The Number of Solutions of a System of Polynomial Equations

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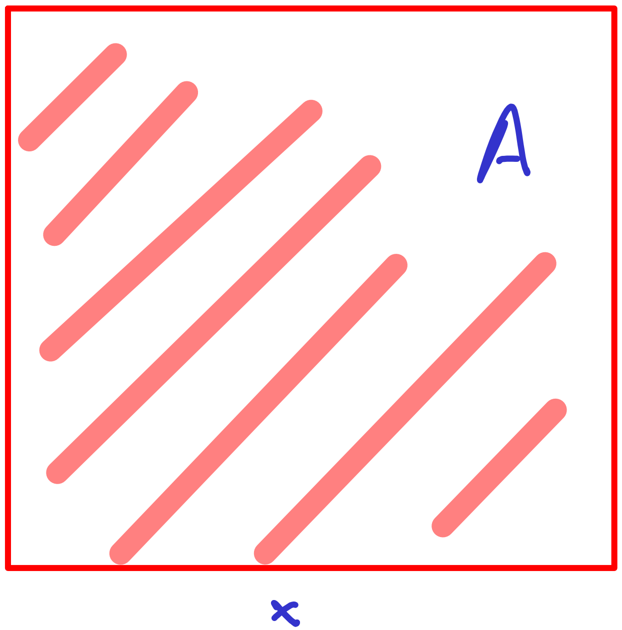
From a Babylonian clay tablet (~1800 BC)

Pb: Add the area and two-third of a square to obtain 0:35.
Which is the side of my square?



MS 3052
Eight mathematical problems with drawings of subdivided trapezoids and triangles.
Babylonia, 1763-1739 BC

$$x^2 + \frac{2}{3}x = \frac{35}{60}$$

Sol: Take 1. Two-thirds of 1 is 0:40. Half of this, 0:20, you multiply by 0:20 and it 0:6:40, you add to 0:35 and the result 0:41:40, has 0:50 as it square root. The 0:20 which you have multiplied by itself, you substract from 0:50, and 0:30 is the side of the square.

D. Burton. The history of mathematics, 1997.

In other words:

$$x = \sqrt{\left(\frac{0:40}{2}\right)^2 + 0:35} - \frac{0:40}{2} = \dots = 0:30 = \frac{30}{60} = \frac{1}{2}$$

THE FUNDAMENTAL THEOREM OF ALGEBRA

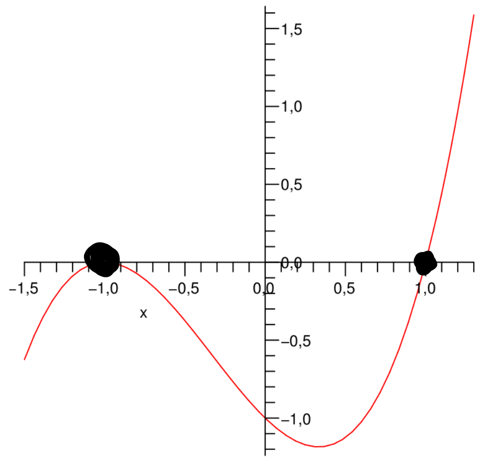
THM (D'ALEMBERT 1746 - GAUSS 1798)

LET $f \in \mathbb{C}[x]$. THEN $f(x)=0$ HAS $\deg(f)$ SOLUTIONS

i.e. $\mathbb{C}$ IS "ALGEBRAICALLY CLOSED"

Ex: $f = x^3 + x^2 - x - 1$

$$V(f) = \{x \in \mathbb{C} \mid f(x) = 0\} = \{\pm 1\}$$



Bézout's theorem (1764)

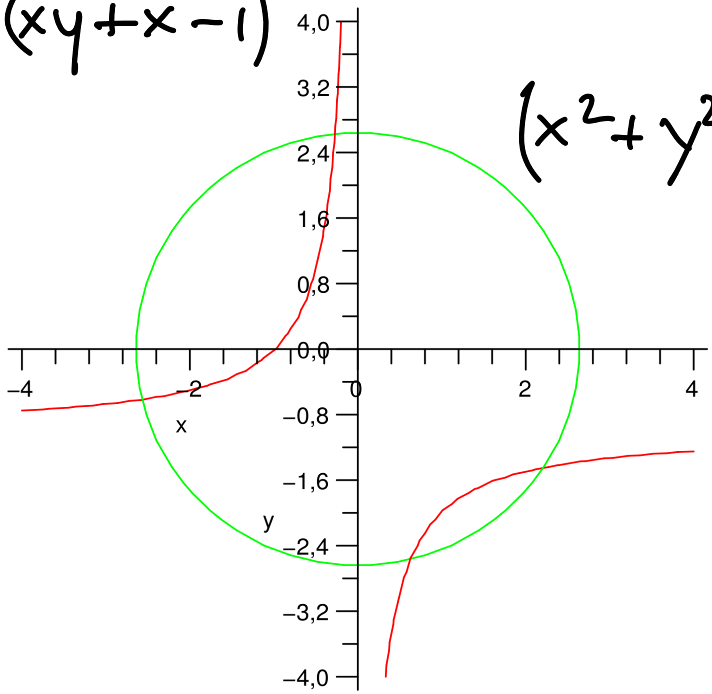
Let $f_1,\dots,f_n \in \mathbb{C}[x_1,\dots,x_n]$ st

