ARITHMETIC POSITIVITY ON TORIC VARIETIES

JOSÉ IGNACIO BURGOS GIL, ATSUSHI MORIWAKI, PATRICE PHILIPPON,
AND MARTÍN SOMBRA

ABSTRACT. We continue the study of the arithmetic geometry of toric varieties started in [BPS14]. In this text, we study the positivity properties of metrized $\mathbb{R}$-divisors in the toric setting. For a toric metrized $\mathbb{R}$-divisor, we give formulae for its arithmetic volume and its χ-arithmetic volume, and we characterize when it is arithmetically ample, nef, big or pseudo-effective, in terms of combinatorial data. As an application, we prove a higher-dimensional analogue of Dirichlet's unit theorem for toric varieties, we give a characterization for the existence of a Zariski decomposition of a toric metrized $\mathbb{R}$-divisor, and we prove a toric arithmetic Fujita approximation theorem.

CONTENTS

INTRODUCTION

The study of the positivity properties of a divisor on an algebraic variety is a central subject in algebraic geometry which has many important results and applications. A modern account about this subject can be found in the book [Laz04].

There are different notions of positivity for a divisor: it can be ample, nef, big, or pseudo-effective. There are also numerical invariants of a divisor related with

---

positivity, like its degree and its volume. The degree of a divisor is the top intersection product of the divisor with itself, while the volume measures the asymptotic growth of the space of global sections of the multiples of the divisor. When the divisor is ample both invariants agree but, in general, the volume is always nonnegative while the degree can be either positive, zero or negative. The different notions of positivity, the degree and the volume of a divisor are invariant under numerical equivalence and can be extended to $\mathbb{R}$-divisors.

Analogues of these invariants and notions of positivity have been introduced in Arakelov geometry and their study has interesting applications to Diophantine geometry. In [Zha95a], Zhang started the study of a theory of arithmetic ampleness and proved an arithmetic Nakai-Moishezon criterion. Using this theory, he obtained the so-called "theorem on successive algebraic minima" relating the minimal height of points in the variety which are Zariski dense with the height of the variety itself. This result plays an important role in Diophantine geometry, for example in the context of the Manin-Mumford conjecture, the Bomogolov and Lehmer questions, and the Zilber-Pink conjecture.

In [Mor09], Moriwaki introduced the notion of arithmetic volume measuring the growth of the number of small sections of the multiples of an arithmetic divisor, and proved the continuity of this invariant. In [Yua08], Yuan studied the basic properties of big arithmetic divisors, that is, arithmetic divisors with strictly positive arithmetic volumes. As an application, he obtained a very general criterion for the equidistribution of points of small height in an arithmetic variety, generalizing the previous equidistribution theorems of Szpiro–Ullmo–Zhang [SUZ97], Bilu [Bil97], Favre–Rivera-Letelier [FR06], Baker–Rumely [BR06] and Chambert-Loir [Cha06].

Given these results, it is interesting to dispose of effective criteria to test the positivity properties of an arithmetic divisor and to be able to calculate the associated invariants in concrete situations. In this direction, Moriwaki has studied a family of twisted Fubini-Study metrics on the hyperplane divisor of $\mathbb{P}^n_{\mathbb{Z}}$. He has obtained criteria for when these metrics define an ample, nef, big or pseudo-effective arithmetic divisor, he also computed the arithmetic volumes of such divisors, proved a Fujita approximation theorem and gave a criteria for when a special type of Zariski decomposition exists [Mor11]. The present text generalizes these results to arbitrary toric (adelically) metrized $\mathbb{R}$-divisors on toric varieties.

Toric varieties can be described in combinatorial terms and many of their algebro-geometric properties can be translated in terms of this description. A proper toric variety X of dimension n over an arbitrary field is given by a complete fan Σ on a vector space $N_{\mathbb{R}} \simeq \mathbb{R}^n$. A toric $\mathbb{R}$-divisor D on X defines a function $\Psi_D \colon N_{\mathbb{R}} \to \mathbb{R}$ which is linear in each cone of the fan Σ. Following the usual terminology in toric geometry, we call such function a "virtual support function". One can also associate to D a polytope Δ_D in the dual space $M_{\mathbb{R}} := N_{\mathbb{R}}^{\vee}$. There is a "toric dictionary" that translates algebro-geometric properties of the pair (X, D) into combinatorial properties of the fan, the virtual support function and the polytope. For instance, the set of points of Δ_D in the dual lattice $M = N^{\vee}$ gives a basis for the space of global sections of $\mathcal{O}(D)$, and the volume of D can be computed as $\mathrm{vol}(X, D) = n! \, \mathrm{vol}_M(\Delta_D)$, where vol_M is the Haar measure on $M_{\mathbb{R}}$ which gives covolume 1 to the lattice M. The divisor D is nef if and only if the function Ψ_D is concave and, if this is the case, its degree coincides with its volume.

In [BPS14], this toric dictionary has been extended to cover some of the arithmetic properties of toric varieties. Let $\mathbb{K}$ be a global field, that is, a field which is either a number field or the field of rational functions of a projective curve, and suppose that X is a toric variety over $\mathbb{K}$. Then, a toric metrized divisor $\overline{D}$ on X defines a family of functions $\psi_{\overline{D}, v} \colon N_{\mathbb{R}} \to \mathbb{R}$ indexed by the set of places $\mathfrak{M}_{\mathbb{K}}$ of

$\mathbb{K}$ such that $\psi_{\overline{D},v} = \Psi_D$ for all v except for a finite number of them. By duality, this family of functions gives rise to a family of concave functions on the polytope $\vartheta_{\overline{D},v} \colon \Delta_D \to \mathbb{R}$, called the local roof functions of $\overline{D}$. The global roof function $\vartheta_{\overline{D}}$ of a metrized divisor $\overline{D}$ is defined as a weighted sum over all places of these local roof functions. The convex subset $\Theta_{\overline{D}} \subset \Delta_D$ is the set of points where $\vartheta_{\overline{D}}$ takes nonnegative values.

These objects encode many of the Arakelov-theoretical properties of $\overline{D}$. For instance, a metrized divisor $\overline{D}$ is called semipositive in [BPS14] if its metrics are uniform limits of semipositive smooth (respectively algebraic) metrics for the Archimedean (respectively non-Archimedean) places. Then, $\overline{D}$ is semipositive if and only if all the functions $\psi_{\overline{D},v}$ are concave. If this is the case, the height of X with respect to $\overline{D}$ can be computed as

$$\mathrm{h}_{\overline{D}}(X) = (n+1)! \int_{\Delta_D} \vartheta_{\overline{D}} \, \mathrm{d\,vol}_M \,.$$

Moreover, we show in the present text how these notions and results extend to toric metrized $\mathbb{R}$-divisors. We refer the reader to §4 for the precise definitions and more details.

The arithmetic volume and the χ-arithmetic volume of a metrized $\mathbb{R}$-divisor $\overline{D}$ measure respectively the growth of the number of small sections of the multiples of $\overline{D}$ and the growth of the Euler characteristic of the space of sections of the multiples of $\overline{D}$ (Definition 3.14). Our first main result in this text are formulae for the arithmetic volume and the χ-arithmetic volume of a toric metrized $\mathbb{R}$-divisor (Theorem 5.6).

Theorem 1. *Let X be a proper toric variety over $\mathbb{K}$ and $\overline{D}$ a toric metrized $\mathbb{R}$-divisor on X. Then the arithmetic volume of $\overline{D}$ is given by*

$$\widehat{\mathrm{vol}}(X,\overline{D}) = (n+1)! \int_{\Delta_D} \max(0,\vartheta_{\overline{D}}) \, \mathrm{d\,vol}_M = (n+1)! \int_{\Theta_{\overline{D}}} \vartheta_{\overline{D}} \, \mathrm{d\,vol}_M,$$

while its χ-arithmetic volume is given by

$$\widehat{\mathrm{vol}}_\chi(X,\overline{D}) = (n+1)! \int_{\Delta_D} \vartheta_{\overline{D}} \, \mathrm{d\,vol}_M \,.$$

The height is defined for DSP metrized $\mathbb{R}$-divisors, that is, differences of semipositive ones, whereas the arithmetic volume and the χ-arithmetic volume are defined for any metrized $\mathbb{R}$-divisor. Observe that when $\overline{D}$ is semipositive, the χ-arithmetic volume agrees with the height and the formula for $\widehat{\mathrm{vol}}_\chi(X,\overline{D})$ coincides with that for $\mathrm{h}_{\overline{D}}(X)$. Nevertheless, we show that the notion of height no longer coincides with that of χ-arithmetic volume for arbitrary DSP $\mathbb{R}$-divisors (Examples 5.9 and 5.11).

Formulae similar to those in Theorem 1 were previously obtained by Yuan [Yua09a], [Yua09b] and by Boucksom and Chen [BC11] for a metrized divisor $\overline{D}$ on a variety over a number field, under the hypothesis that the underlying divisor D is big and that the metrics at the non-Archimedean places are given by a global projective model over the ring of integers of the number field. These formulae are expressed in terms of the integral of a function over the Okounkov body of the divisor. The Okounkov body is a generalization to arbitrary divisors of the polytope that appears in toric geometry. Indeed, the functions introduced by Yuan and by Boucksom and Chen can be seen as a generalization to arbitrary metrized divisors (under the aforementioned hypothesis) of the global roof function.

Our second main result is the following characterization of positive toric metrized $\mathbb{R}$-divisors (Theorem 6.1).

Theorem 2. *Let X be a proper toric variety over $\mathbb{K}$ and $\overline{D}$ a toric metrized $\mathbb{R}$-divisor on X. Then*

(1) *$\overline{D}$ is ample if and only if Ψ_D is strictly concave on Σ, the function $\psi_{\overline{D},v}$ is concave for all $v \in \mathfrak{M}_{\mathbb{K}}$, and $\vartheta_{\overline{D}}(x) > 0$ for all $x \in \Delta_D$;*

(2) *$\overline{D}$ is nef if and only if $\psi_{\overline{D},v}$ is concave for all $v \in \mathfrak{M}_{\mathbb{K}}$ and $\vartheta_{\overline{D}}(x) \geq 0$ for all $x \in \Delta_D$;*

(3) *$\overline{D}$ is big if and only if $\dim(\Delta_D) = n$ and there exists $x \in \Delta_D$ such that $\vartheta_{\overline{D}}(x) > 0$;*

(4) *$\overline{D}$ is pseudo-effective if and only if there exists $x \in \Delta_D$ such that $\vartheta_{\overline{D}}(x) \geq 0$;*

(5) *$\overline{D}$ is effective if and only if $0 \in \Delta_D$ and $\vartheta_{\overline{D},v}(0) \geq 0$ for all $v \in \mathfrak{M}_{\mathbb{K}}$.*

There are several questions one can ask about the relations between the different notions of positivity. An effective metrized $\mathbb{R}$-divisor is also pseudo-effective and, conversely, one can ask if any pseudo-effective metrized $\mathbb{R}$-divisor is linearly equivalent to an effective one. As Moriwaki pointed out, this question can be seen as an extension of Dirichlet's unit theorem to metrized $\mathbb{R}$-divisors on varieties [Mor13].

Another relevant question is whether one can approximate pseudo-effective or big metrized $\mathbb{R}$-divisors by nef or ample ones. A Zariski decomposition of a big metrized $\mathbb{R}$-divisor $\overline{D}$ amounts to its decomposition, up to a birational transformation, into an effective part and a nef part which has the same arithmetic volume as $\overline{D}$. Such a decomposition always exists when the underlying variety is a curve over a number field [Mor12c] but it does not always exist for varieties of higher dimension [Mor13].

In the absence of a Zariski decomposition, one can ask for the existence of an arithmetic Fujita approximation. The existence of an arithmetic Fujita approximation was proved by Yuan [Yua09a] and by Chen [Che10] for the case when $\mathbb{K}$ is a number field, D is a divisor, the metrics at the infinite places are smooth, and those at the finite places come from a common projective model over $\mathcal{O}_{\mathbb{K}}$.

As a consequence of our characterization of the different notions of arithmetic positivity, we give a positive answer to the Dirichlet's unit theorem for toric varieties when the base field $\mathbb{K}$ is an A-field, that is, a number field or the function field of a curve over a finite field. We also give a criterion for when a toric Zariski decomposition exists and we prove a toric Fujita approximation theorem (Theorem 7.2).

Theorem 3. *Let X be a proper toric variety over $\mathbb{K}$ and $\overline{D}$ a toric metrized $\mathbb{R}$-divisor on X.*

(1) *Assume that $\mathbb{K}$ is an A-field. Then $\overline{D}$ is pseudo-effective if and only if there exists $a \in \Delta_D$ and $\alpha \in \mathbb{K}^{\times} \otimes \mathbb{R}$ such that*
$$\overline{D} + \widehat{\operatorname{div}}(\alpha \chi^a) \geq 0.$$

(2) *Assume that $\overline{D}$ is big. Then there exists a birational toric map $\varphi \colon X' \to X$ and toric metrized $\mathbb{R}$-divisors $\overline{P}, \overline{E}$ on X' such that $\overline{P}$ is nef, $\overline{E}$ is effective,*
$$\varphi^* \overline{D} = \overline{P} + \overline{E} \quad and \quad \widehat{\operatorname{vol}}(X', \overline{P}) = \widehat{\operatorname{vol}}(X, \overline{D})$$
if and only if $\Theta_{\overline{D}}$ is a quasi-rational polytope (Definition 4.5).

(3) *Assume that $\overline{D}$ is big. Then, for every $\varepsilon > 0$, there exists a birational toric map $\varphi \colon X' \to X$ and toric metrized $\mathbb{R}$-divisors $\overline{A}, \overline{E}$ on X' such that $\overline{A}$ is ample, $\overline{E}$ is effective,*
$$\varphi^* \overline{D} = \overline{A} + \overline{E} \quad and \quad \widehat{\operatorname{vol}}(X', \overline{A}) \geq \widehat{\operatorname{vol}}(X, \overline{D}) - \varepsilon.$$

A stronger version of the Zariski decomposition asks that the nef part is maximal in a precise sense (Definition 3.27). In the toric setting, one can ask for the existence of a decomposition which is maximal among all toric ones (Definition 7.1).

Indeed, we show that the criterion in Theorem 3(2) extends to pseudo-effective metrized $\mathbb{R}$-divisors if one uses this stronger version of the Zariski decomposition (Theorem 7.2(2)).

A related question in whether the existence of a non-necessarily toric Zariski decomposition of a big toric metrized $\mathbb{R}$-divisor is equivalent to the existence of a toric one. In §8 we give a partial affirmative answer to this question restricting to toric varieties defined over $\mathbb{Q}$ and arithmetic $\mathbb{R}$-divisors (Theorem 8.2). Roughly speaking, arithmetic $\mathbb{R}$-divisors correspond to metrized $\mathbb{R}$-divisors whose metrics at the non-Archimedean places are given by a single integral model and they are closer to the more traditional language of arithmetic varieties, see Example 3.19 for the precise definition and more details.

Since Arakelov geometry can be developed in different frameworks, we discuss briefly the one in the present text. We have chosen to use the adelic language introduced in this context by Zhang in [Zha95b] instead of the language of arithmetic varieties of Gillet and Soulé as in [GS90]. This point of view is more general and flexible, and allows to treat the cases of number fields and of function fields in a uniform way. Moreover, since general proper toric varieties are not necessarily projective nor smooth, we do not add any hypothesis of projectivity or smoothness. Also, we work in the framework of $\mathbb{R}$-divisors since it is the appropriate one for Dirichlet's unit theorem on varieties, and it is also suitable for discussing the Zariski decomposition and Fujita approximation problems.

There are several different definitions in the literature for the various notions of arithmetic positivity, depending on the used framework. Adding the appropriate technical hypothesis, these different definitions are equivalent but, in general, they are not. Due to our choice of working framework, we had to adjust these pre-existing definitions (Definition 3.18). A systematic study of the definitions we propose here, including the openness of the ample and the big cones, the closedness of the nef and the pseudo-effective cones, and the continuity of the arithmetic volume, falls outside the scope of the present text. Nevertheless, our results show that these definitions behave as expected in the toric case.

1. Global fields

Throughout this text, by a *valued field* we mean a field K together with an absolute value $|\cdot|$ that is either Archimedean or associated to a nontrivial discrete valuation. If $(K, |\cdot|)$ is a valued field, then we set $K^\circ = \{x \in K \mid |x| \leq 1\}$. When $|\cdot|$ is non-Archimedean, the unit ball K° is a ring.

Let $\mathbb{K}$ be a field and $\mathfrak{M}$ a family of absolute values on $\mathbb{K}$ with positive real weights. For each $v \in \mathfrak{M}$ we denote by $|\cdot|_v$ the corresponding absolute value, by $n_v \in \mathbb{R}_{>0}$ the weight, and by $\mathbb{K}_v$ the completion of $\mathbb{K}$ with respect to $|\cdot|_v$. We also set

$$\lambda_v = \begin{cases} 1 & \text{if } |\cdot|_v \text{ is Archimedean,} \\ -\log|\varpi_v|_v & \text{otherwise,} \end{cases}$$

where ϖ_v is a uniformizer of the maximal ideal of $\mathbb{K}_v^\circ$.

We say that $(\mathbb{K}, \mathfrak{M})$ is an *adelic field* if the following conditions hold:
 (1) for each $v \in \mathfrak{M}$, the completion $\mathbb{K}_v$ is a valued field;
 (2) for each $\alpha \in \mathbb{K}^\times$, $|\alpha|_v = 1$ except for a finite number of v.

The adelic field $(\mathbb{K}, \mathfrak{M})$ is said to satisfy the *product formula* if, for all $\alpha \in \mathbb{K}^\times$,

$$\sum_{v \in \mathfrak{M}} n_v \log |\alpha|_v = 0.$$

Let $\mathbb{F}$ be a finite extension of $\mathbb{K}$. For each $v \in \mathfrak{M}$, let $\mathfrak{N}_v$ be the set of pairs $w = (|\cdot|_w, n_w)$ where $|\cdot|_w$ is an absolute value on $\mathbb{F}$ that extends $|\cdot|_v$ and

$$(1.1) \qquad\qquad n_w = \frac{[\mathbb{F}_w : \mathbb{K}_v]}{[\mathbb{F} : \mathbb{K}]} n_v.$$

If $\mathfrak{N} = \sqcup_v \mathfrak{N}_v$, then $(\mathbb{F}, \mathfrak{N})$ is an adelic field. For $w \in \mathfrak{N}$, we note $w \mid v$ if $|\cdot|_w$ extends $|\cdot|_v$. By [Lan83, Proposition 4.3], if $(\mathbb{K}, \mathfrak{M})$ satisfies the product formula and $\mathbb{F}$ is separable over $\mathbb{K}$, then $(\mathbb{F}, \mathfrak{N})$ satisfies the product formula too.

Example 1.2. Let $\mathfrak{M}_\mathbb{Q}$ be the set formed by the Archimedean and the p-adic absolute values of $\mathbb{Q}$, normalized in the standard way, with all weights equal to 1. Then $(\mathbb{Q}, \mathfrak{M}_\mathbb{Q})$ is an adelic field that satisfies the product formula. We identify $\mathfrak{M}_\mathbb{Q}$ with the set $\{\infty\} \cup \{\text{primes of } \mathbb{Z}\}$. For a number field $\mathbb{K}$, the construction above gives an adelic field $(\mathbb{K}, \mathfrak{M}_\mathbb{K})$ which satisfies the product formula too.

Example 1.3. Consider the function field $\mathrm{K}(C)$ of a smooth projective curve C over a field k. For each closed point $v \in C$ and $\alpha \in \mathrm{K}(C)^\times$, we denote by $\mathrm{ord}_v(\alpha)$ the order of α in the discrete valuation ring $\mathcal{O}_{C,v}$. We associate to each v the absolute value and weight given by

$$|\alpha|_v = c_k^{-\mathrm{ord}_v(\alpha)}, \quad n_v = [k(v) : k]$$

with

$$c_k = \begin{cases} \#k & \text{if } \#k < \infty, \\ e & \text{if } \#k = \infty. \end{cases}$$

Let $\mathfrak{M}_{\mathrm{K}(C)}$ denote this set of absolute values and weights. The pair $(\mathrm{K}(C), \mathfrak{M}_{\mathrm{K}(C)})$ is an adelic field which satisfies the product formula, since the degree of a principal divisor is zero. In this case, $\lambda_v = \log(c_k)$ for all v.

More generally, let $\mathbb{K}$ be a finite extension of $\mathrm{K}(C)$. Applying the construction in (1.1), we obtain an adelic field $(\mathbb{K}, \mathfrak{M}_{\mathbb{K}/\mathrm{K}(C)})$. In this geometric setting, this construction can be formulated as follows. Let $\pi \colon B \to C$ be a dominant morphism of smooth projective curves over k such that the finite extension $\mathrm{K}(C) \hookrightarrow \mathbb{K}$ identifies with $\pi^* \colon \mathrm{K}(C) \hookrightarrow \mathrm{K}(B)$. For a closed point $v \in C$, the absolute values of $\mathbb{K}$ that extend $|\cdot|_v$ are in bijection with the closed points of the fibre of v. For each closed point $w \in \pi^{-1}(v)$, the corresponding absolute value and weight are given, for $\beta \in \mathrm{K}(B)^\times$, by

$$(1.4) \qquad\qquad |\beta|_w = c_k^{-\frac{\mathrm{ord}_w(\beta)}{e_w}}, \quad n_w = \frac{e_w[k(w) : k]}{[\mathrm{K}(B) : \mathrm{K}(C)]},$$

where e_w is the ramification index of w over v. We have

$$(1.5) \qquad\qquad \lambda_w = \log(c_k)/e_w.$$

Observe that this structure of adelic field on $\mathbb{K}$ depends on the extension and not just on the field $\mathrm{K}(B)$. For instance, $(\mathrm{K}(C), \mathfrak{M}_{\mathrm{K}(C)})$ corresponds to the identity map $C \to C$, but another finite morphism $\pi \colon C \to C$ may give a different structure

of adelic field on $\mathrm{K}(C)$. The projection formula for the map π implies that, for each $v \in \mathfrak{M}_{\mathrm{K}(C)}$, the equation

$$[\mathbb{K} : \mathrm{K}(C)] = \sum_{\substack{w \in \mathfrak{M}_{\mathbb{K}/\mathrm{K}(C)} \\ v|w}} [\mathbb{K}_w : \mathrm{K}(C)_v]$$

is satisfied. From which we obtain that $(\mathbb{K}, \mathfrak{M}_{\mathbb{K}/\mathrm{K}(C)})$ also satisfies the product formula.

Definition 1.6. A *global field* is a finite extension $\mathbb{K}/\mathbb{Q}$ or $\mathbb{K}/\mathrm{K}(C)$ for a smooth projective curve C over a field k, with the structure of adelic field given in examples 1.2 or 1.3, respectively. To lighten the notation, we will usually denote those global fields as $\mathbb{K}$ although, in the function field case, the structure of adelic field depends on the particular extension. In both cases, we will denote by $\mathfrak{M}_{\mathbb{K}}$ the set of places and by $d_{\mathbb{K}}$ the degree of the extension.

Note that our use of the terminology "global field" is slightly more general than the usual one since, in the function field case, we allow an arbitrary base field. The price to pay for this greater generality is that, in the function field case, we can not use the nice topology of the adeles. Instead, we will have to use geometric arguments. Following Weil [Wei74], we will use the terminology "A-field" for the global fields which are either a number field or a finitely generated extension of degree of transcendence 1 of a finite field.

We recall the notion of $\mathfrak{M}_{\mathbb{K}}$-divisor, which can be also found in [Lan94, Chapter V] for the case of number fields.

Definition 1.7. Let $\mathbb{K}$ be a global field. An $\mathfrak{M}_{\mathbb{K}}$-*divisor* is a collection $\mathfrak{c} = \{\mathfrak{c}_v\}_{v \in \mathfrak{M}_{\mathbb{K}}}$ of positive real numbers such that $\mathfrak{c}_v = 1$ for all but finite number of v and such that $\mathfrak{c}_v$ belongs to the image of $|\cdot|_v$ for all non-Archimedean v. We set

$$\widehat{\mathrm{L}}(\mathfrak{c}) = \{\gamma \in \mathbb{K} \mid |\gamma|_v \leq \mathfrak{c}_v \text{ for all } v\}$$

and

$$\widehat{l}(\mathfrak{c}) = \begin{cases} \log(\#\widehat{\mathrm{L}}(\mathfrak{c})) & \text{if } \mathbb{K} \text{ is a number field,} \\ \log(c_k) \dim_k(\widehat{\mathrm{L}}(\mathfrak{c})) & \text{if } \mathbb{K} \text{ is a function field.} \end{cases}$$

We also set $\widehat{\deg}(\mathfrak{c}) = \sum_v d_{\mathbb{K}} n_v \log(\mathfrak{c}_v)$.

