A Macaulay formula for the sparse resultant

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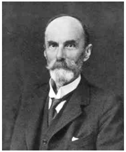
Based on joint work with C. D'Andrea and G. Jeronimo (arXiv:2004.14622)

The homogeneous resultant



For $i = 0, \dots, n$ let $d_i \geqslant 1$ and set

$$F_i = \sum_{|\boldsymbol{a}|=d_i} c_{i,\boldsymbol{a}} \, \boldsymbol{t}^{\boldsymbol{a}}$$

$\boldsymbol{t}^{\boldsymbol{a}} = t_0^{a_0} \dots t_n^{a_n}$

$\mathrm{Res}_{\boldsymbol{d}}$ is a multihomogeneous polynomial in the sets of coeffs $\boldsymbol{c}_i$, $i = 0, \dots, n$, that vanishes iff

$\boldsymbol{d} = (d_0, \dots, d_n)$

$$V_{\mathbb{P}^n}(F_0, \dots, F_n) \neq \varnothing$$

Its partial degrees are

$$\deg_{\boldsymbol{c}_i}(\mathrm{Res}_{\boldsymbol{d}}) = \prod_{j \neq i} d_j$$

Macaulay's formula

Set $\mathbb{K} = \mathbb{C}(\boldsymbol{c})$ and consider the Sylvester linear map

$$\mathbb{K}[\boldsymbol{t}]^{n+1} \longrightarrow \mathbb{K}[\boldsymbol{t}], \quad (G_0, \dots, G_n) \longmapsto \sum_{i=0}^{n} G_i F_i$$

If we restrict this map to finite pieces of equal dimension

$$\bigoplus_{i=0}^{n} V_i \longrightarrow V$$

and let $\mathcal{M}$ be the corresponding Sylvester matrix we have

a general fact: $\mathrm{Res}_{\boldsymbol{d}} \mid \det(\mathcal{M})$

and a particular but more surprising one:

there is $\mathcal{M}$ with a diagonal submatrix $\mathcal{E}$ such that $\mathrm{Res}_{\boldsymbol{d}} = \dfrac{\det(\mathcal{M})}{\det(\mathcal{E})}$

Example

Let $n = 2$ and $\boldsymbol{d} = (1, 2, 2)$: then

$$\begin{aligned} F_0 &= \alpha_0\, t_0 + \alpha_1\, t_1 + \alpha_2\, t_2 \\ F_1 &= \beta_0\, t_0^2 + \beta_1\, t_0\, t_1 + \beta_2\, t_1^2 + \beta_3\, t_0\, t_2 + \beta_4\, t_1\, t_2 + \beta_5\, t_2^2 \\ F_2 &= \gamma_0\, t_0^2 + \gamma_1\, t_0\, t_1 + \gamma_2\, t_1^2 + \gamma_3\, t_0\, t_2 + \gamma_4\, t_1\, t_2 + \gamma_5\, t_2^2 \end{aligned}$$

The corresponding Sylvester matrix and diagonal submatrix are

	t_0^3	$t_0^2\, t_1$	$t_0^2\, t_2$	$t_0\, t_1\, t_2$	$t_0\, t_1^2$	t_1^3	$t_1^2\, t_2$	$t_0\, t_2^2$	$t_1\, t_2^2$	t_2^3
$t_0^2\, F_0$	α_0	α_1	α_2							
$t_0\, t_1\, F_0$		α_0		α_2	α_1					
$t_0\, t_2\, F_0$			α_0	α_1				α_2		
$t_1\, t_2\, F_0$				α_0			α_1		α_2	
$t_0\, F_1$	β_0	β_1	β_3	β_4	β_2			β_5		
$t_1\, F_1$		β_0		β_3	β_1	β_2	β_4		β_5	
$t_2\, F_1$			β_0	β_1			β_2	β_3	β_4	β_5
$t_0\, F_2$	γ_0	γ_1	γ_3	γ_4	γ_2			γ_5		
$t_1\, F_2$		γ_0		γ_3	γ_1	γ_2	γ_4		γ_5	
$t_2\, F_2$			γ_0	γ_1			γ_2	γ_3	γ_4	γ_5

Example (cont.)

