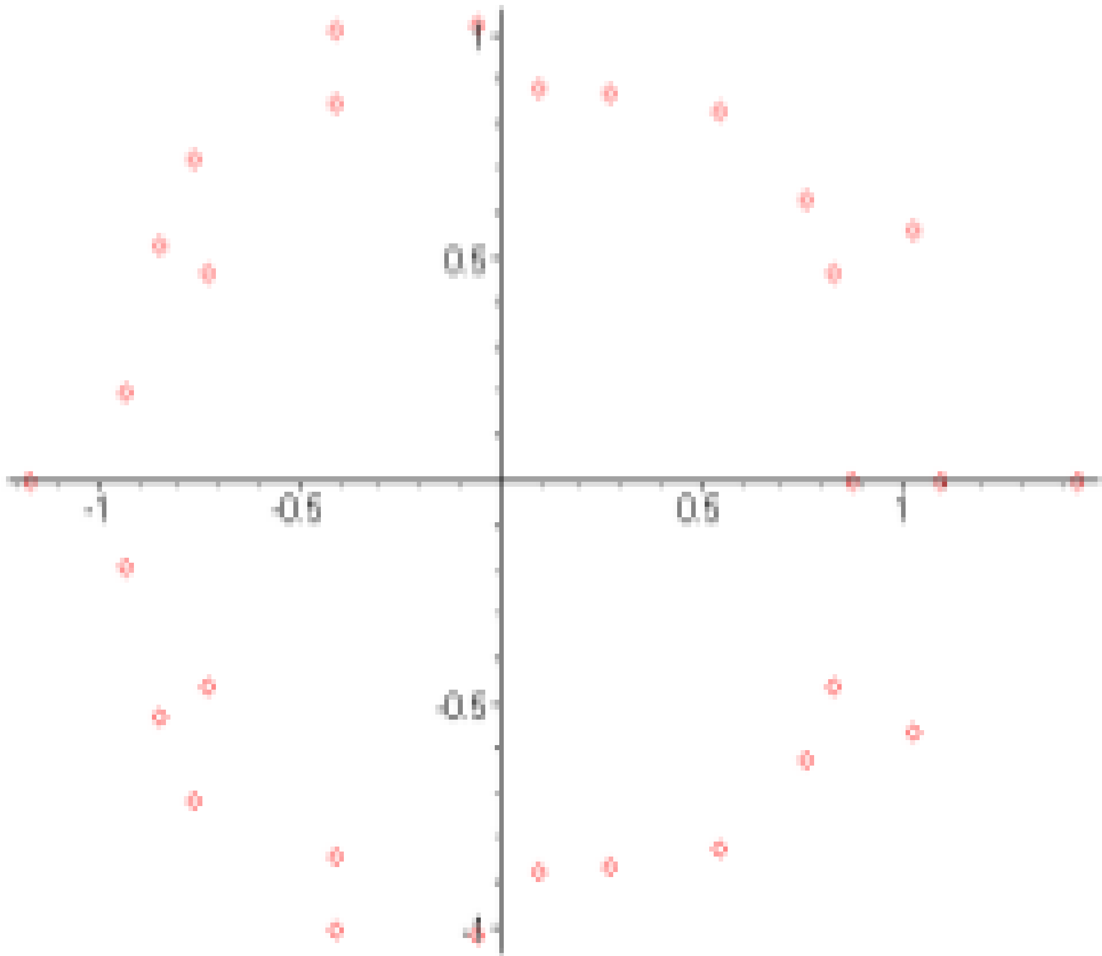
$$\deg P_d = d \gg 0 \quad \& \quad \mathrm{coeffs}(P_d) \subset \{0, \pm 1\}$$

Plot the roots of P_d and see what happens...