Example 1.8. Let $\mathbb{K} = \mathrm{K}(B)/\mathrm{K}(C)$ be an extension of function fields viewed as a global field as in Example 1.3. Let $\mathfrak{c} = (\mathfrak{c}_v)_v$ be an $\mathfrak{M}_{\mathbb{K}}$-divisor. For each closed point $v \in B$, the condition that $\mathfrak{c}_v$ belongs to the image of the absolute value $|\cdot|_v$ is equivalent to $\log(\mathfrak{c}_v)/\lambda_v \in \mathbb{Z}$. Consider the Weil divisor on B given by

$$D(\mathfrak{c}) = \sum_v d_v[v]$$

with $d_v = \log(\mathfrak{c}_v)/\lambda_v$. Let $\mathrm{L}(D(\mathfrak{c}))$ be the associated linear series and $l(D(\mathfrak{c}))$ its dimension. Then

$$(1.9) \qquad \widehat{\mathrm{L}}(\mathfrak{c}) = \mathrm{L}(D(\mathfrak{c})), \quad \widehat{l}(\mathfrak{c}) = \log(c_k)\, l(D(\mathfrak{c})), \quad \widehat{\deg}(\mathfrak{c}) = \log(c_k)\deg(D(\mathfrak{c})).$$

These equalities follow easily from the definitions. For instance, we prove the last equation with the help of (1.4) and (1.5):

$$\widehat{\deg}(\mathfrak{c}) = \sum_v d_{\mathbb{K}} n_v \log(\mathfrak{c}_v) = \sum_v d_{\mathbb{K}} \frac{e_v[k(v):k]}{d_{\mathbb{K}}} \lambda_v d_v$$

$$= \log(c_k) \sum_v d_v[k(v):k] = \log(c_k)\deg(D(\mathfrak{c})).$$

Thus, an $\mathfrak{M}_{\mathbb{K}}$-divisor can be identified with a Weil divisor on the curve B. This identification respects their linear series and the associated invariants, up to the multiplicative constant $\log(c_k)$.

Lemma 1.10. *Let $\mathbb{K}$ be a global field. Then, there exists $\kappa > 0$ depending only on $\mathbb{K}$ such that, for any $\mathfrak{M}_{\mathbb{K}}$-divisor $\mathfrak{c}$,*

$$\left| \widehat{l}(\mathfrak{c}) - \max(0, \widehat{\deg}(\mathfrak{c})) \right| \leq \kappa.$$

Proof. If $\mathbb{K}$ is a number field, then, in the notation of [Lan94, page 101],

$$\widehat{l}(\mathfrak{c}) = \log(\lambda(\mathfrak{c})), \quad \widehat{\deg}(\mathfrak{c}) = \log \|\mathfrak{c}\|_{\mathbb{K}}$$

and [Lan94, Chapter V, Theorem 0] gives the result.

Hence, we only have to consider the case when $\mathbb{K}$ is the function field of a smooth projective curve B. Let $D = D(\mathfrak{c})$ be the Weil divisor associated to the $\mathfrak{M}_{\mathbb{K}}$-divisor $\mathfrak{c}$ as in Example 1.8. If $\deg(D) < 0$, then $l(D) = 0$ and so $\widehat{l}(\mathfrak{c}) = \max(0, \widehat{\deg}(\mathfrak{c}))$ by (1.9), and the lemma is proved in this case. When $\deg(D) \geq 0$, we have that $l(D) \leq \deg(D) + 1$ and, by the Riemann-Roch theorem [Liu02, Theorem 3.17],

$$l(D) \geq \deg(D) - (g(B) - 1)$$

where $g(B)$ is the genus of B. Hence, $|\widehat{l}(\mathfrak{c}) - \max(0, \widehat{\deg}(\mathfrak{c}))| \leq (g(B) - 1) \log(c_k)$, thus proving the lemma. $\qquad\square$

Lemma 1.11. *Let $\mathbb{K}$ be a global field and $S \subset \mathfrak{M}_{\mathbb{K}}$ a finite subset. Let $\{\gamma_v\}_{v \in \mathfrak{M}_{\mathbb{K}}}$ be a collection of positive real numbers such that $\gamma_v = 1$ for all except a finite number of v, and such that*

$$\prod_{v \in \mathfrak{M}_{\mathbb{K}}} \gamma_v^{n_v} < 1.$$

Let $0 < \eta \leq 1$ be a real number. Then, there is an integer $\ell_0 \geq 1$ such that, for all $\ell \geq \ell_0$, there exists $\alpha \in \mathbb{K}^{\times}$ with $|\alpha|_v \gamma_v^{\ell} \leq 1$ for all $v \in \mathfrak{M}_{\mathbb{K}}$ and $|\alpha|_v \gamma_v^{\ell} < \eta$ for all $v \in S$.

Proof. Consider the finite set of places

$$S' = S \cup \{v \in \mathfrak{M}_{\mathbb{K}} \mid \gamma_v \neq 1\} \cup \{v \in \mathfrak{M}_{\mathbb{K}} \mid v \text{ is Archimedean}\}.$$

For each $v \in S'$ we pick $\widetilde{\gamma}_v > \gamma_v$ in such a way that $\prod_v \widetilde{\gamma}_v^{n_v} < 1$ while, for $v \in \mathfrak{M}_{\mathbb{K}} \setminus S'$, we set $\widetilde{\gamma}_v = \gamma_v = 1$. Then for each $v \in S'$ we can find an integer ℓ_v such that

(1.12) $$\ell_v(\log(\widetilde{\gamma}_v) - \log(\gamma_v)) > \lambda_v - \log(\eta).$$

Choose $\ell_0 \geq \max_{v \in S'} \ell_v$ satisfying also that

$$-\ell_0 \sum_v d_{\mathbb{K}} \, n_v \log(\widetilde{\gamma}_v) > \kappa,$$

where κ is the constant in Lemma 1.10. Let $\ell \geq \ell_0$. By (1.12), for each $v \in S'$ the interval $[-\ell \log(\widetilde{\gamma}_v), \log(\eta) - \ell \log(\gamma_v)]$ has length bigger than λ_v. Therefore, we can choose $x_v \in \mathbb{K}_v^{\times}$ with

$$\frac{1}{\widetilde{\gamma}_v^{\ell}} \leq |x_v|_v < \frac{\eta}{\gamma_v^{\ell}}.$$

Set $\mathfrak{c}_v = |x_v|_v$ for $v \in S'$ and $\mathfrak{c}_v = 1$ for $v \notin S'$. Then $\mathfrak{c} = (\mathfrak{c}_v)_v$ is an $\mathfrak{M}_{\mathbb{K}}$-divisor with

$$\widehat{\deg}(\mathfrak{c}) = \sum_v d_{\mathbb{K}} \, n_v \log |x_v|_v \geq -\ell \sum_v d_{\mathbb{K}} \, n_v \log(\widetilde{\gamma}_v) > \kappa.$$

Lemma 1.10 then implies that $L(\mathfrak{c}) \neq \{0\}$. Hence, we can find $\alpha \in \mathbb{K}^{\times}$ such that $|\alpha|_v \leq \mathfrak{c}_v \leq \gamma_v^{-\ell}$ for all $v \in \mathfrak{M}_{\mathbb{K}}$ and $|\alpha|_v \leq \mathfrak{c}_v < \eta \gamma_v^{-\ell}$ for $v \in S'$, proving the result. $\qquad\square$

2. Adelic vector spaces

Let $(K, |\cdot|)$ be a valued field and V a vector space over K. By a *norm* on V we will mean a norm in the usual sense in the Archimedean case and a norm satisfying the ultrametric inequality in the non-Archimedean case. Let $(V, \|\cdot\|)$ be a normed vector space over K. If $E \subset K$ and $F \subset V$ we write

$$|E| = \{|\alpha| \mid \alpha \in E\} \subset \mathbb{R}_{\geq 0}, \quad \|F\| = \{\|x\| \mid x \in F\} \subset \mathbb{R}_{\geq 0}.$$

Let $V^\circ = \{x \in V \mid \|x\| \leq 1\}$ be the *unit ball* of V. When K is non-Archimedean, V° is a K°-module.

Example 2.1. Let K be a valued field and $r \geq 0$. We can give a structure of normed vector space to K^r by considering, if $|\cdot|$ is Archimedean, the Euclidean norm and, if $|\cdot|$ is non-Archimedean, the ℓ^∞-norm. In precise terms, for $x = (x_1, \ldots, x_r) \in K^r$,

$$\|x\| = \begin{cases} \left(\sum_{i=1}^r |x_i|^2 \right)^{1/2} & \text{if } |\cdot| \text{ is Archimedean,} \\ \max_i |x_i|_v & \text{otherwise.} \end{cases}$$

This choice of norm gives the *standard* structure of normed vector space on K^r.

We recall the notion of orthogonality of a basis in a normed vector space.

Definition 2.2. Let K be a valued field and V a normed vector space over K. A set of vectors $\{b_1, \ldots, b_r\}$ of V is *orthogonal* if, for all $\gamma_1, \ldots, \gamma_r \in K$,

$$\left\| \sum_{i=1}^r \gamma_i b_i \right\| \geq \max_i \|\gamma_i b_i\|.$$

An orthogonal set of vectors $\{b_1, \ldots, b_r\}$ is *orthonormal* if $\|b_i\| = 1$ for all i.

For an r-dimensional normed vector space, the presence of an orthogonal basis allows to compare its unit ball with an ellipsoid in the standard normed vector space K^r.

Lemma 2.3. *Let $(K, |\cdot|)$ be a valued field and $(V, \|\cdot\|)$ a normed vector space over K of finite dimension r. Suppose that there is an orthogonal basis $b = \{b_1, \ldots, b_r\}$ of V and let $\phi_b \colon V \to K^r$ be the induced isomorphism.*

(1) *Suppose that $|\cdot|$ is Archimedean and consider the ellipsoid*

$$E = \left\{ (\gamma_1, \ldots, \gamma_r) \in K^r \;\Big|\; \sum_{i=1}^r |\gamma_i|^2 \|b_i\|^2 \leq 1 \right\}.$$

   Then $r^{-1}E \subset \phi_b(V^\circ) \subset \sqrt{r}E$.
(2) *Suppose that $|\cdot|$ is associated to a nontrivial discrete valuation with uniformizer ϖ. Let $\alpha_i \in K$ such that $\|b_i\| \leq |\alpha_i| < |\varpi|^{-1}\|b_i\|$. Then the vectors $\{\alpha_1^{-1}b_1, \ldots, \alpha_r^{-1}b_r\}$ form a basis of V°. In particular, if we consider the set*

$$E = \left\{ (\gamma_1, \ldots, \gamma_r) \in K^r \;\Big|\; \max_i |\gamma_i| \|b_i\| \leq 1 \right\},$$

   then $\phi_b(V^\circ) = E$.

Proof. We consider first the Archimedean case. On the one hand, let $x \in V^\circ$ and write $x = \sum_i \gamma_i b_i$ with $\gamma_i \in K$. Then $\max_i |\gamma_i| \|b_i\| \leq \|x\| \leq 1$, since the basis b is orthogonal. Hence $\sum_i |\gamma_i|^2 \|b_i\|^2 \leq r$, which implies that $\phi_b(V^\circ) \subset \sqrt{r}E$. On the other hand, let $(\gamma_1, \ldots, \gamma_r) \in r^{-1}E$ and write $x = \sum_i \gamma_i b_i$ for the corresponding point of V. We have $\|x\| \leq r \max_i |\gamma_i| \|b_i\| \leq r(\sum_i |\gamma_i|^2 \|b_i\|^2)^{1/2} \leq 1$, which implies that $r^{-1}E \subset V^\circ$.

Now consider the non-Archimedean case. Let $x = \sum_i \gamma_i b_i \in V$. If $x \in V^\circ$, then

$$\max_i |\gamma_i| \, |\alpha_i| < |\varpi|^{-1} \max_i |\gamma_i| \|b_i\| \leq |\varpi|^{-1} \left\| \sum_i \gamma_i b_i \right\| \leq |\varpi|^{-1}.$$

Since the first inequality is strict, this implies that $\max_i |\gamma_i| \, |\alpha_i| \leq 1$, hence $\gamma_i \alpha_i \in K^\circ$ and so $V^\circ \subset \sum_i K^\circ \alpha_i^{-1} b_i$. Conversely, let $x \in \sum_i K^\circ \alpha_i^{-1} b_i$. Then $\|x\| \leq \max_i |\alpha_i^{-1}| \|b_i\| \leq 1$, which proves the reverse inclusion. $\qquad\square$

The notion of orthogonality on general normed vector spaces is delicate in the Archimedean case. By contrast this notion behaves nicely in the non-Archimedean case. For instance, if $(V, \|\cdot\|)$ is a normed vector space of dimension r over a non-Archimedean valued field and $\{b_1, \ldots, b_r\}$ is a orthogonal basis of V, then

$$(2.4) \qquad \left\| \sum_{i=1}^r \gamma_i b_i \right\| = \max_i \|\gamma_i b_i\|.$$

Moreover, orthogonal bases always exist in the non-Archimedean case.

Proposition 2.5. *Let $(V, \|\cdot\|)$ be a normed vector space of dimension r over a non-Archimedean valued field $(K, |\cdot|)$. Then there exists an orthogonal basis of V.*

Proof. When K is locally compact, the proof can be found in [Wei74, Proposition II.3]. For completeness we include a proof for an arbitrary discrete valuation.

Let $|K^\times|$ be the set of nonzero values of K. This is a discrete subgroup of $\mathbb{R}_{>0}$. It can be verified that, if $x_1, \ldots, x_k$ are nonzero vectors of V such that the norms $\|x_i\|$ belong to different cosets with respect to $|K^\times|$, then these vectors are orthogonal. Since orthogonal vectors are linearly independent, we deduce that the set of norms $\|V \setminus \{0\}\|$ is a finite union of at most r cosets of $|K^\times|$. Hence, this is a discrete subset of $\mathbb{R}_{>0}$.

Let now $b_1, \ldots, b_r$ be a basis of V. We construct a orthogonal basis inductively. Put $e_1 = b_1$. For $2 \leq k \leq r-1$, assume that we have already chosen a set $e_1, \ldots, e_k$ of orthogonal vectors that span the same subspace as $b_1, \ldots, b_k$. Choose a vector $e_{k+1} = b_{k+1} + \sum_{j=1}^k \alpha_j e_j$ with the property that

$$(2.6) \qquad \|e_{k+1}\| = \inf \left\{ \left\| b_{k+1} + \sum_{j=1}^k \alpha_j e_j \right\| \, \Big| \, \alpha_1, \ldots, \alpha_k \in K \right\}.$$

This vector exists because of the discreteness of $\|V \setminus \{0\}\|$. Condition (2.6) implies that the set $e_1, \ldots, e_{k+1}$ is orthogonal. $\qquad\square$

Corollary 2.7. *The unit ball V° is a free K°-module of rank r.*

Proof. By Proposition 2.5, V admits an orthogonal basis. Thus, the statement follows from Lemma 2.3(2). $\qquad\square$

In the Archimedean case, a norm is determined by its unit ball. This is not true in the discrete valuation case. For a normed space $(V, \|\cdot\|)$ over a valued field $(K, |\cdot|)$, the *norm associated to the unit ball* is defined, for $x \in V$, as

$$(2.8) \qquad \|x\|_{V^\circ} = \inf\{|\alpha| \mid \alpha \in K, x \in \alpha V^\circ\}$$

In general, $\|x\| \leq \|x\|_{V^\circ}$. Following [Gau09], we say that the normed space $(V, \|\cdot\|)$ is *pure* if $\|x\| = \|x\|_{V^\circ}$ for all $x \in V$. The *purification* of $(V, \|\cdot\|)$ is the normed vector space $(V, \|\cdot\|_{V^\circ})$.

All normed spaces over an Archimedean field are pure. In the non-Archimedean case, we have the following criterion.

Proposition 2.9. *Let $(V, \|\cdot\|)$ be a normed space over a non-Archimedean valued field $(K, |\cdot|)$. Then the following conditions are equivalent:*

(1) $(V, \|\cdot\|)$ *is pure;*
(2) $\|V\| = |K|$;
(3) *there exists an orthonormal basis of V;*
(4) *every K°-basis of V° is orthonormal.*

Proof. Since the valuation is discrete, we have that

$$\|x\|_{V^\circ} = \min\{t \in |K| \mid t \geq \|x\|\}.$$

The equivalence of (1) and (2) follows easily from this. The fact that (3) implies (2) is clear, whereas the reverse implication follows from Proposition 2.5. By Corollary 2.7, V° admits a K° basis, and so (4) implies (3). Thus, it only remains to show that (1) implies (4).

Consider K^r with its standard structure of normed vector space as in Example 2.1. With this structure, the standard basis is orthonormal. Let $b = \{b_1, \ldots, b_r\}$ be a K°-basis of V° and $\phi_b \colon V \to K^r$ the isomorphism given by this basis. The image of V° by this isomorphism is the unit ball of K^r. Therefore, if V is pure, ϕ_b is an isometry and b is an orthonormal basis. $\qquad\square$

Partly following [Gau09], we introduce a notion of adelic vector space. As Gaudron points out, this notion extends that of Hermitian vector bundle, which is at the base of Arakelov geometry.

Definition 2.10. Let $(\mathbb{K}, \mathfrak{M})$ be an adelic field. An *adelic vector space* over $(\mathbb{K}, \mathfrak{M})$ is a pair $\overline{V} = (V, \{\|\cdot\|_v\}_{v \in \mathfrak{M}})$ where V is a vector space over $\mathbb{K}$ and, for each $v \in \mathfrak{M}$, $\|\cdot\|_v$ is a norm on the completion $V_v := V \otimes \mathbb{K}_v$, satisfying, for each $x \in V \setminus \{0\}$, that $\|x\|_v = 1$ for all but a finite number of v.

Let $\overline{V}$ be an adelic vector space over $(\mathbb{K}, \mathfrak{M})$. An element $x \in V$ is *small* if $\|x\|_v \leq 1$ for all $v \in \mathfrak{M}$. A small element $x \in V$ is *strictly small* if $\prod_{v \in \mathfrak{M}} \|x\|_v^{n_v} < 1$. If $S \subset \mathfrak{M}$ is a finite set, then a small element $x \in V$ is *strictly small on S* if $\|x\|_v < 1$ for all $v \in S$.

The adelic vector space $\overline{V}$ is called *pure* if $(V_v, \|\cdot\|_v)$ is pure for all $v \in \mathfrak{M}$. The *purification* of $\overline{V}$ is the adelic vector space $\overline{V}_{\mathrm{pur}} = (V, \{\|\cdot\|_{v, V_v^\circ}\}_v)$. If $\overline{V}$ is finite dimensional, it is called *generically trivial* if there is a K-basis of V that is an orthonormal basis of V_v for all but a finite number of v. Clearly, if $\overline{V}$ is generically trivial, the same is true for its purification.

Note that Gaudron's definition of adelic vector space includes the condition of being generically trivial.

Example 2.11. Let $\mathbb{K}$ be a global field and $r \geq 0$. The *standard* structure of adelic vector space on $\mathbb{K}^r$ is defined by choosing the standard norm on $\mathbb{K}_v^r$ for each place $v \in \mathfrak{M}_{\mathbb{K}}$, as explained in Example 2.1. The obtained adelic vector space is pure and generically trivial.

Example 2.12. Let $\mathbb{K}$ be a global field. A *normed vector bundle* is:
(1) when $\mathbb{K}$ is a number field, a locally free $\mathcal{O}_{\mathbb{K}}$-module $\mathcal{E}$, together with the choice of a norm $\|\cdot\|_v$ on $\mathcal{E} \otimes \mathbb{K}_v$ for each Archimedean place v;
(2) when $\mathbb{K} = \mathrm{K}(B)$ is the function field of a smooth projective curve, a locally free $\mathcal{O}_B$-module $\mathcal{E}$.

To a normed vector bundle $\mathcal{E}$, we associate the adelic vector space $\overline{E} = (E, \{\|\cdot\|_v\})$ given by the vector space $E = \mathcal{E} \otimes \mathbb{K}$, the given norm for each Archimedean place $v \in \mathfrak{M}_{\mathbb{K}}$, and the norm

$$\|x\|_v = \inf\{|\alpha|_v \mid \alpha \in \mathbb{K}, x \in \alpha \mathcal{E}_v\},$$

for each non-Archimedean place v. Clearly, this adelic vector space is pure and generically trivial.

The previous example covers all cases of pure and generically trivial adelic vector spaces over a global field.

Proposition 2.13. *Let $\overline{V}$ be a finite dimensional adelic vector space over a global field. Assume that $\overline{V}$ is pure and generically trivial. Then, it is the adelic vector space associated to a normed vector bundle.*

Proof. We give the proof of this statement for the case of function fields only, the case of number fields being analogous. Let $\mathbb{K} = \mathrm{K}(B)$ for a smooth projective curve B. For each open subset $U \subset B$, we write

$$\mathcal{E}_{\overline{V}}(U) = \{x \in V \mid \|x\|_v \leq 1, \ \forall v \in U\}.$$

Clearly, $\mathcal{E}_{\overline{V}}$ is a sheaf of $\mathcal{O}_B$-modules. Let $v_0 \in B$ and choose a $\mathbb{K}_{v_0}^\circ$-basis b of $V_{v_0}^\circ$. Since $\overline{V}$ is generically trivial, there is a basis e of V that is an orthonormal basis of $\overline{V}_v$ for all but a finite number of places v. Let U be the subset of B containing the generic point, the point v_0, and all the closed points $v \in B$ such that $\det(e/b)$ is a unit of $\mathbb{K}_v^\circ$. Then, U is a neighbourhood of v_0 such that b is a $\mathbb{K}_v^\circ$-basis of V_v° for all closed points $v \in U$. This shows that $\mathcal{E}_{\overline{V}}$ is locally free.

Since $\overline{V}$ is pure, its norms agree with the norms induced by $\mathcal{E}_{\overline{V}}$, which completes the proof. □

Definition 2.14. Let $\overline{V}$ be an adelic vector space over a global field $\mathbb{K}$. The set of small elements of $\overline{V}$ is denoted by $\widehat{H}^0(\overline{V})$. We further write

$$\widehat{h}^0(\overline{V}) = \begin{cases} \log(\#\widehat{H}^0(\overline{V})) & \text{if } \mathbb{K} \text{ is a number field,} \\ \log(c_k)\dim_k(\widehat{H}^0(\overline{V})) & \text{if } \mathbb{K} \text{ is a function field.} \end{cases}$$

If $\mathbb{K}$ is a function field over a finite field, then $\widehat{h}^0(\overline{V}) = \log(\#\widehat{H}^0(\overline{V}))$ since $c_k = \#k$. Thus, both definitions agree for A-fields.

Example 2.15. Let $\mathbb{K}$ be a global field and $\mathfrak{c} = (\mathfrak{c}_v)_v$ an $\mathfrak{M}_{\mathbb{K}}$-divisor. We define a normed vector space $\overline{V}(\mathfrak{c})$, given by $V(\mathfrak{c}) = \mathbb{K}$ and $\|\alpha\|_v = \mathfrak{c}_v^{-1}|\alpha|_v$. Then $\overline{V}(\mathfrak{c})$ is a pure and generically trivial adelic vector space over $\mathbb{K}$. Moreover, $\widehat{H}^0(\overline{V}(\mathfrak{c})) = \widehat{\mathrm{L}}(\mathfrak{c})$ and $\widehat{h}^0(\overline{V}(\mathfrak{c})) = \widehat{l}(\mathfrak{c})$.

When $\mathbb{K}$ is a function field and the adelic vector space comes from a normed vector bundle, the sets $\widehat{H}^0(\overline{V})$ can be interpreted as the space of global sections of the model defining the metric.

Example 2.16. Let $\mathbb{K} = \mathrm{K}(B)$ be the function field of a smooth projective curve with the structure of global field given by Example 1.3. Let $\mathcal{E}$ be a locally free $\mathcal{O}_B$-module and $\overline{E}$ the associated adelic vector space as in Example 2.12. Then there is a canonical isomorphism

$$H^0(B, \mathcal{E}) \simeq \widehat{H}^0(\overline{E}),$$

given by restriction to the generic fibre. In particular,

$$(2.17) \qquad\qquad \widehat{h}^0(\overline{E}) = \log(c_k)h^0(B, \mathcal{E}).$$

We next recall the definition of the Euler characteristic of an adelic vector space. For general function fields, the definition differs from that for A-fields since, in that case, we do not dispose of a Haar measure on the corresponding space of adeles.

Definition 2.18. Let $\mathbb{K}$ be a global field and $\overline{V}$ a generically trivial adelic vector space over $\mathbb{K}$ of finite dimension r.

Assume first that $\mathbb{K}$ is a number field. Let $\mathbb{A}$ be its ring of adeles, set $V_{\mathbb{A}} = V \otimes_{\mathbb{K}} \mathbb{A}$, and consider the adelic unit ball

$$V_{\mathbb{A}}^{\circ} = \prod_v V_v^{\circ} \subset V_{\mathbb{A}}.$$

Let μ be a Haar measure on $\mathbb{K}_{\mathbb{A}}^r$. The *Euler characteristic* of $\overline{V}$ is defined as

$$\widehat{\chi}(\overline{V}) = \log \Big(\frac{\mu(V_{\mathbb{A}}^{\circ})}{\mu(V_{\mathbb{A}}/V)} \Big).$$

This number does not depend on the choice of μ. In other words, the Euler characteristic is the ratio between the volume of the unit ball and the covolume of the lattice $V \subset V_{\mathbb{A}}$.

