FoCM 2014

Point-Hypersurface Incidences

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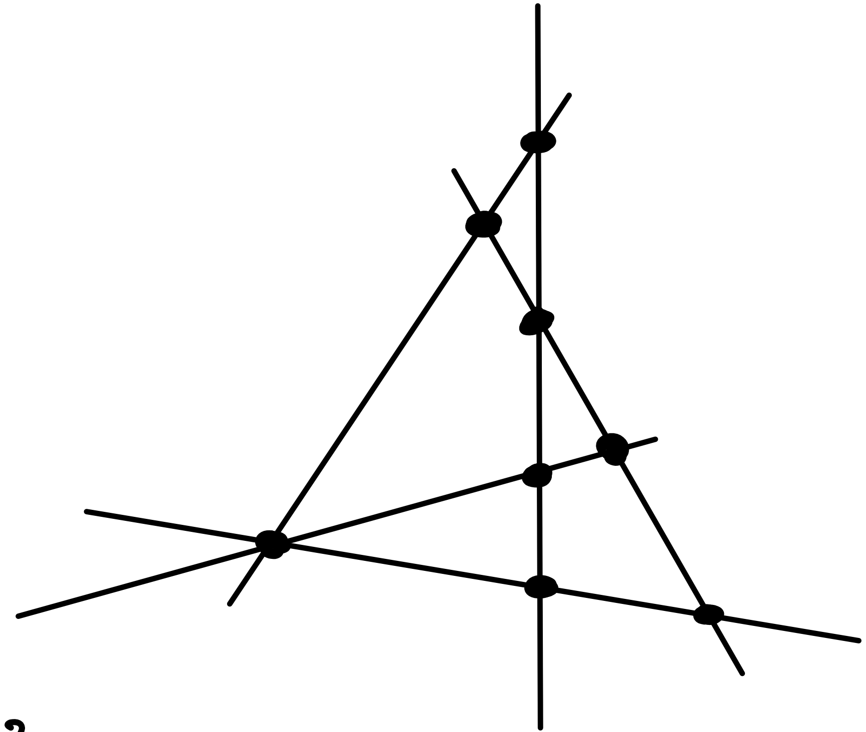
Joint work with S. Basu (Purdue)

Point-Line Incidences



$m = 8 \qquad n = 5$

$I = 17$

$\mathcal{P} \subset \mathbb{R}^2$

$\mathcal{L}$ Lines in $\mathbb{R}^2$

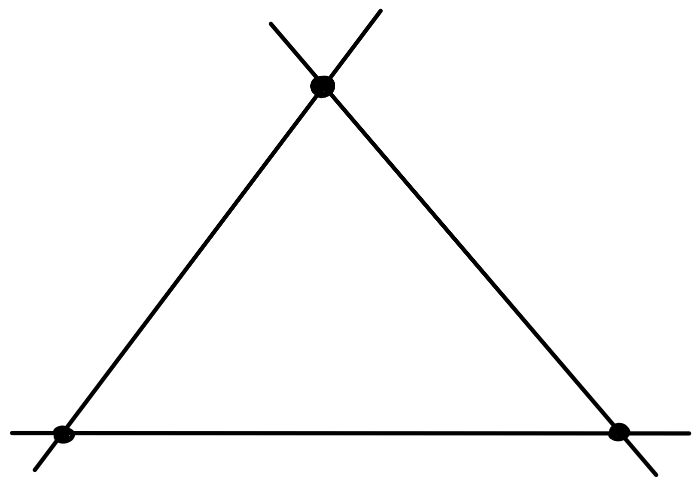
$m = \# \mathcal{P}$

$n = \# \mathcal{L}$

$$\mathcal{J}(\mathcal{P}, \mathcal{L}) = \{ (p, \ell) \in \mathcal{P} \times \mathcal{L} \mid p \in \ell \}$$

$$I(\mathcal{P}, \mathcal{L}) = \# \mathcal{J}(\mathcal{P}, \mathcal{L})$$

Pb (Erdős) Which is the maximum $I(\mathcal{P},\mathcal{L})$ for given m, n?



