Where do the small points go?

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ALGEBRAIC NUMBERS

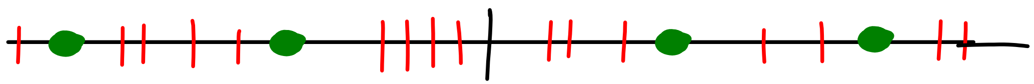
$\overline{\mathbb{Q}}$: solutions of $f=0$ for

$$f = x^d + a_{d-1}x^{d-1} + \dots + a_0 \qquad \text{with } a_j \in \mathbb{Q}$$

$\overline{\mathbb{Z}}$: same, for $a_j \in \mathbb{Z}$

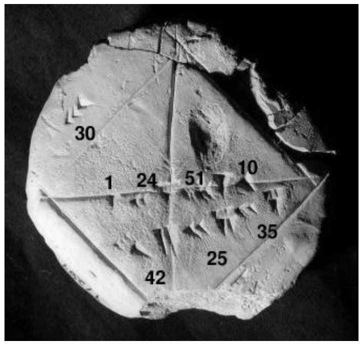
Examples

$\xi \in \mathbb{Q}$ or $\mathbb{Z}$



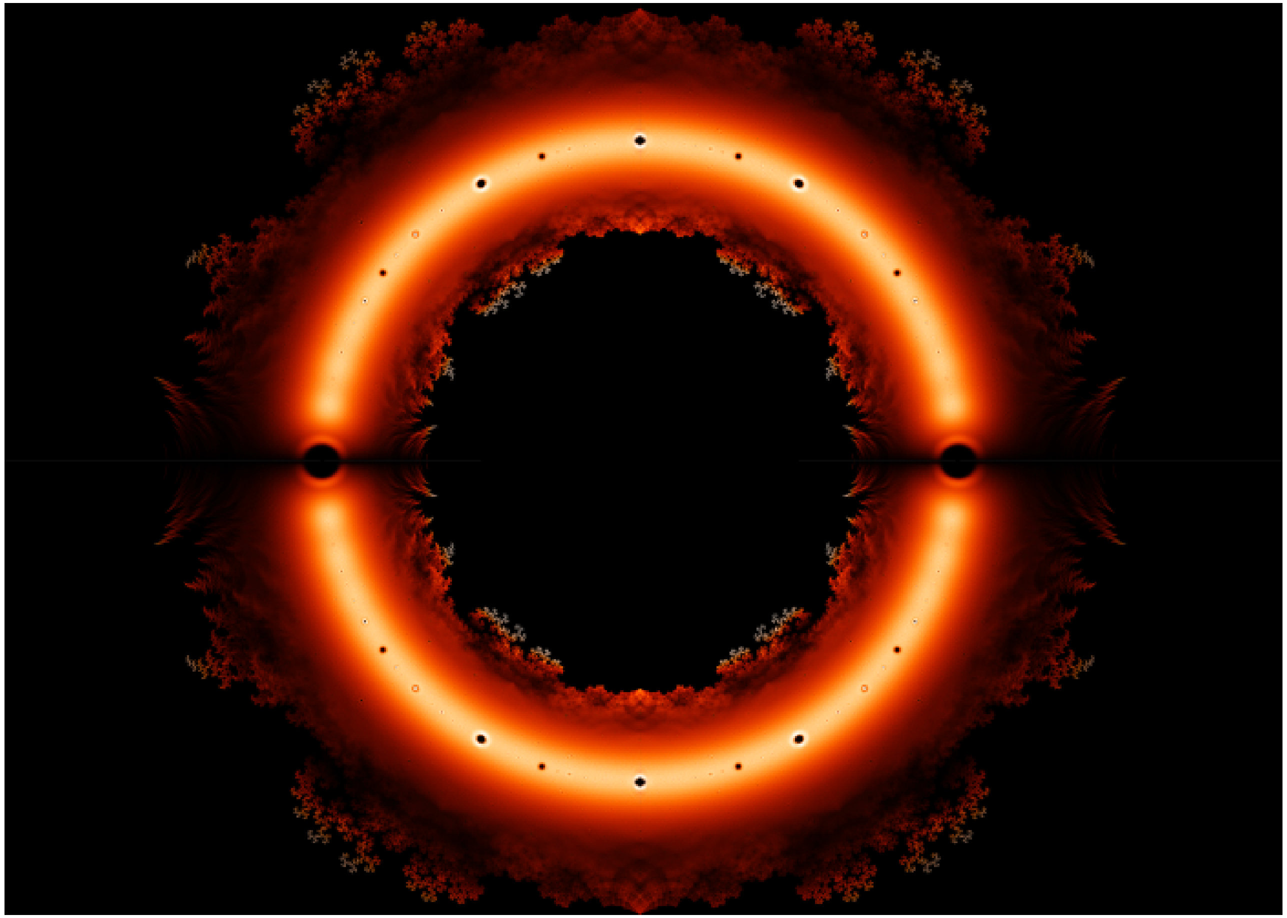
$\varphi = \frac{1+\sqrt{5}}{2} = 1{,}618\dots$

$\sqrt{2} =$



YBC 7282 (~ 1700 b.c.)