Then

$$\mathrm{Res}_{(1,2,2)} = \frac{\det\begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & & & & & & & \\ & \alpha_0 & & \alpha_2 & \alpha_1 & & & & & \\ & & \alpha_0 & \alpha_1 & & & & \alpha_2 & & \\ & & & \alpha_0 & & & \alpha_1 & & \alpha_2 & \\ \beta_0 & \beta_1 & \beta_3 & \beta_4 & \beta_2 & & & \beta_5 & & \\ & \beta_0 & & \beta_3 & \beta_1 & \beta_2 & \beta_4 & & \beta_5 & \\ & & \beta_0 & \beta_1 & & & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ \gamma_0 & \gamma_1 & \gamma_3 & \gamma_4 & \gamma_2 & & & \gamma_5 & & \\ & \gamma_0 & & \gamma_3 & \gamma_1 & \gamma_2 & \gamma_4 & & \gamma_5 & \\ & & \gamma_0 & \gamma_1 & & & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 \end{bmatrix}}{\det\begin{bmatrix} \beta_2 & \beta_5 \\ \gamma_2 & \gamma_5 \end{bmatrix}}$$

$$= \alpha_2^4\beta_2^2\gamma_0^2 - 2\alpha_1\alpha_2^3\beta_2\beta_4\gamma_0^2 + \alpha_1^2\alpha_2^2\beta_4^2\gamma_0^2 + 2\alpha_1^2\alpha_2^2\beta_2\beta_5\gamma_0^2 - 2\alpha_1^3\alpha_2\beta_4\beta_5\gamma_0^2 + \alpha_1^4\beta_5^2\gamma_0^2 - \alpha_2^4\beta_1\beta_2\gamma_0$$

$+$ 227 other monomial terms!

The sparse resultant

(Gelfand, Kapranov and Zelevinsky 1993, Sturmfels 1994)

For $i = 0, \dots, n$ let $\mathcal{A}_i \subset \mathbb{Z}^n$ nonempty and finite, and set

$$F_i = \sum_{\boldsymbol{a} \in \mathcal{A}_i} c_{i,\boldsymbol{a}} \, \boldsymbol{x}^{\boldsymbol{a}}$$

$\mathrm{Res}_{\boldsymbol{\mathcal{A}}}$ is a multihomogeneous polynomial in the $\boldsymbol{c}_i$'s vanishing iff

$$V_{X_{\boldsymbol{A}}}(F_0, \dots, F_n) \neq \emptyset$$

$\boldsymbol{\mathcal{A}} = (\mathcal{A}_0, \dots, \mathcal{A}_n)$

Its partial degrees are

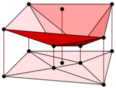
$$\deg_{\boldsymbol{c}_i}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}}) = \mathrm{MV}(\Delta_0, \dots, \Delta_{i-1}, \Delta_{i+1}, \dots, \Delta_n)$$

MV = mixed volume, $\Delta_i = \mathrm{conv}(\mathcal{A}_i)$

The Canny-Emiris formula

For $i = 0, \ldots, n$ choose a convex piecewise affine defined over $\mathcal{A}_i$

$$\rho_i \colon \Delta_i \longrightarrow \mathbb{R}$$



Let

$$\rho = \boxplus_{i=0}^{n} \rho_i \colon \Delta \to \mathbb{R} \quad \text{and} \quad S(\rho)$$

be their inf-convolution and the associated mixed subdivision of Δ

$\Delta = \sum_{i=0}^{n} \Delta_i$

Every n-cell $C \in S(\rho)^n$ has a decomposition $C = \sum_{i=0}^{n} C_i$ and with $C_i \in S(\rho_i)$

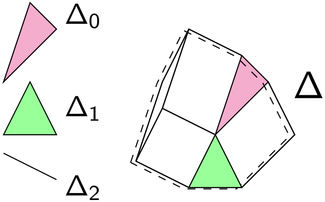
Suppose that for each C

$$\dim(C) = \sum_{i=0}^{n} \dim(C_i)$$

$S(\rho)$ is "tight"

The Canny-Emiris formula (cont.)

