Lecture in Moscou

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A refinement of the Kušnirenko-Bernštein theorem.

(joint work with Martín Sombra)

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Gelfond's method for algebraic independence has been first applied to prove the algebraic independence of at least two values of the exponential function. It exploits the construction of an auxiliary function given as an exponential polynomial, that is a function on a multiplicative torus or, equivalently, a Laurent polynomial. Higher dimensional generalisation of Gelfond's method also makes a heavy use of the so-called Bézout's theorem in various guise : geometric, arithmetic and metric. It turns out that in this context a much more precise estimate, due to Kušnirenko and Bernštein, is known to bound the number of common zeros in a torus of a family of Laurent polynomials. Here, we shall recall Kušnirenko-Bernštein's theorem and present some new improvements on this beautiful result.

Throughout the lecture we will denote by $\mathbf{k}$ a commutative field algebraically closed.

§ 1. **Degrees of intersections**

Let $f, g \in \mathbf{k}[x^{\pm}, y^{\pm}]$ and $Z_0 = Z_0(f,g)$ denote the set of common isolated zeros of f and g in $(\mathbf{k}^{\times})^2$. Set $\Delta_x(f) = -\operatorname{ord}_{1/x}(f) - \operatorname{ord}_x(f)$ and $\Delta_y(f) = -\operatorname{ord}_{1/y}(f) - \operatorname{ord}_y(f)$ and similarly for g .

Bézout (bihomogeneous) theorem tells us that :

$$\operatorname{card}(Z_0) \leq \Delta_x(f)\Delta_y(g) + \Delta_x(g)\Delta_y(f) \ ,$$

with equality for general Laurent polynomials of given bidegrees.

Write $f = \sum_{i,j\in\mathbf{Z}} \alpha_{ij} x^i y^j$ and $g = \sum_{i,j\in\mathbf{Z}} \beta_{ij} x^i y^j$, one associates to these Laurent polynomials convex polytopes in $\mathbf{R}^2$ as follows :

$$\begin{aligned}\tilde{P} &:= \mathrm{Conv}\{(i,j); \alpha_{ij} \neq 0\}\\ \tilde{Q} &:= \mathrm{Conv}\{(i,j); \beta_{ij} \neq 0\}\end{aligned} .$$

We denote by $\tilde{P}+\tilde{Q}$ the Minkowski sum of two convexes.

Bernštein-Kušnirenko's theorem asserts :

$$\mathrm{card}(Z_0) \leq \mathrm{MV}_{\mathbf{R}^2}(\tilde{P},\tilde{Q}) := \mathrm{Vol}_{\mathbf{R}^2}(\tilde{P}+\tilde{Q}) - \mathrm{Vol}_{\mathbf{R}^2}(\tilde{P}) - \mathrm{Vol}_{\mathbf{R}^2}(\tilde{Q}) \ ,$$

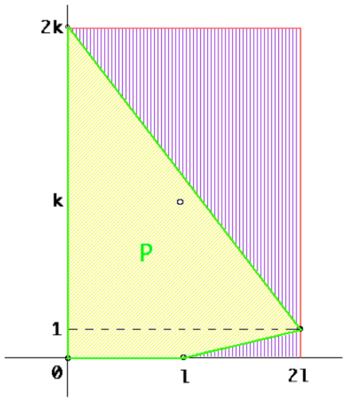
with equality for general Laurent polynomials with given polytopes.

Remark – For $\tilde{P}=\tilde{Q}$ one has $\mathrm{MV}_{\mathbf{R}^2}(\tilde{P},\tilde{Q}) := 2\mathrm{Vol}_{\mathbf{R}^2}(\tilde{P})$.

