Macaulay style formulae for the sparse resultant

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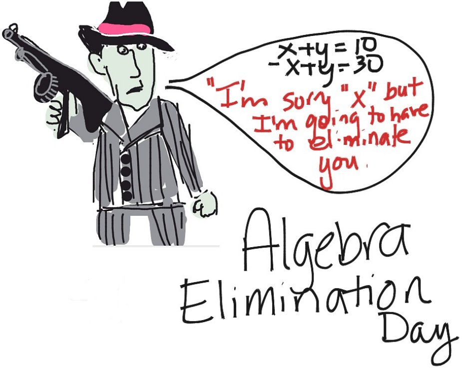
Elimination theory



The frame

Let $\Omega \subset X \times T$ a subvariety and $\pi\colon X \times T \to T$ projection onto the second factor

Problem

Understand (!) $\pi(\Omega)$

A bit of terminology:

- X solution space
- T coefficient space
- Ω incidence variety

The determinant

The condition that the linear system

$$\begin{cases} c_{0,0}x_0 & + \cdots + & c_{0,n}x_n & = & 0 \\ \vdots & & \vdots & & \vdots \\ c_{n,0}x_0 & + \cdots + & c_{n,n}x_n & = & 0 \end{cases}$$

admits a solution $\boldsymbol{\xi} = (\xi_0, \dots, \xi_n) \neq (0, \dots, 0)$ is equivalent to

$$\det(c_{i,j})_{i,j} = 0$$

In this case $\Omega \subset \mathbb{P}^n \times (\mathbb{P}^n)^{n+1}$ and

$$\pi(\Omega) = V(\det) \subset (\mathbb{P}^n)^{n+1}$$

The Sylvester resultant

Let

$$\begin{aligned} F &= a_0 x_0^m + a_1 x_0^{m-1} x_1 + a_2 x_0^{m-2} x_1^2 + \ldots + a_m x_1^m \\ G &= b_0 x_0^k + b_1 x_0^{k-1} x_1 + b_2 x_0^{k-2} x_1^2 + \ldots + b_k x_1^k \end{aligned}$$

Their Sylvester resultant

$$\mathrm{Res}_{m,k}(F, G) \in \mathbb{Z}[a_0, \ldots, a_m, b_0, \ldots, b_k]$$

is the unique (up to $\pm$) irreducible polynomial that vanishes iff $\exists \boldsymbol{\xi} \in \mathbb{P}^1$ s.t. $F(\boldsymbol{\xi}) = G(\boldsymbol{\xi}) = 0$

Indeed

$$\mathrm{Res}_{m,k}(F, G) = \det \begin{pmatrix} a_m & a_{m-1} & \cdots & a_0 & & & \\ & a_m & a_{m-1} & \cdots & a_0 & & \\ & & \ddots & & & \ddots & \\ & & & a_m & a_{m-1} & \cdots & a_0 \\ b_k & b_{k-1} & \cdots & b_0 & & & \\ & b_k & b_{k-1} & \cdots & b_0 & & \\ & & \ddots & & & \ddots & \\ & & & b_k & b_{k-1} & \cdots & b_0 \end{pmatrix}$$

The multivariate resultant

For $i = 0, \dots, n$ let

$$F_i = \sum_{a_0 + \cdots + a_n = d_i} c_{i,\mathbf{a}}\, x_0^{a_0} \dots x_n^{a_n}$$

homogeneous polynomial in the variables $x_0, x_1, \dots, x_n$ of degree d_i

Set $\mathbf{c}_i := (c_{i,\mathbf{a}})_{|\mathbf{a}|=d_i}$. The multivariate resultant

$$\mathrm{Res}_{d_0,\dots,d_n}(F_0, \dots, F_n) \in \mathbb{Z}[\mathbf{c}_0, \dots, \mathbf{c}_n]$$

is the unique irreducible polynomial that vanishes iff $\exists \boldsymbol{\xi} \in \mathbb{P}^n$ s.t.

$$F_1(\boldsymbol{\xi}) = \cdots = F_n(\boldsymbol{\xi}) = 0$$

Properties

- $\mathrm{Res}_{d_0,d_1,\dots,d_n}(F_0,\dots,F_n)$ is multihomogeneous of degree

$$\prod_{j\neq i} d_j$$

in each set of variables $\mathbf{c}_i$

- Poisson's formula:

$$\mathrm{Res}_{d_0,d_1,\dots,d_n}(F_0,F_1,\dots,F_n) = \mathrm{Res}_{d_1,\dots,d_n}(F_1^\infty,\dots,F_n^\infty)^{d_0}\cdot \prod_{F_1(\xi)=\cdots=F_n(\xi)=0} \frac{F_0(\xi)}{\xi_0^{d_0}}$$

with $F_i^\infty := F_i(0,x_1,\dots,x_n)$

Formulae

A known case: if $d_i = 1$ for all i, then

$$\mathrm{Res}_{d_0,\dots,d_n}(F_0,\dots,F_n) = \det(c_{i,j})_{i,j}$$

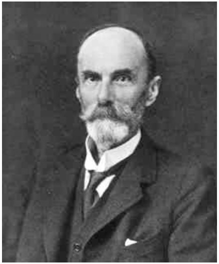
A less trivial case:

$$\begin{aligned}
F_0 &= c_{0,0}x_0 + c_{0,1}x_1 + c_{0,2}x_2\\
F_1 &= c_{1,0}x_0 + c_{1,1}x_1 + c_{1,2}x_2\\
F_2 &= c_{2,0}x_0^2 + c_{2,1}x_0x_1 + c_{2,2}x_0x_2 + c_{2,3}x_1^2 + c_{2,4}x_1x_2 + c_{2,5}x_2^2
\end{aligned}$$