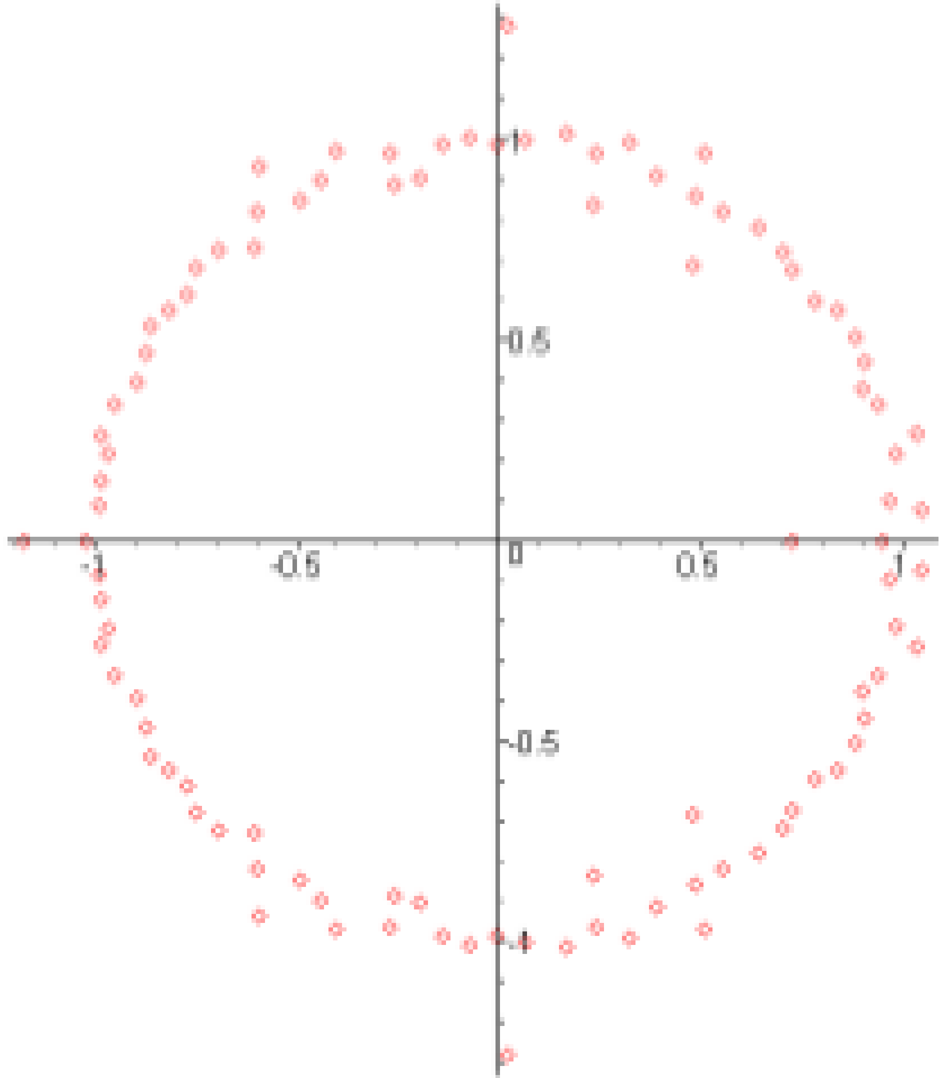
For instance, let $d = 10$ and $f = -x^{10}+x^9+x^8+x^6+x^5-x^4+x^3-x^2+x-1$



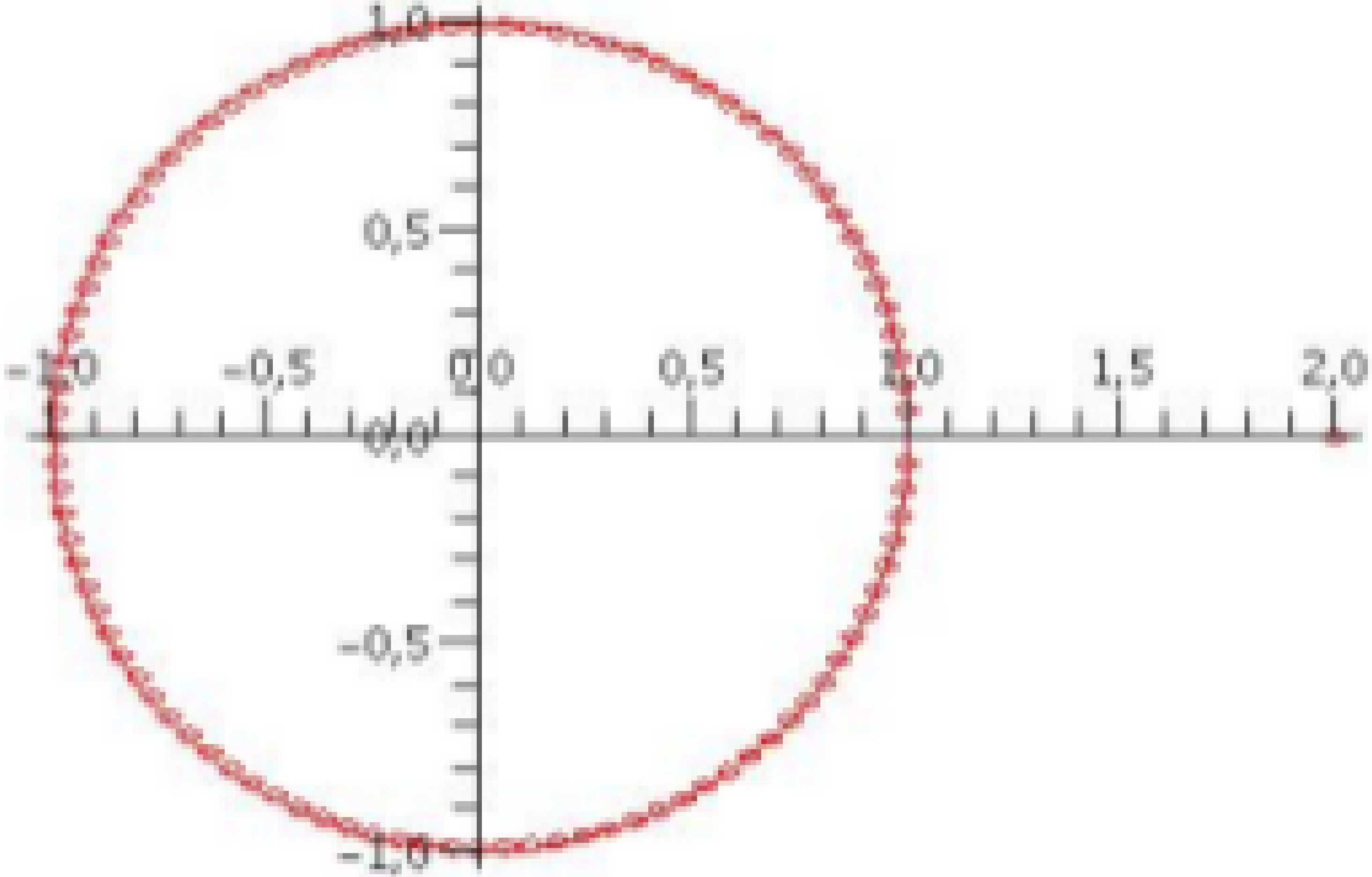
$d = 30$ and $f = x^{30} - x^{29} - x^{28} + x^{26} + x^{25} - x^{24} - x^{23} - x^{22} + x^{21} - x^{20} + x^{19} + \cdots$