Roots of Littlewood Polynomials



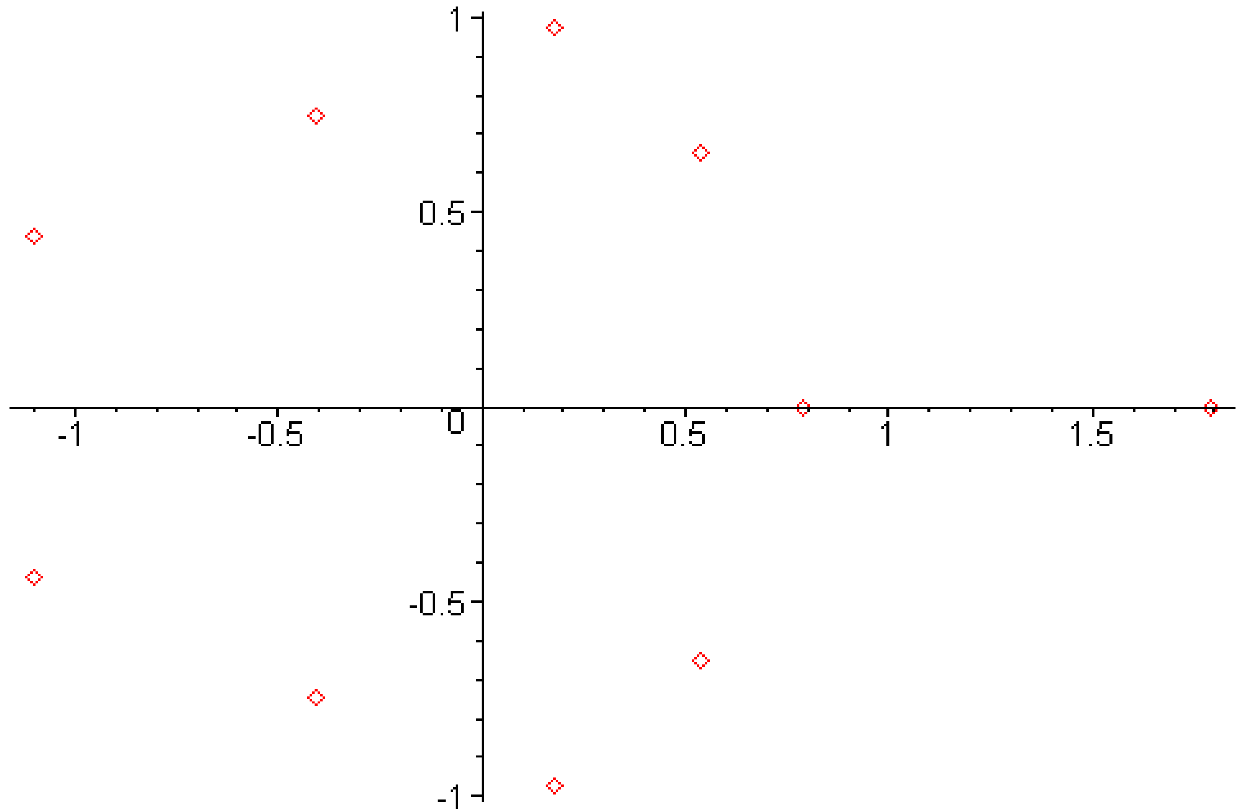
Roots of all polynomials with coefficients ± 1 and degree ≤ 24

by Báez, Christensen and Derbyshire

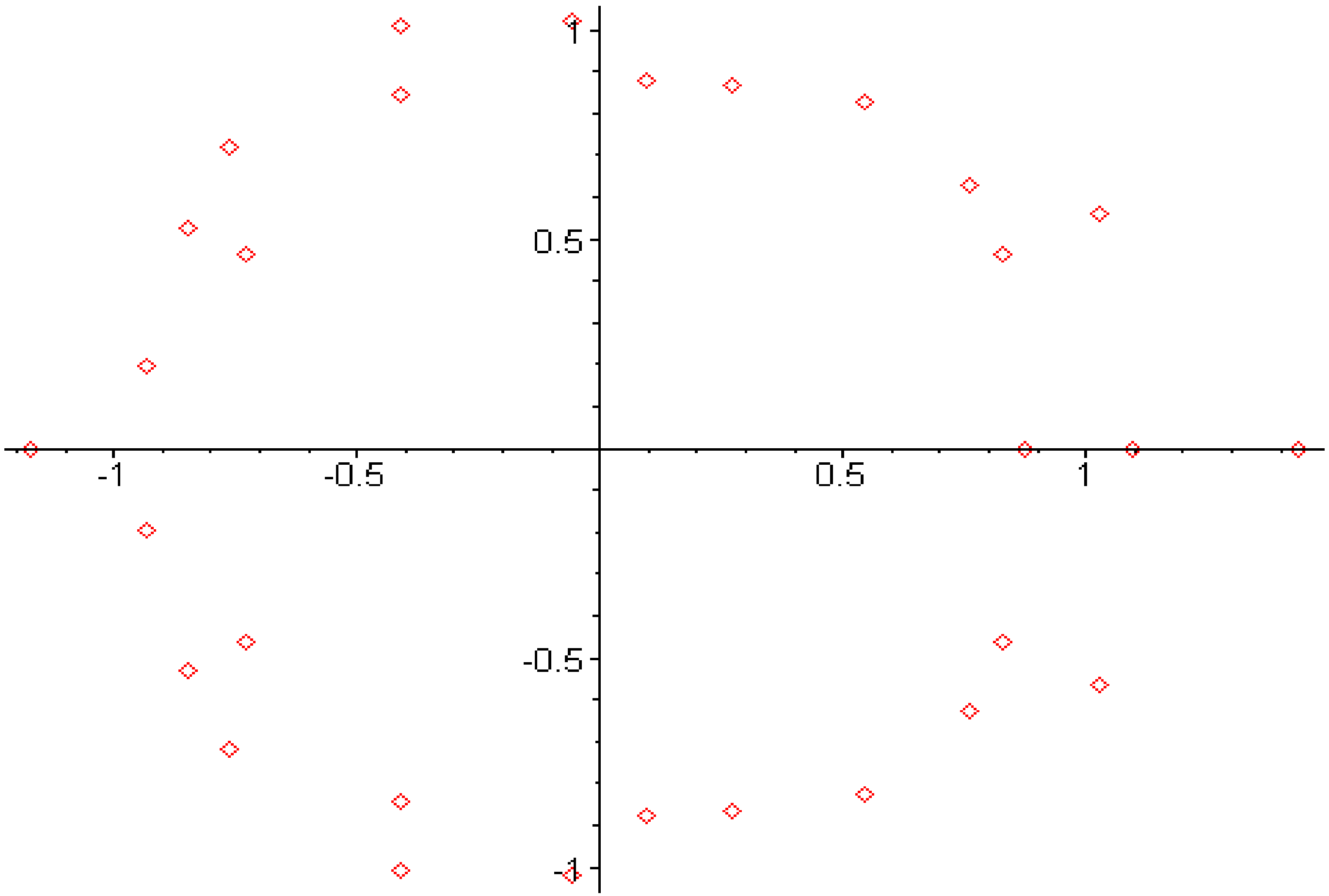
An experiment

Plot the roots of a random polynomial with coefficients $+1, -1$ or 0 and degree $d \gg 0$ and try to find a pattern