$$f_1(x) = \dots = f_n = 0$$

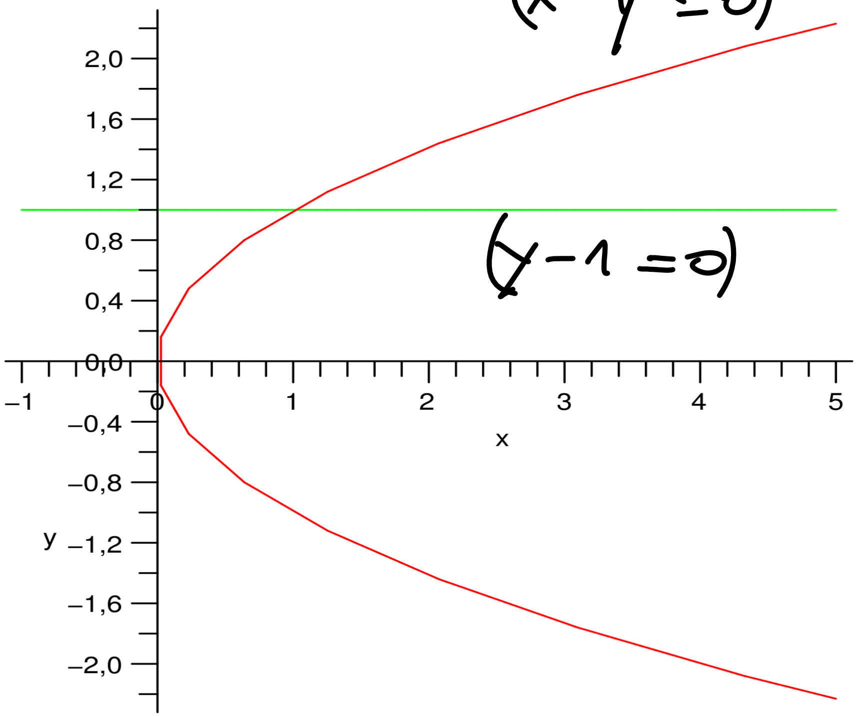
has finite solutions

Then it has $\le \prod_{i=1}^{n} \deg(f_i)$ solutions

$(xy+x-1)$



$(x^2+y^2-7=0)$



$(x-y^2=0)$

$(y-1=0)$

Bézout Theorem on $\mathbb{P}^n$

Given homogeneous $f_1,\dots,f_n \in \mathbb{C}[x_1,\dots,x_n]$

set $V(f_1,\dots,f_n) = \{\underline{x}\in\mathbb{P}^n \mid f_1(\underline{x}) = \dots = f_n(\underline{x}) = 0\}$

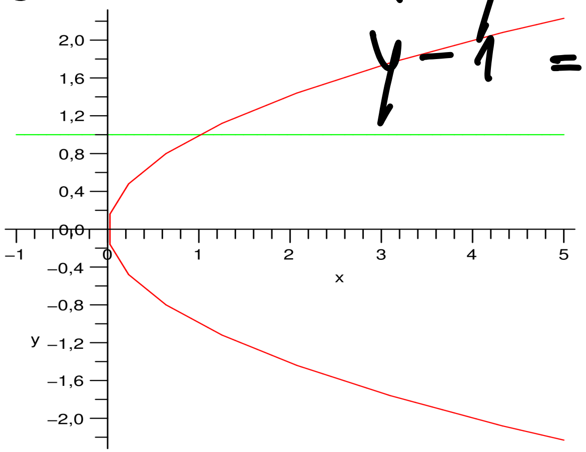
T: If $\# V(f_1,\dots,f_n)<\infty$ then

$$\# V(f_1,\dots,f_n) = \prod_{i=1}^{n} \deg(f_i)$$

Example (cont.): $(z:x:y)\in\mathbb{P}^2$ st $xz-y^2=y-z=0$

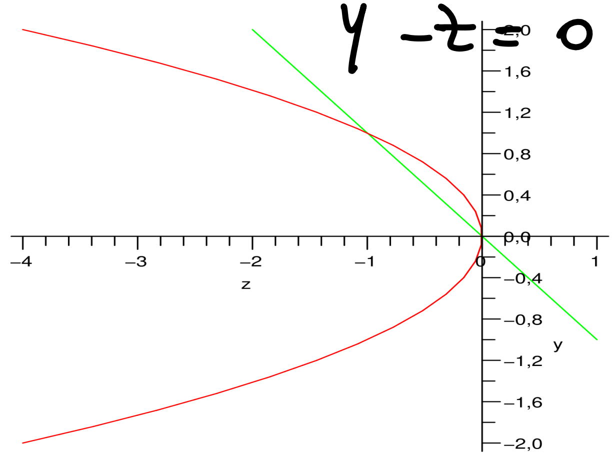
In $z=1$: $x-y^2=0$

$y-1=0$



In $x=1$: $z-y^2=0$

$y-z=0$



DEGREE OF VARIETIES

Let $X^r \subset \mathbb{P}^n$ irreducible variety

The degree of X is

$$\deg(X) = \# X \cap E \in \mathbb{N}^*$$

for E generic linear space of dimension $n-r$

- $r=0$: $\deg(X) = \# X$
- X linear: $\deg(X) = 1$
- $r=n-1$: $X = V(F)$ with $F \in \mathbb{C}[x_0, \dots, x_n]$ irreducible

$$\deg(X) = \deg(F)$$

HILBERT FUNCTION OF IDEALS

Let $I \subset k[x_0, \dots, x_n]$ homogeneous ideal

Then $k[x_0, \dots, x_n]/I = \bigoplus_{d \geq 0} \left(k[\underline{x}]/I\right)_d$