Example – Let $k,\ell \in \mathbf{N}^*$, consider the polynomials

$$\begin{aligned} f &:= (y-1)^{2k} + (y-1)^k x^\ell - 3yx^{2\ell}\\ g &:= -3(y-1)^{2k} + (y-1)^k x^\ell + 3yx^{2\ell}\end{aligned} .$$

One verifies easily that the common zeros of these two polynomials are the points of coordinates $(u^k, \frac{2}{3})$ where u runs through the ℓ-th root of $-\frac{1}{3}$, therefore $\mathrm{card}(Z_0) = \ell$. However, Bézout's theorem only gives the upper bound $\mathrm{card}(Z_0) \leq 8k\ell$ and Bernštein-Kušnirenko's $\mathrm{card}(Z_0) \leq 2\mathrm{Vol}_{\mathbf{R}^2}(\tilde{P}) = 4k\ell + \ell$.



Remark – In the above example, one can replace the middle term $(y-1)^k x^\ell$ by $(y-1)^k x^{\ell-1}$, and then the upper bound given by Bernštein-Kušnirenko becomes : $\mathrm{card}(Z_0) \leq 4k\ell + \ell - 1$.

Write now $f = \sum_{i\in\mathbf{Z}} \alpha_i(y) x^i$ and $g = \sum_{i\in\mathbf{Z}} \beta_i(y) x^i$ with $\alpha,\beta \in \mathbf{k}[y^\pm]$, suppose f and g are of content 1 in $\mathbf{k}[y^\pm]$. Introduce for all $v \in \mathbf{P}_1(\mathbf{k})$ the following convex polytopes in $\mathbf{R}^2$:

$$\begin{aligned} P_v &= \mathrm{Conv}\{(i, -\mathrm{ord}_v(\alpha_i)); \alpha_i \neq 0\}\\ Q_v &= \mathrm{Conv}\{(i, -\mathrm{ord}_v(\beta_i)); \beta_i \neq 0\}\end{aligned}$$

and the intervals in $\mathbf{R}$

$$\begin{aligned} P &= \mathrm{Conv}\{i; \alpha_i \neq 0\} \\ Q &= \mathrm{Conv}\{i; \beta_i \neq 0\} \end{aligned} .$$

Let $\vartheta_{P_v} : P \to \mathbf{R}$ be the function defined by $\vartheta_{P_v}(u) = \max\{t; (u,t) \in P_v\}$ and similarly $\vartheta_{Q_v} : Q \to \mathbf{R}$. This « roof » function also appears in tropical geometry.

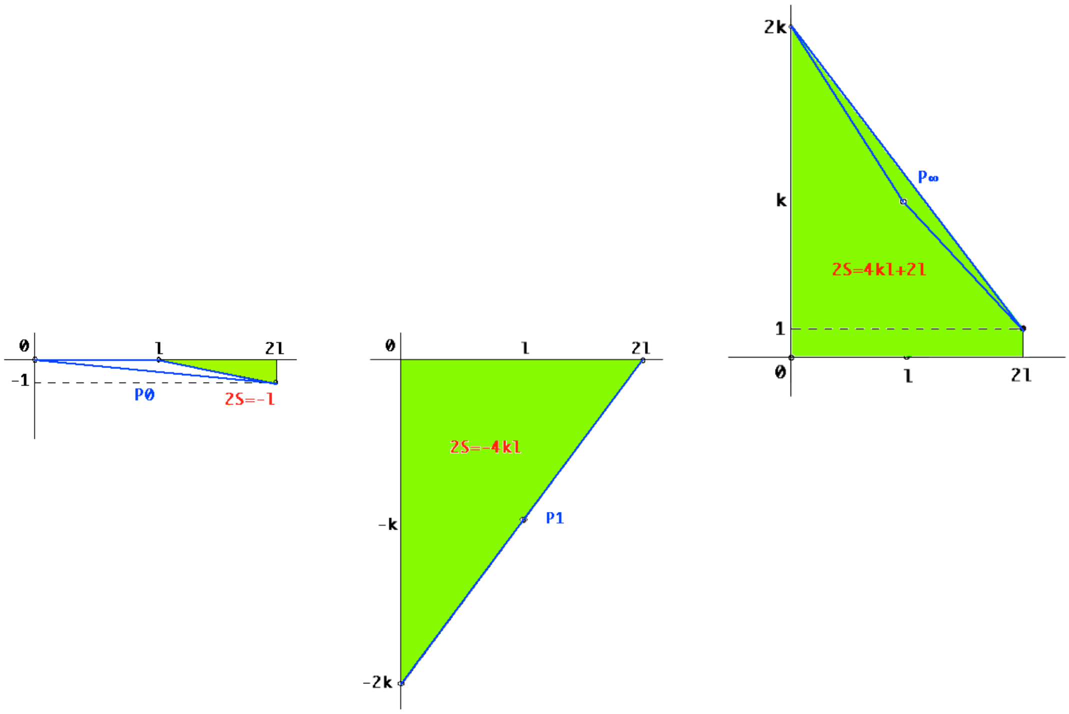
We also denote $\vartheta_{P_v} \boxplus \vartheta_{Q_v} : P+Q \to \mathbf{R}$ the function defined by $u \mapsto \max\{\vartheta_{P_v}(s) + \vartheta_{Q_v}(t); s \in P, t \in Q, s+t = u\}$. This operation is known as the *sup-convolution* in convex analysis.