Next assume that $\mathbb{K}$ is a function field. Let b be a basis of V over $\mathbb{K}$ and, for each $v \in \mathfrak{M}_{\mathbb{K}}$, choose a basis b_v of V_v° over $\mathbb{K}_v^{\circ}$. The *Euler characteristic* of $\overline{V}$ is defined as

$$(2.19) \qquad \widehat{\chi}(\overline{V}) = \sum_v d_{\mathbb{K}} \, n_v \log |\det(b_v/b)|_v \,,$$

where b_v/b denotes the matrix of b_v with respect to the basis b. This quantity does not depend on the choices of bases because of the product formula.

In both cases, the formula makes sense because the adelic vector space is generically trivial.

Remark 2.20. Let $\mathbb{K}$ be a number field and $\overline{V}$ an adelic vector space over $\mathbb{K}$ of finite dimension r. With notations as in Definition 2.18, for each Archimedean place v , we consider the Lebesgue measure μ_v on $\mathbb{K}_v^r$. Therefore, for v real, $\mu_v((\mathbb{K}_v^r)^{\circ}) = 2^r$ whereas, for v complex, $\mu_v((\mathbb{K}_v^r)^{\circ}) = \pi^r$. We choose a basis b of V over $\mathbb{K}$ and, for each non-Archimedean v, we also choose a basis b_v of V_v° over $\mathbb{K}_v^{\circ}$. Then

$$(2.21) \qquad \widehat{\chi}(\overline{V}) = \sum_{v|\infty} \log(\mu_v(\phi_{b,v}(V_v^{\circ}))) + \sum_{v \nmid \infty} d_{\mathbb{K}} \, n_v \log |\det(b_v/b)|_v,$$

where $\phi_{b,v} \colon V_v \to \mathbb{K}_v^r$ is the isomorphism induced by the basis b. This is the analogue for number fields of formula (2.19).

When $\overline{V}$ is an adelic vector space over a function field coming from a normed vector bundle, its Euler characteristic coincides with the Euler characteristic of the associated model, up to an additive constant.

Example 2.22. Let $\pi \colon B \to C$ be a dominant morphism of smooth projective curves over a field k and consider the function field $\mathbb{K} = \mathrm{K}(B)$ with the structure of global field as in Example 1.3. Let $\mathcal{E}$ be a locally free $\mathcal{O}_B$-module of rank r and $\overline{E}$ the associated generically trivial adelic vector space as in Example 2.12. Choose a $\mathbb{K}$-basis b of E and a $\mathbb{K}_v^{\circ}$-basis b_v of E_v° for each $v \in \mathfrak{M}_{\mathbb{K}}$. Then

$$\widehat{\chi}(\overline{E}) = \sum_v d_{\mathbb{K}} \, n_v \log |\det(b_v/b)|_v = \sum_v e_v [k(v) : k] \log \Big(c_k^{\frac{-\mathrm{ord}_v(\det(b_v/b))}{e_v}} \Big)$$

$$= \log(c_k) \sum_v [k(v) : k] \mathrm{ord}_v(\det(b/b_v)) = \log(c_k) \deg(\mathcal{E}),$$

where e_v denotes the ramification index of v over $\pi(v)$. The Riemann-Roch theorem for vector bundles on curves then implies

$$\widehat{\chi}(\overline{E}) = \log(c_k)(\chi(\mathcal{E}) + r(g(B) - 1)),$$

where $\chi(\mathcal{E})$ is the Euler characteristic of $\mathcal{E}$ and $g(B)$ is the genus of B.

The space of small elements and the Euler characteristic of an adelic vector space depend only on its purification, as it follows immediately from the definitions.

Proposition 2.23. *Let $\overline{V}$ be a generically trivial adelic vector space over a global field. Then*

$$\widehat{H}^0(\overline{V}) = \widehat{H}^0(\overline{V}_{\mathrm{pur}}), \quad \widehat{\chi}(\overline{V}) = \widehat{\chi}(\overline{V}_{\mathrm{pur}}).$$

The presence of an orthogonal basis allows us to estimate the Euler characteristic of an adelic vector space.

Proposition 2.24. *Let $\overline{V}$ be a generically trivial adelic vector space over a global field $\mathbb{K}$ of finite dimension r. Let $b = \{b_1, \ldots, b_r\}$ be a basis of V over $\mathbb{K}$ which is an orthogonal basis of V_v for all $v \in \mathfrak{M}_{\mathbb{K}}$. Let $S \subset \mathfrak{M}_{\mathbb{K}}$ be the finite set of non-Archimedean places such that $\|b_i\|_v \neq 1$ for some i. Then*

$$\left| \widehat{\chi}(\overline{V}) - \sum_{v \in \mathfrak{M}_{\mathbb{K}}} \sum_{i=1}^{r} d_{\mathbb{K}}\, n_v \log(\|b_i\|_v^{-1}) \right| \leq d_{\mathbb{K}}\, r\left(\log(\pi r) + \sum_{v \in S} n_v \lambda_v \right).$$

Proof. We consider the case when $\mathbb{K}$ is a number field. We will freely use the notation in Remark 2.20. Let v be an Archimedean place and consider the ellipsoid $E_v = \{(\gamma_1, \ldots, \gamma_r) \in \mathbb{K}_v^r \mid \sum_i |\gamma_i|_v^2 \|b_i\|_v^2 \leq 1\}$. When v is real, the volume of this ellipsoid is $\mu_v(E_v) = 2^r \prod_i \|b_i\|_v^{-d_{\mathbb{K}} n_v}$ whereas, when v is complex, it is $\mu_v(E_v) = \pi^r \prod_i \|b_i\|_v^{-d_{\mathbb{K}} n_v}$. Lemma 2.3(1) then implies

$$(2.25) \quad -r\log(r) \leq \log(\mu_v(\phi_{b,v}(V_v^\circ))) - \sum_i d_{\mathbb{K}} n_v \log(\|b_i\|_v^{-1}) \leq \frac{r\log(r)}{2} + r\log\pi.$$

Now let v be a non-Archimedean place. Let $\alpha_{v,i} \in \mathbb{K}_v$ such that $\|b_i\|_v \leq |\alpha_{v,i}|_v < |\varpi_v|_v^{-1}\|b_i\|_v$ for $v \in S$ and $\alpha_{v,i} = 1$ for $v \notin S$. By Lemma 2.3(2), the vectors $b_{v,i} := \alpha_{v,i}^{-1} b_i$, $i = 1, \ldots, r$, form a basis of V_v° and $|\det(b_v/b)|_v = \prod_i |\alpha_{v,i}|_v^{-1}$. Hence, for $v \in S$,

$$(2.26) \quad\quad\quad -r\lambda_v \leq \log|\det(b_v/b)|_v - \sum_i \log(\|b_i\|_v^{-1}) \leq 0,$$

whereas $\log|\det(b_v/b)|_v = \sum_i \log(\|b_i\|_v^{-1}) = 0$ for $v \notin S$. Adding up (2.25) and (2.26) and using the formula (2.21), we conclude

$$-d_{\mathbb{K}}\, r\left(\log(r) + \sum_{v \in S} n_v \lambda_v \right) \leq \widehat{\chi}(\overline{V}) - \sum_v \sum_i d_{\mathbb{K}}\, n_v \log(\|b_i\|_v^{-1}) \leq d_{\mathbb{K}}\, r\log(\pi r).$$

The case of a function field follows similarly by applying Lemma 2.3(2). The resulting upper bound is better, since it does not have the term $\log(\pi r)$. $\qquad\square$

3. Metrized $\mathbb{R}$-divisors

In this section, we introduce the basic definitions concerning metrized $\mathbb{R}$-divisors and the problems we are interested in. We start by recalling the geometric analogues for $\mathbb{R}$-divisors. Details on the theory of $\mathbb{R}$-divisors can be found in [Laz04].

Let K be a field and X a proper normal variety over K of dimension n. We denote by $\mathrm{Car}(X)$ and $\mathrm{Div}(X)$ the groups of Cartier divisors and of Weil divisors of X, respectively. The spaces of $\mathbb{R}$-divisors and of $\mathbb{R}$-Weil divisors of X are defined as

$$\mathrm{Car}(X)_{\mathbb{R}} = \mathrm{Car}(X) \otimes_{\mathbb{Z}} \mathbb{R}, \quad \mathrm{Div}(X)_{\mathbb{R}} = \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Thus, an $\mathbb{R}$-Cartier divisor on X is a formal linear combination $\sum_i \alpha_i D_i$ with $\alpha_i \in \mathbb{R}$ and $D_i \in \mathrm{Car}(X)$, and similarly for an $\mathbb{R}$-Weil divisor. In this text, we will be mainly concerned with Cartier divisors and $\mathbb{R}$-Cartier divisors and so we will call them just divisors and $\mathbb{R}$-divisors, for short.

Since X is normal, there is an injective morphism $\mathrm{Car}(X) \hookrightarrow \mathrm{Div}(X)$. Then, the fact that $\mathrm{Div}(X)$ is a free Abelian group implies that the maps $\mathrm{Car}(X) \to \mathrm{Car}(X)_{\mathbb{R}}$

and $\mathrm{Car}(X)_{\mathbb{R}} \to \mathrm{Div}(X)_{\mathbb{R}}$ are injective. Given an $\mathbb{R}$-divisor D on X, its *support* is defined as the support of the associated $\mathbb{R}$-Weil divisor and is denoted by $|D|$.

Let $\mathrm{K}(X)^{\times}$ be the multiplicative group of nonzero rational functions of X and set $\mathrm{K}(X)_{\mathbb{R}}^{\times} := \mathrm{K}(X)^{\times} \otimes_{\mathbb{Z}} \mathbb{R}$. The map $\mathrm{div}: \mathrm{K}(X)^{\times} \to \mathrm{Car}(X)$ extends by linearity to a map $\mathrm{K}(X)_{\mathbb{R}}^{\times} \to \mathrm{Car}(X)_{\mathbb{R}}$, also denoted by div.

The spaces of nonzero *global sections* and *global $\mathbb{R}$-sections* of an $\mathbb{R}$-divisor D on X are respectively defined as

$$\Gamma(X, D)^{\times} = \{(f, D) \mid f \in \mathrm{K}(X)^{\times}, \ \mathrm{div}(f) + D \geq 0\},$$

$$\Gamma(X, D)_{\mathbb{R}}^{\times} = \{(f, D) \mid f \in \mathrm{K}(X)_{\mathbb{R}}^{\times}, \ \mathrm{div}(f) + D \geq 0\}.$$

The spaces of nonzero *rational sections* and *rational $\mathbb{R}$-sections* of D are respectively defined as

$$\mathrm{Rat}(X, D)^{\times} := \mathrm{K}(X)^{\times} \times \{D\}, \quad \mathrm{Rat}(X, D)_{\mathbb{R}}^{\times} := \mathrm{K}(X)_{\mathbb{R}}^{\times} \times \{D\}.$$

For $s = (f, D) \in \mathrm{Rat}(X, D)_{\mathbb{R}}^{\times}$, we write

$$\mathrm{div}(s) = \mathrm{div}(f) + D \in \mathrm{Car}(X)_{\mathbb{R}}.$$

Let $\mathrm{L}(D) = \Gamma(X, D)^{\times} \cup \{(0, D)\}$ be the *Riemann-Roch space* of D. Note that, contrary to the usual convention, we have added the label "D" to the elements of $\mathrm{L}(D)$. This is a K-vector space and we set $l(D) = \dim_K(\mathrm{L}(D))$ for its dimension. The *volume* of D is defined as

$$\mathrm{vol}(X, D) = \limsup_{\ell \to \infty} \frac{l(\ell D)}{\ell^n / n!}.$$

Let Y be a d-dimensional subvariety of X and $D_1, \ldots, D_d$ a family of $\mathbb{R}$-divisors on X. The intersection product $(D_1 \cdots D_d \cdot Y)$ is defined by multilinearity from the intersection product of Y with divisors. The *degree* of Y with respect to an $\mathbb{R}$-divisor D is $\deg_D(Y) = (D^d \cdot Y)$.

An $\mathbb{R}$-divisor is *ample* (respectively *big, effective*) if it is a linear combination of ample (respectively big, effective) divisors with positive coefficients. Given divisors D_1 and D_2, the condition that $D_1 - D_2$ is effective is denoted by $D_1 \geq D_2$. An $\mathbb{R}$-divisor D on X is *nef* if $\deg_D(C) \geq 0$ for every curve $C \subset X_{\overline{K}}$. It is *pseudo-effective* if there exists a birational map $\varphi: X' \to X$ of normal proper varieties over K and a divisor E on X' such that $\ell\varphi^* D + E$ is big for all $\ell \geq 1$.

Remark 3.1. The definition of pseudo-effective given here differs from the one in [Laz04, Definition 2.2.25] because we are not assuming that the variety X is projective. For projective normal varieties, both definitions are equivalent.

We now introduce metrized divisors and metrized $\mathbb{R}$-divisors on varieties over global fields. Throughout the rest of this section, X will be a normal proper variety over a global field $\mathbb{K}$ of dimension n. For each place $v \in \mathfrak{M}_{\mathbb{K}}$, we denote by X_v^{an} the v-adic analytification of X. In the Archimedean case, if $\mathbb{K}_v \simeq \mathbb{C}$, then X_v^{an} is an analytic space over $\mathbb{C}$, whereas if $\mathbb{K}_v \simeq \mathbb{R}$, then X_v^{an} is an analytic space over $\mathbb{R}$, that is, an analytic space over $\mathbb{C}$ together with an antilinear involution (see for instance [BPS14, Remark 1.1.5]). In the non-Archimedean case, X_v^{an} is a Berkovich space (see [BPS14, §1.2]). Similarly, a line bundle L on X defines a collection of analytic line bundles $\{L_v^{\mathrm{an}}\}_{v \in \mathfrak{M}_{\mathbb{K}}}$.

Following *loc. cit.*, all considered metrics will be continuous, by definition. For $v \in \mathfrak{M}_{\mathbb{K}}$, a metric $\|\cdot\|_v$ on L_v^{an} is *semipositive* if it is the uniform limit of a sequence of semipositive smooth (respectively algebraic) metrics in the Archimedean (respectively non-Archimedean) case. The metric $\|\cdot\|_v$ is *DSP* if it is the quotient of two semipositive ones. A metric on L is, by definition, an adelic collection of metrics $\|\cdot\|_v$ on L_v^{an}, $v \in \mathfrak{M}_K$. Such a collection is *quasi-algebraic* if there is an

integral model which defines the metric $\| \cdot \|_v$ except for a finite number of v. We refer to [BPS14, Chapter 1] for the precise definitions and more details.

Definition 3.2. For $v \in \mathfrak{M}_{\mathbb{K}}$, a *$v$-adically metrized divisor* on X is a pair $\overline{D} = (D, \| \cdot \|)$ formed by a divisor and a metric on the analytic line bundle $\mathcal{O}(D)_v^{\mathrm{an}}$. We say that $\overline{D}$ is *smooth* (in the Archimedean case), *algebraic* (in the non-Archimedean case), or *semipositive* (smooth or algebraic) if so is the metric $\| \cdot \|$. The *v-adic Green function* of $\overline{D}$ is the function $g_{\overline{D}} \colon X_v^{\mathrm{an}} \setminus |D| \to \mathbb{R}$ given by

$$g_{\overline{D}}(p) = -\log \|s_D(p)\|,$$

where s_D is the canonical section of $\mathcal{O}(D)$. The space of all v-adically metrized divisors on X is denoted by $\widehat{\mathrm{Car}}(X)_v$.

A *quasi-algebraic metrized divisor* on X is a pair $\overline{D} = (D, \{\| \cdot \|_v\}_{v \in \mathfrak{M}_{\mathbb{K}}})$ formed by a divisor D and a quasi-algebraic metric on the line bundle $\mathcal{O}(D)$. The space of quasi-algebraic metrized divisors on X is denoted by $\widehat{\mathrm{Car}}(X)$. For $\overline{D} \in \widehat{\mathrm{Car}}(X)$ and $v \in \mathfrak{M}_{\mathbb{K}}$ we denote by $g_{\overline{D},v}$ the *v-adic Green function* of $\overline{D}$, defined as the Green function of the v-adically metrized divisor $(D, \| \cdot \|_v)$.

Definition 3.3. Let X be a proper normal variety over $\mathbb{K}$. The *space of quasi-algebraic metrized $\mathbb{R}$-divisors* on X is the quotient

$$\widehat{\mathrm{Car}}(X)_{\mathbb{R}} = \widehat{\mathrm{Car}}(X) \otimes_{\mathbb{Z}} \mathbb{R} \Big/ \sim$$

where $\sim$ is the equivalence relation given by $\sum_i \alpha_i \overline{D}_i \sim \sum_j \beta_j \overline{E}_j$ if and only if $\sum_i \alpha_i D_i = \sum_j \beta_j E_j$ and, for each $v \in \mathfrak{M}_{\mathbb{K}}$, there is a dense open subset U_v of X_v^{an} such that

$$\sum_i \alpha_i g_{\overline{D}_i,v}(p) = \sum_j \beta_j g_{\overline{E}_j,v}(p) \quad \text{for } p \in U_v.$$

The function $g_{\overline{D},v} \colon U_v \to \mathbb{R}$ defined by $g_{\overline{D},v}(p) = \sum_i \alpha_i g_{\overline{D}_i,v}$ is called the *v-adic Green function* of $\overline{D}$. An element of $\overline{D} \in \widehat{\mathrm{Car}}(X)_{\mathbb{R}}$ is called a *quasi-algebraic metrized $\mathbb{R}$-divisor* or, for short, a *metrized $\mathbb{R}$-divisor*.

For $v \in \mathfrak{M}_{\mathbb{K}}$, we can similarly define the space of v-adically metrized $\mathbb{R}$-divisors on X, denoted by $\widehat{\mathrm{Car}}(X)_{v,\mathbb{R}}$. A v-adically metrized $\mathbb{R}$-divisor $\overline{D}$ is called *smooth* or *algebraic* if it can be written as $\overline{D} = \sum \alpha_i \overline{D}_i$ with $\overline{D}_i \in \widehat{\mathrm{Car}}(X)_v$ smooth or algebraic, respectively. In each of these cases, it is called *semipositive* (smooth or algebraic) if it can be written as $\overline{D} = \sum \alpha_i \overline{D}_i$, with $\alpha_i > 0$ and $\overline{D}_i \in \widehat{\mathrm{Car}}(X)_v$ semipositive (smooth or algebraic).

Definition 3.4. A rational function $f \in \mathrm{K}(X)^{\times}$ defines a metrized divisor

$$\widehat{\mathrm{div}}(f) = (\mathrm{div}(f), \{\| \cdot \|_{f,v}\}_{v \in \mathfrak{M}_{\mathbb{K}}}),$$

where $\{\| \cdot \|_{f,v}\}_v$ is the metric on the line bundle $\mathcal{O}(\mathrm{div}(f))_v^{\mathrm{an}}$ which is given by $\|f^{-1}(p)\|_{f,v} = 1$ for the section $f^{-1} \in \mathcal{O}(\mathrm{div}(f))_v^{\mathrm{an}}$ and $p \in X_v^{\mathrm{an}}$. This construction defines a group morphism from $\widehat{\mathrm{div}} \colon \mathrm{K}(X)^{\times} \to \widehat{\mathrm{Car}}(X)$, which extends by linearity to a group morphism

$$\widehat{\mathrm{div}} \colon \mathrm{K}(X)_{\mathbb{R}}^{\times} \longrightarrow \widehat{\mathrm{Car}}(X)_{\mathbb{R}}.$$

Let $\overline{D}$ and $\overline{D}'$ be metrized $\mathbb{R}$-divisors. We say that they are *linearly equivalent* if there exists $f \in \mathrm{K}(X)_{\mathbb{R}}^{\times}$ such that

$$\overline{D} - \overline{D}' = \widehat{\mathrm{div}}(f).$$

For $f \in \mathrm{K}(X)_{\mathbb{R}}^{\times}$ and $v \in \mathfrak{M}_{\mathbb{K}}$, the v-adic Green function of $\widehat{\mathrm{div}}(f)$ is the function given by $g_{\widehat{\mathrm{div}}(f),v}(p) = -\log |f(p)|_v$.

Lemma 3.5. *Let $v \in \mathfrak{M}_{\mathbb{K}}$ and $(D, \|\cdot\|)$ a v-adically metrized $\mathbb{R}$-divisor on X. Let $r \geq 1$ and consider a decomposition $D = \sum_{i=1}^{r} \alpha_i D_i$ with $\alpha_i \in \mathbb{R}$ and $D_i \in \mathrm{Car}(X)$. Then there are metrics $\|\cdot\|_i$ on $\mathcal{O}(D_i)_v^{\mathrm{an}}$, $i = 1, \dots, r$, such that*

$$(D, \|\cdot\|) = \sum_{i=1}^{r} \alpha_i (D_i, \|\cdot\|_i).$$

Proof. Set $\overline{D} = (D, \|\cdot\|)$ and let $\overline{D} = \sum_{j=1}^{l} \beta_j \overline{E}_j$ be a decomposition with $\beta_j \in \mathbb{R}$ and $\overline{E}_j \in \widehat{\mathrm{Car}}(X)_v$. We first consider the case when the decomposition is $D = 0$.

Let $\{\gamma_1, \dots, \gamma_r\}$ be a basis of the $\mathbb{Q}$-vector space generated by the numbers $\beta_1, \dots, \beta_l$. Write $\beta_j = \frac{1}{m} \sum_{i=1}^{s} n_{i,j} \gamma_i$ with $n_{i,j} \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 1}$. Then

$$\overline{D} = \frac{1}{m} \sum_{i=1}^{s} \gamma_i \overline{F}_i$$

with $\overline{F}_i = \sum_j n_{i,j} \overline{E}_j \in \widehat{\mathrm{Car}}(X)_v$. Since $D = 0$ and $\gamma_1, \dots, \gamma_s$ are linearly independent over $\mathbb{Q}$, we deduce that $F_i = 0$ for all i. Since $\mathcal{O}(0)_v^{\mathrm{an}} = \mathcal{O}_{X_v^{\mathrm{an}}}$, to give a metric $\|\cdot\|$ on this line bundle is equivalent to give the continuous function $\|1\| \colon X_v^{\mathrm{an}} \to \mathbb{R}_{>0}$. Therefore, we can gather together the different metrics $\|\cdot\|_{F_i}$ on $\mathcal{O}(F_i)_v^{\mathrm{an}}$ with their coefficients, into a single metric on $\mathcal{O}_{X_v^{\mathrm{an}}}$ given by

$$\|1\| = \prod_{i=1}^{r} \|1\|_{F_i}^{\gamma_i/m}.$$

Then $\overline{D} = (0, \|\cdot\|)$, which finishes the proof in this case.

We now consider the general case. Choose metrics $\|\cdot\|_1'$ on $\mathcal{O}(D_1)_v^{\mathrm{an}}$ and $\|\cdot\|_i$ on $\mathcal{O}(D_i)_v^{\mathrm{an}}$, $i = 2, \dots, r$, and denote by $\overline{D}_1'$ and $\overline{D}_i$ the corresponding v-adically metrized divisors. By applying the previous case to the metrized $\mathbb{R}$-divisor $\sum_j \beta_j \overline{E}_j - \alpha_1 \overline{D}_1' - \sum_{i=2}^{r} \alpha_i \overline{D}_i$, we obtain a metric $\|\cdot\|_0$ on the trivial line bundle. Finally, we define a new metric on $\mathcal{O}(D_1)_v^{\mathrm{an}}$ by $\|\cdot\|_1 = \|1\|_0^{1/\alpha_1} \|\cdot\|_1'$ and we set $\overline{D}_1 = (D_1, \|\cdot\|_1)$. Hence $\overline{D} = \sum_{i=1}^{r} \alpha_i \overline{D}_i$, proving the result. $\square$

Let $\overline{D}$ be a v-adically metrized $\mathbb{R}$-divisor on X and $s = (f, D)$ a rational $\mathbb{R}$-section of D. Consider the function $\|s\|$ given by

$$\|s(p)\| = |f(p)|_v \, \mathrm{e}^{-g_{\overline{D}}(p)},$$

where, if $f = \prod_i f_i^{\alpha_i}$, then $|f(p)|_v$ is defined as $\prod_i |f_i(p)|_v^{\alpha_i}$. This function is well-defined on a dense open subset of X_v^{an}. The following result shows that it can be extended everywhere outside the support of $\mathrm{div}(s)$.