then $\mathrm{Res}_{1,1,2}(F_0, F_1, F_2)$ is

$$c_{0,0}^2c_{1,1}^2c_{2,5}c_{0,0}^2c_{1,1}c_{1,2}c_{2,4}+c_{0,0}^2c_{1,2}^2c_{2,3}-2c_{0,0}c_{0,1}c_{1,0}c_{1,1}c_{2,5}+c_{0,0}c_{0,1}c_{1,0}c_{1,2}c_{2,4}$$
$$+c_{0,0}c_{0,1}c_{1,1}c_{1,2}c_{2,2}-c_{0,0}c_{0,1}c_{1,2}^2c_{2,1}+c_{0,0}c_{0,2}c_{1,0}c_{1,1}c_{2,4}2c_{0,0}c_{0,2}c_{1,0}c_{1,2}c_{2,3}$$
$$-c_{0,0}c_{0,2}c_{1,1}2c_{2,2}+c_{0,0}c_{0,2}c_{1,1}c_{1,2}c_{2,1}+c_{0,1}^2c_{1,0}^2c_{2,5}-c_{0,1}^2c_{1,0}c_{1,2}c_{2,2}+c_{0,1}^2c_{1,2}^2c_{2,0}$$
$$-c_{0,1}c_{0,2}c_{1,0}2c_{2,4}+c_{0,1}c_{0,2}c_{1,0}c_{1,1}c_{2,2}+c_{0,1}c_{0,2}c_{1,0}c_{1,2}c_{2,1}-2c_{0,1}c_{0,2}c_{1,1}c_{1,2}c_{2,0}$$
$$+c_{0,2}^2c_{1,0}^2c_{2,3}-c_{0,2}^2c_{1,0}c_{1,1}c_{2,1}+c_{0,2}^2c_{1,1}^2c_{2,0}$$

The Macaulay formula (1916)



$$\mathrm{Res}_{d_0,\dots,d_n} = \frac{\det(\mathbb{M})}{\det(\mathbb{E})}$$

with $\mathbb{M}$ a Sylvester matrix and $\mathbb{E}$ a block diagonal submatrix

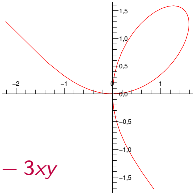
Let $\varphi\colon \mathbb{C} \dashrightarrow \mathbb{C}^2$ a nonconstant rational map given by

$$\varphi(t) = \left(\frac{p(t)}{r(t)}, \frac{q(t)}{s(t)}\right)$$

for $p, q, r, s \in \mathbb{C}[t]$ with $\gcd(p, r) = 1$ and $\gcd(q, s) = 1$
The defining polynomial of $\overline{\varphi(\mathbb{C})}$ is

$$\mathrm{Res}^{(t)}\left(r(t)\,x - p(t), s(t)\,y - q(t)\right) \in \mathbb{C}[x, y]$$

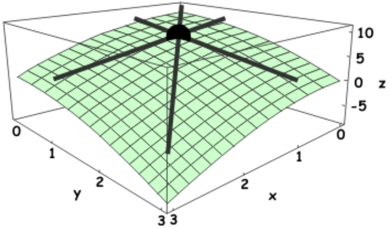
Example. The defining polynomial of the image of $\varphi(t) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3}\right)$ is



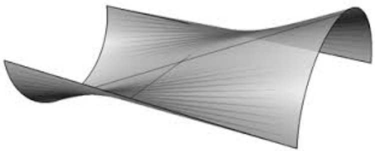
$$\mathrm{Res}^{(t)}\left((1+t^3)\,x - 3t, (1+t^3)\,y - 3t^2\right) = x^3 + y^3 - 3xy$$

Flexes of hypersurfaces

A point ξ of a surface $S \subset \mathbb{P}^3$ is a *flex* if there is a line L with order of contact ≥ 4 at ξ



If L is a line in S, then $L \subset \text{Flex}(S)$ In particular, if S is ruled then $\text{Flex}(S) = S$



The Monge-Cayley-Salmon theorem

S is ruled iff $\text{Flex}(S) = S$

Consequence. If S is not ruled then

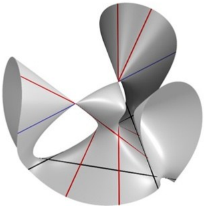
$$\# \text{ lines in } S \leq \deg(\text{Flex}(S))$$

Theorem (Salmon 1862)

If S is not ruled then $\mathrm{Flex}(S)$ is a 1-dimensional subvariety with

$$\deg(\mathrm{Flex}(S)) \leq \deg(S) \cdot (11 \deg(S) - 24)$$

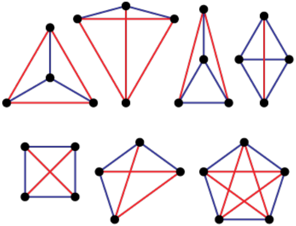
If S is cubic then $\deg(\mathrm{Flex}(S)) \leq 27$



Distinct distances

Conjecture (Erdös 1946)

n points in the plane define at least $\Omega\left(\frac{n}{\sqrt{\log n}}\right)$ *distinct distances*



Guth & Katz (Ann. Math. 2015) proved that they define at least

$$\Omega\left(\frac{n}{\log n}\right)$$

distinct distances

Equations for the flex locus

A point ξ in a hypersurface $S \subset \mathbb{P}^n$ is a *flex* if there is a line L with order of contact $\geq n+1$ at ξ

Let $F \in \mathbb{C}[x_0, \dots, x_n]$ homogeneous s.t. $S = V(F)$. Write

$$F(\mathbf{x} + t\mathbf{y}) = \sum_{i=0}^{n} F_i(\mathbf{x}, \mathbf{y}) t^i + O(t^{n+1})$$