$d = 100$ and $f = -x^{100} - x^{98} + x^{96} + x^{94} - x^{93} + x^{92} - x^{91} - x^{90} + x^{88} - x^{84} + \cdots$



$$f = x^{100} - x^{99} - x^{98} - x^{97} - \cdots - x - 1$$



Zariski density

\+

small height

$\longrightarrow$ analytic density

HEIGHT OF POINTS

- $X^n/\mathbb{Q}$ (proper) algebraic variety
- D Cartier divisor on X

For each $v \in M_{\mathbb{Q}}$

- X_v v-adic analytic space $\begin{cases} X(\mathbb{C}) & (v=\infty) \\ \text{Berkovich space} & (v \neq \infty) \end{cases}$
- $\| \ \|_v$ v-adic metric on $\mathcal{O}(D)_v$
- $\overline{D} = \left(D, (\|\cdot\|_v)_{v \in M_{\mathbb{Q}}}\right)$ metrized Cartier divisor

The height of $p \in X(\mathbb{Q})$ wr to $\overline{D}$ is

$$h_{\overline{D}}(p) = -\sum_{v \in M_{\mathbb{Q}}} \log \|s(p)\|_v$$

for any rational section s regular and $\neq 0$ at p

ESSENTIAL MINIMUM

$$\mu^{ess}_{\overline{D}}(X) = \inf\{\theta \in \mathbb{R} \mid \{p \in X(\overline{\mathbb{Q}}) \mid h_{\overline{D}}(p) \leq \theta\} \text{ Zariski dense}\}$$

__Fact:__ $(p_k)_{k\geq 1}$ generic sequence in $X(\overline{\mathbb{Q}})$

i.e. $\forall\, Y \subsetneqq X \qquad \#\{k \mid p_k \in Y\} < \infty$

Then
$$\underline{\lim}\; h_{\overline{D}}(p_k) \geq \mu^{ess}_{\overline{D}}(X)$$

__Pb:__ For $(p_k)_{k\geq 1}$ generic st $\lim\limits_{k\to\infty} h_{\overline{D}}(p_k) = \mu^{ess}_{\overline{D}}(X)$
study the limit distribution of Gp_k

EQUIDISTRIBUTION OF GALOIS ORBITS OF SMALL POINTS

<u>THM</u> (Yuan 2008 after Szpiro-Ullmo-Zhang, Bilu, Chambert-Loir, Favre-Rivera, Baker-Rumely, ...)

$X^n/\mathbb{Q}$ proper, $\overline{D}$ metrized divisor.
Sup D ample & $\overline{D}$ semipositive.

Let $(p_k)_{k\geq 1}$ generic st

$$h_{\overline{D}}(p_k) \xrightarrow[k\to\infty]{} \frac{h_{\overline{D}}(X)}{(n+1)\deg_D(X)}$$

Then, for $v \in \mathfrak{M}_{\mathbb{Q}}$

$$G\, p_k \xrightarrow[k\to\infty]{} c_1(\|\cdot\|_v)^{\wedge n}$$

proba measure on X_v

By Zhang's theorem on successive algebraic minima

$$\mu^{ess}_{\bar{D}}(X) \leq \frac{h_{\bar{D}}(X)}{\deg_D(X)} \leq (n+1)\,\mu^{ess}_{\bar{D}}(X)$$

if $\bar{D}$ nef

$\Rightarrow$ the equidistribution thm can only be applied when

$$\mu^{ess}_{\bar{D}}(X) = \frac{h_{\bar{D}}(X)}{(n+1)\deg_D(X)}$$

Toric varieties

$\mathbb{T} = \mathbb{G}_m^n$ algebraic torus $/\mathbb{Q}$

A toric variety (with torus $\mathbb{T}$) is a normal variety X st $\mathbb{T} \subset X$ and $\mathbb{T} \circlearrowright X$