Proposition 3.6. *Let $v \in \mathfrak{M}_{\mathbb{K}}$ and $\overline{D}$ a v-adically metrized $\mathbb{R}$-divisor on X. Let $s = (f, D) \in \mathrm{Rat}(X, D)_{\mathbb{R}}^{\times}$. Then $\|s\|$ can be extended to a continuous function from $X_v^{\mathrm{an}} \setminus |\mathrm{div}(s)|$ to $\mathbb{R}_{>0}$.*

In particular, the v-adic Green function $g_{\overline{D}}$ can be extended to a continuous function from $X_v^{\mathrm{an}} \setminus |D|$ to $\mathbb{R}$.

Proof. Write $\overline{D} = \sum_{i=1}^{r} \alpha_i \overline{D}_i$ and $f = \prod_{j=1}^{k} f_j^{\beta_j}$ with $\alpha_i, \beta_j \in \mathbb{R}$, $\overline{D}_i \in \widehat{\mathrm{Car}}(X)_v$ and $f_j \in \mathrm{K}(X)^{\times}$. Let $\{\gamma_l\}_l$ be a basis of the $\mathbb{Q}$-linear space generated by the numbers $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_k$ and write

$$D = \sum_l \frac{\gamma_l}{m} E_l, \quad f = \prod_l g_l^{\gamma_l/m}$$

with $m \in \mathbb{Z}_{\geq 1}$, $E_l \in \mathrm{Car}(X)$ and $g_l \in \mathrm{K}(X)^{\times}$. The pair $s_l = (g_l, E_l)$ defines a rational section of the Cartier divisor E_l. We have $|\mathrm{div}(s)| = \bigcup_l |\mathrm{div}(s_l)|$ because the γ_l are $\mathbb{Q}$-linearly independent.

By Lemma 3.5, there are v-adic metrics $\|\cdot\|_l$ on each E_l such that

$$\overline{D} = \sum_l \frac{\gamma_l}{m} \overline{E}_l$$

with $\overline{E}_l = (E_l, \|\cdot\|_l)$. The function $\|s_l\|_l$ is continuous from $X_v^{\mathrm{an}} \setminus |\operatorname{div}(s_l)|$ to $\mathbb{R}_{>0}$ and there is an open dense subset $U_v \subset X_v^{\mathrm{an}}$ such that, for $p \in U_v$,

$$\|s(p)\| = \prod_l \|s_l(p)\|_l^{\gamma_l/m}.$$

This latter function is well-defined outside $\bigcup_l |\operatorname{div}(s_l)| = |\operatorname{div}(s)|$ and gives the sought extension of $\|s\|$. The second statement follows from the first one applied to the canonical section $s_D = (1, D)$. $\qquad\square$

The definition below introduces the notion of v-adic metric of an $\mathbb{R}$-divisor directly, without passing through v-adic metrics on divisors as in Definition 3.3. The following proposition shows that both points of view are equivalent.

Definition 3.7. Let D be an $\mathbb{R}$-divisor on X and $v \in \mathfrak{M}_{\mathbb{K}}$.

(1) A v-*adic Green function* on D is a continuous function $g \colon X_v^{\mathrm{an}} \setminus |D| \to \mathbb{R}$ such that, for each point $p \in X_v^{\mathrm{an}}$ and a rational $\mathbb{R}$-section $s = (f, D)$ with $p \notin |\operatorname{div}(s)|$, the function $g(p) - \log |f(p)|_v$ can be extended to a continuous function in a neighbourhood of p.

(2) A v-*adic metric* on D is an assignment that, to each rational $\mathbb{R}$-section s associates a continuous function $\|s\| \colon X_v^{\mathrm{an}} \setminus |\operatorname{div}(s)| \to \mathbb{R}_{>0}$ such that, for each $\gamma \in \mathrm{K}(X)_{\mathbb{R}}^{\times}$, it holds $\|(\gamma s)(p)\| = |\gamma(p)|_v \|s(p)\|$ on an open dense subset of X_v^{an}.

Proposition 3.8. *Let D be an $\mathbb{R}$-divisor on X and $v \in \mathfrak{M}_{\mathbb{K}}$. It is equivalent to choose a v-adically metrized $\mathbb{R}$-divisor $\overline{D}$ over D, to choose a v-adic Green function for D, or to choose a v-adic metric on D.*

Proof. Let $\overline{D}$ be a v-adically metrized $\mathbb{R}$-divisor over D. The fact that $g_{\overline{D}}$ is a v-adic Green function for D in the sense of Definition 3.7(1) follows from Proposition 3.6.

Now, if g is a v-adic Green function for D, the assignment that to each $s = (f, D) \in \operatorname{Rat}(X, D)_{\mathbb{R}}^{\times}$ associates the function

$$\|s(p)\|_v = |f(p)|_v \, \mathrm{e}^{-g(p)}$$

is a metric on D in the sense of Definition 3.7(2)

Let now $\|\cdot\|$ be a metric on D. Choose a decomposition $D = \sum_i \alpha_i D_i$ with $\alpha_i \in \mathbb{R}$ and $D_i \in \operatorname{Car}(X)$. For each i, choose a metric $\|\cdot\|_i$ on $\mathcal{O}(D_i)$. Let $f \in \mathrm{K}(X)^{\times}$ and $s_i = (f_i, D_i) \in \operatorname{Rat}(X, D_i)_{\mathbb{R}}^{\times}$ and $s = (f \prod_i f_i^{\alpha_i}, D) \in \operatorname{Rat}(X, D)_{\mathbb{R}}^{\times}$. It is easy to check that the formula

$$\|(f, 0)\| = \|s\| \cdot \prod_i \|s_i\|^{-\alpha_i}$$

defines a metric on the trivial line bundle $\mathcal{O}_{X_v^{\mathrm{an}}}$. Denote by $\overline{0}$ the zero divisor equipped with this metric. Then $\overline{D} = \sum \alpha_i \overline{D}_i + \overline{0}$ is a v-adically metrized $\mathbb{R}$-divisor in the sense of Definition 3.3.

If we apply cyclically this three constructions starting at any point, we obtain the identity, showing the equivalence of the three points of view. $\qquad\square$

Remark 3.9. Since the points of view of metrized $\mathbb{R}$-divisors, Green functions and metrics are equivalent, we will apply the introduced terminology to any of them. For instance, it makes sense to talk of algebraic or smooth metrics on an $\mathbb{R}$-divisor.

Let D be an $\mathbb{R}$-divisor on X and $v \in \mathfrak{M}_{\mathbb{K}}$. There is a natural action of $\mathcal{C}^0(X_v^{\mathrm{an}}, \mathbb{R})$, the space of real-valued continuous functions on X_v^{an}, on the space of v-adic metrics on D. This action is given, for $f \in \mathcal{C}^0(X_v^{\mathrm{an}}, \mathbb{R})$, by the formula $\|\cdot\| \mapsto \mathrm{e}^f \|\cdot\|$. By Proposition 3.8, this action might be equivalently written in terms of v-adic Green functions as $g_{\overline{D}} \mapsto g_{\overline{D}} - f$. From this description, the following corollary follows immediately.

Corollary 3.10. *Let $D \in \mathrm{Car}(X)_{\mathbb{R}}$ and $v \in \mathfrak{M}_{\mathbb{K}}$. The action of $\mathcal{C}^0(X_v^{\mathrm{an}}, \mathbb{R})$ on the space of v-adic metrics on D is free and transitive.*

Given two v-adic metrics $\|\cdot\|$ and $\|\cdot\|'$ on D, the difference of their Green functions $g_{D, \|\cdot\|} - g_{D, \|\cdot\|'}$ extends to a continuous function on X_v^{an}. The *distance* between $\|\cdot\|$ and $\|\cdot\|'$ is defined as

$$\mathrm{dist}(\|\cdot\|, \|\cdot\|') = \sup_{p \in X_v^{\mathrm{an}}} \left| g_{D, \|\cdot\|}(p) - g_{D, \|\cdot\|'}(p) \right|.$$

Corollary 3.11. *Let $D \in \mathrm{Car}(X)_{\mathbb{R}}$ and $v \in \mathfrak{M}_{\mathbb{K}}$. The space of v-adic metrics on D is complete with respect to the topology defined by* dist.

Proof. This follows immediately from Corollary 3.10 and the fact that the space $\mathcal{C}^0(X_v^{\mathrm{an}}, \mathbb{R})$ is complete with respect to the uniform convergence of functions. $\square$

Definition 3.12. Let D be an $\mathbb{R}$-divisor on X. For $v \in \mathfrak{M}_{\mathbb{K}}$, a v-adic metric $\|\cdot\|$ on D is *semipositive* if it is the limit of a sequence of semipositive smooth (in the Archimedean case) or algebraic (in the non-Archimedean case) metrics on D.

A quasi-algebraic metric $\{\|\cdot\|_v\}_{v \in \mathfrak{M}_{\mathbb{K}}}$ on D is *semipositive* if $\|\cdot\|_v$ is a semipositive v-adic metric for all v. A quasi-algebraic metrized $\mathbb{R}$-divisor is *DSP* if there are semipositive metrized $\mathbb{R}$-divisors $\overline{D}_1, \overline{D}_2$ on X such that $\overline{D} = \overline{D}_1 - \overline{D}_2$.

Let Y be a d-dimensional cycle on X and $v \in \mathfrak{M}_{\mathbb{K}}$. Let $\overline{D}_i$, $i = 0, \ldots, d$, be semipositive v-adically metrized $\mathbb{R}$-divisors on X such that $D_0, \ldots, D_d$ meet Y properly in the sense of [BPS14, Definition 1.4.9], that is, for all $I \subset \{0, \ldots, d\}$, every component of the intersection

$$Y \cap \bigcap_{i \in I} |D_i|$$

is of dimension $d - \#I$. The *v-adic height* of Y with respect to $\overline{D}_0, \ldots, \overline{D}_d$, denoted $\mathrm{h}_{v, \overline{D}_0, \ldots, \overline{D}_d}(Y)$, is defined by multilinearity and continuity from the local height of a cycle with respect to semipositive line bundles [BPS14, Definition 1.4.11]. These v-adic heights satisfy the *Bézout formula*

$$(3.13) \quad \mathrm{h}_{v, \overline{D}_0, \ldots, \overline{D}_d}(Y) = \mathrm{h}_{v, \overline{D}_0, \ldots, \overline{D}_{d-1}}(Y \cdot D_d)$$
$$- \int_{X_v^{\mathrm{an}}} \log \|s_{D_d}\|_v \, \mathrm{c}_1(\overline{D}_0) \wedge \cdots \wedge \mathrm{c}_1(\overline{D}_{d-1}) \wedge \delta_Y,$$

where $\mathrm{c}_1(\overline{D}_0) \wedge \cdots \wedge \mathrm{c}_1(\overline{D}_{d-1}) \wedge \delta_Y$ is the measure on X_v^{an} defined by multilinearity and continuity from the corresponding measures associated to semipositive (smooth or algebraic) metrized line bundles.

For semipositive metrized $\mathbb{R}$-divisors $\overline{D}_i \in \mathrm{Car}(X)_{\mathbb{R}}$, $i = 0, \ldots, d$, the *global height* of Y with respect to $\overline{D}_0, \ldots, \overline{D}_d$, denoted by $\mathrm{h}_{\overline{D}_0, \ldots, \overline{D}_d}(Y)$, can be defined from local heights similarly as in Definition 1.5.9 of *loc. cit.*. If $D_0, \ldots, D_d$ meet Y properly, then

$$\mathrm{h}_{\overline{D}_0, \ldots, \overline{D}_d}(Y) = \sum_{v \in \mathfrak{M}_{\mathbb{K}}} n_v \, \mathrm{h}_{v, \overline{D}_0, \ldots, \overline{D}_d}(Y).$$

Finally, the notions of local and global heights of cycles extend to DSP metrized $\mathbb{R}$-divisors by multilinearity.

Next, we define arithmetic linear series, arithmetic volume and χ-arithmetic volume of a metrized $\mathbb{R}$-divisor $\overline{D}$ on X, extending to our framework the previous definitions by Moriwaki and Yuan. Let $s \in \Gamma(X, D)^{\times}$ be a nonzero global section of D. For each place v, the function $\|s\|_v \colon X_v^{\mathrm{an}} \setminus |\operatorname{div}(s)| \to \mathbb{R}_{>0}$ can be extended to a continuous function $X_v^{\mathrm{an}} \to \mathbb{R}_{\geq 0}$ because $\operatorname{div}(s)$ is effective. We set

$$\|s\|_{v,\sup} = \sup_{p \in X_v^{\mathrm{an}}} \|s(p)\|_v.$$

This induces a v-adic norm on the $\mathbb{K}_v$-vector space $\mathrm{L}(D) \otimes \mathbb{K}_v$. The collection of norms $\{\|\cdot\|_{v,\sup}\}_v$ gives a structure of generically trivial adelic vector space on $\mathrm{L}(D)$, that we denote by $\mathrm{L}(\overline{D})$. The small (respectively strictly small, strictly small on a set of places) elements of $\mathrm{L}(\overline{D})$ will be called *small* (respectively *strictly small*, *strictly small on a set of places*) sections. We write $\widehat{\mathrm{L}}(\overline{D}) = \widehat{H}^0(\mathrm{L}(\overline{D}))$ for the set of small sections of $\mathrm{L}(\overline{D})$ and we also set $\widehat{l}(\overline{D}) = \widehat{h}^0(\mathrm{L}(\overline{D}))$ (Definition 2.14). Hence,

$$\widehat{l}(\overline{D}) = \begin{cases} \log(\#\widehat{\mathrm{L}}(\overline{D})) & \text{if } \mathbb{K} \text{ is a number field,} \\ \log(c_k) \dim_k(\widehat{\mathrm{L}}(\overline{D})) & \text{if } \mathbb{K} \text{ is a function field.} \end{cases}$$

Definition 3.14. The *arithmetic volume* and the χ-*arithmetic volume* of a metrized $\mathbb{R}$-divisor $\overline{D}$ on X are respectively defined as

$$\widehat{\operatorname{vol}}(X, \overline{D}) = \frac{1}{d_{\mathbb{K}}} \limsup_{\ell \to \infty} \frac{\widehat{l}(\ell\overline{D})}{\ell^{n+1}/(n+1)!},$$

$$\widehat{\operatorname{vol}}_{\chi}(X, \overline{D}) = \frac{1}{d_{\mathbb{K}}} \limsup_{\ell \to \infty} \frac{\widehat{\chi}(\mathrm{L}(\ell\overline{D}))}{\ell^{n+1}/(n+1)!}.$$

Remark 3.15. For metrized divisors on a variety over a number field, the $\limsup$ in the above definition is actually a limit [BC11, Theorems 2.8 and 3.1]. We will not use this fact in this text.

By extension, a global $\mathbb{R}$-section $s \in \Gamma(X, D)_{\mathbb{R}}^{\times}$ is called *small* if $\|s\|_{v,\sup} \leq 1$ for all $v \in \mathfrak{M}_{\mathbb{K}}$. The set of small elements of $\Gamma(X, D)_{\mathbb{R}}^{\times}$ is denoted by $\widehat{\Gamma}(X, \overline{D})_{\mathbb{R}}^{\times}$.

A small $\mathbb{R}$-section s of $\overline{D}$ is called *strictly small* if $\prod_v \|s\|_{v,\sup}^{n_v} < 1$. If $S \subset \mathfrak{M}_{\mathbb{K}}$ is a finite set of places, the $\mathbb{R}$-section s is *strictly small on* S if $\|s\|_{v,\sup} < 1$ for all $v \in S$.

Lemma 1.11 has the following useful consequence for the existence of small $\mathbb{R}$-sections which are strictly small on a given set of places.

Proposition 3.16. *Let $\overline{D}$ be a metrized $\mathbb{R}$-divisor on X and $s \in \Gamma(X, D)_{\mathbb{R}}^{\times}$ a global $\mathbb{R}$-section such that*

$$\prod_{v \in \mathfrak{M}_{\mathbb{K}}} \|s\|_{v,\sup}^{n_v} < 1.$$

Let $S \subset \mathfrak{M}_{\mathbb{K}}$ be a finite subset and $0 < \eta \leq 1$ a real number. Then there is an integer $\ell_0 \geq 1$ such that, for each $\ell \geq \ell_0$, there exists $\alpha_\ell \in \mathbb{K}^{\times}$ such that $\|\alpha_\ell s^\ell\|_{v,\sup} \leq 1$ for all $v \in \mathfrak{M}_{\mathbb{K}}$ and $\|\alpha_\ell s^\ell\|_{v,\sup} < \eta$ for all $v \in S$.

In particular, $\alpha_\ell s^\ell \in \widehat{\Gamma}(X, \ell\overline{D})_{\mathbb{R}}^{\times}$ is strictly small on S.

Proof. This follows by applying Lemma 1.11 to the real numbers $\gamma_v = \|s\|_{v,\sup}$. $\square$

Lemma 3.17. *Let $\overline{D}$ be a metrized divisor on X. If $\overline{D}$ has a strictly small $\mathbb{R}$-section s, then there exists a positive integer ℓ such that $\ell\overline{D}$ has a strictly small section s_0 with $|\operatorname{div}(s_0)| = |\operatorname{div}(s)|$.*

Proof. Write $s = (f, D)$ with $f = \prod_{i=1}^{d} f_i^{\alpha_i}$ where $\alpha_1, \ldots, \alpha_d$ are $\mathbb{Q}$-linearly independent real numbers and $\alpha_1 \in \mathbb{Q}$. The associated $\mathbb{R}$-Weil divisor can be written

down as $[\text{div}(s)] = \sum_j r_j Z_j$ with $r_j > 0$ and Z_j an irreducible hypersurface. Then

$$|\text{div}(s)| = \bigcup_j Z_j = |\alpha_1 \text{div}(f_1) + D| \cup \bigcup_{i=2}^d |\text{div}(f_i)|.$$

Let $\sigma\colon \mathbb{R}^{d-1} \to \text{Rat}(X,D)_{\mathbb{R}}^{\times}$ be the map defined by

$$\sigma(\beta_2,\ldots,\beta_d) = \Big(f_1^{\alpha_1} \cdot \prod_{i=2}^d f_i^{\beta_j}, D\Big).$$

Then, for each $\beta \in \mathbb{R}^{d-1}$, $[\text{div}(\sigma(\beta))] = \sum_j r_j(\beta) Z_j$ where r_j is an affine function. Consider the open polyhedral set

$$\Lambda = \{\beta \in \mathbb{R}^{d-1} \mid r_j(\beta) > 0 \text{ for all } j\}.$$

This set is nonempty because $(\alpha_2,\ldots,\alpha_d) \in \Lambda$. Let $\tau\colon \Lambda \to \mathbb{R}$ be the function defined by

$$\tau(\beta) = \prod_v \|\sigma(\beta)\|_{\sup,v}^{n_v}.$$

For $v \in \mathfrak{M}_{\mathbb{K}}$, $\beta, \beta' \in \Lambda$ and $0 \leq \theta \leq 1$,

$$\|\sigma(\theta\beta + (1-\theta)\beta')\|_{\sup,v} \leq \|\sigma(\beta)\|_{\sup,v}^{\theta} \|\sigma(\beta')\|_{\sup,v}^{1-\theta}.$$

From this, we deduce that τ is log-concave on Λ and, a fortiori, continuous. Since $\tau(\alpha_2,\ldots,\alpha_d) < 1$, there exists $\beta \in \Lambda \cap \mathbb{Q}^{d-1}$ such that $\tau(\beta) < 1$. By Proposition 3.16, there exists $\gamma \in \mathbb{K}^{\times}$ and a positive integer ℓ such that $s_0 = \gamma\,\sigma(\beta)^{\ell}$ is a strictly small section of $\ell\overline{D}$. By construction,

$$|\text{div}(s_0)| = \bigcup_j Z_j = |\text{div}(s)|,$$

as stated. $\qquad\square$

We introduce the different notions of arithmetic positivity for a metrized $\mathbb{R}$-divisor.

Definition 3.18. Let X be a proper normal variety over $\mathbb{K}$ and $\overline{D}$ a metrized $\mathbb{R}$-divisor on X.

(1) $\overline{D}$ is *generated by small $\mathbb{R}$-sections* if, for each $p \in X(\overline{K})$, there exists $s = (f,D) \in \widehat{\Gamma}(X,\overline{D})_{\mathbb{R}}^{\times}$ such that $p \notin |\text{div}(s)|$.

(2) $\overline{D}$ is *ample* if the following conditions hold:
   (a) the $\mathbb{R}$-divisor D is ample;
   (b) the metric is semipositive;
   (c) for each $\overline{M} \in \widehat{\text{Car}}(X)_{\mathbb{R}}$ there exists an ℓ_0 such that, for all real numbers $\ell \geq \ell_0$, the metrized $\mathbb{R}$-divisor $\overline{M}+\ell\overline{D}$ is generated by small $\mathbb{R}$-sections.

(3) $\overline{D}$ is *nef* if the following conditions hold:
   (a) the $\mathbb{R}$-divisor D is nef;
   (b) the metric is semipositive;
   (c) for every point $p \in X(\overline{\mathbb{K}})$ it holds $h_{\overline{D}}(p) \geq 0$.

(4) $\overline{D}$ is *big* if $\widehat{\text{vol}}(X,\overline{D}) > 0$.

(5) $\overline{D}$ is *pseudo-effective* if there exists a birational map $\varphi\colon X' \to X$ of normal proper varieties over K and a metrized $\mathbb{R}$-divisor $\overline{E}$ on X' such that $\ell\varphi^*\overline{D}+\overline{E}$ is big for all $\ell \geq 1$.

(6) $\overline{D}$ is *effective* if $(1,D) \in \widehat{\text{L}}(\overline{D})$. Given $\mathbb{R}$-divisors $\overline{D}_1, \overline{D}_2$ on X, the fact that $\overline{D}_1 - \overline{D}_2$ is effective is denoted by $\overline{D}_1 \geq \overline{D}_2$.

The notion of metrized $\mathbb{R}$-divisor contains that of arithmetic $\mathbb{R}$-divisor introduced by Moriwaki [Mor12c].

Example 3.19. Let $\mathcal{X}$ be normal projective flat scheme over $\mathbb{Z}$ with smooth generic fiber $X = \mathcal{X} \times \mathrm{Spec}(\mathbb{Q})$. An *arithmetic $\mathbb{R}$-Cartier divisor* on $\mathcal{X}$ is a pair $\overline{\mathcal{D}} = (\mathcal{D}, g)$ where $\mathcal{D}$ is an $\mathbb{R}$-Cartier divisor on $\mathcal{X}$ and g is a locally integrable real function on $X(\mathbb{C})$ that is invariant under complex conjugation. Let D be the restriction of $\mathcal{D}$ to X. The function g is called a *Green function for D of C^0-type* (respectively C^∞-*type, PSH-type*) if, for every $p \in X(\mathbb{C})$, it can be locally written as

$$g(x) = u(x) + \sum_{i=1}^k (-\alpha_i) \log |f_i(x)|^2,$$

where $D = \sum_{i=1}^k \alpha_i D_i$ is the decomposition of D into irreducible components, f_i is a local equation for D_i, and u is a continuous function (respectively a smooth function, a plurisubharmonic function). If g is a Green function of C^0-type (respectively C^∞-type, PSH-type), then $\overline{D} = (\mathcal{D}, g)$ is called an *arithmetic divisor of C^0-type* (respectively C^∞-*type, PSH-type*).

On the one hand, extending the method of Zhang [Zha95b] (see also [BPS14, §1.3]) to $\mathbb{R}$-Cartier divisors, the integral model $\mathcal{D}$ defines an algebraic v-adic metric on D for each place $v \in \mathfrak{M}_\mathbb{Q} \setminus \{\infty\}$. On the other hand, if g is a Green function for D of C^0-type, then the function $\frac{1}{2}g$ is an ∞-adic Green function for D which, by Proposition 3.8, induces an ∞-adic metric on D, The factor $1/2$ comes from the different normalization of Green functions in [GS90] and [Bur97].

Therefore, each arithmetic $\mathbb{R}$-divisor $\overline{\mathcal{D}}$ of C^0-type defines a quasi-algebraic metrized $\mathbb{R}$-divisor on X that we denote by $\overline{D}$. The metrized $\mathbb{R}$-divisors that arise in this way are called *algebraic*.

We now discuss the relationship between the different notions of positivity for arithmetic $\mathbb{R}$-divisors that appear in [Mor12c] and those in the present text. Let $\overline{\mathcal{D}} = (\overline{D}, g)$ be an arithmetic $\mathbb{R}$-divisor of C^0-type and $\overline{D}$ its associated algebraic metrized $\mathbb{R}$-divisor.

(1) $\overline{\mathcal{D}}$ is of *PSH*-type if and only if the ∞-adic metric induced by its Green function is semipositive.
(2) If the $\mathbb{R}$-divisor $\mathcal{D}$ is relatively nef, then the induced v-adic metrics are semipositive for all $v \neq \infty$. The converse of this result is not established yet, except in the case of equal characteristic zero [BFJ11, Theorem 2.17].
(3) $\overline{\mathcal{D}}$ is effective (respectively big, pseudo-effective) in the sense of [Mor12c] if and only if $\overline{D}$ is effective (respectively big, pseudo-effective) in the sense of Definition 3.18.
(4) If $\overline{\mathcal{D}}$ is nef in the sense of [Mor12c, §6.1] then $\overline{D}$ is nef in the sense of Definition 3.18. The converse is not known because the converse of (2) is not known.
(5) If $\overline{\mathcal{D}}$ is of C^∞-type, then it is ample in the sense of [Mor12c, §6.1] if and only if $\overline{D}$ is ample in the sense of the present text. However, the definition of ampleness in *loc. cit.* includes being of C^∞-type, while in the present text an ample metrized $\mathbb{R}$-divisor is not necessarily of C^∞-type.