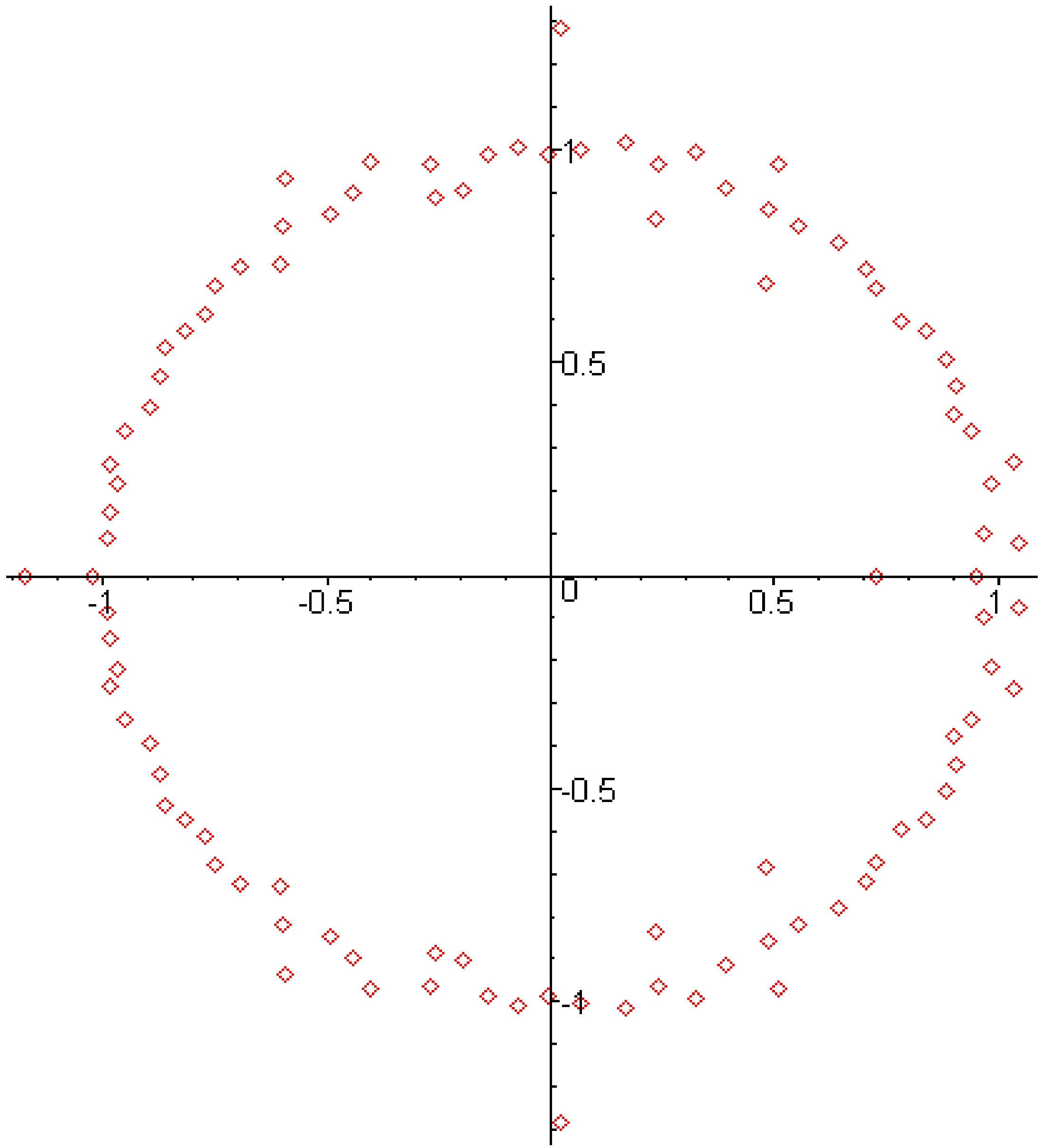
$d = 10$ and $f = -x^{10} + x^9 + x^8 + x^6 + x^5 - x^4 + x^3 - x^2 + x - 1$



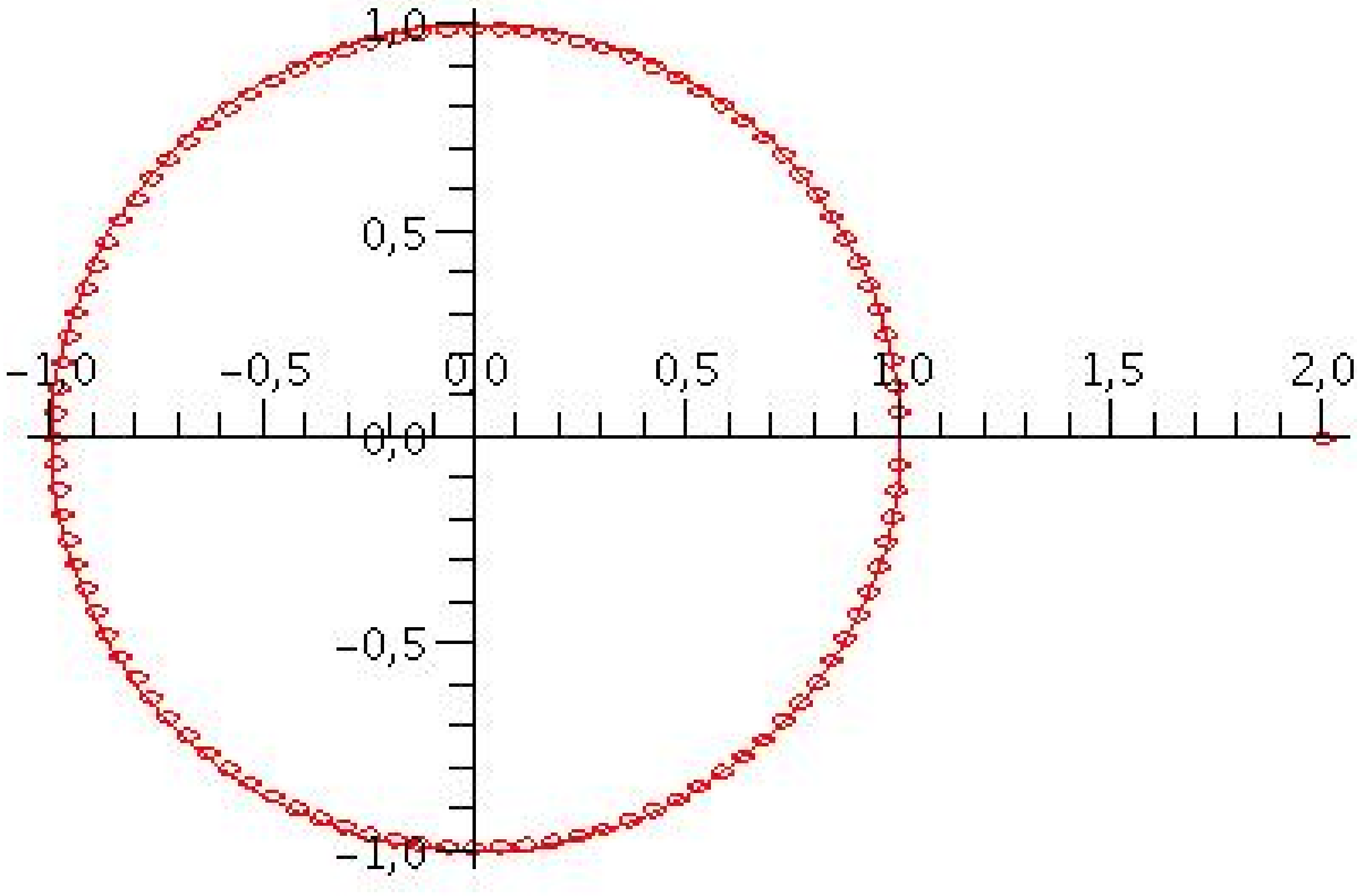
$d = 30$ and $f = x^{30} - x^{29} - x^{28} + x^{26} + x^{25} - x^{24} - x^{23} - x^{22} + x^{21} - x^{20} + x^{19} + \cdots$