Construction

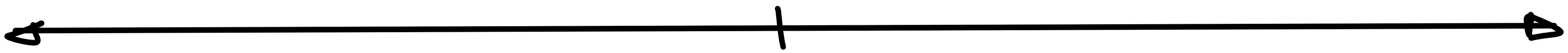
Σ fan on $\mathbb{R}^n$

$\sigma \in \Sigma \rightsquigarrow X_\sigma$ affine toric variety

$\tau \subset \sigma \Longrightarrow X_\tau \hookrightarrow X_\sigma$ open immersion

$$X_\Sigma := \bigcup_{\sigma \in \Sigma} X_\sigma$$

Ex:



$$\begin{array}{ccccc} \mathbb{C} & \hookleftarrow & \mathbb{C}^\times & \hookrightarrow & \mathbb{C} \\ x^{-1} & \leftarrow\!\shortmid & x & \mapsto & x \end{array}$$

$X_\Sigma = \mathbb{P}^1$

Toric Cartier divisors

Assume Σ covers $\mathbb{R}^n$ (equivalently X_Σ proper)

A virtual support function is $\Psi : \mathbb{R}^n \to \mathbb{R}$

st $\quad \Psi|_\sigma = m_\sigma \in (\mathbb{Z}^n)^\vee \qquad \forall \sigma \in \Sigma$

$$\Psi \rightsquigarrow D_\Psi = \left(X_\sigma , x^{-m_\sigma} \right)_{\sigma \in \Sigma}$$ toric Cartier divisor

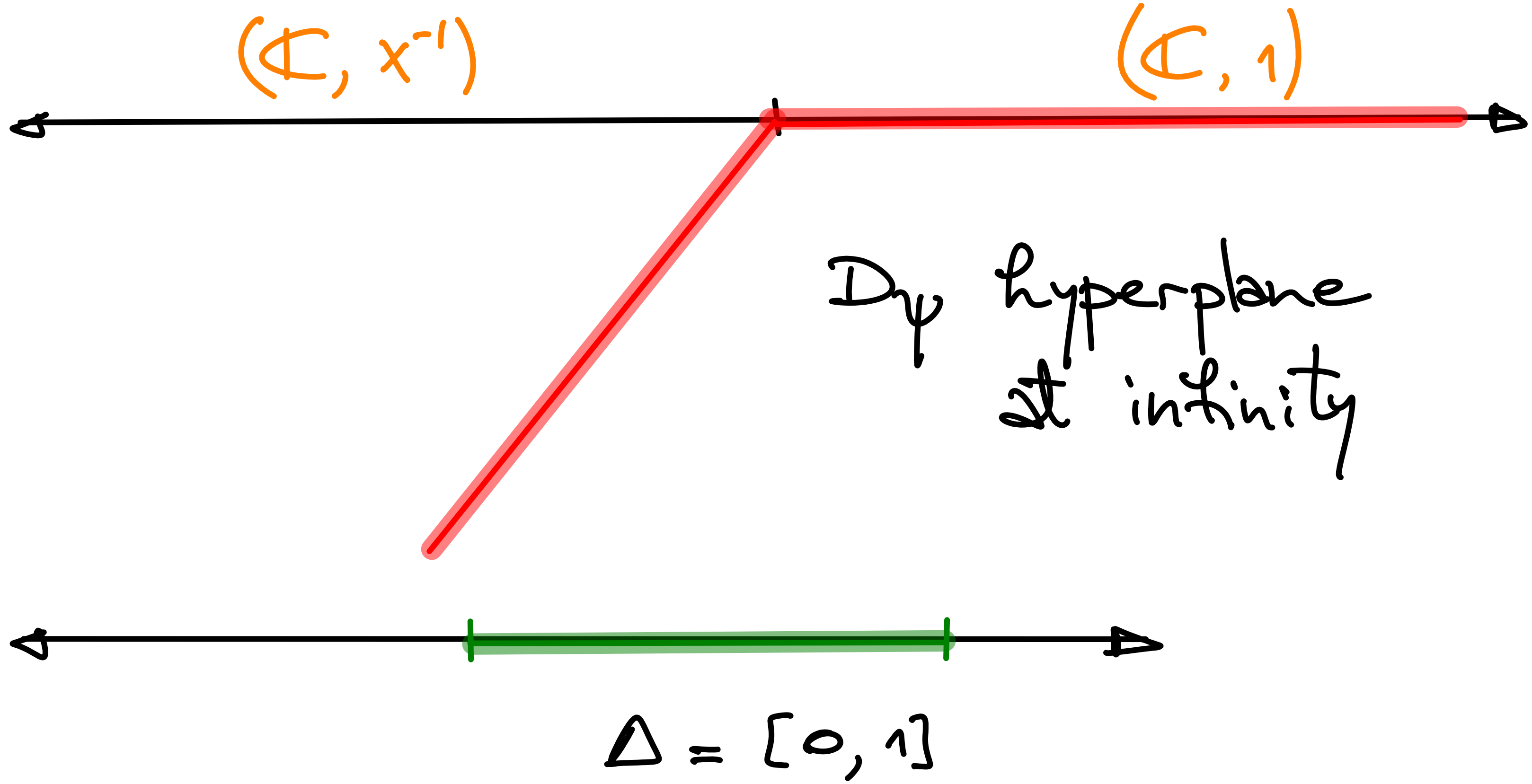
$$\Delta_\Psi = \{ x \in (\mathbb{R}^n)^\vee \mid x \geq \Psi \}$$ polytope

