# The zero set of the independence polynomial of a graph

Martín Sombra

ICREA and Universitat de Barcelona, Spain

Seminario del CMaLP
12 November 2021

Based on joint work with Juan Rivera-Letelier

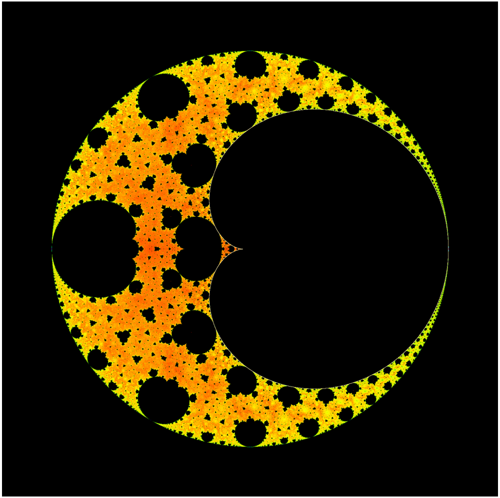


Figure by N. Fagella

# Independence polynomials of graphs

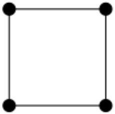
Let $G$ be a finite graph

A subset $I$ of $V_G$ is *independent* if $v \not\sim v'$ for all $v, v' \in V_G$

$V_G$ the vertices of $G$

The *independence polynomial* of $G$ is

$$Z_G = \sum_{\substack{I \subset V_G \\ \text{independent}}} x^{\# I} \in \mathbb{Z}_{\geqslant 0}[x]$$

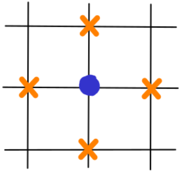


$$Z_G = 1 + 4\,x + 2\,x^2$$

# The neighbour exclusion model

Let $\mathcal{G}$ be a (possibly infinite) graph

The *neighbour exclusion model* or *hardcore lattice gas model* on $\mathcal{G}$ represents the behavior of large particles at the vertices of $\mathcal{G}$ excluding the presence of other particles at the adjacent ones



An *activity parameter* $\lambda \in \mathbb{R}_{\geqslant 0}$ determines the probability that a particle appears at a given vertex

# Hamiltonian and partition function

- Interaction: for $s, s' \in \{0,1\}$

$$U(s,s') = \begin{cases} +\infty & \text{if } s = s' = 1 \\ 0 & \text{otherwise} \end{cases}$$

- Potential: for $s \in \{0,1\}$

$$K(s) = -\log(\lambda)\, s$$

Fix a finite subgraph $G \subset \mathcal{G}$

- Hamiltonian: for a configuration $\sigma \colon G \to \{0,1\}$

$$H(\sigma,\lambda) = \sum_{v \sim v'} U(\sigma(v),\sigma(v')) + \sum_{v} K(\sigma(v))$$

$$= \begin{cases} -\log(\lambda)\,\#\operatorname{supp}(\sigma) & \text{if } \operatorname{supp}(\sigma) \text{ independent} \\ 0 & \text{otherwise} \end{cases}$$

- Partition function:

$$\sum_{\sigma} \exp(-H(\sigma,\lambda)) = Z_G(\lambda)$$

# The pressure function

The *pressure* (or *free energy*) is defined as

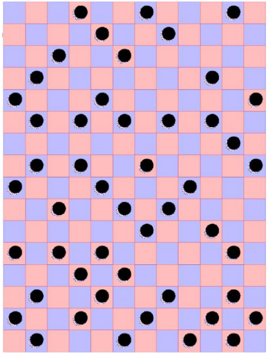
$$\mathcal{P}_{\mathcal{G}}(\lambda) = \lim_{G \to \mathcal{G}} \frac{1}{\# V_G} \log(Z_G(\lambda))$$

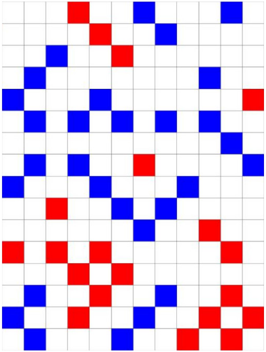
Phase transitions (i.e. bifurcations of Gibbs measures) are related to the lost of analycity of $\mathcal{P}$