$d = 100$ and $f = -x^{100} - x^{98} + x^{96} + x^{94} - x^{93} + x^{92} - x^{91} - x^{90} + x^{88} - x^{84} + \cdots$



$$f = x^{100} - x^{99} - x^{98} - x^{97} - \cdots - x - 1$$



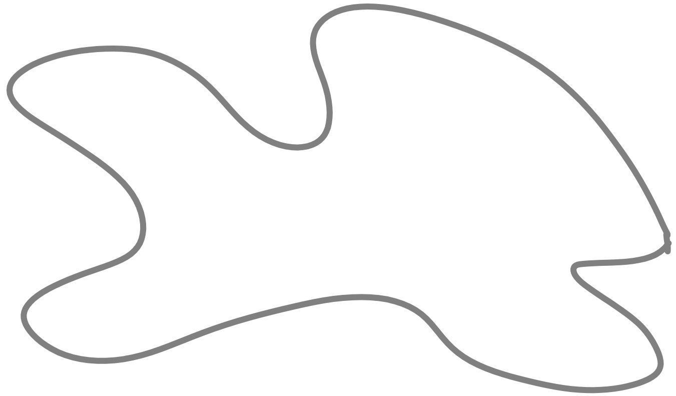
Sets with many Galois orbits in $\mathbb{Z}$

Let $\xi \in \overline{\mathbb{Q}}$

Set $m_\xi \in \mathbb{Q}[x]$ minimal monic polynomial of ξ

$\mathrm{Gal}\,\xi := \{\eta \in \overline{\mathbb{Q}} \mid m_\xi(\eta) = 0\} \hookrightarrow \mathbb{C}$

Fix $K \subset \mathbb{C}$ compact



Pb: Is there an infinite number of Galois orbits of algebraic integers close to K?

A particular case

Set $K = \lambda \mathbb{D}$ for $\mathbb{D} := \{ z \in \mathbb{C} \mid |z| \leq 1 \}$ and $\lambda \in \mathbb{R}$

Let $\xi \in \overline{\mathbb{Z}} \setminus \{0\}$ st $\mathrm{Gal}\,\xi \subset K$

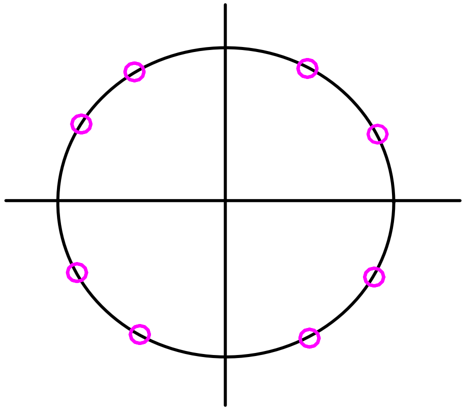
$$m_\xi = \prod_{\eta \in \mathrm{Gal}\,\xi} (x - \eta) = x^d + a_{d-1} x^{d-1} + \dots + a_0 \in \mathbb{Z}[x]$$

Then $\lambda^{\#\mathrm{Gal}\,\xi} \geq \left| \prod \eta \right| = |a_0| \geq 1$

and $\boxed{\lambda \geq 1}$

Indeed

$\mathrm{Gal}\,\xi \subset \mathbb{D}$ iff ξ root of 1



Another particular case

$K = [a,b]$ with $a < b \in \mathbb{R}$

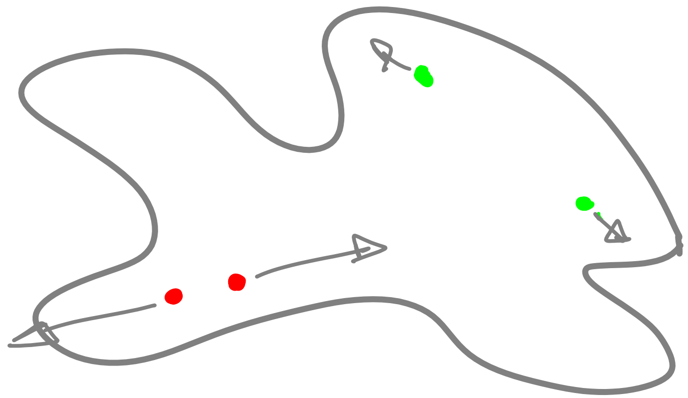
T (Robinson 1964)

If $b-a > 4$, there are ∞ many Galois orbits of totally real algebraic integers contained in $[a,b]$.

If $b-a < 4$, there are $< \infty$ many.