Prop: D_Ψ nef $\iff$ Ψ concave

If so,

$$\deg_D(X) = n! \operatorname{vol}(\Delta)$$

Ex

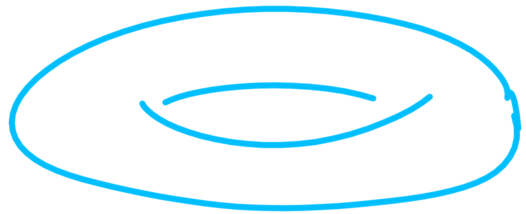


Toric metrics

$v \in \mathfrak{M}_{\mathbb{Q}}$

$\mathbb{S}_v \subset \mathbb{T}_v$ compact torus

Ex: $\mathbb{S}_\infty = \{ (t_1, \dots, t_n) \in (\mathbb{C}^\times)^n \mid |t_i| = 1 \ \forall i \}$



v-adic toric metric on $\mathcal{O}(D_\Psi)_v$:= $\mathbb{S}_v$-invariant

Construction

Suppose $v = \infty$

$$p \mapsto \mathrm{val}_\infty(p) = -\log|p|$$

$$\begin{tikzcd} (\mathbb{C}^\times)^n \arrow[d] \arrow[dr, "\log\|s_\Phi\|_v"] & \\ \mathbb{R}^n \arrow[r, "f_v"'] & \mathbb{R} \end{tikzcd}$$

<u>Thm 1</u>

$$\left\{ \begin{array}{l} f_v : \mathbb{R}^n \to \mathbb{R} \text{ concave} \\ \text{st } |f_v - \psi| \text{ bounded} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} v\text{-adic semipositive} \\ \text{metrics on } \mathcal{O}(D_\psi)_v \end{array} \right\}$$

The v-adic roof function is the Legendre-Fenchel dual

$$\vartheta_v : \Delta \to \mathbb{R} \qquad x \mapsto \inf_{u \in \mathbb{R}^n} \langle x, u \rangle - f_v(u)$$

The global roof function is

$$\vartheta := \sum_v \vartheta_v$$

An abridged toric dictionary

X toric variety with torus $\mathbb{T}$	Σ fan on $\mathbb{R}^n$
D nef toric divisor on X	$\Psi: \mathbb{R}^n \to \mathbb{R}$ concave Σ-linear $\Delta \subset \mathbb{R}^n$ lattice polytope
$\lVert\cdot\rVert_v$ semipositive toric metric on $\mathcal{O}(D)_v$	$\psi_v: \mathbb{R}^n \to \mathbb{R}$ concave st $\lvert\psi_v - \Psi\rvert$ bounded $\vartheta_v: \Delta \to \mathbb{R}$ concave
$\overline{D}$ metrized divisor	$\vartheta = \sum_v \vartheta_v$

Some consequences

Thm 2:

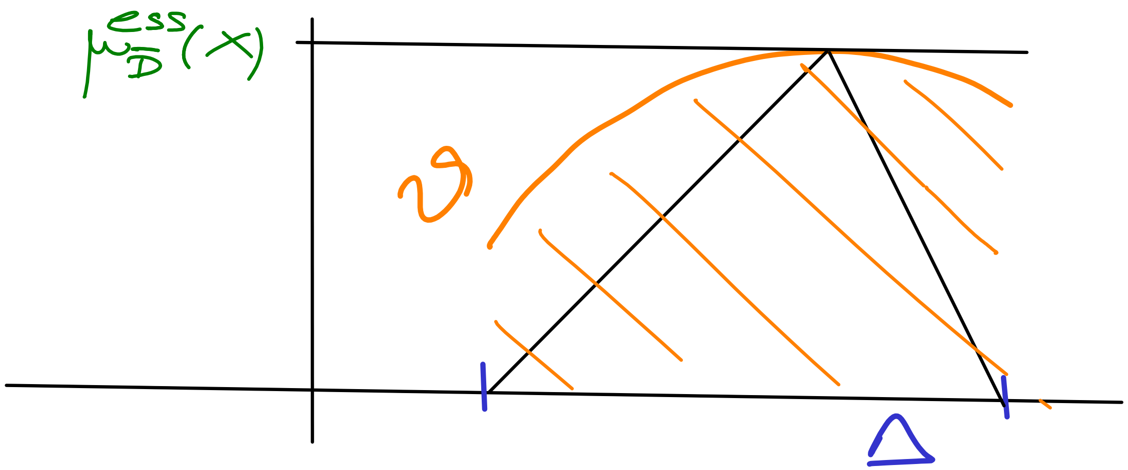
$$\mu^{ess}_{\bar{D}}(X) = \max_{x \in \Delta} \vartheta(x)$$