The following statements contain some of the basic properties of metrized $\mathbb{R}$-divisors.

Proposition 3.20. *Let $\overline{D}$ be an ample metrized $\mathbb{R}$-divisor on X. Then*

(1) *$\overline{D}$ is generated by strictly small $\mathbb{R}$-sections;*
(2) *for all subvarieties Y of $X_{\overline{\mathbb{K}}}$, it holds $\mathrm{h}_{\overline{D}}(Y) > 0$.*

Proof. (1) Let $v_0 \in \mathfrak{M}_\mathbb{K}$ and consider the trivial divisor $0 \in \mathrm{Car}(X)$ with the metric defined by $\|1\|_v = 1$ for $v \neq v_0$ and $\|1\|_{v_0} = 2$. By the ampleness of $\overline{D}$, there exists

$\ell \geq 0$ such that $\overline{0} + \ell\overline{D}$ is generated by small $\mathbb{R}$-sections. These small $\mathbb{R}$-sections are strictly small $\mathbb{R}$-sections of $\ell\overline{D}$, which proves the statement.

(2) We prove this by induction on $d = \dim(Y)$. Consider first the case $d = 0$. Let $Y = \{p\}$ where p is a point defined over a finite extension $\mathbb{F}$ of $\mathbb{K}$. By (1), there is a strictly small $\mathbb{R}$-section s of $\overline{D}$ such that $p \notin |\operatorname{div}(s)|$. Then

$$\mathrm{h}_{\overline{D}}(p) = - \sum_{w \in \mathfrak{M}_{\mathbb{F}}} n_w \log \|s(p)\|_w > 0.$$

Assume now that Y is a subvariety of dimension $d \geq 1$ defined over $\mathbb{F}$. Let s be a strictly small $\mathbb{R}$-section of $\overline{D}$ which meets Y properly. By Bézout's formula (3.13),

$$\mathrm{h}_{\overline{D}}(Y) = \mathrm{h}_{\overline{D}}(Y \cdot \operatorname{div} s) - \sum_{w \in \mathfrak{M}_{\mathbb{F}}} n_w \int_{X_w^{\mathrm{an}}} \log \|s\|_w \, c_1(D, \|\cdot\|_w)^{\wedge d} \wedge \delta_Y.$$

Since D is ample, $Y \cdot \operatorname{div}(s)$ is a nonzero $(d-1)$-dimensional effective cycle. Applying linearity and the induction hypothesis, $\mathrm{h}_{\overline{D}}(Y \cdot \operatorname{div} s) > 0$. The fact that the metric is semipositive implies that, for each $w \in \mathfrak{M}_{\mathbb{F}}$, the signed measure $c_1(D, \|\cdot\|_w)^{\wedge d} \wedge \delta_Y$ is positive and its total mass is $\deg_D(Y)$. Therefore

$$\mathrm{h}_{\overline{D}}(Y) > - \sum_{w \in \mathfrak{M}_{\mathbb{F}}} n_w \int_{X_w^{\mathrm{an}}} \log \|s\|_w \, c_1(D, \|\cdot\|_w)^{\wedge d} \wedge \delta_Y$$

$$\geq - \sum_{w \in \mathfrak{M}_{\mathbb{F}}} n_w \log \|s\|_{w,\sup} \int_{X_w^{\mathrm{an}}} c_1(D, \|\cdot\|_w)^{\wedge d} \wedge \delta_Y > 0,$$

since s is a strictly small $\mathbb{R}$-section and the last integral is equal to $\deg_D(Y)$ for all w. $\qquad\square$

Remark 3.21. In [Zha95a], Zhang proved a Nakai-Moishezon numerical criterion of arithmetic ampleness in the form of a converse to Proposition 3.20(2) for Hermitian line bundles, under some technical hypothesis. It would be interesting to know if such a result is true in full generality for metrized $\mathbb{R}$-divisors. Namely: let $\overline{D}$ be a semipositive metrized $\mathbb{R}$-divisor such that $\mathrm{h}_{\overline{D}}(Y) > 0$ for all effective cycles Y of X. Is it true that $\overline{D}$ is ample?

Lemma 3.22. *Let $\overline{D}$ be a metrized $\mathbb{R}$-divisor on X.*

(1) *Let $\varphi \colon Z \to X$ be a birational morphism of normal proper varieties over $\mathbb{K}$. Then*
$$\widehat{\operatorname{vol}}(X, \overline{D}) = \widehat{\operatorname{vol}}(Z, \varphi^*\overline{D}).$$
In particular, $\overline{D}$ is big if and only if $\varphi^*\overline{D}$ is big.

(2) *If $\overline{D}$ is big and $\overline{E} \in \widehat{\operatorname{Car}}(X)$ has a small section, then $\overline{D} + \overline{E}$ is big.*

(3) *If $\ell_0\overline{D}$ is big for some $\ell_0 \geq 1$, then $\overline{D}$ is big.*

Proof. (1) Let $s = (f, D) \in \Gamma(X, D)^{\times}$. Then $\operatorname{div}(s) = \operatorname{div}(f) + D$ is effective and so $\operatorname{div}(\varphi^*s) = \varphi^* \operatorname{div}(s)$ is also effective. Hence, $\varphi^*s = (f \circ \varphi, \varphi^*D) \in \Gamma(Z, \varphi^*D)^{\times}$. Thus, there is a well-defined map

$$\varphi^* \colon \Gamma(X, D)^{\times} \to \Gamma(Z, \varphi^*D)^{\times}.$$

We claim that this map is a bijection. The injectivity is clear because the map $f \mapsto f \circ \varphi$ is a bijection between $\mathrm{K}(X)^{\times}$ and $\mathrm{K}(Z)^{\times}$. Let now $s = (f \circ \varphi, \varphi^*D) \in \Gamma(Z, \varphi^*D)^{\times}$ and set $s' = (f, D) \in \operatorname{Rat}(X, D)^{\times}$. Let $[\operatorname{div}(s)]$ be the $\mathbb{R}$-Weil divisor associated to $\operatorname{div}(s)$. Since $\operatorname{div}(s)$ is effective, the same is true for $[\operatorname{div}(s)]$. Since X is normal and Z is proper, for each codimension one point $x \in X^{(1)}$ there is a neighbourhood U of x and a section $U \to Z$ of φ [EGA, II, 7.3.5]. This implies that $[\operatorname{div}(s')]$ is effective. By [EGA, IV, (21.6.9.1)], it follows that $\operatorname{div}(s')$ is effective and so $s' \in \Gamma(X, D)^{\times}$, proving the claim.

Given $s \in \Gamma(X,D)^\times$, then $\|s\|_{v,\sup} = \|\varphi^*s\|_{v,\sup}$ for all v and so φ^* induces an isomorphism between $\widehat{L}(\overline{D})$ and $\widehat{L}(\varphi^*\overline{D})$. Hence $\widehat{\mathrm{vol}}(X,\overline{D}) = \widehat{\mathrm{vol}}(Z,\varphi^*\overline{D})$, which proves the first statement.

(2) Let s_0 be a small section of $\overline{E}$. There is an injective map

$$\widehat{L}(\ell\overline{D}) \hookrightarrow \widehat{L}(\ell(\overline{D}+\overline{E}))$$

given by $s \mapsto s_0^\ell s$. Hence, $\widehat{\mathrm{vol}}(X,\overline{D}+\overline{E}) \geq \widehat{\mathrm{vol}}(X,\overline{D})$ and the statement follows.

(3) Assume that $\ell_0\overline{D}$ is big. We have that

$$\frac{1}{d_\mathbb{K}} \limsup_{\ell\to\infty} \frac{\widehat{l}(\ell\overline{D})}{\ell^{n+1}/(n+1)!} \geq \frac{1}{d_\mathbb{K}} \limsup_{\ell\to\infty} \frac{\widehat{l}(\ell_0\ell\overline{D})}{(\ell_0\ell)^{n+1}/(n+1)!}.$$

Hence, $\widehat{\mathrm{vol}}(X,\overline{D}) \geq \widehat{\mathrm{vol}}(X,\ell_0\overline{D})/\ell_0^{n+1} > 0$ and so $\overline{D}$ is big. $\square$

Proposition 3.23. *Let $\overline{D}$ be a pseudo-effective metrized $\mathbb{R}$-divisor on X, $\rho\colon X'' \to X$ a birational map of normal proper varieties over $\mathbb{K}$, and $\overline{A}$ an ample metrized divisor on X''. Then $\ell\rho^*\overline{D} + \overline{A}$ is big for all $\ell \geq 1$.*

Proof. By the pseudo-effectiveness of $\overline{D}$, there is a birational map $\varphi\colon X' \to X$ from a normal proper variety X' and a metrized divisor $\overline{E}$ on X' such that $q\varphi^*\overline{D} + \overline{E}$ is big for all $q \geq 1$. Consider the fibre product of X' and X'' over X and the corresponding commutative diagram of varieties

$$
\begin{array}{ccc}
Y & \xrightarrow{\,p_1\,} & X' \\
{\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \varphi} \\
X'' & \xrightarrow{\,\rho\,} & X
\end{array}
$$

where p_1 and p_2 are birational maps. Set $\phi = \varphi \circ p_1$. The map $\phi\colon Y \to X$ is birational. Since A is ample, p_2^*A is big, which implies that there is an integer $j_0 \geq 1$ such that $j_0 p_2^*A - p_1^*E$ has a nonzero section s_0.

Let $S \subset \mathfrak{M}_\mathbb{K}$ be a finite subset such that $\|s_0\|_{\sup,v} \leq 1$ for all $v \notin S$. Let $\eta = (\sup_{v\in S} \|s_0\|_{v,\sup})^{-1}$. Combining Proposition 3.20(1), Lemma 3.17 and Proposition 3.16, we deduce that there exists $j_1 \geq 1$ and a small section s_1 of $j_1 p_2^*\overline{A}$ such that $\|s_1\|_{v,\sup} < \eta$ for $v \in S$. Hence, $s_1 s_0$ is a small section of $(j_0+j_1)p_2^*\overline{A} - p_1^*\overline{E}$.

Set $j_2 = j_0 + j_1$. Then

$$j_2(\ell\phi^*\overline{D} + p_2^*\overline{A}) = (j_2\ell\phi^*\overline{D} + p_1^*\overline{E}) + (j_2 p_2^*\overline{A} - p_1^*\overline{E}).$$

Since $(j_2\ell\phi^*\overline{D} + p_1^*\overline{E}) = p_1^*(j_2\ell\varphi^*\overline{D} + \overline{E})$, this is a big metrized $\mathbb{R}$-divisor thanks to Lemma 3.22(1). Lemma 3.22(2) and the fact that $j_2 p_2^*\overline{A} - p_1^*\overline{E}$ has a small section imply that $j_2(\ell\phi^*\overline{D} + p_2^*\overline{A})$ is big. By Lemma 3.22(3), $\ell\phi^*\overline{D} + p_2^*\overline{A} = p_2^*(\ell\rho^*\overline{D} + \overline{A})$ is big. By Lemma 3.22(1), we conclude that $\ell\rho^*\overline{D} + \overline{A}$ is big. $\square$

In [Mor13], Moriwaki proposed an extension of Dirichlet's unit theorem to the higher-dimensional case. The following is the natural extension of this question to our more general setting.

Question 3.24. *(Dirichlet's unit theorem) Let $\mathbb{K}$ be an A-field, X a normal proper variety over $\mathbb{K}$ and $\overline{D}$ a metrized $\mathbb{R}$-divisor on X. Are the following conditions (1) and (2) equivalent?*

(1) *$\overline{D}$ is pseudo-effective;*
(2) *there exists $f \in \mathrm{K}(X)_\mathbb{R}^\times$ such that $\overline{D} + \widehat{\mathrm{div}}(f) \geq 0$.*

In the setting of Question 3.24, the classical Dirichlet's unit theorem shows up when considering the zero-dimensional case. Let $\mathbb{K}$ be an A-field and $S \subset \mathfrak{M}_\mathbb{K}$ a finite subset containing the Archimedean places. Let U_S be the group of S-units

of $\mathbb{K}$ and H_S the hyperplane of $\mathbb{R}^S$ defined by $\sum_{v \in S} n_v \xi_v = 0$. The adelic version of Dirichlet's unit theorem states that the regulator map $U_S \otimes_{\mathbb{Z}} \mathbb{R} \to H_S$ given by $u \mapsto (\log |u|_v)_{v \in S}$ is an isomorphism [Wei74, Chapter IV, §4, Theorem 9].

Let now $X = \mathrm{Spec}(\mathbb{K})$, $D = 0$ the zero divisor on X and $(\xi_v)_{v \in S} \in H_S$. We put $\xi_v = 0$ for $v \notin S$. Each ξ_v gives a v-adic Green function for D and we denote by $\overline{D}$ the associated metrized divisor. It can be verified that this metrized divisor is pseudo-effective (see for instance Theorem 6.1(4)). Condition (2) for $\overline{D}$ is equivalent to the existence of $u \in \mathbb{K}^\times \otimes_{\mathbb{Z}} \mathbb{R}$ such that $\xi_v - \log |u|_v \geq 0$ for all v. Since $\sum_v n_v \log |u|_v = 0 = \sum_v n_v \xi_v$ because of the product formula, the previous inequality forces $\log |u|_v = \xi_v$ for all places and, in particular, $u \in U_S \otimes_{\mathbb{Z}} \mathbb{R}$. Hence, the implication (1)$\Rightarrow$(2) in Question 3.24 is equivalent to the surjectivity of the regulator map in Dirichlet's unit theorem.

In higher dimension, the fact that (2) implies (1) follows from the facts that effective metrized $\mathbb{R}$-divisors are pseudo-effective and that pseudo-effectiveness is invariant with respect to linear equivalence. Moriwaki has proven the reverse implication when X is smooth and projective, D is numerically trivial and the metrics at the finite places come from a common normal projective model over $\mathcal{O}_{\mathbb{K}}$ [Mor13].

Remark 3.25. If $\mathbb{K}$ is a general function field, the answer is "no", simply because the analogue of the classical Dirichlet's unit theorem does not hold. The simplest example is the following. Let C be an elliptic curve over a field k of characteristic zero, $\mathbb{K} = \mathrm{K}(C)$ and $X = \mathrm{Spec}(\mathbb{K})$. Let $\mathcal{D}$ be a divisor of degree zero on C whose class in the Picard group is non-torsion and let $\overline{D}$ be the corresponding metrized $\mathbb{R}$-divisor on X. Then $\overline{D}$ is pseudo-effective but there is no $f \in \mathrm{K}(X)_{\mathbb{R}}^\times$ such that $\overline{D} + \widehat{\mathrm{div}}(f) \geq 0$. Assume that such f exists. Then $\mathcal{D} + \mathrm{div}(f)$ would be effective and of degree 0. Hence $\mathcal{D} + \mathrm{div}(f) = 0$. Since $\mathcal{D}$ is a divisor the previous equation implies that we can find a $g \in \mathrm{K}(X)^\times$ and $m \in \mathbb{Z}_{\geq 1}$ with $m\mathcal{D} + \mathrm{div}(g) = 0$. Therefore the class of $\mathcal{D}$ in the Picard group is torsion, contradicting the hypothesis.

In §7 we will show that, for toric varieties, the answer to Question 3.24 is positive.

We now turn our attention towards the approximation of pseudo-effective and big divisors by nef and ample ones.

Definition 3.26. Let X be a normal proper variety over $\mathbb{K}$ and $\overline{D}$ a pseudo-effective metrized $\mathbb{R}$-divisor on X. A *Zariski decomposition* of $\overline{D}$ is a birational map $\varphi \colon X' \to X$ of normal proper varieties over $\mathbb{K}$ and a decomposition

$$\varphi^* \overline{D} = \overline{P} + \overline{E}$$

with $\overline{P}, \overline{E} \in \widehat{\mathrm{Car}}(X')_{\mathbb{R}}$ such that $\overline{P}$ is nef, $\overline{E}$ is effective and $\widehat{\mathrm{vol}}(X', \overline{P}) = \widehat{\mathrm{vol}}(X, \overline{D})$.

Sometimes it is convenient to consider the following variant.

Definition 3.27. Let X be a normal proper variety over $\mathbb{K}$ and $\overline{D}$ a pseudo-effective metrized $\mathbb{R}$-divisor on X. Let $\Upsilon(\overline{D})$ be the set of pairs $(\varphi, \overline{P})$, where $\varphi \colon X' \to X$ is a birational map of normal proper varieties over $\mathbb{K}$ and $\overline{P}$ is a nef metrized $\mathbb{R}$-divisor on X' such that $\varphi^* \overline{D} - \overline{P} \geq 0$. On $\Upsilon(\overline{D})$ we consider the equivalence relation $(\varphi, \overline{P}) \sim (\varphi_1, \overline{P}_1)$ whenever there exists a commutative diagram of birational morphisms

$$\begin{array}{ccc} Z & \xrightarrow{\;\nu\;} & X' \\ {\scriptstyle \nu_1}\big\downarrow & & \big\downarrow{\scriptstyle \varphi} \\ X'_1 & \xrightarrow{\;\varphi_1\;} & X \end{array}$$

such that $\nu^*\overline{P} = \nu_1^*\overline{P}_1$. On the set of equivalence classes $\Upsilon(\overline{D})/\sim$, we consider the order relation given by $[(\varphi, \overline{P})] \leq [(\varphi_1, \overline{P}_1)]$ whenever there is a commutative diagram of birational maps as above with $\nu^*\overline{P} \leq \nu_1^*\overline{P}_1$.

The *strong Zariski decomposition* of $\overline{D}$ is the greatest element of $\Upsilon(\overline{D})/\sim$, if it exists.

If $\overline{D}$ is a big metrized $\mathbb{R}$-divisor on X, then a strong Zariski decomposition of $\overline{D}$ gives a Zariski decomposition in the sense of Definition 3.26 [Mor12b, Proposition B.1].

Question 3.28. *Let $\overline{D}$ be a pseudo-effective metrized $\mathbb{R}$-divisor on X. When does $\overline{D}$ admit a Zariski decomposition or a strong Zariski decomposition?*

In [Mor12c], Moriwaki showed that a strong Zariski decomposition of $\overline{D}$ exists if $\mathbb{K}$ is a number field, X is a curve, $\overline{D}$ is big and the metrics at the finite places come from a common normal projective model over $\mathcal{O}_{\mathbb{K}}$. In higher dimension, it is not true that every big metrized $\mathbb{R}$-divisor admits a Zariski decomposition. Indeed, there are examples of toric big metrized divisors on $\mathbb{P}^2$ that do not admit a Zariski decomposition and, a fortiori, do not admit a strong Zariski decomposition [Mor13]. In §7, we will consider toric Zariski decompositions and toric strong Zariski decompositions, and we will give a criterion for such decompositions to exist. Furthermore, in §8 we will show that, under some hypothesis, the existence of a non-necessarily toric Zariski decomposition of a big toric metrized $\mathbb{R}$-divisor implies the existence of a toric one.

In the absence of a Zariski decomposition, one can ask for the existence of a Fujita approximation.

Question 3.29. *(Fujita approximation) Let $\overline{D}$ be a big metrized $\mathbb{R}$-divisor on X. Let $\varepsilon > 0$ be a positive real number. Does there exist a birational proper map $\varphi\colon X' \to X$ and metrized $\mathbb{R}$-divisors $\overline{A}, \overline{E} \in \widehat{\mathrm{Car}}(X')_{\mathbb{R}}$ such that $\overline{A}$ is ample, $\overline{E}$ is effective,*

$$\varphi^*\overline{D} = \overline{A} + \overline{E} \quad and \quad \widehat{\mathrm{vol}}(X', \overline{A}) \geq \widehat{\mathrm{vol}}(X, \overline{D}) - \varepsilon?$$

The existence of an arithmetic Fujita approximation was independently obtained by Yuan [Yua09a] and by Chen [Che10] in the case when $\mathbb{K}$ is a number field, D is a divisor and the metrics at the infinite places are smooth and those at the finite places come from a common projective model over $\mathcal{O}_{\mathbb{K}}$. More recently, Boucksom and Chen [BC11] have given a more elementary proof of this fact.

In §7 we will give a proof of the Fujita approximation theorem for big toric metrized $\mathbb{R}$-divisors on a toric variety.

4. TORIC METRIZED $\mathbb{R}$-DIVISORS

In this section, we recall the necessary background on the algebraic and arithmetic geometry of toric varieties from [BPS14] and we extend some of the results in this reference to toric metrized $\mathbb{R}$-divisors. We will follow the notations and conventions in [BPS14, Chapters 3 and 4].

Let $N \simeq \mathbb{Z}^n$ be a lattice and $M = N^\vee$ the dual lattice. Set $N_{\mathbb{R}} = N \otimes \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes \mathbb{R}$. The pairing between $x \in M_{\mathbb{R}}$ and $u \in N_{\mathbb{R}}$ is denoted by $\langle x, u \rangle$.

Let K be a field and set $\mathbb{T} = \mathrm{Spec}(K[M]) \simeq \mathbb{G}_m^n$ for the split torus over K corresponding to N. Let Σ be a complete (rational) fan on $N_{\mathbb{R}}$ and X_{Σ} the proper toric variety over K defined by Σ. We write $X = X_{\Sigma}$ for short. This is a normal variety of dimension n with an open dense immersion $\mathbb{T} \hookrightarrow X$ and an action of $\mathbb{T}$ on X that extends the action of $\mathbb{T}$ on itself by translations.

The toric variety X has a distinguished point x_0 in its principal open subset X_0, corresponding to the unit of the torus $\mathbb{T}$. A toric line bundle is a line bundle

L on X together with the choice of a nonzero point $z_0 \in L_{x_0}$ [BPS14, Definition 3.3.4]. A toric section of L is a section s that is regular and nowhere vanishing on the principal open subset X_0, and such that $s(x_0) = z_0$. There is a bijection between toric divisors and isomorphism classes of toric line bundles with a toric section. If (L, s) is a toric line bundle with a toric section, then $\mathrm{div}(s)$ is a toric divisor [BPS14, Theorem 3.3.7]. Conversely, given a toric divisor D on X, the line bundle $\mathcal{O}(D)$ is a subsheaf of $\mathcal{K}_X$ and the rational function $1 \in \mathrm{K}(X)$ provides a distinguished rational section s_D of $\mathcal{O}(D)$ such that $\mathrm{div}(s_D) = D$. This section does not vanish on X_0 and so $(\mathcal{O}(D), s_D(x_0))$ is a toric line bundle. The correspondence $D \mapsto ((\mathcal{O}(D), s_D(x_0)), s_D)$ is the inverse of the correspondence defined by $(L, s) \mapsto \mathrm{div}(s)$. Thus, the languages of toric line bundles with toric sections and that of toric divisors are equivalent. In the sequel, we will mostly use the latter and, more generally, that of $\mathbb{R}$-divisors. We denote by $\mathrm{Car}_{\mathbb{T}}(X)$ the group of toric divisors on the toric variety X.

Definition 4.1. An $\mathbb{R}$-*virtual support function* on Σ is a function Ψ on $N_{\mathbb{R}}$ such that, for each cone $\sigma \in \Sigma$, there exists $m_\sigma \in M_{\mathbb{R}}$ such that $\Psi(u) = \langle m_\sigma, u \rangle$ for all $u \in \sigma$. A set of functionals $\{m_\sigma\}_{\sigma \in \Sigma}$ as above is called a set of *defining vectors* of Ψ. If we can choose $m_\sigma \in M$ for all σ, then Ψ is called a *virtual support function* on Σ. We respectively denote by $\mathrm{VSF}(\Sigma)$ and by $\mathrm{VSF}(\Sigma)_{\mathbb{R}}$ the spaces of virtual support functions and of $\mathbb{R}$-virtual support functions.

A concave virtual support function (respectively, $\mathbb{R}$-virtual support function) on Σ is called a *support function* (respectively, an $\mathbb{R}$-*support function*) on Σ. We denote by $\mathrm{SF}(\Sigma)$ the semigroup of support functions on Σ and by $\mathrm{SF}(\Sigma)_{\mathbb{R}}$ the convex cone of $\mathbb{R}$-support functions on Σ.

To a toric divisor D one associates a virtual support function, denoted by Ψ_D, in the following way: for each $\sigma \in \Sigma$, there is $m_\sigma \in M$ such that $D = \mathrm{div}(\chi^{-m_\sigma})$ on the affine open set X_σ. Then, $\{m_\sigma\}_\sigma$ is a set of defining vectors for Ψ_D. The correspondence $D \mapsto \Psi_D$ is an isomorphism of $\mathbb{Z}$-modules between $\mathrm{Car}_{\mathbb{T}}(X)$ and $\mathrm{VSF}(\Sigma)$.