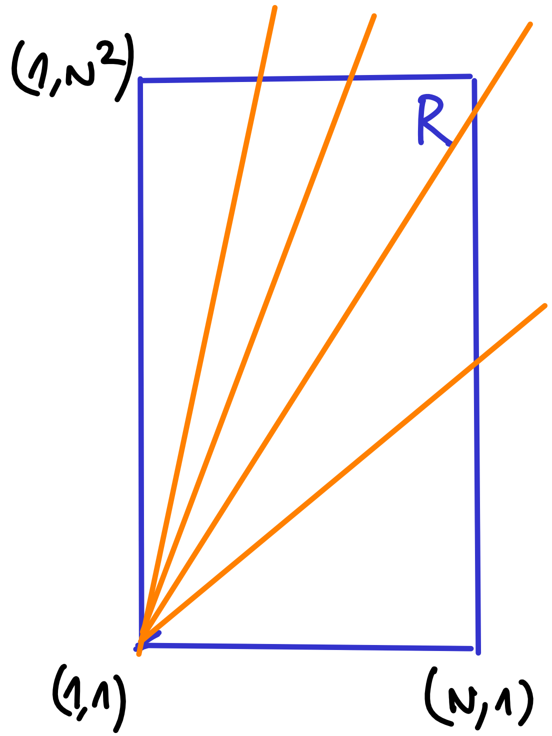
$m = n = 3 \Rightarrow \max I(\mathcal{P},\mathcal{L}) = 6$

$\underline{T}$ (Szemerédi - Trotter 1983)

$$I(\mathcal{P}, \mathcal{L}) \lesssim m^{2/3} n^{2/3} + m + n$$

$\underline{Ex}$: $N \geq 1$

$\mathcal{P} = R \cap \mathbb{Z}^2$ $\qquad m = 2N^3$



For $a, b \in \mathbb{Z}$

$$l_{a,b} = \{(t, at+b) \mid t \in \mathbb{R}\}$$

$$\mathcal{L} = \{ l_{a,b} \mid 1 \leq a \leq N, 1 \leq b \leq N^2 \}$$

$n = N^3$

$$\mathcal{P} \cap L_{a,b} = \{ (i, ai+b) \mid 1 \leq i \leq N \}$$

$$\Rightarrow I(\mathcal{P}, \mathcal{L}) = N^4 \sim m^{2/3} n^{2/3}$$

Applications of ST

- "Sum-product" theorems:

Given $A \subset \mathbb{R}$ finite, how far is it from being a subring?

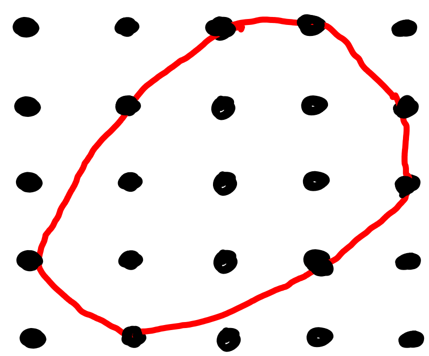
$$\max \#(A+A), \#(A \cdot A) \gtrsim \#A^{5/4}$$ Elekes 1997

- Number of lattice points in a strictly convex curve

$C \subset [0,N]^2$ strictly convex

$$\# C \cap \mathbb{Z}^2 \lesssim N^{2/3}$$

Iosevich 2004



Incidences in Higher Dimensions

Conj 1: Let $d, k \geq 2$, $c > 0$.

Let $\mathcal{P} \subset \mathbb{R}^d$, Σ hypersurfaces of $\mathbb{R}^d$ st

(a) $\deg(\sigma) \leq c \quad \forall \sigma \in \Sigma$

(b) $\forall \sigma_1, \ldots, \sigma_d \in \Sigma \qquad \# \sigma_1 \cap \cdots \cap \sigma_d < \infty$

(c) $\forall p_1, \ldots, p_k \in \mathcal{P} \qquad \# \{\sigma \mid p_1, \ldots, p_k \in \sigma\} < c$

Set $m = \# \mathcal{P} \qquad n = \# \Sigma$

Then

$$I(\mathcal{P}, \Sigma) \lesssim_{d,k,c} m^{1 - \frac{k-1}{dk-1}} \, n^{1 - \frac{d-1}{dk-1}} + m + n$$

Cases :