Thm 3: If $\bar{D}$ semipositive

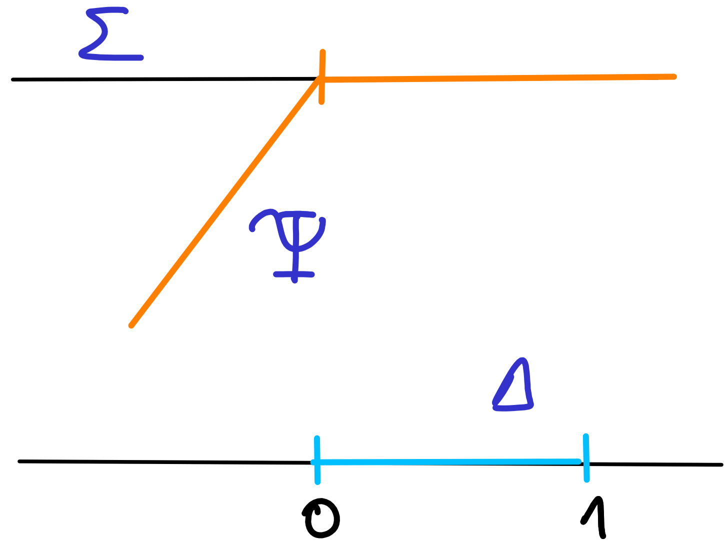
$$h_{\bar{D}}(X) = (n+1)! \int_\Delta \vartheta(x)\, d\mathrm{vol}$$

Thm 4: $\bar{D}$ nef $\iff$ ψ_v concave $(\forall v)$ & $\vartheta \geq 0$

Cor 1 (successive algebraic minima)



Some examples



$X = \mathbb{P}^1_{\mathbb{Q}}$ $\qquad D = (0:1)$

1) Weil height

For $v \in \mathcal{M}_{\mathbb{Q}}$ set

$$\|s_\ell(p)\|_{v,\mathrm{can}} = \frac{|\ell(p)|_v}{\max(|p_0|_v, |p_1|_v)}$$

"canonical" metric

for $p \in \mathbb{P}^1(\mathbb{C}_v)$ and $\ell \in \mathbb{C}_v[x_0, x_1]_1$

$$\Longrightarrow \quad h_{\overline{D}} = \text{Weil height}$$

- $\psi_v = \Psi$ & $\vartheta_v \equiv 0$ $(\forall v)$, $\vartheta \equiv 0$

$$\Rightarrow \quad h_{\overline{D}}(\mathbb{P}^1) = \mu^{\mathrm{ess}}_{\overline{D}}(\mathbb{P}^1) = 0$$

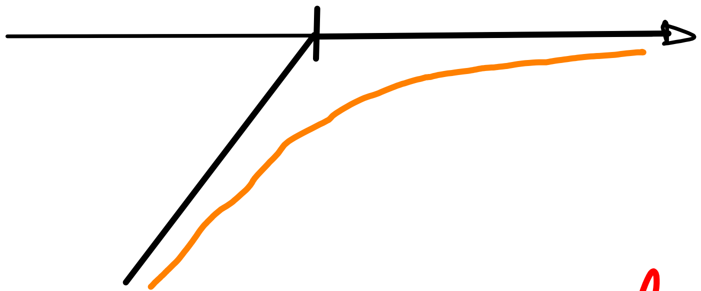
2) Fubini-Study height

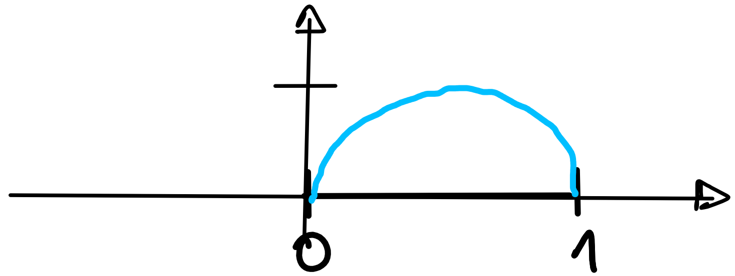
$$\begin{cases} \|s_\ell(p)\|_v = \dfrac{|\ell(p)|_\infty}{\sqrt{|p_0|_\infty^2+|p_1|_\infty^2}} \\[2ex] \|\cdot\|_v = \|\cdot\|_{v,\mathrm{can}} \qquad (v\neq\infty)\end{cases}$$

If $\xi = \frac{a}{b} \in \mathbb{Q}^\times$ then $h_{FS}(\xi) = \log\sqrt{a^2+b^2}$

$\psi_v(u) = \frac{1}{2}\log(1+e^{-2u})$

$\vartheta(x) = \vartheta_v(x) = -\frac{1}{2}\left(x\log x + (1-x)\log(1-x)\right)$