POTENTIAL THEORY ON $\mathbb{C}$

Let μ probability measure on $\mathbb{C}$



$$\mathcal{E}(\mu) := \iint \log\left(\frac{1}{|z-t|}\right) d\mu(t)\, d\mu(z)$$

energy of μ

$$V_K := \inf_{\mathrm{supp}(\mu) \subset K} \mathcal{E}(K)$$

Robin constant of K

$$\mathrm{cap}(K) := e^{-V_K}$$

capacity of K

I: If $\mathrm{cap}(K) > 0$, there is a unique μ supported in K with $V_K = \mathcal{E}(\mu)$

$\rightsquigarrow$ μ_K harmonic measure of K

Examples

- $K = \lambda \mathbb{D}$

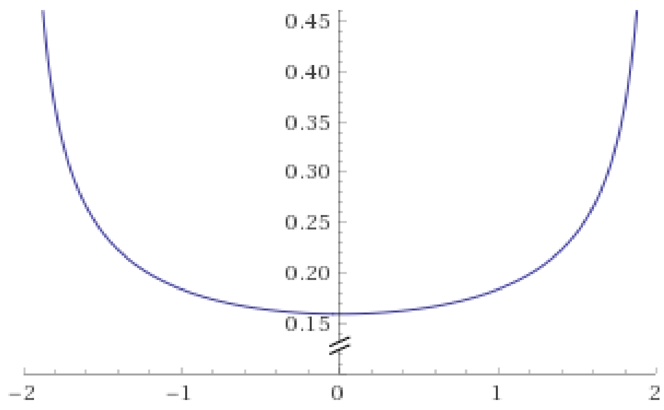
  $\mathrm{cap}(K) = \lambda$

  $\mu_K =$ uniform probability measure on λS^1

- $K = [a,b]$

  $\mathrm{cap}(K) = \dfrac{b-a}{4}$

  $d\mu_K = \dfrac{dx}{\pi\sqrt{(x-a)(b-x)}}$ on K



The Fekete-Szegö theorem

I (Fekete 1923) If cap(K) < 1, there is $U \supset K$ open

$$\#\{\xi \in \overline{\mathbb{Z}} \mid \mathrm{Gal}\,\xi \subset U\} < \infty$$

II (Fekete-Szegö 1955)

If cap(K) ≥ 1 and $\overline{K} = K$ for all $U \supset K$ open

$$\#\{\xi \in \overline{\mathbb{Z}} \mid \mathrm{Gal}\,\xi \subset U\} = \infty$$

If cap(K) = 1 and $\xi_\ell \in \overline{\mathbb{Z}}$, $\ell \geq 1$, with $\deg \xi_\ell \xrightarrow[\ell\to\infty]{} \infty$

and $\mathrm{Gal}\,\xi_\ell \subset B(K,\varepsilon)$ $\forall \varepsilon$ and $\ell \gg 0$

$$\lim_{\ell\to\infty} \mathrm{Gal}\,\xi_\ell = \mu_K$$ (weak-* convergence of measures)

Proof (Fekete thm)

Let $U \supset K$ open with $\operatorname{cap}(\overline{U}) < 1$

Let $\xi \in \overline{\mathbb{Z}} \setminus \{0\}$ with $\operatorname{Gal}\xi \subset U$ and $d = \deg \xi \gg 0$

$$\operatorname{discr}(\xi) = \prod_{\substack{\eta, \eta' \in \operatorname{Gal}\xi \\ \eta \neq \eta'}} (\eta - \eta') \in \mathbb{Z} \setminus \{0\}$$

$$1 \leq \underline{\lim} \, |\operatorname{discr}(\xi_\ell)|^{1/n(n-1)} \leq \max_{z_1, \dots, z_d} \Big(\prod_{i<j} |z_i - z_j| \Big)^{2/d(d-1)} = \delta_d(U)$$

By definition $\delta_d(U) \xrightarrow[d \to \infty]{} \tau(K)$ transfinite diameter of K

By PT "fundamental theorem" $\tau(K) = \operatorname{cap}(K)$

Hence $\operatorname{cap}(K) \geq 1$ ↯

The Weil height

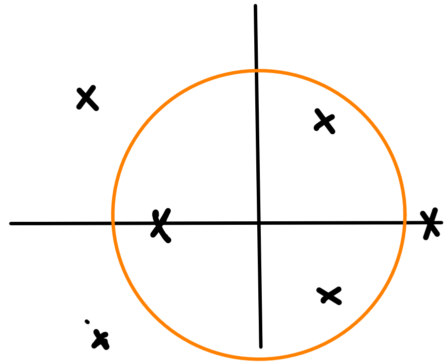
usual

Set $M_{\mathbb{Q}} = \{\text{absolute values on } \mathbb{Q}\} = \{|\cdot|_\infty\} \cup \{|\cdot|_p \mid p \text{ prime}\}$

p-adic

$v \in M_{\mathbb{Q}} \rightsquigarrow \mathbb{C}_v$: complete and algebraically closed field field extension of $\mathbb{Q}$

For $\xi \in \overline{\mathbb{Q}}^\times$



$$h_W(\xi) := \sum_{v \in M_{\mathbb{Q}}} \frac{1}{\#\mathrm{Gal}\,\xi} \sum_{\eta \in \mathrm{Gal}_v \xi} \log \max(1, |\eta|_v)$$

If $\xi = \frac{a}{b} \in \mathbb{Q}^\times$ then $h_W(\xi) = \log \max(|a|, |b|)$

$\approx$ bit complexity of ξ

Basic property (Kronecker): $h_W(\xi) = 0$ iff ξ root of 1

An equidistribution theorem

$\underline{\mathrm{T}}$ (Bilu 1997) $\quad p_\ell \in \overline{\mathbb{Q}}^\times$, $\ell \geq 1$, s.t.

- $\forall p \in \overline{\mathbb{Q}}^\times \quad \#\{\ell \mid p_\ell = p\} < \infty$: $\quad (p_\ell)_{\ell \geq 1}$ is generic
- $\lim\limits_{\ell \to \infty} h_W(p_\ell) = 0$: $(p_\ell)_{\ell \geq 1}$ is small

Then

$$\lim_{\ell \to \infty} \mathrm{Gal}_\infty \xi = S^1$$

A comparison: $\xi_\ell \in \overline{\mathbb{Q}}^\times$ root of $f_\ell \in \mathbb{Z}[x]$ irreducible with coefficients $+1, -1$ or 0 and degree d_ℓ

$$h_W(\xi_\ell) \leq \frac{\log d_\ell}{d_\ell}$$

Hence $\lim\limits_{\ell \to \infty} \mathrm{Gal}_\infty \xi_\ell = \mu_{S^1}$

An open problem

Are almost all polynomials with coefficients $+1, -1$ or 0 irreducible?