The Hilbert function of I is

$$H_I : \mathbb{N} \to \mathbb{N} \qquad d \mapsto \dim_k \left(k[\underline{x}]/I\right)_d$$

$\underline{T}$ (Hilbert 1893) $\exists\, P_I \in \mathbb{Q}[t]$ and $d_0 \geq 0$ st

$$H_I(d) = P_I(d) \qquad \text{for } d \geq d_0$$

Moreover $\deg(P_I) = \dim(V(I))$

If $I = I(X)$

$$P_I = \deg(X) \frac{t^r}{r!} + O(t^{r-1})$$

Bézout Theorem III

Prop Let $X^r \subset \mathbb{P}^n$ irred variety and $H \subset \mathbb{P}^n$ hypersurface st $X \not\subset H$. Then

$$\deg(X \cap H) = \deg(X)\deg(H)$$

Cor (classical Bézout)
Let $H_1, \dots, H_n \subset \mathbb{P}^n$ st $\# H_1 \cap \dots \cap H_n < \infty$. Then

$$\#(H_1 \cap \dots \cap H_n) = \prod_{i=1}^{n} \deg H_i$$

Pf: Write $H = V(f)$ with $f \in \mathbb{C}[\underline{x}]_\delta$ irred

$\exists$ exact sequence of graded algebras

$$0 \to \mathbb{C}[\underline{x}]/I(X) \xrightarrow{\cdot f} \mathbb{C}[\underline{x}]/I(X) \to \mathbb{C}[\underline{x}]/I(X)+(f) \to 0$$

$$\Rightarrow H_{I(X)}(d) = H_{I(X)}(d-\delta) + H_{I(X)+(f)}(d)$$

$$\deg(X)\,\frac{d^r}{r!} = \deg(X)\,\frac{(d-\delta)^r}{r!} + \deg(X \cap V(f))\,\frac{d^{r-1}}{(r-1)!} + O(d^{r-2})$$

$$\Rightarrow \quad \deg(X \cap H) = \delta \cdot \deg(X) \qquad \boxtimes$$

AFFINE VARIETIES

Let $V^r \subset \mathbb{C}^n$ irred affine variety

Set $i: \mathbb{C}^n \longrightarrow \mathbb{P}^n \quad (z_1, \dots, z_n) \mapsto (1 : z_1 : \dots : z_n)$

$\overline{V} \subset \mathbb{P}^n$ the closure of $i(V)$.

Set

$$\deg(V) := \deg(\overline{V}) = \# V \cap E$$

for $E \subset \mathbb{C}^n$ generic affine space of dimension $n-r$

$$H_{I(V)}(d) = H_{I(\overline{V})}(d) = \dim\left(\mathbb{C}[x_1, \dots, x_n]_{\leq d} \Big/ I(V)_{\leq d} \right)$$

Affine Bézout

Let $V \subset \mathbb{C}^n$ and write

$$V = \bigcup_i V_i \quad \text{irreducible decomposition}$$

Set

$$\deg(V) := \sum_i \deg(V_i) \qquad \text{(Heintz 1983)}$$

I. Let $V, W \subset \mathbb{C}^n$. Then

$$\deg(V \cap W) \leq \deg(V)\deg(W)$$

set-theoretic intersection!

Example: $f \in \mathbb{C}[x,y,z]$ squarefree

C curve contained in $\mathrm{Sing}(V(f))$

$$\Longrightarrow \quad \deg(C) \leq \deg(f)(\deg(f) - 1)$$

BACK TO SYSTEMS OF EQUATIONS

$f = 1 + x - y + 2xy - x^2y + xy^2 \qquad g = 2 - x + y - 7xy + x^2y + xy^2$

Bezout predicts $3 \cdot 3 \leq 9$ solutions in $\mathbb{C}^2$

There are 5.

Why this discrepancy?

Possible answer:

There are also two double roots $(0:1:0)$, $(0:0:1)$ at the "line at infinity" $(z=0)$

ALTERNATIVE ANSWER

Do not apply classical Bézout but the "intermediate version"

Consider the monomial map

$$\varphi : \mathbb{C}^2 \hookrightarrow \mathbb{P}^5 \qquad (x,y) \mapsto (1 : x : y : xy : x^2y : xy^2)$$

$$X := \overline{\mathrm{im}(\varphi)} \subset \mathbb{P}^5 \qquad \deg(X) = 5$$

$$\Rightarrow V(f,g) = \varphi^{-1}\left(X \cap V(\ell_1,\ell_2)\right)$$

with $\ell_1 = y_0 + y_1 - y_2 + 2y_3 - y_4 + y_5$

$\ell_2 = 2y_0 - y_1 + y_2 - 7y_3 + y_4 + y_5$

$$\Rightarrow \# V(f,g) \leq \deg(X) = 5$$

no "lost points"!