Let $\delta \in \mathbb{R}^n$ generic and set $\mathcal{B} = (\Delta + \delta) \cap \mathbb{Z}^n$



This data allows to pick suitable finite pieces of the Sylvester map

$$\mathbb{K}[\boldsymbol{x}^{\pm 1}]^{n+1} \longrightarrow \mathbb{K}[\boldsymbol{x}^{\pm 1}], \quad (G_0, \dots, G_n) \longmapsto \sum_{i=0}^{n} G_i F_i$$

giving $\mathcal{M} \in \mathbb{K}^{\#\mathcal{B} \times \#\mathcal{B}}$ and a diagonal submatrix $\mathcal{E}$

Conjecture (Canny and Emiris 1993)

There are ρ_i, $i = 0, \dots, n$, such that $\mathrm{Res}_{\boldsymbol{\mathcal{A}}} = \frac{\det(\mathcal{M})}{\det(\mathcal{E})}$.

The Canny-Emiris formula (cont.)

Canny and Emiris 1993, Sturmfels 1994: $\det(\mathcal{M})$ is a nonzero multiple of the sparse resultant with

$$\deg_{\boldsymbol{c}_0}(\det(\mathcal{M})) = \mathrm{MV}(\Delta_1, \dots, \Delta_n)$$

Permuting variables produces Canny-Emiris matrices $\mathcal{M}_i$ with

$$\mathrm{Res}_{\boldsymbol{\mathcal{A}}} = \gcd(\det(\mathcal{M}_0), \dots, \det(\mathcal{M}_n))$$

D'Andrea 2002: alternative contruction giving another Sylvester matrix $\widetilde{\mathcal{M}}$ with a diagonal submatrix $\widetilde{\mathcal{E}}$ such that

$$\mathrm{Res}_{\boldsymbol{\mathcal{A}}} = \frac{\det(\mathcal{M})}{\det(\mathcal{E})}$$

Other related results: Sturmfels and Zelevinsky 1994, Weyman and Zelevinsky 1994, Dickenstein and Emiris 2003, Khetan 2003, Emiris and Konaxis 2011, Bender, Faugère, Mantzaflaris and Tsigaridas 2018, Groh 2020, Busé, Mantzaflaris and Tsigaridas 2020...

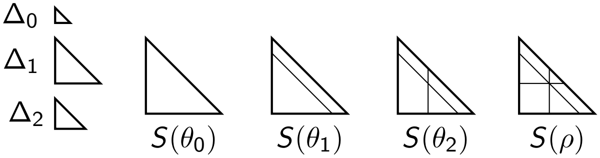
Incremental chains

A chain of mixed subdivisions of Δ

$$S(\theta_0) \preceq \cdots \preceq S(\theta_n) \preceq S(\rho)$$

with $\theta_k = \boxplus_{i=0}^{n} \theta_{k,i}$ is incremental if

$$\theta_{k,i} = 0 \text{ for } k \geqslant i$$



$S(\rho)$ is admissible if there is an incremental chain as above satisfying a technical condition

Main results

Theorem 1 (D'Andrea, Jeronimo, S 2020)

Let ρ_i, $i = 0, \dots, n$, such that $S(\rho)$ is admissible. Then

$$\mathrm{Res}_{\mathcal{A}} = \frac{\det(\mathcal{M})}{\det(\mathcal{E})}$$

Theorem 2 (D'Andrea, Jeronimo, S 2020)

Let $d_i \geqslant 1$, $i = 0, \dots, n$. There are ρ_i, $i = 0, \dots, n$, such that $S(\rho)$ is admissible and whose associated Sylvester matrix and diagonal submatrix coincide with Macaulay's

Strategy

Based on toric degenerations and on the functoriality of sparse resultants and Canny-Emiris matrices with respect to them:

$$\frac{\det(\mathcal{M}_{\boldsymbol{\mathcal{A}}})}{\mathrm{Res}_{\boldsymbol{\mathcal{A}}}} = \frac{\mathrm{init}_{\boldsymbol{\omega}}(\det(\mathcal{M}_{\boldsymbol{\mathcal{A}}}))}{\mathrm{init}_{\boldsymbol{\omega}}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}})} = \prod_{D} \frac{\det(\mathcal{M}_{\boldsymbol{\mathcal{A}}_D})}{\mathrm{Res}_{\boldsymbol{\mathcal{A}}_D}}$$

This allows to reduce to the basic cases where the formula is easy to prove directly

Questions and remarks

- Characterize (or at least give a test) to show that a given mixed subdivision of Δ is admissible
- How many of the possible mixed subdivisions of Δ are admissible?
- How many of them satisfy the Canny-Emiris formula?
- Can you find a better/simpler condition?
- Extend Macaulay's formula to the multiprojective setting

Thanks!