Then $\xi \in \mathrm{Flex}(S)$ iff $\exists \eta \neq \xi$ s.t.

$$F_0(\xi, \eta) = \cdots = F_n(\xi, \eta) = 0$$

Theorem (Busé, D'Andrea, S, Weimann 2018)

There is $H \in \mathbb{C}[x_0, \dots, x_n]$ s.t. for any linear form ℓ,

$$\mathrm{Res}^{(\mathbf{y})}_{1,\dots,n,1}(F_0(\mathbf{x},\mathbf{y}), \dots, F_n(\mathbf{x},\mathbf{y}), \ell(\mathbf{y})) = \ell(\mathbf{x})^{n!} \cdot H \pmod F$$

and

$$\mathrm{Flex}(S) = S \cap V(H)$$

If S has no ruled component then $\mathrm{Flex}(S)$ is a subscheme of codimension 1 and degree

$$\deg(S) \cdot \left(\left(\sum_{i=1}^{n} \frac{n!}{i} \right) \deg(S) - (n+1)! \right)$$

Not the end of the story?

When $n = 2$ we can take

$$H = \text{Hess}(f) = \det \begin{pmatrix} \frac{\partial^2 F}{\partial x_0^2} & \frac{\partial^2 F}{\partial x_0 \partial x_1} & \frac{\partial^2 F}{\partial x_0 \partial x_2} \\ \frac{\partial^2 F}{\partial x_0 \partial x_1} & \frac{\partial^2 F}{\partial x_1^2} & \frac{\partial^2 F}{\partial x_1 \partial x_2} \\ \frac{\partial^2 F}{\partial x_0 \partial x_2} & \frac{\partial^2 F}{\partial x_1 \partial x_2} & \frac{\partial^2 F}{\partial x_2^2} \end{pmatrix}$$

What about $n \geq 3$?

Sparse polynomial systems

For $i = 1, \ldots, n$ let $\mathcal{A}_i \subset \mathbb{Z}^n$ a finite set and

$$f_i = \sum_{\mathbf{a} \in \mathcal{A}_i} c_{i,\mathbf{a}} \, \mathbf{x}^{\mathbf{a}}$$

a Laurent polynomial in $\mathbf{x} = (x_1, \ldots, x_n)$ with exponents in $\mathcal{A}_i$

Set $Q_i = \operatorname{conv}(\mathcal{A}_i)$ convex polytope in $\mathbb{R}^n$ and

$$\mathrm{MV}(Q_1, \ldots, Q_n)$$

the mixed volume

The Bernstein-Kushnirenko-Khovanskii theorem

The number of isolated solutions in $(\mathbb{C}^\times)^n$ of

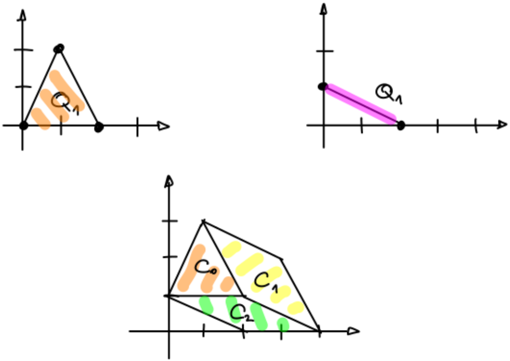
$$f_1 = \cdots = f_n = 0$$

is $\leq \mathrm{MV}(Q_1, \ldots, Q_n)$

Example.

$$f_1 = 1 - x_1 x_2^2 + 2x^2 \quad \text{and} \quad f_2 = x_2 - x_1^2$$

The associated polytopes and Minkowski sum are



The mixed volume is the sum of the volumes of the mixed cells:

$$MV(Q_1, Q_2) = \text{vol}(C_2) + \text{vol}(C_3) = 5$$

Sparse elimination theory

For $i = 0, \dots, n$ let $\mathcal{A}_i \subset \mathbb{Z}^n$ be a finite set and

$$f_i = \sum_{\mathbf{a} \in \mathcal{A}_i} c_{i,\mathbf{a}} \, \mathbf{x}^{\mathbf{a}}$$

a Laurent polynomial in $\mathbf{x} = (x_1, \dots, x_n)$ with exponents in $\mathcal{A}_i$.
Set $\boldsymbol{\mathcal{A}} = (\mathcal{A}_0, \dots, \mathcal{A}_n)$ and

$$\Omega_{\boldsymbol{\mathcal{A}}} = V(f_0, \dots, f_n) \subset (\mathbb{C}^\times)^n \times \prod_{i=0}^{n} \mathbb{P}(\mathbb{C}^{\mathcal{A}_i})$$

Definitions

- The $\boldsymbol{\mathcal{A}}$-eliminant is the irreducible polynomial $\mathrm{Elim}_{\boldsymbol{\mathcal{A}}} \in \mathbb{Z}[\mathbf{c}_0, \dots, \mathbf{c}_n]$ defining $\overline{\pi(\Omega)}$, if it has codimension 1, or $\mathrm{Elim}_{\boldsymbol{\mathcal{A}}} = 1$ otherwise
- The $\boldsymbol{\mathcal{A}}$-resultant is the primitive polynomial $\mathrm{Res}_{\boldsymbol{\mathcal{A}}} \in \mathbb{Z}[\mathbf{c}_0, \dots, \mathbf{c}_n]$ defining $\pi_* \Omega_{\boldsymbol{\mathcal{A}}}$