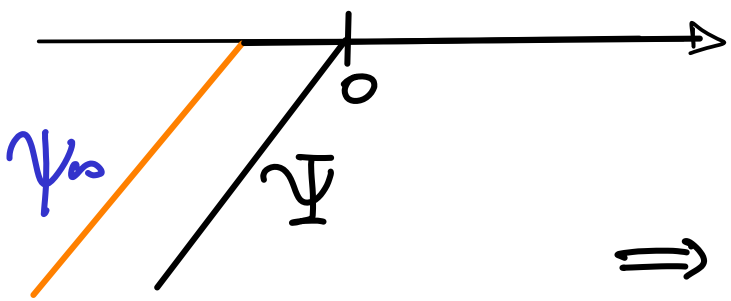


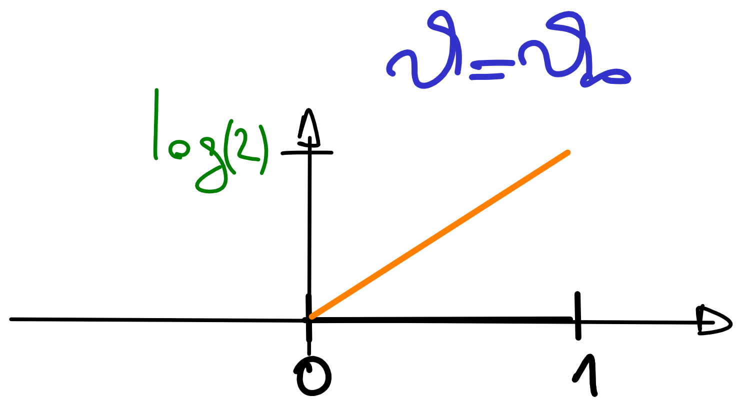
$$\Rightarrow h_D(\mathbb{P}^1) = \frac{1}{2}, \quad \mu^{ess}_D(\mathbb{P}^1) = \frac{\log 2}{2}$$

3) Twisted Weil height

$$\begin{cases} \| s_\ell(p) \|_\infty = \dfrac{|\ell(p)|_\infty}{\max(|p_0|_\infty, 2|p_1|_\infty)} \\ \| \cdot \|_v = \| \cdot \|_{v,\mathrm{can}} \qquad (v \neq \infty) \end{cases}$$

$$h_{\mathrm{Weil}}(\xi) = \log \max(|a|, 2|b|)$$





$$\Longrightarrow h_D(\mathbb{P}^1) = \mu^{\mathrm{ess}}_D = \log 2$$

Cor 2: X toric variety, $\overline{D}$ semipositive with D ample

$$\mu^{\mathrm{ess}}_{\overline{D}}(X) = \frac{h_{\overline{D}}(X)}{(n+1)\deg_D(X)} \iff \vartheta \equiv \text{constant}$$

$\leadsto$ Yuan's thm $\big|_{\text{toric case}}$ = Bilu's thm (for $v = \infty$)

THM 5 $(X, \overline{D})$ toric with D ample & $\overline{D}$ semipositive.
Let $x_{max} \in \Delta$ st $\vartheta(x_{max}) = \max_{x \in \Delta} \vartheta(x)$

TFAE:

(1) 0 vertex of $\partial_{x_{max}} \vartheta$

(2) $\forall v \in \mathfrak{M}_{\mathbb{Q}}$ $\exists$ μ_v proba measure on X_v st.
$\forall (p_k)_{k \geq 1}$ generic st $\lim_{k \to \infty} h_{\overline{D}}(p_k) = \mu^{ess}_{\overline{D}}(X)$

$$G p_k \xrightarrow[k \to \infty]{} \mu_v$$

If so, $\exists ! \, (u_v)_v$ with $u_v \in \partial_{x_{max}} \vartheta_v$ and $\sum_v u_v = 0$
and μ_v "Haar" measure on $\mathrm{val}_v^{-1}(u_v)$

EXAMPLES REVISITED

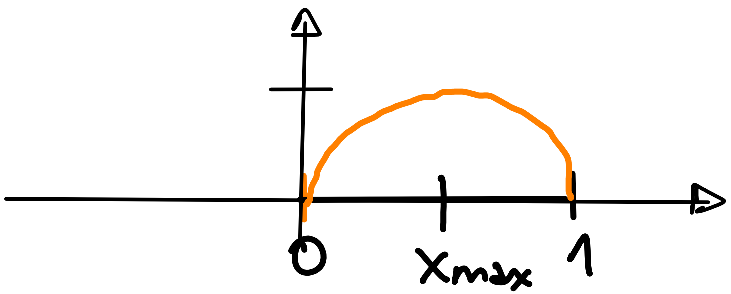
$X = \mathbb{P}^1$, $D = (0:1)$, $(p_k)_{k\geq 1}$ s.t. $\lim_{k\to\infty} \frac{f}{D}(p_k) = \mu_D^{ess}(X)$, $\nu = \infty$