$\rightsquigarrow$ Ehrenfest classification

# Random independent subsets

$\mathcal{G} = \mathbb{Z}^2$

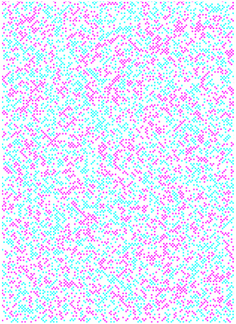




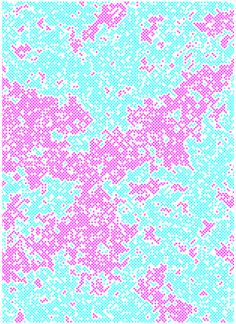
Figures by P. Winkler

# Phase transition

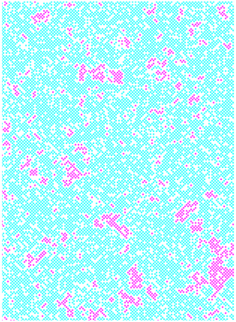
$\lambda = 1$



$\lambda = 3.787$



$\lambda = 3.792$



Figures by P. Winkler

# Regular rooted trees

For $d \geqslant 2$ let $\mathcal{T}$ be the *regular rooted tree* with branching $d$

The phase transition (birth of a "period 2" Gibbs measure) of the neighbour exclusion model on $\mathcal{T}$ appears at

$$\lambda_{\text{cr}} = \frac{d^d}{(d-1)^{d+1}}$$

**Theorem (Rivera-Letelier and S.)**

The pressure function $\mathcal{P}_{\mathcal{T}}$ is real analytic on $(0, +\infty) \setminus \{\lambda_{\text{cr}}\}$, and infinitely differentiable at $\lambda_{\text{cr}}$

The neighbour exclusion model on $\mathcal{T}$ has a phase transition of infinite order at the point $\lambda_{\text{cr}}$

# Recursion

For $k \geqslant -1$ let $T_k$ be the subtree of $\mathcal{T}$ at depth $\leqslant k$

$k = 0$ $\qquad$ $k = 1$ $\qquad$ $k = 2$

Set

$$R_k = \frac{Z_{T_k}}{Z_{T_{k-1}}^d} \in \mathbb{Q}(x)$$

We have that

$$Z_{T_{-1}} = 1, \quad Z_{T_0} = 1 + x, \quad Z_{T_k} = Z_{T_{k-1}}^d + x\, Z_{T_{k-2}}^{d^2} \text{ for } k \geqslant 1$$

which implies that

$$R_0 = 1 + x \quad \text{and} \quad R_k = 1 + \frac{x}{R_{k-1}^d} \quad \text{for } k \geqslant 1$$

A. Scott and A. Sokal, 2006

For $\lambda \in \mathbb{C}^\times$ set

$$f_\lambda : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}, \qquad z \longmapsto 1 + \frac{\lambda}{z^d}$$

$\overline{\mathbb{C}}$ the Riemann sphere

- Critical points: 0 and $\infty$
- $0 \longmapsto \infty \longmapsto 1 \longmapsto 1 + \lambda$
- $f_\lambda^{-1}(\infty) = \{0\}$

Hence

$$R_k(\lambda) = f_\lambda^k(1+\lambda) = f_\lambda^{k+3}(0)$$

$$f^k = \overbrace{f \circ \cdots \circ f}^{k}$$

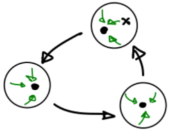
and so

$$\Omega_k := (Z_{T_k} = 0) = \{\lambda \in \mathbb{C} \mid f_\lambda^{k+3}(0) = 0\}$$

$$\Omega := \bigcup_{k \geqslant 0} \Omega_k = \{\lambda \in \mathbb{C} \mid 0 \in \mathrm{Per}(f_\lambda)\}$$

# Hyperbolic components and zero free regions

A parameter $\lambda \in \mathbb{C}^\times$ is *hyperbolic* if all the critical points of $f_\lambda$ converge to attracting cycles

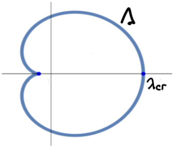


In our case, $\lambda$ is hyperbolic if and only if $f_\lambda$ has and attracting cycle (Fatou's theorem). Hence if $\lambda$ is attracting but not superattracting then $\lambda \notin \Omega$

For instance, if we set

$$\Lambda := \{\lambda \mid f_\lambda \text{ has an attracting fixed point}\} = \left\{ \frac{-d^d \alpha}{(d+\alpha)^{d+1}} \,\middle|\, \alpha \in \mathbb{D} \right\}$$

then $\Lambda \cap \Omega = \varnothing$




Peters and Regts, 2019

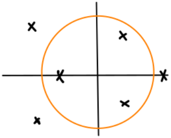
# Heights of algebraic numbers

Let $\xi \in \overline{\mathbb{Q}}^{\times}$ be an algebraic number of degree $n$ and consider its minimal polynomial

$$\gamma \prod_{i=1}^{n} (x - \xi_i) \in \mathbb{Z}[x]$$

The *Weil height* of $\xi$ is

$$\mathrm{h}_{\mathrm{Weil}}(\xi) = \frac{1}{n} \left( \log |\gamma| + \sum_{i=1}^{n} \log^{+} |\xi_i| \right) \in \mathbb{R}_{\geqslant 0}$$