Example. $n = 1$, $\mathcal{A}_0 = \mathcal{A}_1 = \{0, 2\}$,

$$f_0 = c_{0,0} + c_{0,2}x^2 \quad \text{and } f_1 = c_{1,0} + c_{1,2}x^2$$

$$\text{Elim}_{\boldsymbol{\mathcal{A}}} = \det \begin{pmatrix} c_{0,2} & c_{1,2} \\ c_{0,0} & c_{1,0} \end{pmatrix} = c_{0,2}c_{1,0} - c_{1,2}c_{0,0}$$

$$\text{Res}_{\boldsymbol{\mathcal{A}}} = \det \begin{pmatrix} c_{0,2} & 0 & c_{1,2} & 0 \\ 0 & c_{0,2} & 0 & c_{1,2} \\ c_{0,0} & 0 & c_{1,0} & 0 \\ 0 & c_{0,0} & 0 & c_{1,0} \end{pmatrix}$$

$$= (c_{0,2}c_{1,0} - c_{1,2}c_{0,0})^2 = \text{Elim}_{\boldsymbol{\mathcal{A}}}{}^2$$

Indeed, f_0 and f_1 generic such that $\text{Elim}_{\boldsymbol{\mathcal{A}}}(f_0, f_1) = 0$ have 2 common zeros.

Essential families

For $I \subset \{0, \dots, n\}$ set $\boldsymbol{\mathcal{A}}_I = (\mathcal{A}_i)_{i \in I}$ and

$$L_{\boldsymbol{\mathcal{A}}_I} = \sum_{i \in I} \mathbb{Z} \cdot (\mathcal{A}_i - \mathcal{A}_i) \subset \mathbb{Z}^n$$

$\boldsymbol{\mathcal{A}}_I$ is essential if

- $\#I = \mathrm{rank}(L_{\boldsymbol{\mathcal{A}}_I}) + 1$
- $\#I' \leq \mathrm{rank}(L_{\boldsymbol{\mathcal{A}}_{I'}})$ for all $I' \subsetneq I$.

Fact.(Sturmfels) $\mathrm{codim}\, \pi(\Omega_{\boldsymbol{\mathcal{A}}}) = 1$ iff $\exists!$ essential subfamily of $\boldsymbol{\mathcal{A}}$

Example. $\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2 = \{(0,0), (1,0)\} \subset \mathbb{Z}^2$. Then

$$(\mathcal{A}_0, \mathcal{A}_1), \quad (\mathcal{A}_0, \mathcal{A}_2), \quad (\mathcal{A}_1, \mathcal{A}_2)$$

are essential, and $\mathrm{codim}(\pi(\Omega_{\boldsymbol{\mathcal{A}}})) = 2$

Properties of the $\mathcal{A}$-resultant

- $\mathrm{Res}_{\mathcal{A}}$ multihomogeneous of degree

$$\mathrm{MV}(Q_0, \dots, Q_{i-1}, Q_{i+1}, \dots, Q_n)$$

in each set of variables $\mathbf{c}_i$

- Poisson's formula:

$$\mathrm{Res}_{\mathcal{A}}(f_0, \dots, f_n) = \prod_{v \in \mathbb{Z}^n \text{ primitive}} \mathrm{Res}_{\overline{\mathcal{A}}_v}(f_{1,v}, \dots, f_{n,v})^{-h_{\mathcal{A}_0}(v)} \cdot \prod_{\boldsymbol{\xi} \in V(f_1, \dots, f_n)} f_0(\boldsymbol{\xi})$$

with

- $\overline{\mathcal{A}}_v = (\mathcal{A}_{1,v}, \dots, \mathcal{A}_{n,v})$, $\mathcal{A}_{i,v} = \{\mathbf{a} \in \mathcal{A}_i \mid \langle \mathbf{a}, v \rangle \text{ minimum}\}$
- $h_{\mathcal{A}_0}(v) = \min\{\langle \mathbf{a}, v \rangle \mid \mathbf{a} \in \mathcal{A}_0\}$

Let $\boldsymbol{\omega} = (\boldsymbol{\omega}_0, \dots, \boldsymbol{\omega}_n) \in \prod_{i=0}^{n} \mathbb{R}^{\mathcal{A}_i}$ with $\boldsymbol{\omega}_i = (\omega_{i,\mathbf{a}})_{\mathbf{a}\in\mathcal{A}_i}$ for $i = 0, \dots, n$

If $\mathrm{Res}_{\boldsymbol{\mathcal{A}}} = \sum_{\boldsymbol{\beta}} p_{\boldsymbol{\beta}} \mathbf{c}^{\boldsymbol{\beta}}$, the initial part of $\mathrm{Res}_{\boldsymbol{\mathcal{A}}}$ in the direction of $\boldsymbol{\omega}$ is the sum all monomials in $\mathrm{Res}_{\boldsymbol{\mathcal{A}}}$ with the minimum weight with respect to $\boldsymbol{\omega}$:

$$\mathrm{init}_{\boldsymbol{\omega}}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}}) = \sum_{\langle \boldsymbol{\beta}_0, \boldsymbol{\omega}\rangle = \min\{\langle \boldsymbol{\beta}, \boldsymbol{\omega}\rangle : p_{\boldsymbol{\beta}} \neq 0\}} p_{\boldsymbol{\beta}_0} \mathbf{c}^{\boldsymbol{\beta}_0}$$

For $i = 0, \dots, n$ consider the lifted polytope

$$Q_{i,\omega_i} = \mathrm{conv}(\{(\mathbf{a}, \omega_{i,\mathbf{a}}) \mid \mathbf{a} \in \mathcal{A}_i\}) \subset \mathbb{R}^{n+1}$$