(Odlyzco - Poonen 1993)

HEIGHTS FROM ARAKELOV GEOMETRY

- $X^n/\mathbb{Q}$ (proper) algebraic variety
- D divisor on X

For each $v \in \mathfrak{M}_{\mathbb{Q}}$

- X_v v-adic analytic space $<$ $X(\mathbb{C})$ $(v=\infty)$; Berkovich space $(v \neq \infty)$
- $\| \ \|_v$ v-adic metric on $\mathcal{O}(D)_v$
- $\overline{D} = (D, (\|\cdot\|_v)_{v\in \mathfrak{M}_{\mathbb{Q}}})$ metrized divisor

The height of $p \in X(\overline{\mathbb{Q}})$ wrt to $\overline{D}$ is

$$h_{\overline{D}}(p) = \sum_{v \in \mathfrak{M}_{\mathbb{Q}}} \int -\log \|s(q)\|_v \, d\,\mathrm{Gal}_v p$$

for any section s of $\mathcal{O}(D)$ regular and $\neq 0$ at p

ESSENTIAL MINIMUM AND SMALL POINTS

$$\mu_{\overline{D}}^{ess}(X) := \inf \left\{ \theta \in \mathbb{R} \;\middle|\; \{ p \in X(\overline{\mathbb{Q}}) \mid h_{\overline{D}}(p) \leq \theta \} \text{ Zariski dense} \right\}$$

<u>Fact</u>: $(p_k)_{k\geq 1}$ generic sequence in $X(\overline{\mathbb{Q}})$

Then

$$\underline{\lim}\; h_{\overline{D}}(p_k) \geq \mu_{\overline{D}}^{ess}(X)$$

<u>Pb</u>: For $(p_k)_{k\geq 1}$ generic and small and $v \in \mathfrak{M}_{\mathbb{Q}}$

study the limit distribution of $\mathrm{Gal}_v\, p_k$

EQUIDISTRIBUTION OF GALOIS ORBITS OF SMALL POINTS

THM (Yuan 2008 after Szpiro-Ullmo-Zhang, Bilu, ...)

$X^n/\mathbb{Q}$ proper, $\overline{D}$ quasi-canonical metrized divisor:

$$\mu^{ess}_{\overline{D}}(X) = \frac{h_{\overline{D}}(X)}{(n+1)\deg_D(X)}$$

with D ample.

Let $(p_k)_{k\geq 1}$ generic and small

Then, for $v \in \mathfrak{M}_{\mathbb{Q}}$

$$\lim_{k\to\infty} \mathrm{Gal}_v\, p_k = c_1(\|\cdot\|_v)^{\wedge n}$$

proba measure on X_v

By Zhang's theorem on successive algebraic minima

$$\mu^{ess}_{\overline{D}}(X) \leq \frac{h_{\overline{D}}(X)}{\deg_D(X)} \leq (n+1)\,\mu^{ess}_{\overline{D}}(X)$$

CENTRAL EXAMPLE: DYNAMICAL HEIGHTS

Let

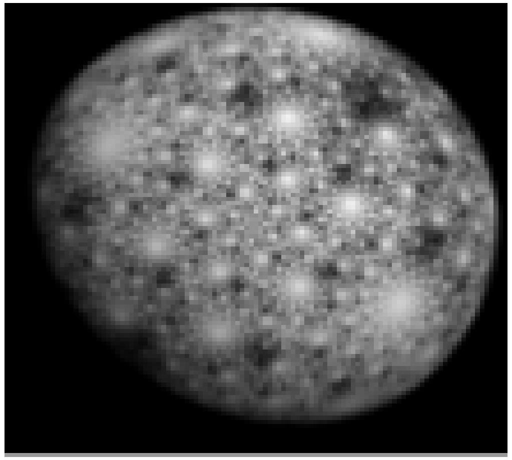
$$\varphi : X \to X$$

s.t. $\varphi^* \mathcal{O}(D) \simeq \mathcal{O}(D)^{\otimes q}$ $\quad q \geq 2$

$\rightsquigarrow$ dynamical height s.t. $h(\varphi(p)) = q \cdot h(p)$

In particular $h(p) = 0$ iff p is preperiodic

In this case $h(X) = \mu^{ess}(X) = 0$



For $(p_\ell)_{\ell \geq 1}$ generic and small and $v \in \mathcal{M}_{\mathbb{Q}}$

$$\lim_{\ell \to \infty} \mathrm{Gal}_v \, p_\ell = v\text{-adic equilibrium measure of } \varphi$$

ARAKELOV GEOMETRY OF TORIC VARIETIES

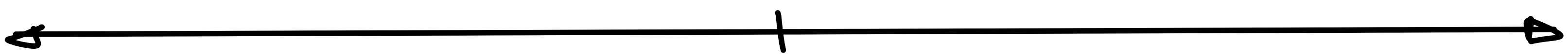
joint with Burgos (Madrid) and Philippon (Paris)

$\mathbb{T} = (\overline{\mathbb{Q}}^\times)^n$ algebraic torus $/\mathbb{Q}$

A toric variety (with torus $\mathbb{T}$) is a normal variety X st $\mathbb{T} \subset X$ and $\mathbb{T} \circlearrowright X$

Σ fan on $\mathbb{R}^n \rightsquigarrow X_\Sigma$ toric variety

Ex:



$X_\Sigma = \mathbb{P}^1$

$$\begin{array}{ccccc} \mathbb{C} & \hookleftarrow & \mathbb{C}^\times & \hookrightarrow & \mathbb{C} \\ x^{-1} & \leftarrow\!\shortmid & x & \mapsto & x \end{array}$$