For instance, for $\xi = \frac{a}{b} \in \mathbb{Q}^{\times}$ with $a, b \in \mathbb{Z}$ coprime

$$\mathrm{h}_{\mathrm{Weil}}(\xi) = \log \max(|a|, |b|)$$

# Arithmetic equidistribution

**Theorem (Bilu, 1997)**

Let $\xi_k \in \overline{\mathbb{Q}}^\times$, $k \geqslant 1$, such that $\xi_k \neq \xi_{k'}$ for all $k \neq k'$ and

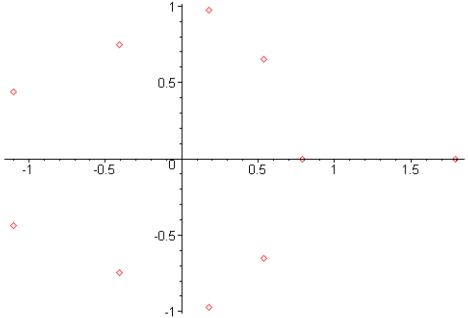
$$\lim_{k \to +\infty} \mathrm{h}_{\mathrm{Weil}}(\xi_k) = 0$$

Then

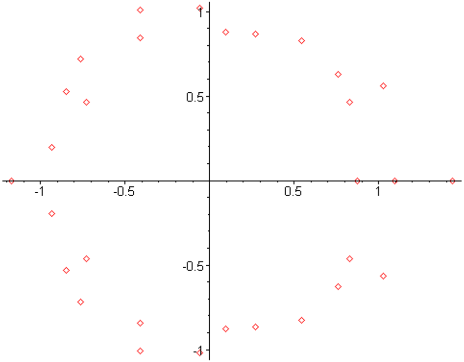
$$\lim_{k \to +\infty} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot \xi_k = \mathrm{Haar}_{S^1}$$

Toric version of Szpiro-Ullmo-Zhang equidistibution on Abelian varieties (1997)

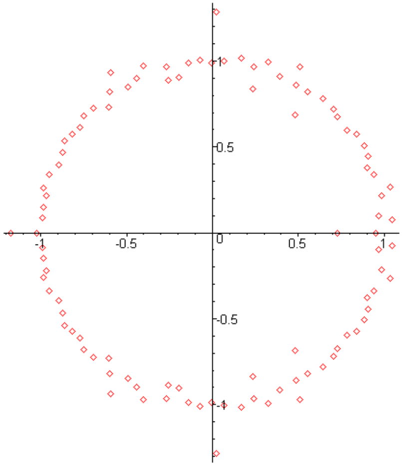
For instance, let $d = 10$ and $f = -x^{10} + x^9 + x^8 + x^6 + x^5 - x^4 + x^3 - x^2 + x - 1$



$d = 30$ and $f = x^{30} - x^{29} - x^{28} + x^{26} + x^{25} - x^{24} - x^{23} - x^{22} + x^{21} - x^{20} + x^{19} + \cdots$



$d = 100$ and $f = -x^{100} - x^{98} + x^{96} + x^{94} - x^{93} + x^{92} - x^{91} - x^{90} + x^{88} - x^{84} + \cdots$



# The bifurcation locus

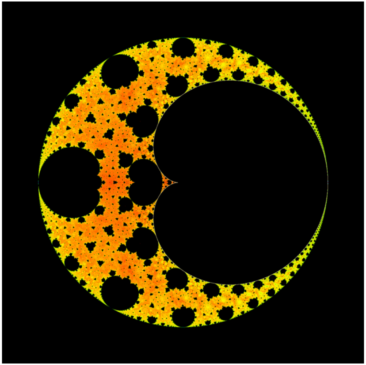
The *bifurcation locus* of the family $(f_\lambda)_{\lambda \in \mathbb{C}^\times}$ is

$$B(f_\lambda) = \{\lambda \in \mathbb{C} \mid \lambda \mapsto J(f_\lambda) \text{ is not continuous}\}$$

It is equipped with the probability measure

$$\mu_{\mathrm{bif}} = \lim_{k \to +\infty} \frac{f_\lambda^k(0)^* \omega}{\deg(f_\lambda^k(0))} \qquad \text{for } \omega \text{ uniform measure on } \overline{\mathbb{C}}$$