Theorem in dimension 2 [**1**] – *With the above notations one has*

$$\mathrm{card}(Z_0) \leq \sum_{v \in \mathbf{P}_1(\mathbf{k})} \mathrm{MI}(\vartheta_{P_v}, \vartheta_{Q_v}) := \int_{P+Q} (\vartheta_{P_v} \boxplus \vartheta_{Q_v})(u)du - \int_P \vartheta_{P_v}(u)du - \int_Q \vartheta_{Q_v}(u)du \ ,$$

with equality for general polynomials with given polytopes P_v *and* Q_v *for all* $v \in \mathbf{P}_1(\mathbf{k})$ *.*

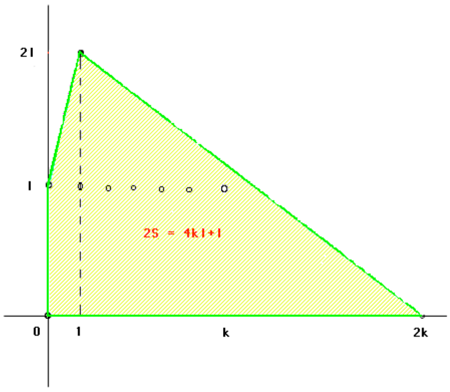
Example (continuation) – The only non trivial polytopes (*i.e.* $P_v \neq P \times \{0\}$ and $Q_v \neq Q \times \{0\}$) show at places $v = 0, 1$ and ∞ .



Consequently, one obtain $\ell = \mathrm{card}(Z_0) \leq -\ell - 4k\ell + 4k\ell + 2\ell = \ell$.

Remark – Replacing again the middle term $(y-1)^k x^\ell$ by $(y-1)^k x^{\ell-1}$, the upper bound given by our theorem above is : $\mathrm{card}(Z_0) \leq 2k + \ell - 1$, which is actually an equality.

On an other hand, writing our polynomials $f = \sum_{j\in \mathbf{Z}} \alpha_j(x)y^j$ and $g = \sum_{j\in \mathbf{Z}} \beta_j(x)y^j$, the application of theorem theorem in dimension 2 [1] doesnot improve on Bernštein-Kušnirenko's estimate. Indeed, one checks that the only place showing a non trivial polytope is the place $v=\infty$, where the polytope appears to be :



and the upper bound provided is $\mathrm{card}(Z_0) \leq 4k\ell + \ell$, the same as with Bernštein-Kušnirenko's theorem.

Remarks – 1) $\mathrm{MI}(\vartheta_{P_v}, \vartheta_{Q_v}) \leq 0$ for all $v \neq 0, \infty$.

2) $\mathrm{MI}(\vartheta_{P_0}, \vartheta_{Q_0}) + \mathrm{MI}(\vartheta_{P_\infty}, \vartheta_{Q_\infty}) = \mathrm{MV}(\tilde{P}, \tilde{Q})$.

3) $\mathrm{MI}(\vartheta, \vartheta) = 2\int_Q \vartheta(u)du$.