THE DEGREE OF A TORIC VARIETY

Let $a_0, a_1, \dots, a_N \in \mathbb{Z}^n$ with $a_0 = 0$ and $\sum_j \mathbb{Z} a_j = \mathbb{Z}^n$

Consider the monomial map

$$(\mathbb{C}^\times)^n \xrightarrow{\varphi} \mathbb{P}^N \qquad \underline{x} \mapsto (\underline{x}^{a_0} : \underline{x}^{a_1} : \dots : x^{a_N})$$

the (projective) toric variety $X = \overline{\mathrm{im}(\varphi)} \subset \mathbb{P}^N$

and the lattice polytope

$$\Delta = \mathrm{conv}(a_0, a_1, \dots, a_N) \subset \mathbb{R}^n$$

I (Teissier 1979) $\qquad \deg(X) = n!\,\mathrm{vol}(\Delta)$

Pf: Consider the morphism of algebras

$$\mathbb{C}[y_0, \dots, y_N]/I(X) \overset{\varphi^*}{\hookrightarrow} \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \qquad y_j \mapsto \underline{x}^{a_j}$$

$$H_{I(X)}(d) = \# \sum_{\substack{\lambda_j \in \mathbb{N} \\ \sum_j \lambda_j = d}} \lambda_j a_j \simeq \#(d\Delta \cap \mathbb{Z}^n) \simeq \mathrm{vol}(d\Delta) = \mathrm{vol}(\Delta)\, d^n$$

$\Rightarrow$ $\deg(X) = n!\,\mathrm{vol}(\Delta)$ $\boxtimes$

Cor (Bernstein-Kushnirenko thm, "unmixed" version)

Let $\Delta \subset \mathbb{R}^n$ lattice polytope and $f_1, \dots, f_n \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ st $N(f_i) \subset \Delta \ \forall i$ and $\# V(f_1, \dots, f_n) < \infty$. Then

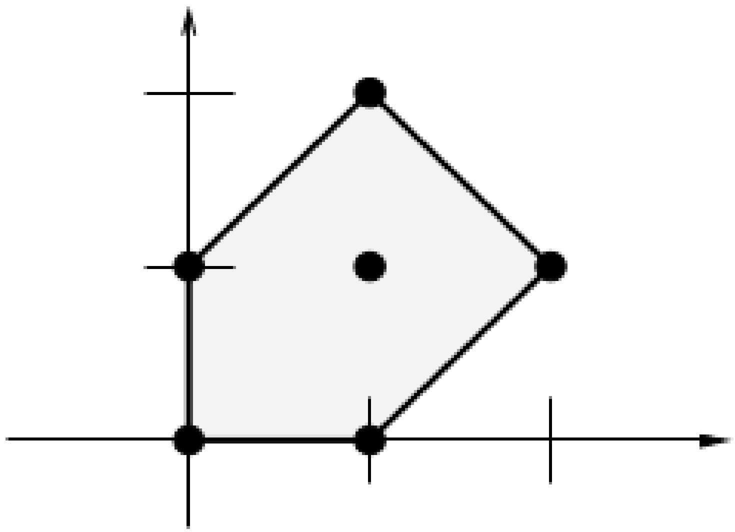
$$\# V(f_1, \dots, f_n) \leq n! \operatorname{vol}(\Delta)$$

Example (cont)

$f = 1 + x - y + 2xy - x^2y + xy^2$

$g = 2 - x + y - 7xy + x^2y + xy^2$

$\Rightarrow \# V(f, g) \leq 5$



A FURTHER EXAMPLE

$f = (s-1) + (s-1)^2 x - 3 s x^2 \qquad g = -7(s-1) + (s-1)^2 x + 3 s x^2$

$f = g = 0$ has two solutions: $(4,1)$ simple, $\left(-\frac{1}{2}, -2\right)$ double

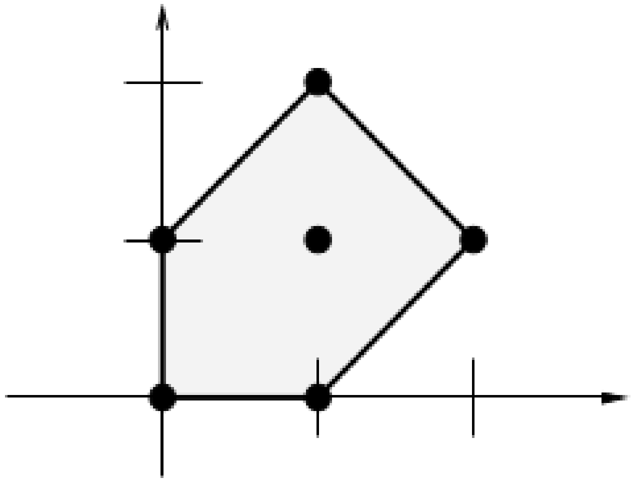
$\Rightarrow \# V(f,g) = 3$

Bounds:

- Bezout: $\# V \leq \deg(f) \deg(g) = 9$
- Bihomogeneous Bezout:

  $\# V \leq \deg_x(f) \deg_y(g) + \deg_y(f) \deg_x(g)$

  $= 2 \cdot 2 + 2 \cdot 2 = 8$
- BK: $\# V \leq 2! \, \mathrm{vol}(\Delta) = 5$



v-ADIC NEWTON POLYTOPES

Let $f = \sum_{j=0}^{N} \alpha_j(s)\, \underline{x}^{a_j} \in \mathbb{C}(s)[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$