De Marco, 2001



# A dynamical height

For $\lambda \in \overline{\mathbb{Q}}^{\times}$ set

$$\mathrm{h}(\lambda) = \lim_{k \to +\infty} \frac{\mathrm{h}_{\mathrm{Weil}}(f_\lambda^k(0))}{\deg(f_\lambda^k)} \in \mathbb{R}_{\geqslant 0}$$

**Theorem (RL-S)**

This limit defines a function $\mathrm{h} \colon \overline{\mathbb{Q}}^{\times} \to \mathbb{R}_{\geqslant 0}$ with

$$\mathrm{h}(\lambda) = 0 \text{ if and only if } 0 \in \mathrm{PrePer}(f_\lambda)$$

Moreover

$$h(\lambda) = \rho(\lambda) + \sum_{p \text{ prime}} \log^+ |\lambda|_p$$

with $\rho \colon \mathbb{C}^{\times} \to \mathbb{R}$ the potential of $\mu_{\mathrm{bif}}$, the normalized bifurcation measure of $(f_\lambda)_{\lambda \in \mathbb{C}^{\times}}$, and $\rho - \log^+$ is continuous on $\mathbb{C}$

$\rightsquigarrow$ h is a height function (Arakelov geometry)

# Quantitative equidistribution

By arithmetic equidistribution (Favre and Rivera-Letelier, 2006):

## Corollary

There is a constant $c > 0$ such that for every Lipschitz function $\varphi \colon \mathbb{C} \to \mathbb{R}$ with compact support we have that

$$\left| \frac{1}{\Omega_k} \sum_{\lambda \in \Omega_k} \varphi(\lambda) - \int \varphi \, d\, \mu_{\mathrm{bif}} \right| \leqslant c \left( \frac{\log(k)}{k} \right)^{1/2} \mathrm{Lip}(\varphi)$$

In particular

$$\lim_{k \to +\infty} \Omega_k = \mu_{\mathrm{bif}}$$

$d = 2$

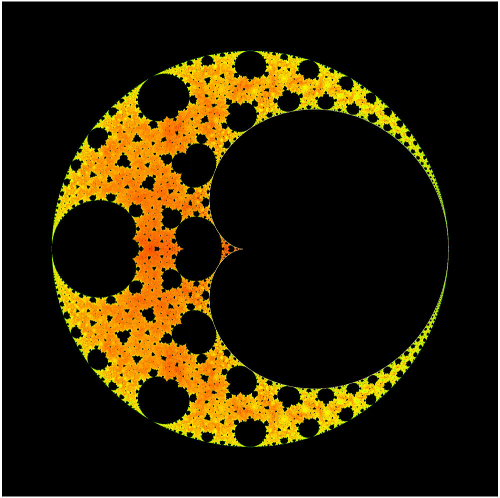


Figure by N. FAGELLA

$d = 3$

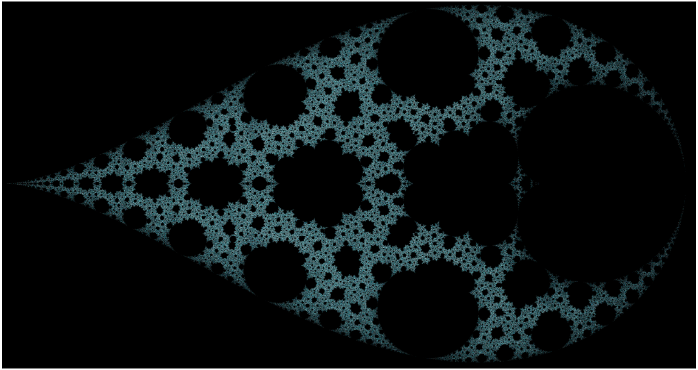


Figure by B. Espigule

# Conclusion

This quantitative equidistribution result implies that

$$\mathcal{P}_{\mathcal{T}}(\lambda) = \lim_{k\to+\infty} \frac{1}{\# T_k} \log(Z_{T_k}(\lambda)) = \int_{\overline{\mathbb{C}}} \log|\lambda - \zeta| \, d\,\mu_{\mathrm{bif}}(\zeta)$$

This is used to prove:

**Proposition (RL-S)**

There is $c > 0$ such that for all $\varepsilon > 0$

$$\mu_{\mathrm{bif}}(B(\lambda_{\mathrm{cr}}, \varepsilon)) < \exp\left(\frac{-c}{\varepsilon}\right)$$

which in turn implies the theorem

**Theorem (RL-S)**

$\mathcal{P}_{\mathcal{T}}$ is real analytic on $(0, +\infty)\backslash\{\lambda_{\mathrm{cr}}\}$, and infinitely differentiable at $\lambda_{\mathrm{cr}}$

**Thanks!**