NORMALIZED RESULTANTS(*)

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The Sylvester Resultant

$$f = \alpha_0 + \alpha_1 x + \cdots + \alpha_d x^d$$

$$g = \beta_0 + \beta_1 x + \cdots + \beta_e x^e \qquad \in \mathbb{C}[x]$$

$$\mathrm{Res}_{d,e}(f,g) = \det \begin{array}{c} \overbrace{\hphantom{\alpha_0 \quad \alpha_0 \quad \alpha_0 \quad}}^{e}\;\overbrace{\hphantom{\beta_0 \quad \beta_0 \quad \beta_0 \quad}}^{d} \\ \begin{pmatrix} \alpha_0 & & & \beta_0 & & \\ \alpha_1 & \alpha_0 & & \beta_1 & \beta_0 & \\ \vdots & \alpha_1 & \ddots & \vdots & \beta_1 & \ddots \\ & \vdots & \alpha_0 & \vdots & \vdots & \beta_0 \\ \alpha_d & & \alpha_1 & \beta_e & & \beta_1 \\ & \alpha_d & \vdots & & \beta_e & \vdots \\ & & \alpha_d & & & \beta_e \end{pmatrix} \end{array} \in \mathbb{C}$$

- $\mathrm{Res}_{de}(f,g) = 0 \iff \begin{cases} \text{either: } \exists\ \xi \in \mathbb{C}^\times & f(\xi) = g(\xi) = 0 \\ \text{or: } & \alpha_0 = \beta_0 = 0 \\ & \alpha_d = \beta_e = 0 \end{cases}$

- Degrees: $\deg_{\underline{\alpha}}(\mathrm{Res}_{de}) = e, \quad \deg_{\beta}(\mathrm{Res}_{de}) = d$

- Poisson formula: $\mathrm{Res}_{de}(f,g) = \alpha_d^e \prod_{f(\xi)=0} g(\xi)$

- Applies to pb in elimination theory:

Ex: $\eta : \mathbb{C} \longrightarrow \mathbb{C}^2 \qquad \eta(x) = (f(x), g(x))$

$\mathrm{Res}_{de}(u - f(x), v - g(x))$ equation for the curve $\mathrm{Im}(\eta)$

THE BKK THEOREM

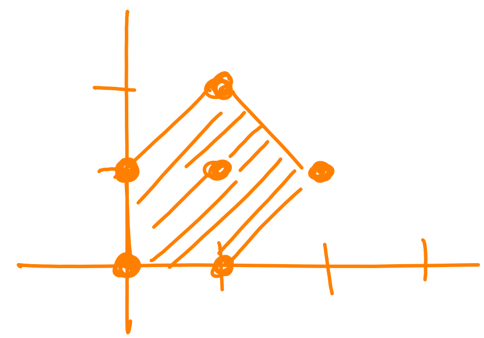
T (Kušnirenko 1975) $\Delta \subset \mathbb{R}^n$ politopo

$f_1, \dots, f_n \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ t.q. $N(f_i) \subset \Delta$

$$\#\{x \in (\mathbb{C}^\times)^n \mid f_1(x) = \dots = f_n(x)\}_0 \leq n!\,\mathrm{vol}(\Delta)$$

$$f = 2 + x - y - 2xy + 3x^2y - xy^2$$

$$g = 1 - x + 2y - 3xy + 7x^2y + xy^2$$



$$\Longrightarrow \#\{f = g = 0\} \leq 2!\,\mathrm{vol}(\Delta) = 5 \quad (< \deg(f) \cdot \deg(g))$$

THE SPARSE RESULTANT

$\mathcal{A}_0, \dots, \mathcal{A}_{n+1} \subset \mathbb{Z}^n$ finite sets

$\Delta_i = \operatorname{conv} \mathcal{A}_i \subset \mathbb{R}^n$ polytope

$F_i = \sum\limits_{\alpha \in \mathcal{A}_i} \mu_{i\alpha} x^\alpha$ "generic" poly with support $\mathcal{A}_i$