For $v \in \mathbb{P}^1$, the v-adic Newton polytope is

$$N_v(f) = \operatorname{conv}\left((a_j, -\operatorname{ord}_v(\alpha_j))\right)_j \subset \mathbb{R}^{n+1}$$

with $\operatorname{ord}_v(\alpha_j)$ order of zero/pole of α_j at v

Projects onto

$$N(f) = \operatorname{conv}(a_0, \dots, a_N) \subset \mathbb{R}^n$$

The v-adic roof function is

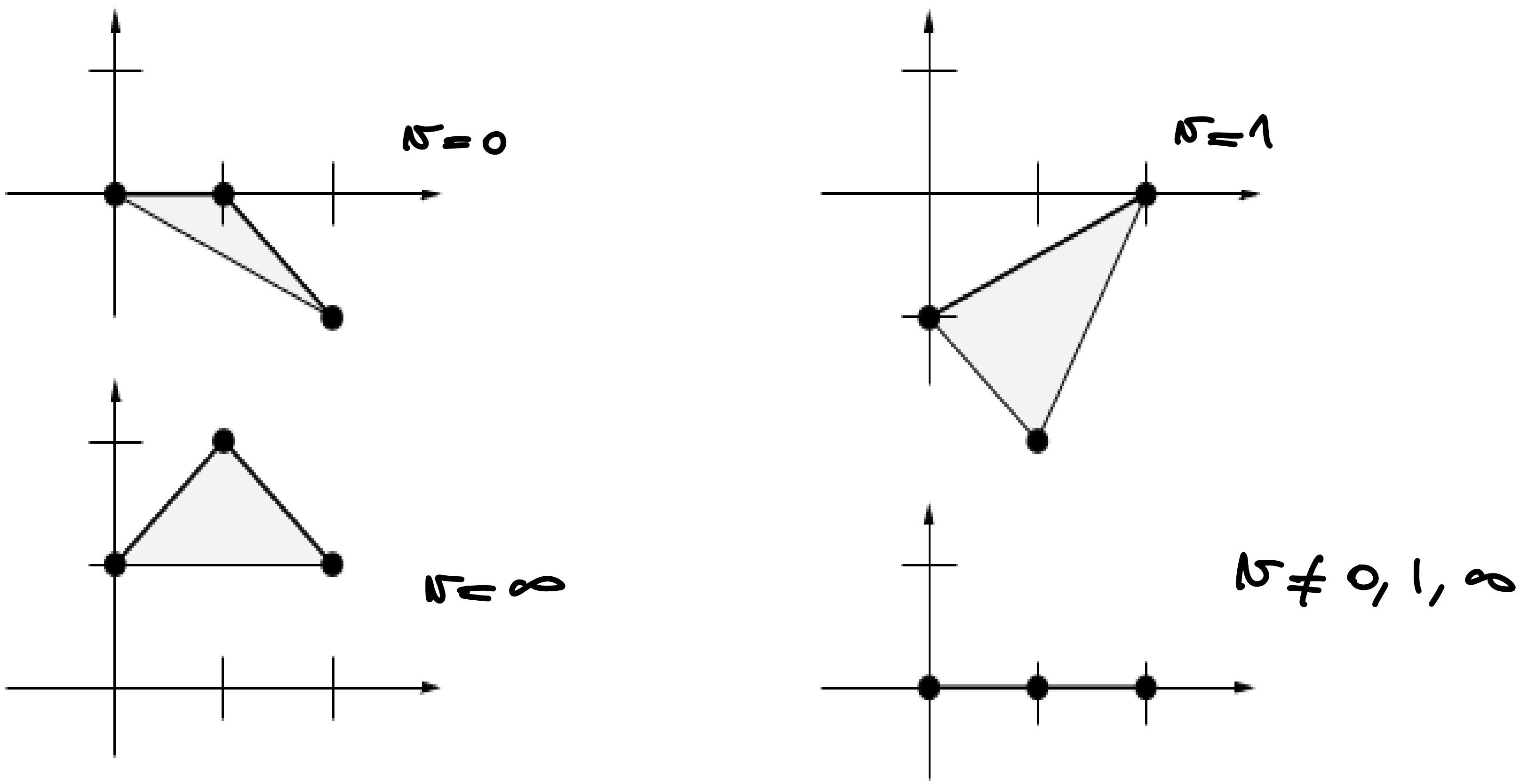
$$\vartheta_v : N(f) \to \mathbb{R}$$

parametrizing the upper envelope of $N_v(f)$

EXAMPLE (CONT.)

$$f = (s-1) + (s-1)^2 x - 3 s x^2$$

$$g = -7(s-1) + (s-1)^2 x + 3 s x^2$$



$\underline{T}$ (Philippon - S 2008)

$f_0, \dots, f_n \in \mathbb{C}[s][x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ without common factors

st $\# V(f_0, \dots, f_s) \subset \mathbb{C} \times (\mathbb{C}^\times)^n < \infty$ in $\mathbb{C}[s]$

Let $\Delta \subset \mathbb{R}^n$ and $\vartheta_v : \Delta \to \mathbb{R}$ $(v \in \mathbb{P}^1)$