Definition 4.2. A *toric $\mathbb{R}$-divisor* on X is a finite linear combination

$$D = \sum_i \alpha_i D_i$$

with $\alpha_i \in \mathbb{R}$ and D_i a toric divisor. To a toric $\mathbb{R}$-divisor D as above, we associate the $\mathbb{R}$-virtual support function $\Psi_D = \sum_i \alpha_i \Psi_{D_i}$. We set

$$\mathrm{Car}_{\mathbb{T}}(X)_{\mathbb{R}} = \mathrm{Car}_{\mathbb{T}}(X) \otimes_{\mathbb{Z}} \mathbb{R}$$

for the linear space of toric $\mathbb{R}$-divisors on X.

There is a group morphism $M \to \mathrm{K}(X)^\times$ given by $m \mapsto \chi^m$, where $\chi^m \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ is the character corresponding to m. By linearity, we can extend it to a group morphism $M_{\mathbb{R}} \to \mathrm{K}(X)^\times_{\mathbb{R}}$. We also denote by χ^m the image of m under this map. Composing with the map div, each element $m \in M_{\mathbb{R}}$ gives rise to a toric $\mathbb{R}$-divisor $\mathrm{div}(\chi^m)$. For $m \in M_{\mathbb{R}}$, we set $s_m = (\chi^m, D)$ for the corresponding rational $\mathbb{R}$-section of D.

Proposition 4.3. *The correspondence $D \mapsto \Psi_D$ is an isomorphism of linear spaces between $\mathrm{Car}_{\mathbb{T}}(X)_{\mathbb{R}}$ and $\mathrm{VSF}(\Sigma)_{\mathbb{R}}$.*

Proof. The correspondence $D \mapsto \Psi_D$ is an isomorphism of $\mathbb{Z}$-modules between $\mathrm{Car}_{\mathbb{T}}(X)$ and $\mathrm{VSF}(\Sigma)$. Hence, it also defines an isomorphism between $\mathrm{Car}_{\mathbb{T}}(X)_{\mathbb{R}}$ and $\mathrm{VSF}(\Sigma) \otimes_{\mathbb{Z}} \mathbb{R}$. The space $\mathrm{VSF}(\Sigma)_{\mathbb{R}}$ can be identified with the linear subspace of $\prod_{\sigma \in \Sigma^n} M_{\mathbb{R}}$ defined by

(4.4) $$\{(m_\sigma)_\sigma \mid m_\sigma - m_\tau \in (\sigma \cap \tau)^\perp \text{ for all } \sigma, \tau \in \Sigma^n\}.$$

This subspace is defined over $\mathbb{Q}$ because the fan Σ is rational, and its restriction to $\prod_{\sigma \in \Sigma^n} M$ agrees with $\mathrm{VSF}(\Sigma)$. Hence, $\mathrm{VSF}(\Sigma)_{\mathbb{R}} = \mathrm{VSF}(\Sigma) \otimes_{\mathbb{Z}} \mathbb{R}$, which proves the statement. $\square$

Definition 4.5. A nonempty compact subset $C \subset M_{\mathbb{R}}$ is called a *quasi-rational* polytope if there are $u_j \in N_{\mathbb{Q}}$ and $\gamma_j \in \mathbb{R}$, $j = 1, \ldots, l$, such that

$$C = \{x \in M_{\mathbb{R}} \mid \langle x, u_j \rangle \geq \gamma_j, j = 1, \ldots, l\}.$$

Let Σ_C denote the normal fan of C. We say that C is *compatible with* Σ whenever Σ refines Σ_C.

To a toric $\mathbb{R}$-divisor D on X we associate the subset of $M_{\mathbb{R}}$ defined as

$$\Delta_D = \mathrm{stab}(\Psi_D),$$

the stability set of Ψ_D (Definition A.1). This set is either empty or a quasi-rational polytope compatible with Σ. It encodes a lot of information about the geometry of the pair (X, D). For instance, each element $m \in \Delta_D \cap M$ gives a toric section $s_m = (\chi^m, D) \in \Gamma(X, D)^{\times}$. Analogously, each $m \in \Delta_D$ defines a toric $\mathbb{R}$-section $s_m \in \Gamma(X, D)_{\mathbb{R}}^{\times}$. The set $\{s_m\}_{m \in \Delta_D \cap M}$ is a basis of $\mathrm{L}(D)$. The proofs of these statements are the same as those for Cartier divisors [Ful93, §3].

Proposition 4.6. *Let D be a toric $\mathbb{R}$-divisor on X. Then*

$$\mathrm{vol}(X, D) = n! \, \mathrm{vol}_M(\Delta_D),$$

where vol_M is the Haar measure of $M_{\mathbb{R}}$ normalized so that M has covolume 1.

Proof. We have that

$$\mathrm{vol}(X, D) = \lim_{\ell \to \infty} \frac{l(\ell D)}{\ell^n / n!} = n! \lim_{\ell \to \infty} \frac{\#(\ell \Delta_D \cap M)}{\ell^n} = n! \, \mathrm{vol}_M(\Delta_M).$$

$\square$

Proposition 4.7. *Let D be a toric $\mathbb{R}$-divisor on X. Then*
  (1) *D is ample if and only if Ψ_D is strictly concave.*
  (2) *The following conditions are equivalent:*
      (a) *D is nef;*
      (b) *Ψ_D is concave;*
      (c) *there exists a finite number of nef divisors D_i on X and $\alpha_i > 0$ such that $D = \sum_i \alpha_i D_i$.*
  (3) *If D is nef, then $\deg_D(X) = n! \, \mathrm{vol}_M(\Delta_D)$.*

Proof. (1) A toric divisor is ample if and only if its corresponding function is strictly concave. Hence, a toric $\mathbb{R}$-divisor D is ample if and only if Ψ_D is a linear combination, with positive real coefficients, of strictly concave support functions. For each cone $\sigma \in \Sigma^n$, choose $u_{\sigma} \in \mathrm{ri}(\sigma) \cap N$, where $\mathrm{ri}(\sigma)$ denotes the relative interior of σ. Using the identification in (4.4), we see that

$$\mathrm{SF}(\Sigma)_{\mathbb{R}} = \{(m_{\sigma})_{\sigma} \mid \langle m_{\sigma} - m_{\tau}, u_{\sigma} \rangle \leq 0 \text{ for all } \sigma, \tau \in \Sigma^n\}$$

is a convex rational polyhedral cone in $\mathrm{VSF}(\Sigma)_{\mathbb{R}}$. Its interior

$$\mathrm{SF}(\Sigma)_{\mathbb{R}}^{\circ} = \{(m_{\sigma})_{\sigma} \mid \langle m_{\sigma} - m_{\tau}, u_{\sigma} \rangle < 0 \text{ for all } \sigma, \tau \in \Sigma^n \text{ such that } \sigma \neq \tau\}$$

can be identified with the subset of $\mathbb{R}$-support functions that are strictly concave on Σ. Hence, any strictly concave $\mathbb{R}$-support function can be written as a linear combination with positive real coefficients of strictly concave support functions, which proves the statement.

(2) The equivalence of (2a) and (2b) can be proved as in the case of divisors, see for instance [CLS11, Theorem 6.1.12]. The equivalence between (2b) and (2c)

follows from the facts that a toric divisor is nef if and only if its corresponding function lies in $\mathrm{SF}(\Sigma)$ and that $\mathrm{SF}(\Sigma)_{\mathbb{R}}$ is a convex rational polyhedral cone.

(3) This formula is well-known for toric divisors. Using (2), the general case follows from this one together with the multilinearity of the mixed degree and of the mixed volume. $\qquad\square$

The set of quasi-rational polytopes of $M_{\mathbb{R}}$ which are compatible with Σ forms a convex cone with respect to the multiplication by scalars in $\mathbb{R}_{\geq 0}$ and the Minkowski sum of sets.

Proposition 4.8. *The correspondence $D \mapsto \Delta_D$ gives an isomorphism between the convex cone of nef toric $\mathbb{R}$-divisors on X and the convex cone of quasi-rational polytopes compatible with Σ.*

Proof. Let $\alpha \in \mathbb{R}_{\geq 0}$ and Ψ, Φ two conic concave functions on $N_{\mathbb{R}}$ (Appendix A). Then
$$\mathrm{stab}(\alpha\Psi) = \alpha\,\mathrm{stab}(\Psi), \quad \mathrm{stab}(\Psi + \Phi) = \mathrm{stab}(\Psi) + \mathrm{stab}(\Phi).$$

Since Ψ is recovered from $\mathrm{stab}(\Psi)$ as the Legendre-Fenchel dual of the indicator function of this convex set (see [BPS14, Example 2.2.1]), the map $\Psi \to \mathrm{stab}(\Psi)$ is an isomorphism between the convex cone of conic concave functions on $N_{\mathbb{R}}$ and that of convex bodies. This isomorphism sends $\mathrm{SF}(\Sigma)_{\mathbb{R}}$ onto the convex cone of quasi-rational polytopes compatible with Σ. The statement follows easily from this since, by propositions 4.3 and 4.7(2), the convex cone of nef toric $\mathbb{R}$-divisors is in one-to-one correspondence with $\mathrm{SF}(\Sigma)_{\mathbb{R}}$. $\qquad\square$

Proposition 4.9. *Let D be a toric $\mathbb{R}$-divisor on X.*
  (1) *The following conditions are equivalent:*
      (a) *D is big;*
      (b) *there exists $m \in M_{\mathbb{R}}$ such that $\Psi_D(u) < \langle m, u \rangle$ for all $u \in N_{\mathbb{R}} \setminus \{0\}$;*
      (c) *$\dim(\Delta_D) = n$.*
  (2) *The following conditions are equivalent:*
      (a) *D is pseudo-effective;*
      (b) *there exists $m \in M_{\mathbb{R}}$ such that $\Psi_D(u) \leq \langle m, u \rangle$ for all $u \in N_{\mathbb{R}}$;*
      (c) *$\Delta_D \neq \emptyset$.*
  (3) *D is effective if and only if $\Psi_D \leq 0$ or, equivalently, if and only if $0 \in \Delta_D$.*
  (4) *Let P be a nef toric $\mathbb{R}$-divisor on X. Then $D \geq P$ if and only if $\Delta_D \supset \Delta_P$.*

Proof. (1) Clearly, (1b) and (1c) are equivalent. In case D is a divisor, Proposition 4.6, implies that (1a) and (1c) are equivalent.

Assume D is a big $\mathbb{R}$-divisor and write $D = \sum_{\alpha_i} \alpha_i D_i$ with $\alpha_i > 0$ and D_i big. Then there exist $m_i \in M_{\mathbb{R}}$ such that $\Psi_{D_i}(u) < \langle m_i, u \rangle$ for all $u \in N_{\mathbb{R}} \setminus \{0\}$. Setting $m = \sum_i \alpha_i m_i$, we have that $\Psi_D(u) < \langle m, u \rangle$ for all $u \in N_{\mathbb{R}} \setminus \{0\}$, which proves (1b).
Conversely, the set
$$C = \{\Psi \in \mathrm{VSF}(\Sigma)_{\mathbb{R}} \mid \exists m \in M_{\mathbb{R}} \text{ such that } \Psi(u) < \langle m, u \rangle \text{ for all } u \in N_{\mathbb{R}} \setminus \{0\}\}$$

is an open convex cone. Let D be an $\mathbb{R}$-divisor such that $\Psi_D \in C$. Since $\mathrm{VSF}(\Sigma)$ is dense in $\mathrm{VSF}(\Sigma)_{\mathbb{R}}$, there exist a finite number of functions $\Psi_i \in C \cap \mathrm{VSF}(\Sigma)$ and positive real numbers α_i such that $\Psi_D = \sum_i \alpha_i \Psi_i$. For each i, let D_i be the divisor corresponding to Ψ_i. Then $D = \sum_{\alpha_i} \alpha_i D_i$. Hence, D is big since each D_i is big. This proves the statement.

(2) Let $\varphi \colon X' \to X$ be a birational toric map and B a toric effective big $\mathbb{R}$-divisor on X'.

Suppose first that $\Delta_{\varphi^*D} = \Delta_D \neq \emptyset$ and let $\ell > 0$. By Lemma A.2(1),
$$\Delta_{\ell\varphi^*D+B} \supset \Delta_{\ell\varphi^*D} + \Delta_B.$$

By (1), the polytope Δ_B has dimension n and so does $\Delta_{\ell\varphi^* D+B}$. Hence, $\ell\varphi^* D + B$ is big for all integers $\ell \geq 0$ and so D is pseudo-effective.

Conversely, suppose that D is pseudo-effective. By an argument similar to the one in the proof of Proposition 3.23, one can verify that the $\mathbb{R}$-divisor $\ell\varphi^* D + B$ is big for all $\ell \geq 1$. Hence, $\Delta_{\ell\varphi^* D+B}$ is of dimension n and, in particular, nonempty. By Lemma A.5(1),

$$\Delta_D = \Delta_{\varphi^* D} = \bigcap_{\ell > 0} \Delta_{\varphi^* D + \frac{1}{\ell} B}.$$

Hence, this polytope is nonempty, since it is the intersection of nested compact sets.

(3) Assume that D is effective and write $D = \sum_i \alpha_i D_i$ with $\alpha_i > 0$ and D_i an effective divisor. Then $0 \in \Delta_{D_i}$ for all i. By Lemma A.2(1),

$$\sum_i \alpha_i \Delta_{D_i} = \sum_i \alpha_i \operatorname{stab}(\Psi_{D_i}) \subset \operatorname{stab}(\Psi_D) = \Delta_D.$$

Hence, $0 \in \Delta_D$ or, equivalently, $\Psi_D \leq 0$.

Conversely, the set of functions $\Psi \in \operatorname{VSF}(\Sigma)_\mathbb{R}$ such that $\Psi \leq 0$ forms a rational convex cone. Hence, if $\Psi_D \leq 0$ then there is a finite number of functions $\Psi_i \in \operatorname{VSF}(\Sigma)$ such that $\Psi_i \leq 0$ and $\alpha_i > 0$ such that $\Psi_D = \sum_i \alpha_i \Psi_i$. Each Ψ_i corresponds to a toric divisor D_i and $D = \sum_i \alpha_i D_i$. Each D_i is effective and so is D.

(4) Suppose that $D \geq P$. Then $\Psi_{D-P} \leq 0$, which is equivalent to $\Psi_D \leq \Psi_P$. Hence, $\Delta_D \supset \Delta_P$. Conversely, suppose that $\Delta_D \supset \Delta_P$. Then $\Delta_D \neq \emptyset$, since P is nef and Δ_D contains Δ_P. Hence, the support function Ψ_{Δ_D} coincides with the concave envelope of Ψ_D. Then $\Psi_D \leq \Psi_{\Delta_D} \leq \Psi_{\Delta_P} = \Psi_P$, which proves $\Psi_{D-P} \leq 0$ and $D \geq P$. $\qquad\square$

In the toric case, the geometric analogues of Dirichlet's unit theorem (Question 3.24), Zariski decomposition (Question 3.28) and Fujita approximation theorem (Question 3.29) are easy to treat, and all three admit a positive answer. Note however that the notion of toric strong Zariski decomposition that appears in the proposition below is weaker than the strong Zariski decomposition.

Proposition 4.10. *Let D be a toric $\mathbb{R}$-divisor on X.*

(1) *D is pseudo-effective if and only if there exists $m \in \Delta_D$ such that*

$$D + \operatorname{div}(\chi^m) \geq 0.$$

(2) *Assume that D is pseudo-effective. Then there exist a birational toric map $\varphi \colon X' \to X$ of proper toric varieties and toric divisors $P, E \in \operatorname{Car}_\mathbb{T}(X')_\mathbb{R}$ such that P is nef, E is effective,*

$$\varphi^* D = P + E \quad and \quad \operatorname{vol}(X, P) = \operatorname{vol}(X, D).$$

Moreover, for any other birational toric map $\varphi_1 \colon X'_1 \to X$ of proper toric varieties and a decomposition $\varphi_1^* D = P_1 + E_1$ with $P_1, E_1 \in \operatorname{Car}_\mathbb{T}(X'_1)_\mathbb{R}$ such that P_1 is nef and E_1 is effective, there are proper birational toric maps $\nu \colon X'' \to X'$ and $\nu_1 \colon X'' \to X'_1$ of proper toric varieties satisfying $\nu^* P \geq \nu_1^* P_1$.

(3) *Assume that D is big and let $\varepsilon > 0$. Then there exist a birational toric map $\varphi \colon X' \to X$ of proper toric varieties and $A, E \in \operatorname{Car}_\mathbb{T}(X')_\mathbb{R}$ such that A is ample, E is effective,*

$$\varphi^* D = A + E \quad and \quad \operatorname{vol}(X, A) \geq \operatorname{vol}(X, D) - \varepsilon.$$

Proof. (1) By Proposition 4.9(2), D is pseudo-effective if and only if $\Psi_D - m \leq 0$ for some $m \in M_\mathbb{R}$. But $\Psi_D - m = \Psi_{D + \mathrm{div}(\chi^m)}$ and, by Proposition 4.9(3), the previous condition is equivalent to the fact that $D + \mathrm{div}(\chi^m) \geq 0$.

(2) By Proposition 4.9(2), $\Delta_D \neq \emptyset$. Let Σ' be a refinement of Σ compatible with Δ_D. Let $\varphi \colon X' \to X$ be the corresponding birational toric map of proper toric varieties. Let P be the nef toric $\mathbb{R}$-divisor on X' associated to Δ_D under the correspondence in Proposition 4.8. Set $E = \varphi^* D - P$. By Proposition 4.9(4), E is effective and, by Proposition 4.6, $\mathrm{vol}(X', P) = \mathrm{vol}(X, D)$. Furthermore, let X_1', P_1 and E_1 as in the statement. Let Σ_1 be the fan that determines X_1'. Let Σ'' be a common refinement of Σ, Σ' and Σ_1' and let $\nu \colon X'' \to X'$ and $\nu_1 \colon X'' \to X_1'$ be the associated proper birational toric maps. By Proposition 4.9(4), $\Delta_{P_1} \subset \Delta_D = \Delta_P$ and, by the same result, $\nu^* P \geq \nu_1^* P_1$.

(3) Let Σ' be a regular refinement of Σ. The toric variety $X' := X_{\Sigma'}$ is projective and there is a birational toric map $\varphi \colon X' \to X$. Let D' be an ample toric $\mathbb{R}$-divisor on X'. Let P be the nef $\mathbb{R}$-divisor on X given by (2), for which we have $\Delta_P = \Delta_D$. Let $0 \leq \gamma < 1$ and $\delta > 0$ such that

$$(4.11) \qquad \gamma \Delta_D + \delta \Delta_{D'} \subset \Delta_D, \quad \mathrm{vol}_M(\gamma \Delta_D + \delta \Delta_{D'}) \geq \mathrm{vol}_M(\Delta_D) - \frac{\varepsilon}{n!}.$$

Set $A = \gamma \varphi^* P + \delta D'$ and $E = \varphi^* D - A$. By Proposition 4.8, $\Delta_A = \gamma \Delta_{\varphi^* P} + \delta \Delta_{D'} \subset \Delta_D = \Delta_{\varphi^* D}$. Proposition 4.9(4) then implies that E is effective. The virtual support function corresponding to A is $\gamma \Psi_P + \delta \Psi_{D'}$, which is strictly concave on Σ'. Hence, A is ample. Finally, Proposition 4.6 together with (4.11) show $\mathrm{vol}(X', A) \geq \mathrm{vol}(X, D) - \varepsilon$. $\qquad\square$

Let $\mathbb{K}$ be a global field and X a proper toric variety over $\mathbb{K}$ of dimension n. For each place $v \in \mathfrak{M}_\mathbb{K}$, we associate to the algebraic torus $\mathbb{T}$ an analytic space $\mathbb{T}_v^{\mathrm{an}}$. We denote by $\mathbb{S}_v^{\mathrm{an}}$ its compact torus. In the Archimedean case, it is isomorphic to $(S^1)^n$. In the non-Archimedean case, it is a compact analytic group, see [BPS14, §4.2] for a description. We denote by $\mathrm{val}_v \colon X_{0,v}^{\mathrm{an}} \to N_\mathbb{R}$ the valuation map associated to the place v as in [BPS14, (4.1.2)].

Definition 4.12. A v-adically metrized $\mathbb{R}$-divisor $\overline{D} = (D, \| \cdot \|)$ on X is *toric* if D is a toric $\mathbb{R}$-divisor and its Green function is invariant with respect to the action of $\mathbb{S}_v^{\mathrm{an}}$. To a toric v-adically metrized $\mathbb{R}$-divisor $\overline{D}$ as above, we associate the function $\psi_{\overline{D}} \colon N_\mathbb{R} \to \mathbb{R}$ defined, for $u \in N_\mathbb{R}$, by

$$\psi_{\overline{D}}(u) = -g_{\overline{D}}(p)$$

for any $p \in X_{0,v}^{\mathrm{an}}$ such that $\mathrm{val}_v(p) = u$. We will alternatively denote this function by $\psi_{D, \| \cdot \|}$.

A quasi-algebraic metrized $\mathbb{R}$-divisor $\overline{D}$ on X is *toric* if $(D, \| \cdot \|_v)$ is a toric v-adically metrized $\mathbb{R}$-divisor for all $v \in \mathfrak{M}_\mathbb{K}$. For each place v, we denote by $\psi_{\overline{D},v}$ the function associated to the toric v-adically metrized $\mathbb{R}$-divisor $(D, \| \cdot \|_v)$. A toric quasi-algebraic metrized $\mathbb{R}$-divisor is also called a *toric metrized $\mathbb{R}$-divisor*, for short.

We denote by $\widehat{\mathrm{Car}}_\mathbb{T}(X)_{\mathbb{R},v}$ and $\widehat{\mathrm{Car}}_\mathbb{T}(X)_\mathbb{R}$ the spaces of toric v-adically metrized $\mathbb{R}$-divisors on X and of toric metrized $\mathbb{R}$-divisors on X.

Let $D_i \in \widehat{\mathrm{Car}}_\mathbb{T}(X)_\mathbb{R}$ and $\alpha_i \in \mathbb{R}$, $i = 1, 2$, and $v \in \mathfrak{M}_\mathbb{K}$. It is immediate from the definitions that

$$(4.13) \qquad\qquad \psi_{\alpha_1 \overline{D}_1 + \alpha_2 \overline{D}_2, v} = \alpha_1 \psi_{\overline{D}_1, v} + \alpha_2 \psi_{\overline{D}_2, v}.$$

Remark 4.14. When $\overline{D}$ is the metrized $\mathbb{R}$-divisor associated to the toric line bundle with section $(\overline{L}, s)$, the function $\psi_{\overline{D},v}$ corresponds to the function $\psi_{\overline{L}, s, v}$ in the notation of [BPS14, Definition 4.3.5].

Let D be a toric divisor on X and $v \in \mathfrak{M}_{\mathbb{K}}$. Recall that D has a canonical v-adic metric, denoted $\|\cdot\|_{D,v,\mathrm{can}}$ [BPS14, Proposition-Definition 4.3.15]. The function associated to this metric agrees with the virtual support function Ψ_D. We extend this construction to toric $\mathbb{R}$-divisors.

Definition 4.15. Let D be a toric $\mathbb{R}$-divisor on X and $v \in \mathfrak{M}_{\mathbb{K}}$. Write $D = \sum_i \alpha_i D_i$ with $\alpha_i \in \mathbb{R}$ and D_i a toric divisor. For each i, let $\|\cdot\|_{D_i,v,\mathrm{can}}$ be the canonical v-adic metric on D_i and write $\overline{D}_{i,v,\mathrm{can}} = (D_i, \|\cdot\|_{D_i,v,\mathrm{can}})$. We define

$$\overline{D}_{v,\mathrm{can}} = \sum_i \alpha_i \overline{D}_{i,v,\mathrm{can}},$$

and we denote by $\|\cdot\|_{D,v,\mathrm{can}}$ the corresponding *v-adic canonical metric* on D. This is a toric v-adic metric on D and $\psi_{\overline{D}_{v,\mathrm{can}}} = \Psi_D$. In particular, it is independent of the chosen decomposition of D.

The following result extends [BPS14, Proposition 4.3.10(2) and Proposition 4.9.2(1)] to toric $\mathbb{R}$-divisors. As explained in [BPS14, §4.1], the variety with corners N_Σ is a compactification of the vector space $N_{\mathbb{R}}$ and, for each $v \in \mathfrak{M}_{\mathbb{K}}$, there is a proper map of topological spaces $\mathrm{val}_v \colon X_v^{\mathrm{an}} \to N_\Sigma$.