For $\mathbf{v} \in \mathbb{Z}^{n+1}$ let $\mathcal{A}_{i,\mathbf{v}} \subset \mathcal{A}_i$ the part of minimal $\mathbf{v}$-weight and

$$f_{i,\mathbf{v}} = \sum_{\mathbf{a} \in \mathcal{A}_{i,\mathbf{v}}} \mathbf{c}_{i,\mathbf{a}} \mathbf{x}^{\mathbf{a}}$$

the "restriction" of f_i to $\mathcal{A}_{i,v}$. Set $\boldsymbol{\omega} = (\boldsymbol{\omega}_0, \dots, \boldsymbol{\omega}_n)$

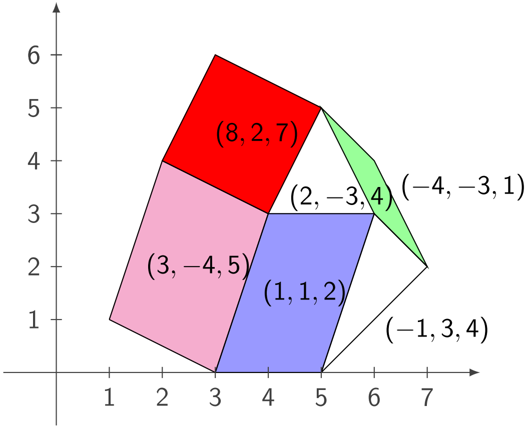
Theorem (D'Andrea, Jeronimo, S 2018)

$$\mathrm{init}_{\boldsymbol{\omega}}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}}) = \prod_{\mathbf{v}} \mathrm{Res}_{\mathcal{A}_{0,\mathbf{v}}, \dots, \mathcal{A}_{n,\mathbf{v}}}(f_{0,\mathbf{v}} \ \dots, f_{n,\mathbf{v}})$$

product over all $\mathbf{v} \in \mathbb{Z}^{n+1}$ primitive inner normals to the facets of the lower envelope of $Q_{0,\boldsymbol{\omega}_0} + \cdots + Q_{n,\boldsymbol{\omega}_n}$.

Example

$\mathcal{A}_0$={(0,0),(1,3),(2,2)}, $\mathcal{A}_1$={(0,0),(1,2),(2,0))}, $\mathcal{A}_2$={(1,1),(3,0)}



Example (cont.)

For $\omega = ((1,-1,0),(0,1,-1),(1,-1))$

$$\mathrm{init}_{\omega}(\mathrm{Res}_{\mathcal{A}}) = u_{0,13}^{5}\, u_{1,00}^{7}\, u_{2,30}^{7}$$

$\mathbf{v}$	$\mathrm{Res}_{\mathcal{A}_{\mathbf{v}}}(\mathbf{f}_{v})$
$(1,1,2)$	$u_{2,30}^{6}$
$(-4,-3,1)$	$u_{2,30}^{1}$
$(3,-4,5)$	$u_{1,00}^{7}$
$(8,2,7)$	$u_{0,13}^{5}$
$(2,-3,4)$	1
$(-1,3,4)$	1

Example (cont.)

$$\mathcal{A}_0=\{(0,0),(1,3),(2,2)\},\quad \mathcal{A}_1=\{(0,0),(1,2),(2,0))\},\quad \mathcal{A}_2=\{(1,1),(3,0)\}$$

Indeed

$$\begin{aligned}
\mathrm{Res}_{\mathcal{A}} = & -u_{1,12}\, u_{1,00}\, u_{0,22}\, u_{0,13}^2\, u_{1,20}^5\, u_{2,11}^5\, u_{2,30}^2\, u_{0,00}^2 + 3\, u_{1,12}^3\, u_{0,22}^2\, u_{1,20}^4\, u_{2,11}^5\, u_{2,30}^2 \\
& +5\, u_{1,12}^3\, u_{1,00}^4\, u_{0,13}^2\, u_{0,22}\, u_{2,11}\, u_{2,30}^6\, u_{0,00}^2 - 7\, u_{1,12}\, u_{1,00}^5\, u_{0,13}^4\, u_{1,20}\, u_{2,11}\, u_{2,30}^6\, u_{0,00} \\
& +2\, u_{1,12}\, u_{1,00}^4\, u_{0,13}^2\, u_{0,22}^2\, u_{1,20}^2\, u_{2,11}^3\, u_{2,30}^4\, u_{0,00} \\
& -2\, u_{1,12}\, u_{100}^3\, u_{0,22}^4\, u_{1,20}^3\, u_{2,11}^5\, u_{2,30}^2\, u_{0,00} \\
& +u_{1,12}^7\, u_{2,11}\, u_{2,30}^6\, u_{0,00}^5 - 13\, u_{0,13}\, u_{0,22}\, u_{1,00}^2\, u_{1,12}^4\, u_{1,20}\, u_{2,11}^2\, u_{2,30}^5\, u_{0,00}^3 \\
& -2\, u_{0,13}^3\, u_{0,22}\, u_{1,00}^3\, u_{1,20}^4\, u_{2,11}^4\, u_{2,30}^3\, u_{0,00} + u_{1,12}\, u_{1,00}^6\, u_{0,22}^5\, u_{2,11}^3\, u_{2,30}^4 \\
& +6\, u_{1,12}^3\, u_{1,00}^3\, u_{0,22}^3\, u_{1,20}\, u_{2,11}^3\, u_{2,30}^4\, u_{0,00}^2 - 7\, u_{1,12}^3\, u_{1,00}\, u_{0,13}^2\, u_{1,20}^3\, u_{2,11}^3\, u_{2,30}^4\, u_{0,00}^3 \\
& +u_{0,13}^5\, u_{1,00}^7\, u_{2,30}^7 \\
& +u_{1,12}\, u_{0,22}^3\, u_{1,20}^6\, u_{2,11}^7\, u_{0,00}^2 - 5\, u_{0,13}\, u_{0,22}^3\, u_{1,00}^5\, u_{1,12}^2\, u_{2,11}^2\, u_{2,30}^5\, u_{0,00} \\
& +u_{0,13}^3\, u_{0,22}^2\, u_{1,00}^6\, u_{1,20}\, u_{2,11}^2\, u_{2,30}^5 + 14\, u_{0,13}^3\, u_{1,00}^3\, u_{1,12}^2\, u_{1,20}^2\, u_{2,11}^2\, u_{2,30}^5\, u_{0,00}^2 \\
& -u_{0,13}\, u_{0,22}^2\, u_{1,00}^2\, u_{1,12}^2\, u_{1,20}^3\, u_{2,11}^4\, u_{2,30}^3\, u_{0,00}^2 + u_{0,13}^3\, u_{1,20}^7\, u_{2,11}^6\, u_{2,30}\, u_{0,00}^2 \\
& +3\, u_{1,12}^5\, u_{0,22}\, u_{1,20}^2\, u_{2,11}^3\, u_{2,30}^4\, u_{0,00}^4
\end{aligned}$$