st $N(f_i) \subset \Delta$ and $\vartheta_{i,v} \leq \vartheta_v$

with $\vartheta_{i,v} : N(f_i) \to \mathbb{R}$ v-adic roof of f

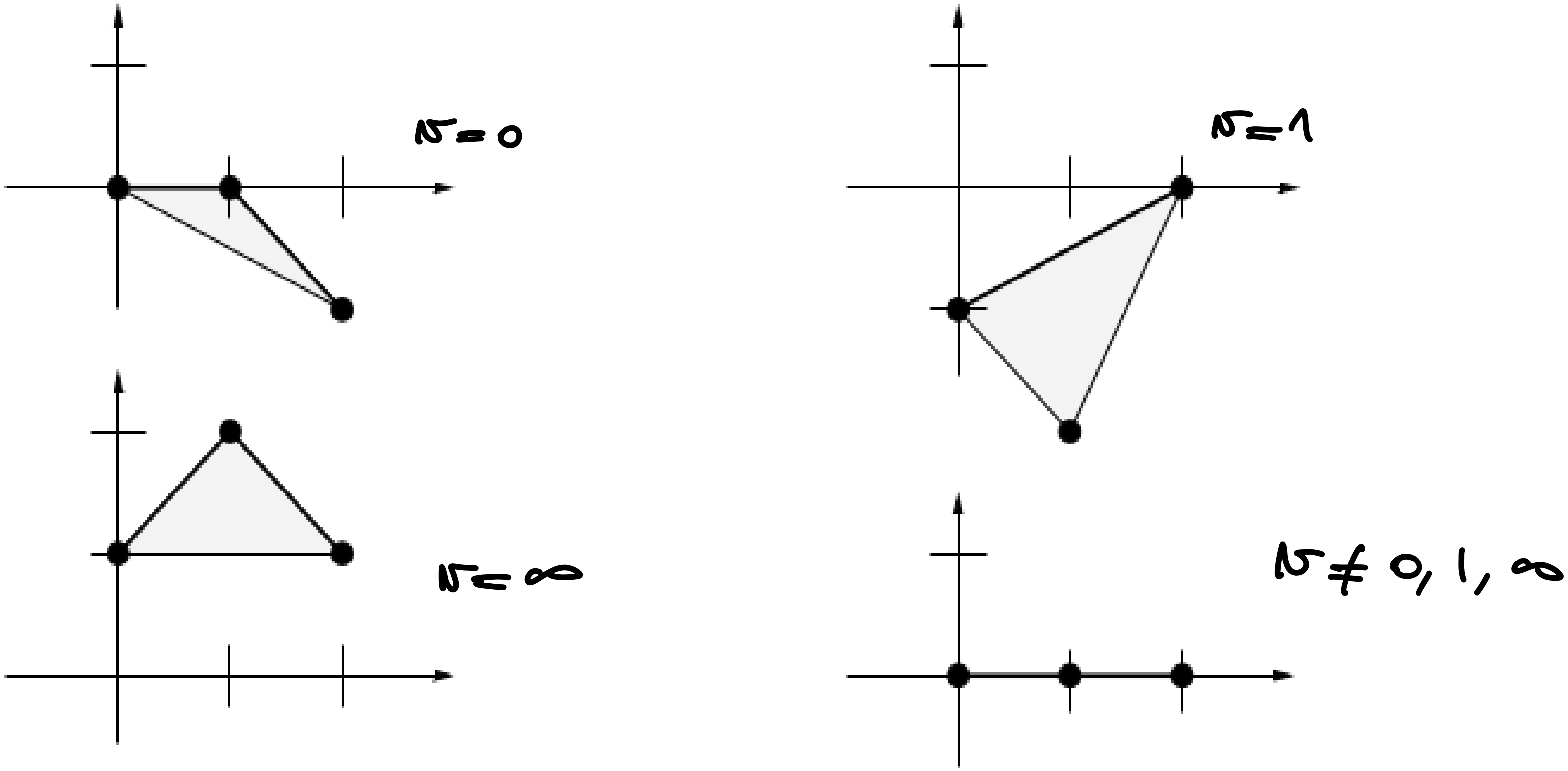
Then

$$\# V(f_0, \dots, f_s) \leq (n+1)! \sum_{v \in \mathbb{P}^1} \int_\Delta \vartheta_v(u)\, du$$

Example (cont.)

$$f = (s-1) + (s-1)^2 x - 3 s x^2 \qquad g = -7(s-1) + (s-1)^2 x + 3 s x^2$$

$$N(f) = N(g) = [0,2]$$



$$\# V(f,g) \leqslant 2! \left(\int_0^2 \vartheta_0 \, du + \int_0^2 \vartheta_1 \, du + \int_0^2 \vartheta_\infty \, du \right) = 2\left(-\frac{1}{2} - 1 + 3\right) = 3$$

"PROOF"

$$f = (s-1) + (s-1)^2 x - 3 s x^2 \qquad g = -7(s-1) + (s-1)^2 x + 3 s x^2$$

Let

$$\varphi : \mathbb{C} \times \mathbb{C}^\times \to \mathbb{P}^1 \times \mathbb{P}^2 \quad (s,t) \mapsto \left((1:s), (s-1 : (s-1)^2 t : s t^2)\right)$$

$$X := \overline{\operatorname{im}(\varphi)} \subset \mathbb{P}^1 \times \mathbb{P}^2$$