AN ABRIDGED TORIC DICTIONARY

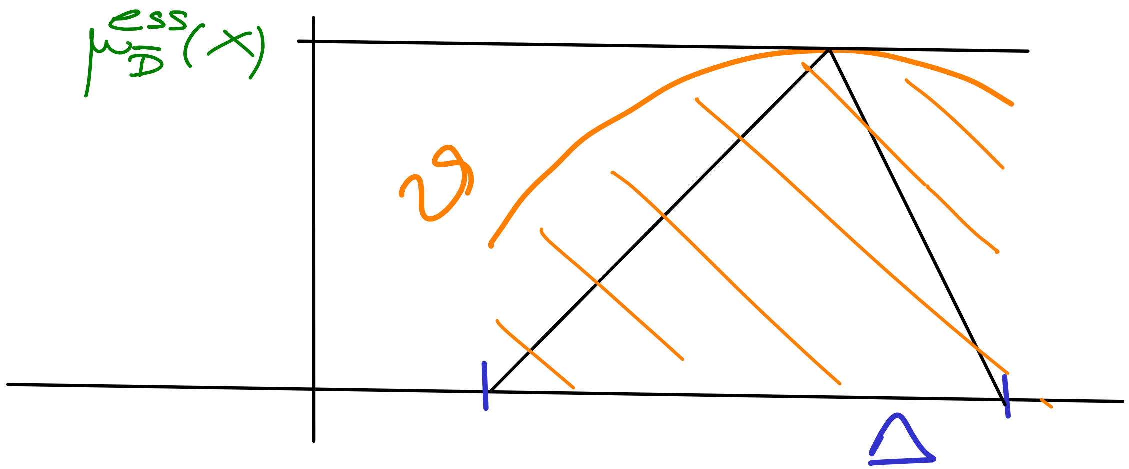
X toric variety with torus $\mathbb{T}$	Σ fan on $\mathbb{R}^n$
D nef toric divisor on X	$\Psi: \mathbb{R}^n \to \mathbb{R}$ concave Σ-linear
	$\Delta \subset \mathbb{R}^n$ lattice polytope
$\lVert\cdot\rVert_v$ toric metric on $\mathcal{O}(D)_v$	$\psi_v: \mathbb{R}^n \to \mathbb{R}$ concave st $\lvert\psi_v - \Psi\rvert$ bounded
	$\vartheta_v: \Delta \to \mathbb{R}$ concave
$\overline{D}$ metrized divisor	$\vartheta = \sum_v \vartheta_v$

SOME MORE ENTRIES

$$\mu^{ess}_{\bar{D}}(X) = \max_{x\in\Delta} \vartheta(x)$$

$$h_{\bar{D}}(X) = (n+1)! \int_{\Delta} \vartheta(x)\, d\mathrm{vol}$$

Successive algebraic minima:



<u>Cor</u>: $\mu^{ess}_{\bar{D}}(X) = \dfrac{h_{\bar{D}}(X)}{(n+1)\deg_D(X)} \iff \vartheta \equiv \text{constant}$

$\rightsquigarrow$ Yuan's thm $\Big|_{\substack{\text{toric case} \\ v=\infty}}$ = Bilu's thm

THM (BPS + Rivera-Letelier)

$(X, \mathcal{D})$ toric with D ample

Let $x_{max} \in \Delta$ st $\vartheta(x_{max}) = \max_{x \in \Delta} \vartheta(x)$

TFAE:

(1) 0 vertex of $\partial_{x_{max}} \vartheta$

(2) $\forall v \in M_{\mathbb{Q}}$ $\exists$ μ_v proba measure on X_v st.

$\forall (p_k)_{k \geq 1}$ generic st $\lim_{k \to \infty} h_{\mathcal{D}}(p_k) = \mu_{\mathcal{D}}^{ess}(X)$

$$\lim_{k \to \infty} \mathrm{Gal}_v \, p_k = \mu_v$$

If so, $\exists!$ $(u_v)_v$ with $u_v \in \partial_{x_{max}} \vartheta_v$ and $\sum_v u_v = 0$

and μ_v "Haar" measure on $\mathrm{val}_v^{-1}(u_v)$

SOME EXAMPLES

$X = \mathbb{P}^1_{\mathbb{Q}}$ $\qquad D = (0:1)$

1) WEIL HEIGHT

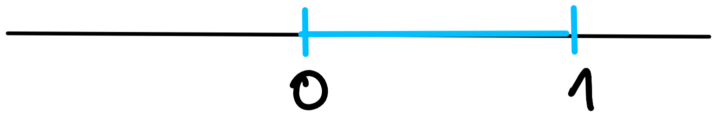
For $v \in \mathcal{M}_{\mathbb{Q}}$ set $\|s_\ell(p)\|_{v,\mathrm{can}}$ "canonical" metric

$$\rightsquigarrow h_{\overline{D}} = \text{Weil height}$$

We have $h_{\overline{D}}(\mathbb{P}^1) = \mu^{ess}_{\overline{D}}(\mathbb{P}^1) = 0$: $\overline{D}$ quasi-canonical

$\vartheta \equiv 0$ diff at any $x_{max} \in (0,1)$

$$\lim_{k\to\infty} \mathrm{Gal}_\infty\, p_k = S^1$$



2) Fubini-Study height

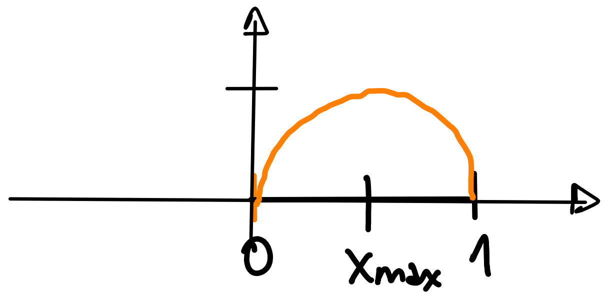
Let

$$h_{FS}$$

height associated to the Fubini-Study metric

If $\xi = \frac{a}{b} \in \mathbb{Q}^*$ then $h_{FS}(\xi) = \log\sqrt{a^2+b^2}$

We have



$$\vartheta(x) = \vartheta_{\sigma}(x) = -\frac{1}{2}\left(x\log x + (1-x)\log(1-x)\right)$$

$$\frac{h_{\overline{D}}(\mathbb{P}^1)}{2\deg_D(\mathbb{P}^1)} = \frac{1}{4} < \mu^{ess}_{\overline{D}}(\mathbb{P}^1) = \frac{\log 2}{2}$$

ϑ diff at $x_{max} = \frac{1}{2} \Rightarrow \lim_{k\to\infty} \mathrm{Gal}_{\infty} p_k = S^1$

Beyond the toric case

Zagier height : $\xi \in \overline{\mathbb{Q}}^{\times}$

$$h_Z(\xi) := h_W(\xi) + h_W(1-\xi)$$

Equivalent to height of points induced by

$$C = (x+y=1) \subset \mathbb{A}^2 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$$