Multivariate homogeneous resultants.

- Macaulay (1916)

$$\mathrm{Res}_{d_0,\dots,d_n} = \frac{\det(\mathbb{M})}{\det(\mathbb{E})}$$

with $\mathbb{M}$ Sylvester matrix and $\mathbb{E}$ block diagonal submatrix

Sparse eliminants.

- Canny-Emiris (1993), Sturmfels (1994): $(\mathbb{M})$ a nonzero multiple of $\mathrm{Elim}_{\mathcal{A}}$ with

$$\deg_{\mathbf{c}_0}(\det(\mathbb{M})) = \deg_{\mathbf{c}_0}(\mathrm{Elim}_{\mathcal{A}})$$

- D'Andrea (2002): Macaulay formula for $\mathrm{Elim}_{\mathcal{A}}$

Sparse eliminants.

- D'Andrea, Jeronimo, S (2018): Macaulay formula for $\mathrm{Res}_{\mathcal{A}}$

Sylvester matrices

Let $\mathcal{E} \subset \mathbb{Z}^n$ a finite subset and RC a *row content* function on $\mathcal{E}$: for $\mathbf{b} \in \mathcal{E}$

$$\mathrm{RC}(\mathbf{b}) = (i, \mathbf{a})$$

with $0 \leq i \leq n$ and $\mathbf{a} \in \mathcal{A}_i$ s.t. $\mathbf{b} - \mathbf{a} + \mathcal{A}_i \subset \mathcal{E}$

For $\mathbf{b}, \mathbf{b}' \in \mathcal{E}$ set

$$\mathbb{M}_{\mathbf{b},\mathbf{b}'} = \text{ coefficient of } \mathbf{x}^{\mathbf{b}'} \text{ in } \mathbf{x}^{\mathbf{b}-\mathbf{a}} f_i$$

Then

$$\det(\mathrm{Elim}_{\boldsymbol{\mathcal{A}}}) \mid \det(\mathbb{M})$$

Proof. Let $\mathbf{f} = (f_0, \dots, f_n)$ generic with $\mathrm{Res}_{\boldsymbol{\mathcal{A}}}(\mathbf{f}) = 0$, and $\boldsymbol{\xi} \in V(f_0, \dots, f_n)$. Then

$$(\boldsymbol{\xi}^{\mathbf{b}})_{\mathbf{b} \in \mathcal{E}} \in \ker(\mathbb{M})$$

and so $\det(\mathbb{M}) = 0$ $\square$

The case $n = 1$

Let

$$\mathcal{A}_0 = \{0, 1, \dots, m\} \quad \text{and} \quad \mathcal{A}_1 = \{0, 1, \dots, k\}$$

Set $\mathcal{E} = \{0, 1, \dots, m + k - 1\}$ with row content function defined, for $b \in \mathcal{E}$, by

$$\mathrm{RC}(b) = \begin{cases} (0, b) & \text{if } 0 \leq b \leq k - 1 \\ (1, b - k) & \text{if } k \leq b \leq m + k - 1 \end{cases}$$

Then

$$\mathbb{M} = \begin{pmatrix} a_m & a_{m-1} & \cdots & a_0 & & & \\ & a_m & a_{m-1} & \cdots & a_0 & & \\ & & \ddots & & & \ddots & \\ & & & a_m & a_{m-1} & \cdots & a_0 \\ b_k & b_{k-1} & \cdots & b_0 & & & \\ & b_k & b_{k-1} & \cdots & b_0 & & \\ & & \ddots & & & \ddots & \\ & & & b_k & b_{k-1} & \cdots & b_0 \end{pmatrix}$$

and $\det(\mathbb{M}) = \mathrm{Res}_{m,k}$, the Sylvester resultant

A Macaulay style formula for $\mathrm{Res}_{\mathcal{A}}$

We simplify and generalize D'Andrea's formula to compute $\mathrm{Res}_{\mathcal{A}}$ without imposing the conditions

- $\boldsymbol{\mathcal{A}} = (\mathcal{A}_0, \dots, \mathcal{A}_n)$ essential
- $L_{\boldsymbol{\mathcal{A}}} = \mathbb{Z}^n$

Produced by a recursive procedure with input

- $\boldsymbol{\mathcal{A}} = (\mathcal{A}_0, \dots, \mathcal{A}_n)$
- $I \subset \{0, \dots, n\}$ s.t. $\boldsymbol{\mathcal{A}}_I$ is essential
- $\boldsymbol{\delta} \in \mathbb{Q}^n$ generic