The system $f = g = 0$ on $\mathbb{C} \times \mathbb{C}^\times$ is equiv to

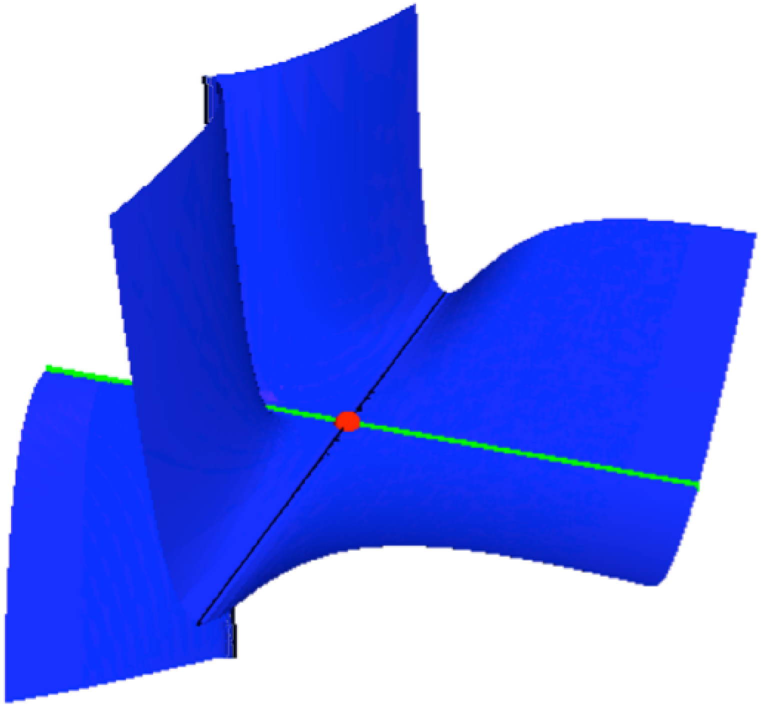
$y_0 + y_1 - 3 y_2 = -7 y_0 + y_1 + 3 y_2 = 0$ on X

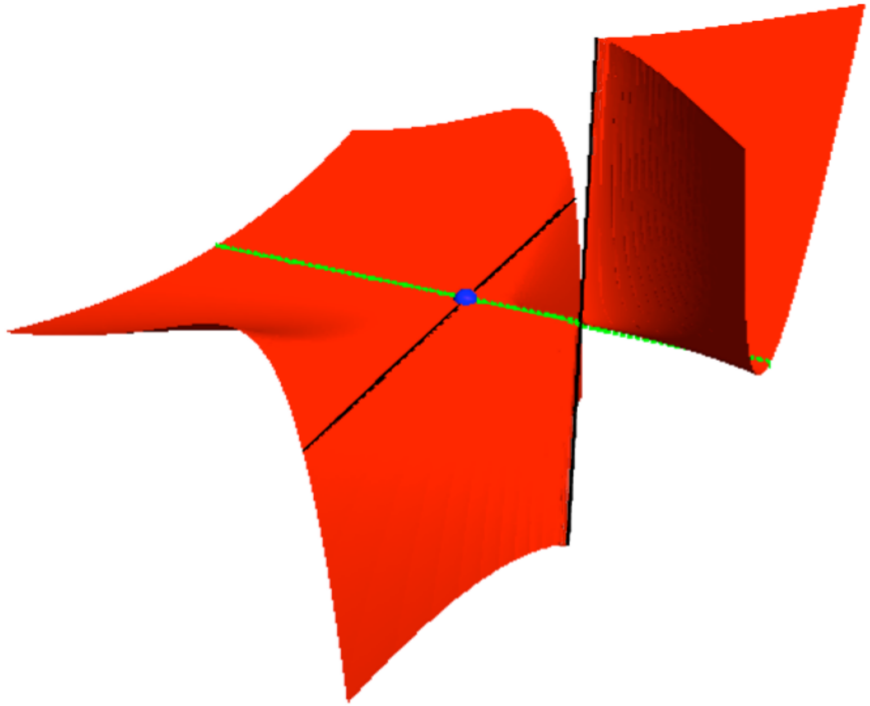
Can show $\deg_L(X) = 3$

hence $\# V(f,g) \leq 3$ $\boxtimes$

Pics of X

In $(s_1 \neq 0, x_0 \neq 0)$ In $(s_0 \neq 0, x_0 \neq 0)$





X is a toric curve over $\mathbb{P}^1$

In particular, it is a T-variety of complexity 1

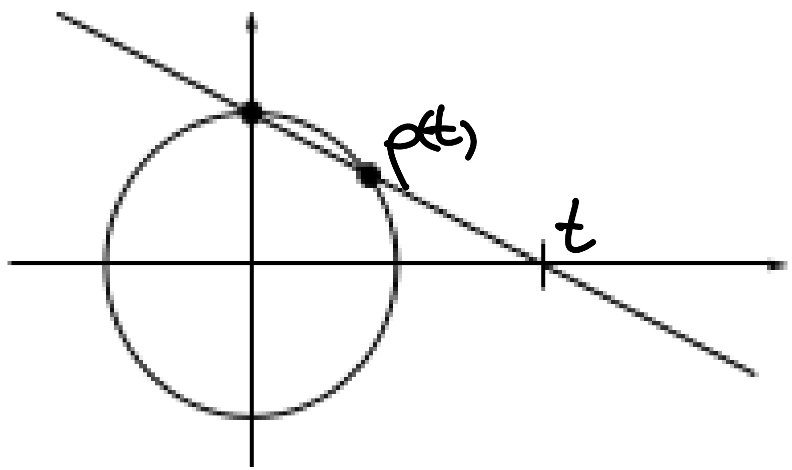
THE NEWTON POLYGON OF A RATIONAL CURVE

Let $\rho : \mathbb{C} \dashrightarrow \mathbb{C}^2$ rtl map

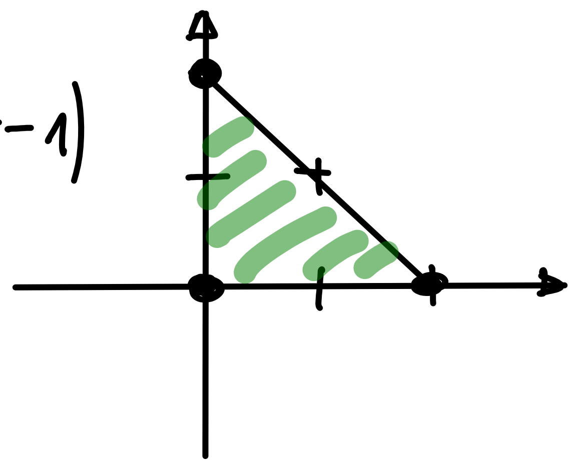
$\Rightarrow \quad C = \overline{\mathrm{im}(\rho)}$ rtl plane curve