Consider $\Omega_{\mathcal{A}} = \{F_0 = \dots = F_n = 0\} \subset \mathbb{C}^\times \times \prod\limits_i \mathbb{P}(\mathbb{C}^{\mathcal{A}_i})$

$$\begin{array}{c} \prod\limits_i \mathbb{P}(\mathbb{C}^{\mathcal{A}_i}) \\ \big\downarrow \pi \\ \prod\limits_i \mathbb{P}(\mathbb{C}^{\mathcal{A}_i}) \end{array}$$

Def (GKZ, Sturmfels '90)

$$\mathrm{Res}_{\underline{A}} = \begin{cases} \text{primitive eq of } \overline{\pi(\Omega_{A})} & \text{if } \operatorname{codim}\left(\overline{\pi(\Omega_{A})}\right) = 1 \\ 1 & \text{otherwise} \end{cases}$$

Is a multihomogeneous poly in $\mathbb{Z}[u_0, \dots, u_n]$

Generalizes previous constructions:

- $n=1$, $A_0 = \{0, \dots, d\}$, $A_1 = \{0, \dots, e\} \rightsquigarrow$ Sylvester
- $A_i = \{0, e_1, \dots, e_n\} \rightsquigarrow$ determinant
- $A_i = d_i \Delta^n \cap \mathbb{Z} \rightsquigarrow$ Cayley/Macaulay

THE DEGREE OF THE SPARSE RESULTANT

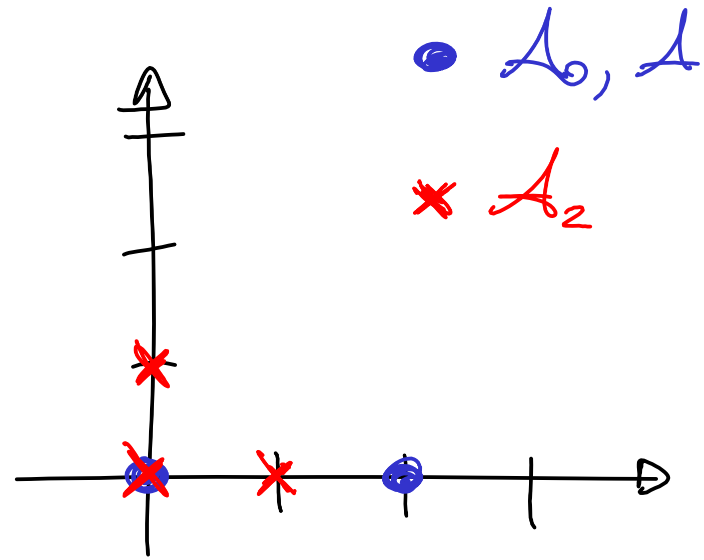
<u>Prop</u> (Sturmfels) $\mathrm{Res}_{\mathcal{A}} \neq 1 \iff \exists!$ "essential" $I \subset \{0,\dots,n\}$

If this is the case, $\mathrm{Res}_{\mathcal{A}} \in \mathbb{Z}[(u_j)_{j\in I}]$ and

$$\deg_{u_i}(\mathrm{Res}_{\mathcal{A}}) = MV_{L_I}(\Delta_j)_{j\in I\setminus\{i\}} \qquad i\in I$$

<u>Ex</u>

$$F_0 = \alpha_0 + \alpha_1 x^2$$
$$F_1 = \beta_0 + \beta_1 x^2$$
$$F_2 = \gamma_0 + \gamma_1 x + \gamma_2 y$$



$$\mathrm{Res} = \pm \det\begin{pmatrix} \alpha_0 & \beta_0 \\ \alpha_1 & \beta_1 \end{pmatrix}$$

$$\deg_{u_0} = \deg_{u_1} = 1 \qquad \deg_{u_2} = 0$$

The Poisson formula

I (Pedersen - Sturmfels 1993) Sup $\{0,\dots,n\}$ essential

$$\mathrm{Res}_{\underline{\mathcal{A}}}(\underline{f}) = \prod_{v\in S^n} \mathrm{Res}_{\mathcal{A}_1^v,\dots,\mathcal{A}_n^v}(f_1^v,\dots,f_n^v)^{c_v} \cdot \prod_{f_1(\xi)=\dots=f_n(\xi)} f(\xi)^{m_\xi}$$

Wrong in "degenerated" cases

Ex $\quad F_0 = U_{00}\, x_1$

$\quad F_1,\dots,F_n$ dense of degree $d \geq 2$

In this case

$\quad \mathrm{Res} = U_{00}$ but $\#\{F_1 = \dots = F_n = 0\} = d^n$!