Proposition 4.16. *Let D be a toric $\mathbb{R}$-divisor on X.*

(1) *Let $v \in \mathfrak{M}_{\mathbb{K}}$. The correspondence $\|\cdot\| \mapsto \psi_{D,\|\cdot\|}$ is a bijection between the set of toric v-adic metrics on D and the set of functions $\psi \colon N_{\mathbb{R}} \to \mathbb{R}$ such that $\psi - \Psi_D$ extends to a continuous function on N_Σ.*

(2) *The correspondence $\{\|\cdot\|_v\}_{v \in \mathfrak{M}_{\mathbb{K}}} \mapsto \{\psi_{D,\|\cdot\|_v}\}_{v \in \mathfrak{M}_{\mathbb{K}}}$ is a bijection between the set of quasi-algebraic toric metrics on D and the set of families of functions $\psi_v \colon N_{\mathbb{R}} \to \mathbb{R}$ such that $\psi_v - \Psi_D$ extends to a continuous function on N_Σ for all v, and $\psi_v = \Psi_D$ for all but a finite number of v.*

Proof. For the local case, Corollary 3.10 together with the properties of the canonical v-adic metric of D implies that the map

$$\|\cdot\| \longmapsto g_{D,\|\cdot\|} - g_{D,v,\mathrm{can}}$$

gives a bijection between the space of toric v-adic metrics on D and that of continuous $\mathbb{S}_v^{\mathrm{an}}$-invariant functions on X_v^{an}. We have that $g_{D,\|\cdot\|} - g_{D,v,\mathrm{can}} = -(\psi_{D,\|\cdot\|} - \Psi_D) \circ \mathrm{val}_v$. Since the map $\mathrm{val}_v \colon X_v^{\mathrm{an}} \to N_\Sigma$ is proper, the $\mathbb{S}_v^{\mathrm{an}}$-invariant functions on X_v^{an} are in one-to-one correspondence with continuous functions on N_Σ.

The global case follows from this and [BPS14, Proposition 4.9.2(1)]. $\square$

Let $\overline{D}$ be a toric metrized $\mathbb{R}$-divisor on X and $v \in \mathfrak{M}_{\mathbb{K}}$. By Proposition 4.16(1), the function $\psi_{\overline{D},v}$ is asymptotically conic (Definition A.3) and its stability set is Δ_D, since it agrees with that of Ψ_D.

Definition 4.17. Let $\overline{D}$ be a toric metrized $\mathbb{R}$-divisor on X. For each $v \in \mathfrak{M}_{\mathbb{K}}$, the *$v$-adic roof function* of $\overline{D}$ is the concave function on Δ_D defined as

$$\vartheta_{\overline{D},v} = \psi_{\overline{D},v}^\vee,$$

see Definition A.1. The *(global) roof function* of $\overline{D}$ is defined as

$$\vartheta_{\overline{D}} = \sum_{v \in \mathfrak{M}_{\mathbb{K}}} n_v \vartheta_{\overline{D},v}.$$

Remark 4.18. In [BPS14, Definition 5.1.4], the v-adic roof function is defined only for toric semipositive metrized line bundles with a toric section. Definition 4.17 extends this definition to arbitrary toric metrized $\mathbb{R}$-divisors.

The following extends [BPS14, Theorem 4.8.1 and Proposition 4.9.2(2)] to toric $\mathbb{R}$-divisors.

Proposition 4.19. *Let D be a toric $\mathbb{R}$-divisor on X.*

(1) *Let $v \in \mathfrak{M}_{\mathbb{K}}$. The maps $\| \cdot \| \mapsto \psi_{D,\|\cdot\|}$ and $\| \cdot \| \mapsto \vartheta_{D,\|\cdot\|}$ are bijections between the set of toric semipositive v-adic metrics on D and, on one hand, the set of concave functions $\psi \colon N_{\mathbb{R}} \to \mathbb{R}$ such that $|\psi - \Psi_D|$ is bounded and, on the other hand, the set of continuous concave functions on Δ_D.*

(2) *The maps $\{\| \cdot \|_v\}_v \mapsto \{\psi_{D,\|\cdot\|_v}\}_v$ and $\{\| \cdot \|_v\}_v \mapsto \{\vartheta_{D,\|\cdot\|_v}\}_v$ are bijections between the set of toric semipositive metrics on D and, on one hand, the set of families of concave functions $\{\psi_v \colon N_{\mathbb{R}} \to \mathbb{R}\}_v$ such that $|\psi_v - \Psi_D|$ is bounded for all v and $\psi_v = \Psi_D$ for all but a finite number of v and, on the other hand, the set of families of continuous concave functions $\{\vartheta_v \colon \Delta_D \to \mathbb{R}\}_v$ such that $\vartheta_v \equiv 0$ for all but a finite number of v.*

Proof. By [BPS14, Propositions 2.5.20(2) and 2.5.23], the first and the second part of both statements are equivalent.

For the local case, given a toric semipositive v-adic metric $\| \cdot \|$ on D, choose $\varepsilon > 0$ and let $\| \cdot \|'$ be a (non necessarily toric) semipositive (smooth or algebraic) metric on D such that

$$\mathrm{dist}(\| \cdot \|, \| \cdot \|') < \varepsilon.$$

Set $\overline{D}' = (D, \| \cdot \|')$ and write $\overline{D}' = \sum_i \alpha_i \overline{D}'_i$ with $\alpha_i > 0$ and $\overline{D}'_i$ a semipositive (smooth or algebraic) metrized divisor on X.

For each i, let $\| \cdot \|'_{i,\mathrm{tor}}$ be the toric metric on D'_i obtained from the metric of $\overline{D}'_i$ by averaging of the metric along the fibres of the map $\mathrm{val}_v \colon X_{0,v}^{\mathrm{an}} \to N_\Sigma$ as in [BPS14, Definition 4.3.3]. Write $\overline{D}'_{i,\mathrm{tor}} = (D_i, \| \cdot \|'_{i,\mathrm{tor}})$. This is a toric semipositive (smooth or algebraic) metrized divisor and so

$$\overline{D}'_{\mathrm{tor}} := \sum_i \alpha_i \overline{D}'_{i,\mathrm{tor}}$$

is a toric semipositive (smooth or algebraic) metrized divisor. Since $\| \cdot \|$ is $\mathbb{S}_v^{\mathrm{an}}$-invariant, it results that $\mathrm{dist}(\| \cdot \|, \| \cdot \|'_{\mathrm{tor}}) < \varepsilon$. Hence,

$$\sup_{u \in N_\Sigma} \left| \psi_{\overline{D}}(u) - \psi_{\overline{D}'_{\mathrm{tor}}}(u) \right| < \varepsilon.$$

By propositions 4.4.1 and 4.7.1 in *loc. cit.*, the functions $\psi_{\overline{D}'_{i,\mathrm{tor}}}$ are concave, and so is $\psi_{\overline{D}'_{\mathrm{tor}}}$ since $\alpha_i > 0$ for all i. Letting $\varepsilon \to 0$, we conclude that $\psi_{\overline{D}}$ is concave.

Conversely, let $\psi \colon N_\Sigma \to \mathbb{R}$ be a concave function such that $\psi - \Psi_D$ is bounded. Denote by $\| \cdot \|$ the toric v-adic metric associated to ψ by Proposition 4.16(1) and let $\varepsilon > 0$. By [BPS14, Proposition 2.5.23(2)], there is a piecewise affine concave function $\zeta \colon \Delta_D \to \mathbb{R}$ such that

$$\sup_{x \in \Delta_D} \left| \psi^\vee(x) - \zeta(x) \right| < \varepsilon.$$

By the density of $\mathbb{Q}$, we can choose ζ defined by a finite number of affine maps with rational coefficients. Its Legendre-Fenchel dual $\phi = \zeta^\vee$ is a piecewise affine concave function on a rational polyhedral complex Π such that

$$\sup_{u \in N_\Sigma} \left| \psi(u) - \phi(u) \right| < \varepsilon.$$

Arguing as in the proof of [BPS14, Theorem 3.7.3], we can assume that the recession of Π agrees with Σ.

Let $\mathcal{P}(\Pi)$ denote the space of piecewise affine functions on Π and $\mathcal{C}(\Pi) \subset \mathcal{P}(\Pi)$ the subset of concave functions. The space $\mathcal{P}(\Pi)$ can be identified with the linear subspace of $\prod_{\Lambda \in \Pi^n} (M_{\mathbb{R}} \times \mathbb{R})$ given by

$$\left\{ (m_\Lambda, \gamma_\Lambda)_{\Lambda \in \Pi^n} \mid \langle m_\Lambda, u \rangle + \gamma_\Lambda = \langle m_{\Lambda'}, u \rangle + \gamma_{\Lambda'} \text{ for all } \Lambda, \Lambda' \in \Pi^n \text{ and } u \in \Lambda \cap \Lambda' \right\}.$$

This is a finite-dimensional linear subspace defined over $\mathbb{Q}$. For each $\Lambda \in \Pi^n$, choose a point $u_\Lambda \in \mathrm{ri}(\Lambda) \cap N_{\mathbb{Q}}$. Then

$$\mathcal{C}(\Pi) = \left\{ (m_\Lambda, \gamma_\Lambda)_\Lambda \in \mathcal{P}(\Pi) \mid \langle m_\Lambda, u_\Lambda \rangle + \gamma_\Lambda \leq \langle m_{\Lambda'}, u_\Lambda \rangle + \gamma_{\Lambda'} \text{ for all } \Lambda, \Lambda' \in \Pi^n \right\}.$$

This is a cone of $\mathcal{P}(\Pi)$ given by a rational H-representation as in [BPS14, (2.1.1)]. Hence, it admits a rational V-representation. This implies that there exist a finite number of H-lattice piecewise affine concave functions $\phi_i \in \mathcal{C}(\Pi)$ and positive real numbers α_i such that

$$\phi = \sum_i \alpha_i \phi_i.$$

For each i, set $\Phi_i = \mathrm{rec}(\phi_i)$, the recession function of ϕ_i (Definition A.3). This is a support function on Σ and so it corresponds to a toric divisor on X, that we denote by D_i. We observe that $\sum_i \alpha_i \Phi_i = \Psi_D$, hence $D = \sum_i \alpha_i D_i$ by Proposition 4.3.

In the non-Archimedean case, [BPS14, Corollary 4.5.9] implies that there exists a semipositive algebraic metric $\|\cdot\|_i$ on D_i such that $\psi_{D_i, \|\cdot\|_i} = \phi_i$. Set $\overline{D}_i = (D_i, \|\cdot\|_i)$ and $\overline{D}' = \sum_i \alpha_i \overline{D}_i$. This gives a semipositive algebraic v-adic metric $\|\cdot\|'$ on D and

$$\mathrm{dist}(\|\cdot\|, \|\cdot\|') = \sup_{u \in N_\Sigma} \left| \psi(u) - \phi(u) \right| < \varepsilon.$$

In the Archimedean case, Theorem 4.8.1 in *loc. cit.* implies that there exists a semipositive smooth metric $\|\cdot\|_i$ on D_i such that $|\phi_i - \psi_{D_i, \|\cdot\|_i}| < \varepsilon / \sum_i \alpha_i$. Let $\|\cdot\|'$ denote the induced semipositive smooth metric on D. Then

$$\mathrm{dist}(\|\cdot\|, \|\cdot\|') = \sup_{u \in N_\Sigma} \left| \psi(u) - \sum_i \alpha_i \psi_{D_i, \|\cdot\|_i}(u) \right| < 2\varepsilon.$$

In both cases, as ε tends to 0, this and Definition 3.12 show that the metric $\|\cdot\|$ is semipositive.

The global statement follows from the local one and Proposition 4.16(2). $\square$

Let Y be a d-dimensional cycle on X and $\overline{D}_i$, $i = 0, \ldots, d$, semipositive toric metrized $\mathbb{R}$-divisors on X. For a place $v \in \mathfrak{M}_{\mathbb{K}}$, the v-*adic toric height* of Y with respect to $\overline{D}_0, \ldots, \overline{D}_d$, denoted $\mathrm{h}^{\mathrm{tor}}_{v, \overline{D}_0, \ldots, \overline{D}_d}(Y)$, is defined from the v-adic toric height of a cycle ([BPS14, Definition 5.1.1]) with respect to semipositive toric line bundles by multilinearity and continuity.

The global height of Y can be computed as

$$\mathrm{h}_{\overline{D}_0, \ldots, \overline{D}_d}(Y) = \sum_{v \in \mathfrak{M}_{\mathbb{K}}} n_v \, \mathrm{h}^{\mathrm{tor}}_{v, \overline{D}_0, \ldots, \overline{D}_d}(Y).$$

This follows easily from the analogous statement for semipositive toric line bundles, see Proposition 5.2.4 in *loc. cit.*.

Recall that there is a one-to-one, dimension reversing, correspondence between the cones of Σ and the orbits of the action of $\mathbb{T}$ on X. Given a cone $\sigma \in \Sigma$, we denote by $O(\sigma)$ the corresponding orbit and by $V(\sigma)$ its closure. Recall also that, given a support function Ψ on Σ, we can associate to each cone $\sigma \in \Sigma$ a face, denoted F_σ, of the polytope $\mathrm{stab}(\Psi)$.

Proposition 4.20. *Let $\overline{D}$ be a toric semipositive metrized $\mathbb{R}$-divisor on X and $\sigma \in \Sigma$ a cone of codimension d. Then, for $v \in \mathfrak{M}_{\mathbb{K}}$,*

$$\mathrm{h}^{\mathrm{tor}}_{v,\overline{D}}(V(\sigma)) = (d+1)! \int_{F_\sigma} \vartheta_{\overline{D},v} \, \mathrm{dvol}_{M(F_\sigma)}$$

and

$$\mathrm{h}_{\overline{D}}(V(\sigma)) = (d+1)! \int_{F_\sigma} \vartheta_{\overline{D}} \, \mathrm{dvol}_{M(F_\sigma)},$$

where $M(F_\sigma)$ is the lattice induced by M on the affine space generated by F_σ and $\mathrm{vol}_{M(F_\sigma)}$ is the Haar measure on that affine space such that the covolume of $M(F_\sigma)$ is one.

In particular,

$$\mathrm{h}_{\overline{D}}(X) = (n+1)! \int_{\Delta_D} \vartheta_{\overline{D}} \, \mathrm{dvol}_M \, .$$

Proof. For the local case, let $v \in \mathfrak{M}_{\mathbb{K}}$. Consider first the case when the metric $\|\cdot\|_v$ is smooth semipositive, if v is Archimedean, or algebraic semipositive, if v is ultrametric. Write

$$\overline{D} = \sum_i \alpha_i \overline{D}_i$$

with $\overline{D}_i$ a $\mathbb{T}$-divisor on X with a semipositive (smooth or algebraic) metric, and $\alpha_i > 0$. For each $\overline{D}_i$, the formula follows from [BPS14, Proposition 5.1.11]. It follows for $\overline{D}$ by the multilinearity of the toric local height and of the mixed integral of concave functions with respect to the sup-convolution.

For a semipositive $\overline{D}$, the formula follows from the continuity of the toric local height with respect to dist, which follows from *loc. cit.*, Theorem 1.4.17(4), and from the continuity of the Legendre-Fenchel duality with respect to the uniform convergence (*loc. cit.*, Proposition 2.2.3).

The global case follows by adding up all local toric heights. $\qquad\square$

Remark 4.21. As in [BPS14, Corollary 5.1.9 and Theorem 5.2.5] one can express the mixed v-adic toric height and the mixed global height of toric metrized $\mathbb{R}$-divisors in terms of mixed integrals.

5. Small sections, arithmetic and χ-arithmetic volumes of toric metrized $\mathbb{R}$-divisors

In this section, we give formulae for the arithmetic and the χ-arithmetic volumes of a toric metrized $\mathbb{R}$-divisor in terms of its global roof function. We keep the notations of the previous section. In particular, X is a proper toric variety of dimension n over a global field $\mathbb{K}$.

We show first how the positivity of the roof function determines the existence of small toric sections.

Proposition 5.1. *Let $\overline{D}$ be a toric metrized $\mathbb{R}$-divisor on X. For $\ell \geq 1$ and $m \in \ell\Delta_D$ we set $s_m \in \Gamma(X, \ell D)^{\times}_{\mathbb{R}}$ for the corresponding $\mathbb{R}$-section.*

(1) Let $v \in \mathfrak{M}_{\mathbb{K}}$. Then $-\log \|s_m\|_{v,\sup} = \ell\vartheta_{\overline{D},v}(m/\ell)$.

(2) If $\vartheta_{\overline{D}}(m/\ell) > 0$, then there exists $e \geq 1$ and $\gamma \in \mathbb{K}^{\times}$ such that $\gamma s_m^e \in \widehat{\Gamma}(X, e\ell\overline{D})^{\times}_{\mathbb{R}}$.

Proof. We compute

$$\vartheta_{\overline{D},v}(m/\ell) = \psi_{\overline{D},v}^{\vee}(m/\ell)$$
$$= \inf_{u \in N_{\mathbb{R}}} \left(\langle m/\ell, u \rangle - \psi_{\overline{D},v}(u) \right)$$
$$= \frac{1}{\ell} \inf_{x \in X_0} \left(-\log |\chi^m(x)|_v + g_{\ell\overline{D},v}(x) \right)$$
$$= \frac{-1}{\ell} \sup_{x \in X_0} \log \|s_m(x)\|_v$$
$$= \frac{-1}{\ell} \log \|s_m\|_{v,\sup},$$

which proves the first statement. The second statement follows from the first, using Lemma 1.11 as in the proof of Proposition 3.16. $\square$

Next we show that, for every place, the basis of toric global sections is orthogonal with respect to the sup-norm.

Proposition 5.2. *Let $\overline{D}$ be a toric metrized $\mathbb{R}$-divisor. Then, for all $v \in \mathfrak{M}_{\mathbb{K}}$, the set $\{s_m\}_{m \in \Delta_D \cap M}$ is an orthogonal basis of $\mathrm{L}(D)$ with respect to the norm $\|\cdot\|_{v,\sup}$.*

Proof. Fix an integral basis of N so that there is an isomorphisms $N \simeq \mathbb{Z}^n \simeq M$. Let $v \in \mathfrak{M}_{\mathbb{K}}$ and choose an integer $r \geq 1$ such that $\Delta_D \subset [-r,r]^n$ and, if v is non-Archimedean, such that $|2r+1|_v = 1$. After possibly taking a finite extension of $\mathbb{K}$, we may assume that $\mathbb{K}_v$ contains a primitive root of 1 of order $2r+1$, which we denote by ω. For every $a = (a_1, \ldots, a_n) \in [-r,r]^n \cap N$ we consider the element $t_a = (\omega^{a_1}, \ldots, \omega^{a_n}) \in \mathbb{S}_v^{\mathrm{an}}$. Such an element determines an automorphism of X_v^{an}. Let $s = (f, D) \in \mathrm{L}(D)$ be a global section. Then $t_a^* s \in \mathrm{L}(D)$ since D is toric. For $p \in X_v^{\mathrm{an}}$,

$$\|t_a^* s\|_v(p) = |f(t_a * p)|_v \, \mathrm{e}^{-g_{\overline{D},v}(p)} = |f(t_a * p)|_v \, \mathrm{e}^{-g_{\overline{D},v}(t_a * p)} = \|s\|_v(t_a * p)$$

because of the $\mathbb{S}_v^{\mathrm{an}}$-invariance of $\|\cdot\|_v$. Hence,

$$(5.3) \qquad \|t_a^* s\|_{v,\sup} = \sup_{p \in X_v^{\mathrm{an}}} \|t_a^* s\|_v(p) = \sup_{p \in X_v^{\mathrm{an}}} \|s\|_v(t_a * p) = \|s\|_{v,\sup}.$$

Write

$$s = \sum_{m \in \Delta_D \cap M} \gamma_m s_m$$

with $\gamma_m \in \mathbb{K}$. For convenience, we also put $\gamma_m = 0$ for $m \in ([-r,r]^n \setminus \Delta_D) \cap M$. Then, for $a \in [-r,r]^n \cap N$,

$$t_a^* s = \sum_{m \in [-r,r]^n \cap M} \gamma_m t_a^* s_m = \sum_m \chi^m(t_a) \gamma_m s_m = \sum_m \omega^{\langle a,m \rangle} \gamma_m s_m.$$

Consider the matrix $\Omega = (\omega^{\langle a,m \rangle})_{a,m}$. Its inverse is given by

$$\Omega^{-1} = (2r+1)^{-n} \left(\omega^{-\langle a,m \rangle} \right)_{m,a}.$$

Therefore, for $m \in [-r,r]^n \cap M$,

$$\gamma_m s_m = \sum_{a \in [-r,r]^n \cap N} (2r+1)^{-n} \omega^{-\langle a,m \rangle} t_a^* s.$$

If v is Archimedean,

$$\|\gamma_m s_m\|_{v,\sup} \leq (2r+1)^n \max_a \|(2r+1)^{-n} \omega^{-\langle a,m \rangle} t_a^* s\|_{v,\sup} = \max_a \|t_a^* s\|_{v,\sup}.$$

while, if v is non-Archimedean,

$$\|\gamma_m s_m\|_{v,\sup} \leq \max \|(2r+1)^{-n} \omega^{-\langle a,m \rangle} t_a^* s\|_{v,\sup} = \max_a \|t_a^* s\|_{v,\sup}.$$

In both cases, we deduce from (5.3) that

$$\max_{m \in \Delta_D \cap M} \|\gamma_m s_m\|_{v,\sup} \leq \|s\|_{v,\sup}.$$

Hence, $\{s_m\}_{m \in \Delta_D \cap M}$ is an orthogonal basis of $L(D)$ with respect to $\|\cdot\|_{v,\sup}$ for all v. $\qquad\square$

Corollary 5.4. *Let $\overline{D}$ be a toric $\mathbb{R}$-divisor on X and $s \in L(D)$. Write*

$$s = \sum_{m \in \Delta_D \cap M} \gamma_m s_m$$

with $\gamma_m \in \mathbb{K}$. If v is Archimedean, then

$$\max_{m \in \Delta_D \cap M} \|\gamma_m s_m\|_{v,\sup} \leq \|s\|_{v,\sup} \leq \#(\Delta_D \cap M) \max_{m \in \Delta_D \cap M} \|\gamma_m s_m\|_{v,\sup},$$

while, if v is non-Archimedean,

$$\|s\|_{v,\sup} = \max_{m \in \Delta_D \cap M} \|\gamma_m s_m\|_{v,\sup}.$$

Proof. The upper bound for $\|s\|_{v,\sup}$ follows from the triangle inequality, in the Archimedean case, and from the ultrametric inequality, in the non-Archimedean case. The remaining inequality follows from Proposition 5.2. $\qquad\square$

Corollary 5.5. *Let $\overline{D}$ be a toric metrized $\mathbb{R}$-divisor and $s \in \widehat{L}(\overline{D})$. Write*

$$s = \sum_{m \in \Delta_D \cap M} \gamma_m s_m$$

with $\gamma_m \in \mathbb{K}$. Then $\gamma_m s_m \in \widehat{L}(\overline{D})$ for all $m \in \Delta_D \cap M$.

Proof. This follows immediately from Corollary 5.4. $\qquad\square$

Let $\overline{D}$ be a toric $\mathbb{R}$-divisor on X and consider the set

$$\Theta_{\overline{D}} = \{x \in \Delta_D \mid \vartheta_{\overline{D}}(x) \geq 0\}.$$

If $\Theta_{\overline{D}} \neq \emptyset$, it is a convex subset of Δ_D.