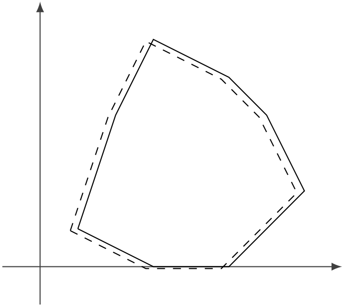
and output a Sylvester matrix $\mathbb{M}$ and a block diagonal submatrix $\mathbb{E}$ of $\mathbb{M}$ s.t.

$$\mathrm{Res}_{\boldsymbol{\mathcal{A}}} = \frac{\det(\mathbb{M})}{\det(\mathbb{E})}$$

The construction

Set

$$\mathcal{E} = (Q_0 + \cdots + Q_n + \boldsymbol{\delta}) \cap \mathbb{Z}^n$$



$$\mathcal{E} = \{(3,0); (4,0); (3,1); (4,1); (3,2); (1,1); (2,1); (2,2); (2,3); (3,3); (4,3); (3,4); (4,4); (3,5); (4,5); (5,4); (2,4); (4,2); (5,1); (5,2); (5,3); (6,2); (6,3)\}$$

Recursive definition of RC, $\mathbb{M}$ and $\mathbb{E}$

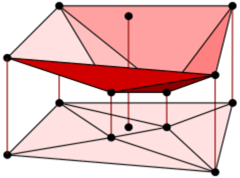
Suppose w.l.o.g. that $(\mathcal{A}_0, \ldots, \mathcal{A}_k)$ is the essential subfamily

$\boxed{k=0}$ $\mathcal{A}_0 = \{\mathbf{a}_0\}$ so that $\mathcal{E} = (\mathbf{a}_0 + Q_1 + \cdots + Q_n + \boldsymbol{\delta}) \cap \mathbb{Z}^n$

Choose generic liftings

$$\widetilde{\omega}_i : \mathcal{A}_i \to \mathbb{R}$$

defining regular polyhedral subdivisions of the Q_i's and of $Q_1 + \cdots + Q_n$



Source: De Loera, Rambau and Santos, *Triangulations*, 2011

For each cell $C = C_1 + \cdots + C_n$ and $\mathbf{b} \in (C + \boldsymbol{\delta}) \cap \mathbb{Z}^n$ set

$$\mathrm{RC}(\mathbf{b}) = \begin{cases} (i, \mathbf{a}) & \text{if } C_i = \{\mathbf{a}\} \text{ and } \dim(C_j) > 0 \text{ for } j < i \\ (0, \mathbf{a}_0) & \text{otherwise} \end{cases}$$

- RC defines a Sylvester matrix $\mathbb{M}$
- $\mathbb{E}$ given by $\{\mathbf{b} \in \mathcal{E} \mid \mathrm{RC}(\mathbf{b}) = (i, a) \text{ with } i \neq 0\}$

In this case

$$\mathrm{Res}_{\mathcal{A}} = c_0^{\mathrm{MV}(Q_1,\ldots,Q_n)} = \frac{\det(\mathbb{M})}{\det(\mathbb{E})}$$

$\boxed{k > 0}$

Choose $\mathbf{a}_0 \in \mathcal{A}_0$ and $\boldsymbol{\omega} = (\omega_0, \omega_1, \dots, \omega_n)$ given by

- $\omega_0(\mathbf{a}_0) = 0$ and $\omega_0(\mathbf{a}) = 1$ for $\mathbf{a} \in \mathcal{A}_0$, $\mathbf{a} \neq \mathbf{a}_0$
- $\omega_i(\mathbf{a}) = 1$ for $\mathbf{a} \in \mathcal{A}_i$ and $i = 1, \dots, n$

Let $\mathbf{v}_0, \dots, \mathbf{v}_N \in \mathbb{Z}^{n+1}$ primitive inner normals to the facets of the lower envelope of $Q_{0,\omega_0} + \cdots + Q_{n,\omega_n}$. Then

- if $\mathbf{v}_0 = (\mathbf{0}, 1)$, $\mathcal{A}_{0,\mathbf{v}_0} = \{\mathbf{a}_0\}$ is an essential subfamily of $(\mathcal{A}_{0,\mathbf{v}_0}, \dots, \mathcal{A}_{n,\mathbf{v}_0})$
- if $\mathbf{v}_j \neq (\mathbf{0}, 1)$, there is an essential subfamily contained in $(\mathcal{A}_{1,\mathbf{v}_j}, \dots, \mathcal{A}_{k,\mathbf{v}_j})$

For $\mathbf{b} \in \mathcal{E}$ in the cell of $\mathbf{v}_j$, define $\mathrm{RC}(\mathbf{b})$ from the RC function associated to $(\mathcal{A}_{0,\mathbf{v}_j}, \dots, \mathcal{A}_{n,\mathbf{v}_j})$ and this essential subfamily

RC defines the Sylvester matrix $\mathbb{M}$

For $j = 0, \dots, N$, let $\mathbb{M}_{\mathbf{v}_j}$ be the matrix associated to $(\mathcal{A}_{0,\mathbf{v}_j}, \dots, \mathcal{A}_{n,\mathbf{v}_j})$ and its marked essential subfamily, and $\mathbb{E}_{\mathbf{v}_j}$ its corresponding submatrix.