Normalization of sparse resultants

Recall

$$
\begin{array}{c}
\Omega_{\underline{A}} = \{F_0 = \cdots = F_n = 0\} \subset \mathbb{C}^{\times} \times \prod_i \mathbb{P}(\mathbb{C}^{A_i}) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \big\downarrow \pi \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \prod_i \mathbb{P}(\mathbb{C}^{A_i})
\end{array}
$$

__Def:__ (Esterov) $\widetilde{\mathrm{Res}}_{\underline{A}}$ is a primitive equation in $\mathbb{Z}[u_0, \dots, u_n]$ for the divisor $\pi_*(\Omega_{\underline{A}})$

I.e. $\widetilde{\mathrm{Res}}_{\underline{A}} = 1 \iff \mathrm{Res}_{\underline{A}} = 1$

otherwise

$$\widetilde{\mathrm{Res}}_{\underline{A}} = \pm \, \mathrm{Res}_{\underline{A}}^{\deg(\pi|_{\Omega_{\underline{A}}})}$$

SPARSE RESULTANTS AS MULTIPROJECTIVE RESULTANTS

Consider the monomial map

$$\varphi_{\underline{\mathcal{A}}} : (\mathbb{C}^\times)^n \longrightarrow \prod_{i=1}^{n} \mathbb{P}(\mathbb{C}^{\mathcal{A}_i}), \quad \varphi_{\underline{\mathcal{A}}}(x) = (x^a)_{a \in \mathcal{A}_i, 0 \leq i \leq n}$$

and the toric variety $X_{\underline{\mathcal{A}}} := \overline{\mathrm{Im}(\varphi_{\underline{\mathcal{A}}})}$

Prop (D'ANDREA - S.)

$$\widetilde{\mathrm{Res}}_{\underline{\mathcal{A}}} = \mathrm{Res}_{e_0, \dots, e_n}(X_{\underline{\mathcal{A}}})$$

Rémond resultant (2001)

$$\rightsquigarrow \deg_{u_i}(\widetilde{\mathrm{Res}}_{\underline{\mathcal{A}}}) = MV_{\mathbb{Z}^n}(\Delta_0, \dots, \Delta_{i-1}, \Delta_{i+1}, \dots, \Delta_n)$$

$$\mathrm{I}: \ \widetilde{\mathrm{Res}}_{\underline{A}}(\underline{f}) = \prod_{v} \widetilde{\mathrm{Res}}_{A_1^v,\dots,A_n^v}(f_1^v,\dots,f_n^v)^{-h_{A_0}(v)} \cdot \prod_{f_1(\xi)=\cdots=f_n(\xi)} f(\xi)^{m_\xi}$$

where $v \in \mathbb{Z}^n \setminus \{0\}$ primitive and

$$h_{A_0}(v) = \min_{a \in A_0} \langle v, a \rangle$$

COMPUTING AN ELIMINANT POLYNOMIAL

Cor: Let $\chi_b : (\mathbb{C}^\times)^n \to \mathbb{C}^\times$ character assoc to $b \in \mathbb{Z}^n$ and $P_b \in \mathbb{C}[t^{\pm 1}]$ "balanced" equation of

$$\chi^b_*(\{f_1 = \cdots = f_n = 0\})$$

$$\Rightarrow \quad P_b = \gamma \cdot \widetilde{\mathrm{Res}}_{\{0,b\},A_1,\ldots,A_n}(z - x^b, f_1, \ldots, f_n)$$

with $\gamma = \overline{\prod}$ facet resultants

$\rightsquigarrow$ applies to the distribution of roots of systems of polynomials (talk by D'Andrea at the Algebra session)