Pb: Compute the Newton polygone of an equation of C

Ex $\quad \rho(t) = \left(\dfrac{2t}{t^2-1} , \dfrac{t^2-1}{t^2+1} \right) \qquad C = V(x^2+y^2-1)$



$N(x^2+y^2-1)$



For $v \in \mathbb{P}^1$ set

$$\mathrm{ord}_v(\varphi) = \left(\mathrm{ord}_v(\varphi_1),\ \mathrm{ord}_v(\varphi_2)\right) \in \mathbb{Z}^2$$

order of zero / pole of φ at v

- $\mathrm{ord}_v(\varphi) = (0,0)$ $\quad \forall^{\circ} v$
- $\sum_{v \in \mathbb{P}^1} \mathrm{ord}_v(\varphi) = (0,0)$

T (Dickenstein-Fleitcher-Sturmfels-Teveler 2007)

$N(E_c)$ constructed by rotating $-90°$ the $\mathrm{ord}_v(\varphi)$ and concatenating counterclockwise.

Pf: uses tropical geometry.

CONNECTION WITH INTERSECTION THEORY

The support function of a convex $Q \subset \mathbb{R}^2$ is

$$w_Q : \mathbb{R}^2 \to \mathbb{R} \qquad \tau \mapsto \max\{\langle \tau, (x_1,x_2)\rangle \mid (x_1,x_2) \in Q\}$$

"weight" of Q in the τ-direction.

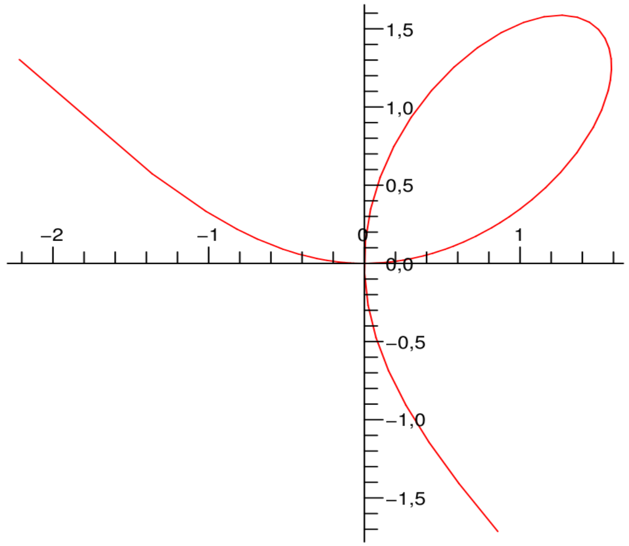
Prop (D'Andrea-S. 2007) Given $\tau = (\tau_1, \tau_2) \in \mathbb{N}^2$

$$w_{N(E_c)} = \# V\left(x_1^{\tau_1} - \rho_1(t),\ x_2^{\tau_2} - \rho_2(t),\ \ell_0 + \ell_1 x_1 + \ell_2 x_2\right)$$

for $\ell_i \in \mathbb{C}$ generic.

This number of solutions can be computed exactly and gives another proof of the DFST theorem.

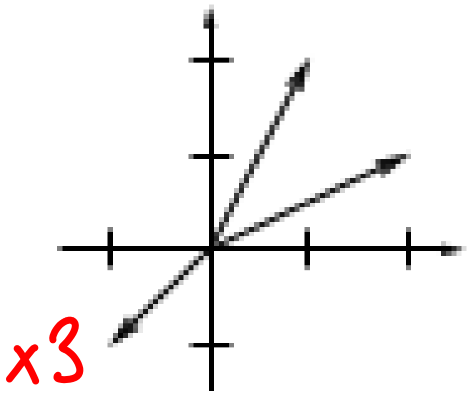
The Descartes' Folium

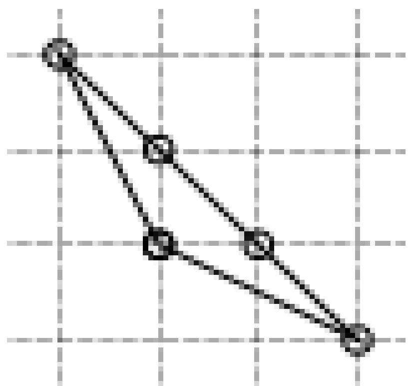


$$\rho : t \mapsto \left(\frac{3t^2}{t^3+1} , \frac{3t}{t^3+1} \right)$$

- $\mathrm{ord}_0(\rho) = (2,1)$
- $\mathrm{ord}_\infty(\rho) = (1,2)$
- $\mathrm{ord}_\omega = (-1,-1)$ for $\omega = 1, \ \frac{-1}{2} \pm \frac{\sqrt{3}}{2}$

Polygon of C and inner normals





$$E_c = x^3 + y^3 - 3xy$$

THANK YOU !