More generally, one defines the mixed integral of $n+1$ concave functions $\varrho_i : Q_i \to \mathbf{R}$, $i = 0, \ldots, n$, defined on some convex sets in $\mathbf{R}^n$, by the formula :

$$\mathrm{MI}(\varrho_0, \ldots, \varrho_n) := \sum_{j=0}^{n} (-1)^{n-j} \sum_{0\leq i_0 < \ldots < i_j \leq n} \int_{Q_{i_0}+\ldots+Q_{i_j}} (\varrho_{i_0} \boxplus \ldots \boxplus \varrho_{i_j})(u) du_1 \ldots du_n \ .$$

We then prove [1] :

Theorem in higher dimensions [1] – *Let $f_0, \ldots, f_n \in \mathbf{k}[y^\pm][x_1^\pm, \ldots, x_n^\pm]$ be of content 1 in $\mathbf{k}[y^\pm]$ and P_{iv} their associated polytopes for each $v \in \mathbf{P}_1(\mathbf{k})$, then*

$$\mathrm{card}(Z(f_0, \ldots, f_n)_0) \leq \sum_{v\in \mathbf{P}_1(\mathbf{k})} \mathrm{MI}(\vartheta_{P_{0v}}, \ldots, \vartheta_{P_{nv}})$$

with equality for $f_0, \ldots, f_n$ general among the polynomials of given associated polytopes $P_{0v}, \ldots, P_{nv}$, $v \in \mathbf{P}_1(\mathbf{k})$.

Let C be a smooth complete curve equipped with a family of ample line bundles $\mathcal{L}_i$ for $0 \leq i \leq n$. Set

$$\Gamma(C;\mathcal{L}_i)[\mathbf{t}^{\pm 1}] := \Gamma(C;\mathcal{L}_i) \otimes_{\mathbf{k}} \mathbf{k}[\mathbf{t}^{\pm 1}] = \bigoplus_{a \in \mathbf{Z}^n} \Gamma(C;\mathcal{L}_i) \otimes_{\mathbf{k}} \mathbf{t}^a \ ,$$

for the $\mathbf{k}$ -vector space of Laurent polynomials with coefficients global sections of $\mathcal{L}_i$, and for each i consider an element $f_i \in \Gamma(C;\mathcal{L}_i)[\mathbf{t}^{\pm 1}]$. The set of zeros (*resp.* isolated zeros) in $C \times (\mathbf{k}^\times)^n$ of such a system is well-defined and, as before, we denote it by $Z(f_0,\ldots,f_n)$ (*resp.* $Z(f_0,\ldots,f_n)_0$). If we fix a non-zero section $s_i \in \Gamma(C;\mathcal{L}_i) \setminus \{0\}$, then $s_i^{-1} f_i$ is a Laurent polynomial with coefficients in $\mathbf{k}(C)$, and we can extend to this setting the notions of v -adic Newton polytope $\mathrm{NP}_v(s_i;f_i)$ and the corresponding functions $\vartheta_{i,v}$, for $s \in C$.

It may happen that the coefficients of one f_i have a non-empty common locus of zeros, which we denote $B(f_i)$. For a system $\mathbf{f} = (f_0,\ldots,f_n)$ as above, we then set $B_i := B(f_i) \setminus \cup_{k \neq i} B(f_k)$ and finally we put $B_{\mathbf{f}} := \cup_{i=0}^n B_i \subset C$. In this general setting we have the following extension of Theorem theorem in higher dimensions [1] :

Theorem on curves [1] – *Let C be a smooth complete curve equipped with ample line bundles $\mathcal{L}_i$ for $0 \leq i \leq n$. Let $f_i \in \Gamma(C;\mathcal{L}_i)[t_1^{\pm 1},\ldots,t_n^{\pm 1}]$ be Laurent polynomials in the $\mathbf{t}$ -variables with coefficients in $\Gamma(C;\mathcal{L}_i)$ and $s_i \in \Gamma(C;\mathcal{L}_i) \setminus \{0\}$ a non-zero global section. For $v \in C$ let $\vartheta_{i,v} : \mathrm{NP}(f) \to \mathbf{R}$ denote the parametrization of the upper envelope of $\mathrm{NP}_v(s_i;f_i)$, then*

$$\sum_{\xi \in Z(f_0,\ldots,f_n)_0 \setminus (B_{\mathbf{f}} \times (\mathbf{k}^\times)^n)} \mathrm{mult}(\xi | Z(f_0,\ldots,f_n)) \leq \sum_{v \in C} \mathrm{MI}(\vartheta_{0,v},\ldots,\vartheta_{n,v}) \ .$$

Furthermore, this is an equality for general polynomials with given functions $(\vartheta_{i,v} : 0 \leq i \leq n, v \in C)$.