1) WEIL HEIGHT

$\vartheta \equiv 0$ diff at any $x_{max} \in (0,1)$

$\Rightarrow$ $G_{p_k} \xrightarrow[k\to\infty]{} S^1$

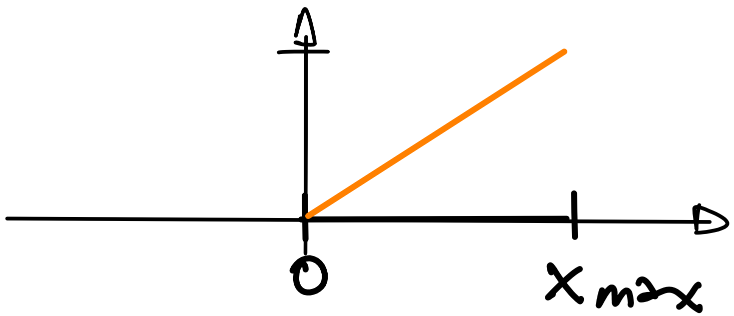
2) FUBINI-STUDY HEIGHT



ϑ diff at $x_{max} = \frac{1}{2}$

$\Rightarrow$ $G_{p_k} \xrightarrow[k\to\infty]{} S^1$

3) Twisted Weil height



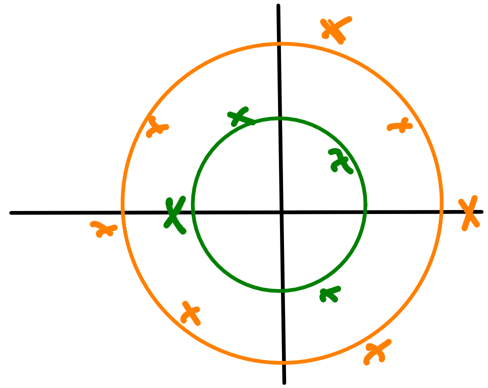
o not a vertex of $\partial_1 \vartheta = (-\infty, 1]$

Recall that

$$h_{\mathrm{Weil}}(\varphi) = \log\max(|p_0|_\infty, 2|p_1|_\infty) + \sum_{v\neq\infty} \log\max(|p_0|_v, |p_1|_v)$$

Take ω_k roots of 1 and set

$$p_k = (1:\omega_k) \qquad q_k = \left(1 : \frac{\omega_k}{2}\right)$$



$$h(p_k) = h(q_k) = \log 2 = \mu^{ess}_{\overline{D}}(X)$$

$\Rightarrow$ no equidistribution in this case

Thm 7 $X, \bar{D}, x_{max}$ as before and fix $v_0 \in M_Q$

Suppose $\exists!\ \mu_{v_0} \in \partial_{x_{max}} \vartheta_{v_0}$ that can be completed to $(\mu_v)_v$ with $\mu_v \in \partial_{x_{max}} \vartheta_v$ and $\sum_v \mu_v = 0$

Then $\forall (p_k)_{k \geq 1}$ generic st $\lim_{k \to \infty} h_{\bar{D}}(p_k) = \mu^{ess}_{\bar{D}}(X)$

$$(val_v)_* G p_k \xrightarrow[k \to \infty]{} \mu_v$$

Pf: Use the variational principle:

equidistribution $\Leftarrow$ μ^{ess} Gateaux diff at $\bar{D}$

+ Thm 2. $\boxtimes$

ADELIC MODULUS CONCENTRATION $\Rightarrow$ EQUIDISTRIBUTION

(THM 7 $\Rightarrow$ THM 6)

Suppose $\exists (\mu_v)_v$ with $\mu_v \in \partial_{x_0} \vartheta_v$ & $\sum_v \mu_v = 0$

$$\Rightarrow \quad (\mathrm{val}_v)_* G_{p_k} \xrightarrow[k\to\infty]{} \mu_v$$

and $(p_k)_{k\geq 1}$ $\overline{D}'$-small for $\|\cdot\|'_v = \gamma_v \tau^* \|\cdot\|_{v,\mathrm{can}}$

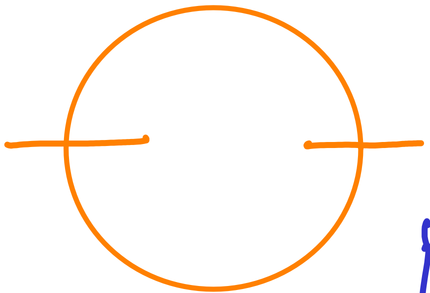
with $\gamma_v \in \mathbb{R}_{>0}$ and τ translation by $p_v \in \mathrm{val}_v^{-1}(\mu_v)$

$\Rightarrow$ reduces to Bilu's thm applied to $\overline{D}'$ $\boxtimes$

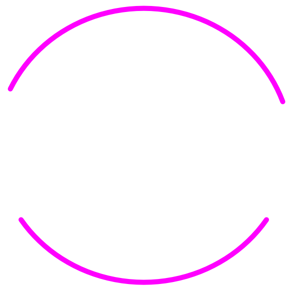
NON EQUIDISTRIBUTION

Non modulus concentration at one v_0

$\Rightarrow$ non equidistribution $\forall v$



μ_{v_0}



μ_v

Pf: Adelic Fekete-Szegö thm (Rumely) $\boxtimes$

WHICH ARE THE POSSIBLE LIMITS ?

Set $P = z^2 - 1 \in \mathbb{C}[z]$

For $k \geq 1$

ξ_k root of $\overbrace{P \circ \cdots \circ P}^{2k+1}$

- $\frac{1}{2} \leq |\eta|_\infty \leq 8 \quad \forall \eta \in G\xi_k$

$\Rightarrow ((1:\xi_k))_{k\geq 1}$ small for D twisted canonical

- $G(1:\xi_k) \xrightarrow{k\to\infty}$ equilibrium measure of Julia(P)

(Ljubich thm)

THANK YOU!