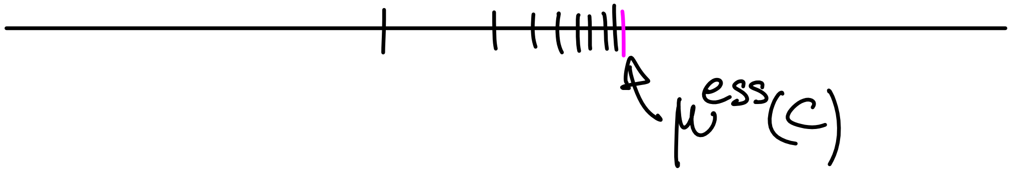
T (Zagier 1995) $h_Z(\xi) = 0$ iff $\xi = 0, 1, \frac{1 \pm \sqrt{3}\, i}{2}$

Else

$$h_Z(\xi) \geq \frac{1}{2} \log\left(\frac{1+\sqrt{5}}{2}\right) = 0{,}2406$$

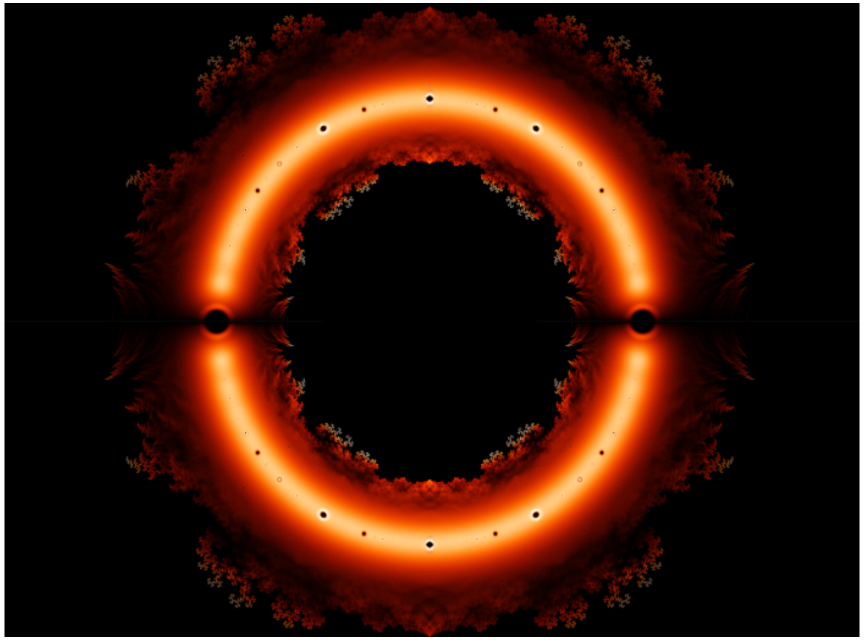
Indeed

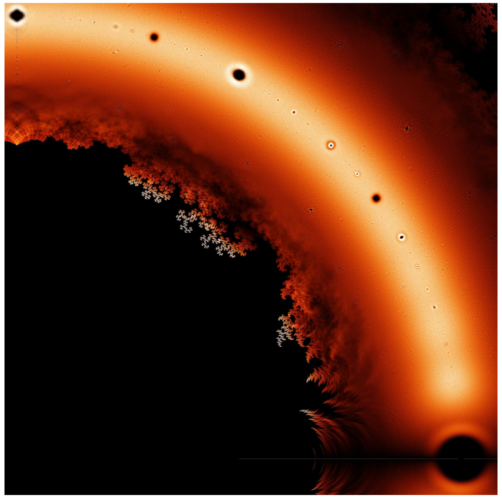
$$0{,}2482 \leq \mu^{ess}(C) \leq 0{,}2544$$ Doche 1999



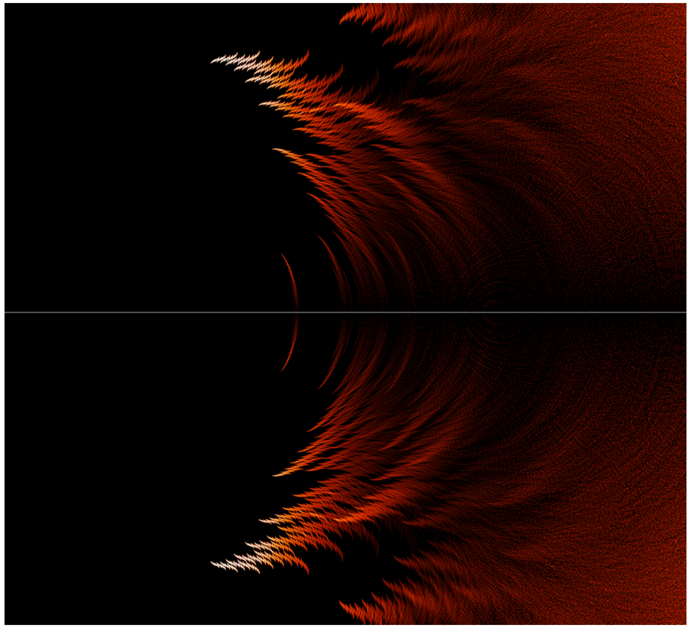
FURTHER OPEN PROBLEMS

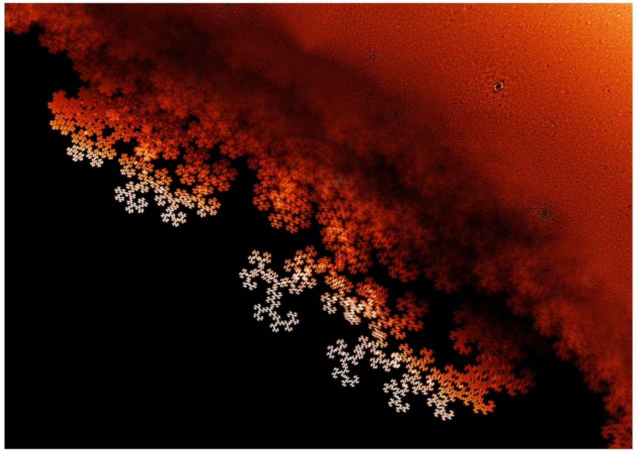
- Compute $\mu^{ess}(C)$.
- Is there an infinite number of height values below $\mu^{ess}(C)$?
- How do the Galois orbits of small points distribute?





GRACIAS !





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