Consider an extension of an elliptic curve E by a torus $\mathbf{k}^\times$ of dimension 1 . The complementary set of the fibres above the origin $0 \in E$ and the point $u_0 \in E$ parametrizing the extension, is identified with $(E \setminus \{0,u_0\}) \times (\mathbf{k}^\times)$. On this open set, the algebra of functions gives an embedding of the extension in $\mathbf{P}^2 \times \mathbf{P}^2$ through the functions (we denote $\Lambda := \mathbf{Z}(\omega_1,\eta_1 u_0) + \mathbf{Z}(\omega_2,\eta_2 u_0) + \mathbf{Z}(0,2\mathrm{i}\pi) \subset \mathbf{C}^2$)

$$\begin{array}{ccc} \mathbf{C}^2/\Lambda & \longrightarrow & \mathbf{P}^2 \times \mathbf{P}^2 \\ (z,t) & \longmapsto & \Big((1 : \wp(z) : \wp'(z)) , \Big(\frac{\wp'(z)+\wp'(u_0)}{\wp(z)-\wp(u_0)} : F(z,t) : F(z,t)^{-1} \Big) \Big) \end{array} \ .$$

Here, $\wp$ stands for the Weierstrass elliptic function and $F = \frac{\sigma(z-u_0)}{\sigma(z)} \mathrm{e}^t$ with σ the Weierstrass sigma function. The polynomials occuring in this situation are of the shape

$$f = \sum_{j=0}^{N} A_j \left(\wp(z), \wp'(z), \frac{\wp'(z)+\wp'(u_0)}{\wp(z)-\wp(u_0)} \right) \times F^{a_j}$$

with A_i polynomials in the three functions indicated and $a_i \in \mathbf{Z}$. One can apply our main theorem theorem on curves [1] to estimate from above the number of common zeros of two such polynomials in $(E \setminus \{0, u_0\}) \times (\mathbf{k}^\times)$. We see that the only positive contributions to the estimate come from the extended Newton polytopes at the places 0 and u_0 of E , whereas the other places can bring negative contributions. In particular, we note that for polynomials of the shape $f(\wp, \wp', F)$ the upper bound does not depend on the extension itself (namely on u_0). Even more particularly, consider integers d , D , L and two polynomials f_1 and f_2 of the special shape ($u_1 \neq 0, \pm u_0$) :

$$f_i = A_{i,0}(\wp) \times (\wp - \wp(u_1))^{dD} + \sum_{j=1}^{d} A_{i,j}(\wp) \times (\wp - \wp(u_1))^{(d-j)D} F^{jL-1} \ ,$$

with $A_{i,j} \in \mathbf{C}[s]$, $d^\circ A_{i,j} \leq d$ and $(\wp - \wp(u_1)) \nmid A_{i,d}(\wp)$ for $i = 1, 2$. Then the number of common isolated zeros of f_1 and f_2 in $(E \setminus \{0\}) \times (\mathbf{k}^\times)$ is bounded above by $4d(d-1)L + 2(d-1)D$, which is most interesting when d is significantly smaller than D and L .

Remark – Replacing the exponents $jL-1$ by jL in the expression of the f_i 's, one get the upper bound $4d(d-1)L$, which is easily seen to be exact since $(\wp - \wp(u_1)) \nmid A_{i,d}(\wp)$ for $i = 1, 2$.

Reference(s)

[1] P. Philippon & M. Sombra, *Mixed integrals and an analog of Bernštein theorem in dimension* 1 , tapuscrit.