1) Σ Lines in $\mathbb{R}^2$: $d=k=2$

$$I \lesssim m^{1-1/3} n^{1-1/3} + m+n$$

ST 1983

2) Σ curves in $\mathbb{R}^2$: $d=2$, k any

$$I \lesssim m^{1-\frac{k-1}{2k-1}} n^{1-\frac{1}{2k-1}} + m + n$$

Pach – Sharir 1998

3) Unit spheres in $\mathbb{R}^3$: $d=k=3$

$$I \lesssim m^{1-1/4} n^{1-1/4} + m+n$$

Zahl 2011

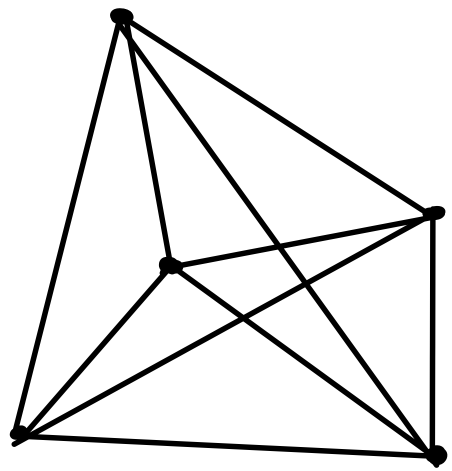
Kaplan – Matousek – Sharir – Safernova 2011

4) Hypersurfaces in $\mathbb{R}^3$: $d=3$, k any

Zahl 2011

POLYNOMIAL PARTITIONING IN $\mathbb{R}^d$

$\mathcal{P} \subset \mathbb{R}^2$ $\qquad m = \# \mathcal{P}$



Pb (Erdős) Which is the minimal number of distances $g(m)$ defined by $\mathcal{P}$?

In general $\qquad g(m) \lesssim \dfrac{m}{\sqrt{\log m}}$

T (Guth - Katz, Ann Math 2015) $\qquad g(m) \gtrsim \dfrac{m}{\log m}$

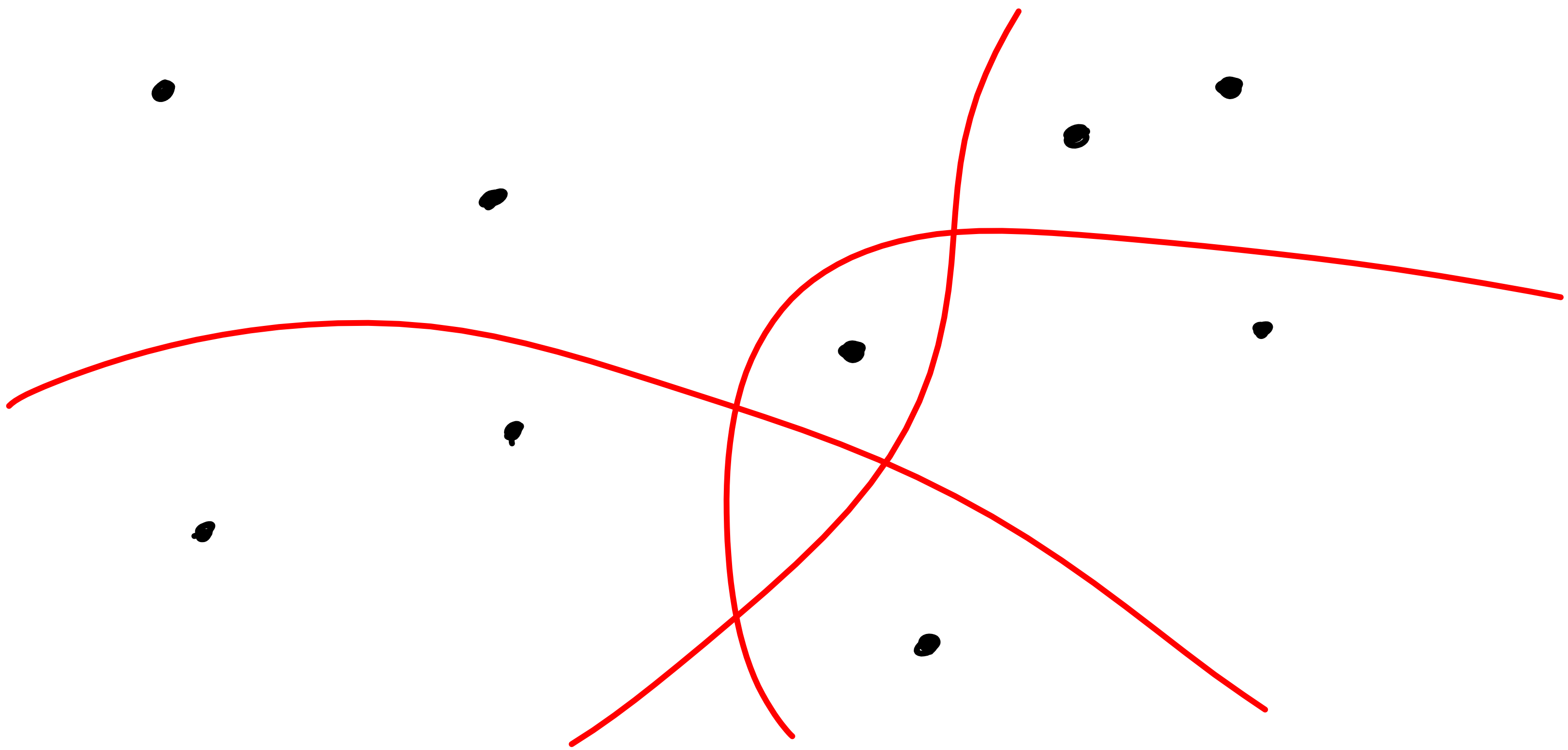
Two ingredients:

- Elekes' framework
- Polynomial partitioning

$\underline{\text{T}}$ (Guth-Katz) $\quad \mathcal{P} \subset \mathbb{R}^d, \quad D \geq 1$

$\Rightarrow \exists f \in \mathbb{R}[x_1, \dots, x_d]_{\leq D} \setminus \{0\} \quad$ st $\quad \forall \gamma \in cc(\mathbb{R}^d \setminus V(f))$

$$\#(\mathcal{P} \cap \gamma) \lesssim \frac{\#\mathcal{P}}{D^d}$$



Polynomial partitioning allows an algebraic implementation of "divide and conquer"

Observations:

- $b_0(\mathbb{R}^2 \setminus V(f)) \lesssim D^2$ Oleinik-Petrovski-Milnor-Thom
- A line $\ell \subset \mathbb{R}^2$ can cross at most $D+1$ cells
  Bézout

Gives another proof of the ST theorem

POLYNOMIAL PARTITIONING ON VARIETIES

T (Basu - S. 2014) Let $X^e \subset \mathbb{C}^d$ irreducible locally set-theoretically defined at degree δ

Let $\mathcal{P} \subset X(\mathbb{R})$ and $D \geq d\delta$

$\Longrightarrow \exists f \in \mathbb{R}[x_1, \dots, x_d]_{\leq D} \setminus \{0\}$ st $\forall \gamma \in cc(X(\mathbb{R}) \setminus V(f))$

$$\#(\mathcal{P} \cap \gamma) \lesssim \frac{\# \mathcal{P}}{\deg(X) \cdot D^e}$$

Main ingredient: sharp lower bounds for Hilbert functions (Chardin - Philippon 1998)

Connected Components of SemiAlgebraic Sets

Current bounds for # connected components depend on the degree of defining equations:

$$b_0(X(\mathbb{R}) \setminus V(g)) \lesssim \delta^{d-e} D^e$$

Barone-Basu 2013

Not sufficient for the application when

$$\deg(X) \ll \delta^{d-e}$$

Conj 2 $\qquad b_0(X(\mathbb{R}) \setminus V(f)) \lesssim \deg(X)\, D^e$

We show Conj 2 $\Rightarrow$ Conj 1

Moreover, Conj 2 true for $e = d-2$

T (Basu - S. 2014) Let $k \geq 2$, $c > 0$.

Let $\mathcal{P} \subset \mathbb{R}^4$, Σ hypersurfaces of $\mathbb{R}^4$ st

(a) $\deg(\sigma) \leq c \quad \forall \sigma \in \Sigma$

(b) $\forall \sigma_1, \dots, \sigma_4 \in \Sigma \qquad \# \sigma_1 \cap \cdots \cap \sigma_4 < \infty$

(c) $\forall p_1, \dots, p_k \in \mathcal{P} \qquad \# \{ \sigma \mid p_1, \dots, p_k \in \sigma \} < c$

Set $\quad m = \# \mathcal{P} \qquad n = \# \Sigma$

Then

$$I(\mathcal{P}, \Sigma) \lesssim m^{1 - \frac{k-1}{4k-1}} n^{1 - \frac{3}{4k-1}} + m + n$$

THANKS !