Set $\mathbb{E}$ as the submatrix of $\mathbb{M}$ with rows and columns are indexed by the points in $\mathcal{E}$ which index the $\mathbb{E}_{\mathbf{v}_j}$'s

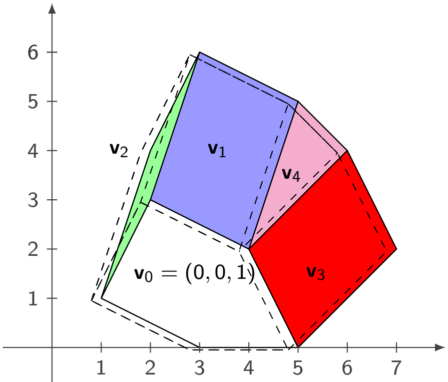
Theorem (D'Andrea, Jeronimo, S 2015)

$$\mathrm{Res}_{\mathcal{A}} = \frac{\det(\mathbb{M})}{\det(\mathbb{E})}$$

Example

$\mathcal{A}_0=\{(0,0),(1,3),(2,2)\}, \quad \mathcal{A}_1=\{(0,0),(1,2),(2,0))\}, \quad \mathcal{A}_2=\{(1,1),(3,0)\}$

$f_0=a_0+a_1xy^3+a_2x^2y^2, \quad f_1=b_0+b_1xy^2+b_2x^2, \quad f_2=c_0xy+c_1x^3$



$\mathcal{E}:=\{(3,0);(4,0);(3,1);(4,1);(3,2);(1,1);(2,1);(2,2);(2,3);(3,3);(4,3);(3,4);(4,4);(3,5);(4,5);$

$(5,4);$**(2,4);**$(4,2);(5,1);(5,2);(5,3);(6,2);(6,3)\}$

$$
\mathbb{M} = \begin{pmatrix}
a_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & 0 \\
0 & a_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & a_2 & 0 & 0 \\
0 & 0 & a_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 \\
0 & 0 & 0 & a_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_2 & a_1 \\
0 & 0 & 0 & 0 & a_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_2 \\
c_1 & 0 & 0 & 0 & 0 & c_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c_1 & 0 & 0 & 0 & 0 & c_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_1 & 0 & 0 & 0 & c_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & b_2 & 0 & 0 & b_0 & 0 & 0 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & b_2 & 0 & 0 & b_0 & 0 & 0 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & b_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 & b_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b_0 & 0 & 0 & 0 & b_1 & 0 & 0 & 0 & 0 & b_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & 0 & 0 & 0 & 0 & b_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_0 & 0 & b_2 & 0 & 0 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_0 & 0 & 0 & 0 & 0 & b_1 & 0 & 0 & 0 & 0 & b_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & c_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_0 & 0 & 0 & 0 & b_2 & 0 & b_1
\end{pmatrix}
$$

$$
\begin{aligned}
\det(\mathbb{M}) = \overbrace{b_1 c_0^3}^{\det(\mathbb{E})} \cdot \Big(& b_0^7 a_1^5 c_1^7 + a_1^3 a_0^2 b_2^7 c_0^6 c_1 - 2 c_1^2 c_0^5 a_2^4 b_0^3 a_0 b_1 b_2^3 + c_0^7 a_2^3 a_0^2 b_1 b_2^6 - c_1^2 c_0^5 a_1^2 b_0 a_0^2 b_1 b_2^5 a_2 \\
& - 2 b_0^3 a_1^3 a_2 a_0 b_2^4 c_0^4 c_1^3 + b_0^6 a_1^3 a_2^2 b_2 c_0^2 c_1^5 + 2 b_0^4 a_1^2 a_2^2 a_0 b_1 b_2^2 c_0^3 c_1^4 - b_0^2 a_1 a_2^2 a_0^2 b_1^2 b_2^3 c_0^4 c_1^3 \\
& + 14 b_0^3 a_0^2 b_1^2 c_0^2 c_1^5 a_1^3 b_2^2 - 5 b_0^5 a_0 b_1^2 c_0^2 c_1^5 a_2^3 a_1 + 6 b_0^3 a_0^2 b_1^3 c_0^3 c_1^4 a_2^3 b_2 - 7 b_0 a_0^3 b_1^3 c_0^3 c_1^4 a_1^2 b_2^3 \\
& + 5 b_0^4 a_0^2 b_1^3 c_0 c_1^6 a_2 a_1^2 - 13 b_0^2 a_0^3 b_1^4 c_0^2 c_1^5 b_2 a_1 a_2 + 3 a_0^3 b_1^3 c_0^5 c_1^2 a_2^2 b_2^4 - 7 b_0^5 a_0 b_1 c_0 c_1^6 a_1^4 b_2 \\
& + 3 a_0^4 b_1^5 c_0^3 c_1^4 b_2^2 a_2 + c_1^4 c_0^3 a_2^5 b_0^6 b_1 + a_0^5 b_1^7 c_0 c_1^6 \Big)
\end{aligned}
$$

Proof (sketch)

Proposition

- $\mathrm{init}_{\boldsymbol{\omega}}(\det(\mathbb{M})) = \prod_{j=0}^{N} \det(\mathbb{M}_{\mathbf{v}_j})$.
- $\det(\mathbb{E}) = \prod_{j=0}^{N} \det(\mathbb{E}_{\mathbf{v}_j})$.
- $\det(\mathbb{M}) = P \cdot \mathrm{Res}_{\boldsymbol{\mathcal{A}}}$ with $P \in \mathbb{Z}[\mathbf{c}_1, \dots, \mathbf{c}_n]$

Hence

$$\frac{\det(\mathbb{M})}{\mathrm{Res}_{\boldsymbol{\mathcal{A}}}} = \frac{\mathrm{init}_{\boldsymbol{\omega}}(\det(\mathbb{M}))}{\mathrm{init}_{\boldsymbol{\omega}}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}})} = \frac{\prod_{j=0}^{N} \det(\mathbb{M}_{\mathbf{v}_j})}{\prod_{j=0}^{N} \mathrm{Res}_{\boldsymbol{\mathcal{A}}_{\mathbf{v}_j}}} = \prod_{j=0}^{N} \det(\mathbb{E}_{\mathbf{v}_j}) = \det(\mathbb{E})$$

Thanks!