Theorem 5.6. *Let $\overline{D}$ be a toric metrized $\mathbb{R}$-divisor. Then the arithmetic volume of $\overline{D}$ is given by*

$$\widehat{\mathrm{vol}}(X, \overline{D}) = (n+1)! \int_{\Delta_D} \max(0, \vartheta_{\overline{D}}) \, d\mathrm{vol}_M = (n+1)! \int_{\Theta_{\overline{D}}} \vartheta_{\overline{D}} \, d\mathrm{vol}_M,$$

while the χ-arithmetic volume of $\overline{D}$ is given by

$$\widehat{\mathrm{vol}}_\chi(X, \overline{D}) = (n+1)! \int_{\Delta_D} \vartheta_{\overline{D}} \, d\mathrm{vol}_M.$$

Proof. We consider first the case when $\mathbb{K}$ is a number field. Let $\ell \geq 1$ and set $\Delta = \Delta_D$ for short. For each $m \in \ell\Delta \cap M$, we consider the $\mathfrak{M}_{\mathbb{K}}$-divisors $\mathfrak{c}_m$ and $\mathfrak{c}'_m$ given by, for $v|\infty$,

$$\log(\mathfrak{c}_{m,v}) = -\log\|s_m\|_{v,\sup}, \quad \log(\mathfrak{c}'_{m,v}) = -\log\|s_m\|_{v,\sup} - \log(\#(\ell\Delta \cap M))$$

and, for $v \nmid \infty$,

$$\log(\mathfrak{c}_{m,v}) = \log(\mathfrak{c}'_{m,v}) = \lambda_v \left\lfloor \frac{-\log\|s_m\|_{v,\sup}}{\lambda_v} \right\rfloor.$$

By Corollary 5.4,

$$\bigoplus_{m \in \ell\Delta \cap M} \widehat{L}(\mathfrak{c}'_m) s_m \subset \widehat{L}(\ell\overline{D}) \subset \bigoplus_{m \in \ell\Delta \cap M} \widehat{L}(\mathfrak{c}_m) s_m.$$

Observe that, by the quasi-algebricity of the metric, there exists a finite set $S \subset \mathfrak{M}_{\mathbb{K}}$ such that, for all $v \notin S$, we have that $\mathfrak{c}_{m,v} = \mathfrak{c}'_{m,v} = 1$ for all $\ell \geq 1$ and $m \in \ell\Delta \cap M$. With Proposition 5.1(1), this implies that

$$\widehat{\deg}(\mathfrak{c}_m) = d_{\mathbb{K}}\ell\vartheta_{\overline{D}}(m/\ell) + O(1), \quad \widehat{\deg}(\mathfrak{c}'_m) = d_{\mathbb{K}}\ell\vartheta_{\overline{D}}(m/\ell) + O(\log(\ell)).$$

Applying Lemma 1.10, we deduce that

$$\left|\widehat{l}(\ell\overline{D}) - d_{\mathbb{K}}\ell \sum_{m \in \ell\Delta \cap M} \max(0, \vartheta_{\overline{D}}(m/\ell))\right| = O(\ell^n \log(\ell)).$$

Therefore,

$$\begin{aligned}
\widehat{\mathrm{vol}}(X, \overline{D}) &= \frac{1}{d_{\mathbb{K}}} \limsup_{\ell \to \infty} \frac{\widehat{l}(\ell\overline{D})}{\ell^{n+1}/(n+1)!} \\
&= (n+1)! \limsup_{\ell \to \infty} \left(\frac{1}{\ell^n} \sum_{m \in \ell\Delta \cap M} \max(0, \vartheta_{\overline{D}}(m/\ell)) + O\left(\frac{\log(\ell)}{\ell}\right)\right) \\
&= (n+1)! \int_{\Delta} \max(0, \vartheta_{\overline{D}}) \, \mathrm{dvol}_M.
\end{aligned}$$

We next prove the formula for the χ-arithmetic volume. Using propositions 5.2, 2.24 and 5.1(1), we see that

$$\begin{aligned}
\frac{1}{d_{\mathbb{K}}} \widehat{\chi}(\mathrm{L}(\ell\overline{D})) &= \sum_{v} \sum_{m \in \ell\Delta \cap M} n_v \log(\|s_m\|_{v,\sup}^{-1}) + O(\#(\ell\Delta \cap M)\log(\#(\ell\Delta \cap M))) \\
&= \sum_{v} \sum_{m \in \ell\Delta \cap M} n_v \ell\vartheta_{\overline{D},v}(m/\ell) + O(\ell^n \log(\ell)) \\
&= \sum_{m \in \ell\Delta \cap M} \ell\vartheta_{\overline{D}}(m/\ell) + O(\ell^n \log(\ell)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\widehat{\mathrm{vol}}_{\chi}(X, \overline{D}) &= (n+1)! \limsup_{\ell \to \infty} \left(\frac{1}{\ell^n} \sum_{m \in \ell\Delta \cap M} \vartheta_{\overline{D}}(m/\ell) + O\left(\frac{\log(\ell)}{\ell}\right)\right) \\
&= (n+1)! \int_{\Delta} \vartheta_{\overline{D}} \, \mathrm{dvol}_M.
\end{aligned}$$

We consider next the function field case. Let $\pi \colon B \to C$ be a dominant morphism of smooth projective curves and consider the function field $\mathbb{K} = \mathrm{K}(B)$ with the adelic structure defined in Example 1.3. Let $\ell \geq 1$. For each $m \in \ell\Delta \cap M$, consider the $\mathfrak{M}_{\mathbb{K}}$-divisor given by

$$\log(\mathfrak{c}_{m,v}) = \lambda_v \left\lfloor \frac{-\log\|s_m\|_{v,\sup}}{\lambda_v} \right\rfloor$$

and let $D(\mathfrak{c}_m)$ be the associated Weil divisor of B. Observe that, for $\gamma \in \mathbb{K}$, we have that $\log|\gamma|_v \leq -\log\|s_m\|_{v,\sup}$ if and only if $\mathrm{ord}_v(\gamma) + \lfloor \frac{-\log\|s_m\|_{v,\sup}}{\lambda_v} \rfloor \geq 0$. Hence, this condition holds for all v if and only if $\mathrm{div}(\gamma) + D(\mathfrak{c}_m) \geq 0$, namely, if and only if $\gamma \in \mathrm{L}(D(\mathfrak{c}_m))$. Corollary 5.4 and equation (1.9) then show

$$\widehat{\mathrm{L}}(\ell\overline{D}) = \bigoplus_{m \in \ell\Delta \cap M} \mathrm{L}(D(\mathfrak{c}_m))s_m = \bigoplus_{m \in \ell\Delta \cap M} \widehat{\mathrm{L}}(\mathfrak{c}_m)s_m.$$

As in the number field case, there is a finite subset $S \subset \mathfrak{M}_{\mathbb{K}}$ independent of ℓ, that contains the support of $D(\mathfrak{c}_m)$ for all m. Therefore, Proposition 5.1(1) imply

$$\widehat{\deg}(\mathfrak{c}_m) = -d_{\mathbb{K}} \sum_{v} n_v \log\|s_m\|_{v,\sup} + O(1) = d_{\mathbb{K}}\ell\vartheta_{\overline{D}}(m/\ell) + O(1).$$

Applying Lemma 1.10, we deduce that

$$\left| \widehat{l}(\ell\overline{D}) - d_{\mathbb{K}}\ell \sum_{m \in \ell\Delta \cap M} \max(0, \vartheta_{\overline{D}}(m/\ell)) \right| = O(\ell^n).$$

Therefore, $\widehat{\text{vol}}(X, \overline{D}) = (n+1)! \int_\Delta \max(0, \vartheta_{\overline{D}}) \, d\text{vol}_M$, as stated. The formula for $\widehat{\text{vol}}_\chi(X, \overline{D})$ can be proved using the same approach as for the case of a number field. $\qquad\square$

Remark 5.7. Usually, the computation of the arithmetic volume is done by comparing the sup-norm with the L^2-norm with the help of Gromov's inequality, and then using the fact that the L^2-norm is a Hermitian norm, see for instance [GS88]. Here, we bypass the use of the L^2-norm with the fact that the basis of toric sections is orthogonal with respect to the sup-norm for all places, both Archimedean or ultrametric.

Corollary 5.8. *Let $\overline{D}$ be a toric metrized $\mathbb{R}$-divisor on X.*

(1) *$\widehat{\text{vol}}_\chi(X, \overline{D}) \leq \widehat{\text{vol}}(X, \overline{D})$, with equality when $\vartheta_{\overline{D}}(x) \geq 0$ for all $x \in \Delta_D$.*
(2) *If D is big, then $\widehat{\text{vol}}_\chi(X, \overline{D}) = \widehat{\text{vol}}(X, \overline{D})$ if and only if $\vartheta_{\overline{D}}(x) \geq 0$ for all $x \in \Delta_D$.*
(3) *If the metric is semipositive, then $\widehat{\text{vol}}_\chi(X, \overline{D}) = \text{h}_{\overline{D}}(X)$.*

Proof. This follows easily from Theorem 5.6, together with Proposition 4.20 for (3). $\qquad\square$

The next examples shows that the χ-arithmetic volume and the height may differ when $\overline{D}$ is not semipositive.

Example 5.9. Let $X = \mathbb{P}^1_{\mathbb{Q}}$ and $\overline{D} = \overline{0}$ the zero divisor equipped with a smooth toric metric at the Archimedean place and with the canonical metric at the non-Archimedean places. The associate functions are

$$\psi_{\overline{D},v} = \begin{cases} f & \text{if } v = \infty, \\ 0 & \text{if } v \neq \infty, \end{cases}$$

for a bounded smooth function $f \colon \mathbb{R} \to \mathbb{R}$.

We have that $\Delta_D = \{0\}$ and so $\widehat{\text{vol}}_\chi(X, \overline{D}) = 0$ thanks to Theorem 5.6. For the height, we first note that $\lim_{|u| \to \infty} f'(u) = 0$, which follows from the fact that the metric corresponding to the Archimedean place is smooth. Then, the Bézout formula gives

$$\text{h}_{\overline{D}}(X) = \text{h}_{\overline{D}}(\text{div}(s)) - \sum_v \int_{X_v^{\text{an}}} \log \|s\| \, c_1(\mathcal{O}(D), \|\cdot\|_v) = \int_\mathbb{R} ff'' \, du.$$

Integrating by parts, we obtain that

$$\int_\mathbb{R} ff'' \, du = \left[ff' \right]_{-\infty}^\infty - \int_\mathbb{R} (f')^2 \, du = -\int_\mathbb{R} (f')^2 \, du.$$

Hence,

$$\text{h}_{\overline{D}}(X) = -\int_\mathbb{R} (f')^2 \, du \leq 0,$$

with equality if and only f is constant. Since f is bounded on $\mathbb{R}$, this last condition is equivalent to the fact that f is concave or, equivalently, that $\overline{D}$ is semipositive. Therefore,

(5.10) $$\text{h}_{\overline{D}}(X) \leq \widehat{\text{vol}}_\chi(X, \overline{D}),$$

with equality if and only if $\overline{D}$ is semipositive.

Example 5.11. Let $X = \mathbb{P}^2_{\mathbb{Q}}$ and $D = 0$. Let $g \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the support function of the standard simplex of Δ^2. Let $\tau \in \mathbb{R}_{\geq 0}$ and $f_\tau \colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f_\tau(u_1, u_2) = g(u_1 - \tau, u_2 - \tau)$ for $(u_1, u_2) \in \mathbb{R}^2$.

Let v_1, v_2 be two different places of $\mathbb{Q}$ and $\tau_1, \tau_2 \in \mathbb{R}_{\geq 0}$. Consider the toric metric on D given, under the correspondence in Proposition 4.16(2), by

$$
\psi_v = \begin{cases} g - f_{\tau_1} & \text{if } v = v_1, \\ f_{\tau_2} - g & \text{if } v = v_2, \\ 0 & \text{otherwise.} \end{cases}
$$

The obtained metric is DSP, since each ψ_v is a difference of concave functions.

By Theorem 5.6, $\widehat{\mathrm{vol}}_\chi(X, \overline{D}) = 0$ since the polytope associated to D is a point. For the height, we have $\mathrm{h}_{\overline{D}}(X) = \mathrm{h}^{\mathrm{tor}}_{v_1, \overline{D}}(X) + \mathrm{h}^{\mathrm{tor}}_{v_2, \overline{D}}(X)$. By [BPS14, Remark 5.1.10], the quantity $\mathrm{h}^{\mathrm{tor}}_{v_1, \overline{D}}(X)$ can be computed as

$$(5.12) \quad \mathrm{MI}(g^\vee, g^\vee, g^\vee) - 3\,\mathrm{MI}(f^\vee_{\tau_1}, g^\vee, g^\vee) + 3\,\mathrm{MI}(f^\vee_{\tau_1}, f^\vee_{\tau_1}, g^\vee) - \mathrm{MI}(f^\vee_{\tau_1}, f^\vee_{\tau_1}, f^\vee_{\tau_1}),$$

where MI denotes the mixed integral of concave functions, see [BPS14, Definition 2.7.16]. We have that $f^\vee_{\tau_1}(x_1, x_2) = \tau_1(x_1 + x_2)$ and $g^\vee(x_1, x_2) = 0$ for $(x_1, x_2) \in \Delta^2$. Hence,

$$
\mathrm{MI}(g^\vee, g^\vee, g^\vee) = 3! \int_{\Delta^2} g^\vee = 0, \quad \mathrm{MI}(f^\vee_{\tau_1}, f^\vee_{\tau_1}, f^\vee_{\tau_1}) = 3! \int_{\Delta^2} f^\vee_{\tau_1} = 2\tau_1
$$

and, using the definition of the mixed integral and computing the relevant sup-convolutions, we can verify that

$$
\mathrm{MI}(f^\vee_{\tau_1}, g^\vee, g^\vee) = \tau_1, \quad \mathrm{MI}(f^\vee_{\tau_1}, f^\vee_{\tau_1}, g^\vee) = 2\tau_1.
$$

The formula (5.12) then implies that $\mathrm{h}^{\mathrm{tor}}_{v_1, \overline{D}}(X) = \tau_1$. Similarly, $\mathrm{h}^{\mathrm{tor}}_{v_2, \overline{D}}(X) = (-1)^3 \tau_2 = -\tau_2$. Hence,

$$
\mathrm{h}_{\overline{D}}(X) = -\tau_1 + \tau_2.
$$

Varying τ_1 and τ_2, this height can be any real number.

Remark 5.13. It would be interesting to see if the inequality (5.10) holds for any toric DSP metrized $\mathbb{R}$-divisor on $\mathbb{P}^1_{\mathbb{Q}}$. Example 5.11 shows that in dimension 2 and in the absence of approachability, there is no relation between the height and the χ-arithmetic volume.

6. Positivity properties of toric metrized $\mathbb{R}$-divisors

In this section, we give criteria for the different positivity conditions for toric metrized $\mathbb{R}$-divisors.

Theorem 6.1. *Let $\overline{D}$ be a toric metrized $\mathbb{R}$-divisor on X and $\vartheta_{\overline{D}} \colon \Delta_D \to \mathbb{R}$ its global roof function.*

(1) *$\overline{D}$ is ample if and only if Ψ_D is strictly concave on Σ, the function $\psi_{\overline{D}, v}$ is concave for all $v \in \mathfrak{M}_{\mathbb{K}}$, and $\vartheta_{\overline{D}}(x) > 0$ for all $x \in \Delta_D$;*

(2) *$\overline{D}$ is nef if and only if $\psi_{\overline{D}, v}$ is concave for all $v \in \mathfrak{M}_{\mathbb{K}}$ and $\vartheta_{\overline{D}}(x) \geq 0$ for all $x \in \Delta_D$;*

(3) *$\overline{D}$ is big if and only if $\dim(\Delta_D) = n$ and there exists $x \in \Delta_D$ such that $\vartheta_{\overline{D}}(x) > 0$;*

(4) *$\overline{D}$ is pseudo-effective if and only if there exists $x \in \Delta_D$ such that $\vartheta_{\overline{D}}(x) \geq 0$;*

(5) *$\overline{D}$ is effective if and only if $0 \in \Delta_D$ and $\vartheta_{\overline{D}, v}(0) \geq 0$ for all $v \in \mathfrak{M}_{\mathbb{K}}$.*

Proof. Write $\Psi = \Psi_D$ and $\Delta = \mathrm{stab}(\Psi)$ for short. We start by proving (1). By Proposition 4.7(1), D is (geometrically) ample if and only if Ψ is strictly concave on Σ and, by Proposition 4.19(2), the metric of $\overline{D}$ is semipositive if and only if $\psi_{\overline{D},v}$ is concave for all v. Thus it only remains to prove that, under the assumption that D is ample and $\overline{D}$ semipositive, the ampleness of $\overline{D}$ is equivalent to the positivity of $\vartheta_{\overline{D}}$.

Assume that $\overline{D}$ is ample. Let m_0 be a vertex of Δ and $p_0 \in X(\mathbb{K})$ its corresponding $\mathbb{T}$-invariant point. By propositions 3.20(2) and 4.20,

$$0 < \mathrm{h}_{\overline{D}}(p_0) = \vartheta_{\overline{D}}(m_0).$$

By the concavity of $\vartheta_{\overline{D}}$, we deduce that $\vartheta_{\overline{D}}(x) > 0$ for all $x \in \Delta$.

Conversely, assume that $\vartheta_{\overline{D}}$ is positive on Δ. Let $\overline{M}$ be a metrized $\mathbb{R}$-divisor on X. Since D is ample, there is a positive integer ℓ' such that $M + \ell' D$ is generated by $\mathbb{R}$-sections. Let $s_1, \ldots, s_r$ be a set of generating $\mathbb{R}$-sections. By the quasi-algebraicity of the metrics, $\|s_i\|_{v,\sup} = 1$ for all v outside a finite subset $S \subset \mathfrak{M}_{\mathbb{K}}$. Put

$$\eta = \min_i \min_{v \in S} \|s_i\|_{v,\sup}^{-1}.$$

Let $m_1, \ldots, m_l$ be the vertices of Δ and s_{m_j} the $\mathbb{R}$-section of D corresponding to m_j. By Proposition 5.1(1) and the hypothesis that $\vartheta_{\overline{D}}$ is positive,

$$\prod_v \|s_{m_j}\|_{v,\sup}^{n_v} = \mathrm{e}^{-\vartheta_{\overline{D}}(m_j)} < 1.$$

Therefore, we can apply Proposition 3.16 to each s_{m_j} to obtain $\ell_0 \geq 1$ and $\alpha_j \in \mathbb{K}^\times$ such that, for each $\ell \geq \ell_0 + \ell'$ and $j = 1, \ldots, r$,

$$\|\alpha_j s_{m_j}^{\ell-\ell'}\|_{v,\sup} \begin{cases} \leq 1 & \text{if } v \notin S, \\ < \eta & \text{if } v \in S. \end{cases}$$

The set of global sections $\alpha_j s_i s_{m_j}^{\ell-\ell'}$, $i = 1, \ldots, r$, $j = 1, \ldots, l$, generates the $\mathbb{R}$-divisor $M + \ell D = (M + \ell' D) + (\ell - \ell')D$. Moreover, for $v \in S$,

$$\|\alpha_j s_i s_{m_j}^{\ell-\ell'}\|_{v,\sup} \leq \|s_i\|_{v,\sup} \|\alpha_j s_{m_j}^{(\ell-\ell')}\|_{v,\sup} < \eta^{-1}\eta = 1,$$

while $\|\alpha_j s_i s_{m_j}^{\ell-\ell'}\|_{v,\sup} \leq 1$, for $v \notin S$. Thus $\overline{M} + \ell\overline{D}$ is generated by small $\mathbb{R}$-sections and $\overline{D}$ is ample, finishing the proof of (1).

We next prove (2). By Proposition 4.7(2), the $\mathbb{R}$-divisor D is nef if and only if Ψ is concave and, by Proposition 4.19(2), $\overline{D}$ is semipositive if and only if the functions $\psi_{\overline{D},v}$ are concave for all v. We are reduced to show that, under the assumption that D is nef and that $\overline{D}$ is semipositive, $\overline{D}$ being nef is equivalent to the nonnegativity of $\vartheta_{\overline{D}}$.

Assume that $\overline{D}$ is nef. As in the first part of the ample case, let m_0 be a vertex of Δ and $p_0 \in X(\mathbb{K})$ a $\mathbb{T}$-invariant closed point corresponding to m_0. By Proposition 4.20,

$$0 \leq \mathrm{h}_{\overline{D}}(p_0) = \vartheta_{\overline{D}}(m_0).$$

By the concavity of $\vartheta_{\overline{D}}$ we deduce $\vartheta_{\overline{D}}(x) \geq 0$ for all $x \in \Delta$.

Conversely, assume that $\vartheta_{\overline{D}}$ is nonnegative and let $p \in X(\mathbb{F})$ be a $\mathbb{F}$-rational point for a finite extension $\mathbb{F}$ of $\mathbb{K}$. Suppose that p lies in an orbit $O(\sigma)$ for a cone $\sigma \in \Sigma$. Choose a point m_σ in the face F_σ of Δ corresponding to σ. Then $\mathrm{div}(s_{m_\sigma}) = D + \mathrm{div}(\chi^{m_\sigma})$ meets the closure $V(\sigma)$ properly, and so we can restrict this $\mathbb{R}$-divisor to $V(\sigma)$. Set $\overline{D}' = \overline{D} + \widehat{\mathrm{div}}(\chi^{m_\sigma})$. Then

$$\mathrm{h}_{\overline{D}}(p) = \mathrm{h}_{\overline{D}'}(p) = \mathrm{h}_{\overline{D}'|_{V(\sigma)}}(p).$$

Furthermore, the roof function $\vartheta_{\overline{D}'|_{V(\sigma)}}$ coincides, up to translation by m_σ, with the restriction $\vartheta_{\overline{D}}\big|_{F_\sigma}$, by the analogue for $\mathbb{R}$-divisors of [BPS14, Proposition 4.8.8].

Thus, we are reduced to prove the nonnegativity of the height for points in the principal orbit. Let $m \in \Delta$ and s_m be the corresponding monomial $\mathbb{R}$-section. Since p lies in the principal orbit, it is not contained in the support of $\operatorname{div}(s_m)$. Then, using Proposition 5.1(1),

$$\mathrm{h}_{\overline{D}}(p) = \sum_{w \in \mathfrak{M}_\mathbb{F}} -n_w \log \|s_m(p)\|_w \geq \sum_{w \in \mathfrak{M}_\mathbb{F}} -n_w \log \|s_m\|_{w,\sup}$$

$$= \sum_{w \in \mathfrak{M}_\mathbb{F}} n_w \vartheta_{\overline{D},w}(m) = \sum_{v \in \mathfrak{M}_\mathbb{K}} n_v \vartheta_{\overline{D},v}(m) = \vartheta_{\overline{D}}(m) \geq 0,$$

which concludes the proof of (2).

The statement (3) is a direct consequence of Theorem 5.6 and the definition of big metrized $\mathbb{R}$-divisor.

We now prove (4). By the toric Chow lemma, there is a birational toric map $\varphi\colon X' \to X$ with X' projective. Let A be an effective ample toric divisor on X' and write $\Phi = \Psi_A$ and $\Gamma = \Delta_A = \operatorname{stab}(\Phi)$ for the corresponding support function and polytope. By Proposition 4.9(3), $\Phi \leq 0$.

Choose $v_0 \in \mathfrak{M}_\mathbb{K}$ and consider the toric metric on A given by the family of functions

$$\phi_v = \begin{cases} \Phi - 1 & \text{if } v = v_0, \\ \Phi & \text{otherwise.} \end{cases}$$

Denote by $\overline{A}$ the obtained toric metrized divisor. We have that $\phi_v^\vee \equiv 1$ if $v = v_0$ and $\phi_v^\vee \equiv 0$ otherwise. Hence, for any $x \in \Gamma$,

$$\vartheta_{\overline{A}}(x) = \sum_v n_v \phi_v^\vee(x) = n_{v_0} > 0.$$

By (1), $\overline{A}$ is ample.

Suppose that $\overline{D}$ is pseudo-effective. By Proposition 3.23, the metrized $\mathbb{R}$-divisor $\ell\varphi^*\overline{D} + \overline{A}$ is big for all $\ell \geq 1$. The virtual support function and the polytope corresponding to this divisor are $\ell\Psi + \Phi$ and $\operatorname{stab}(\ell\Psi + \Phi)$, respectively. Write $\psi_v = \psi_{\overline{D},v}$ for convenience. For $v \in \mathfrak{M}_\mathbb{K}$, the function associated to the v-adic metric on this $\mathbb{R}$-divisor is $\ell\psi_v + \phi_v$, and the v-adic roof function is $(\ell\psi_v + \phi_v)^\vee$.

Set $\Delta_\ell = \frac{1}{\ell}\operatorname{stab}(\ell\Psi + \Phi) = \operatorname{stab}\left(\psi_v + \frac{1}{\ell}\phi_v\right)$ and $\vartheta_{v,\ell} = (\psi_v + \frac{1}{\ell}\phi_v)^\vee$. Then, for $x \in \Delta_\ell$,

$$(\ell\psi_v + \phi_v)^\vee(\ell x) = \ell\vartheta_{v,\ell}(x).$$

Set $\vartheta_\ell = \sum_v n_v \vartheta_{v,\ell}$. Since $\ell\varphi^*\overline{D} + \overline{A}$ is toric and big, by (3), there exists a $x_\ell \in \Delta_\ell$ such that $\vartheta_\ell(x_\ell) > 0$.

By Proposition 4.16(1), the functions ψ_v and ϕ_v are asymptotically conic and, by construction, $0 \in \operatorname{stab}(\phi_v)$. Since by Lemma A.5(1) one has $\Delta_\ell \supset \Delta_{\ell+1}$ for $\ell \geq 1$, the sequence $(x_\ell)_\ell$ lies in the polytope Δ_1. By choosing a convergent subsequence, we can assume that it converges to a point $x_\infty \in \bigcap_\ell \Delta_\ell$. By Lemma A.5(1), $x_\infty \in \Delta$ and, by Lemma A.5(2), for $\ell' \geq \ell$,

$$\vartheta_{v,\ell}(x_{\ell'}) \geq \vartheta_{v,\ell'}(x_{\ell'}) + \left(\frac{1}{\ell} - \frac{1}{\ell'}\right)\phi_v^\vee(0) \geq \vartheta_{v,\ell'}(x_{\ell'}).$$

Hence, $\vartheta_\ell(x_{\ell'}) \geq \vartheta_{\ell'}(x_{\ell'}) \geq 0$. By continuity, $\vartheta_\ell(x_\infty) = \lim_{\ell' \to \infty} \vartheta_\ell(x_{\ell'}) \geq 0$. Applying again Lemma A.5(2),

$$\vartheta_{\overline{D}}(x_\infty) = \sum_v n_v \psi_v^\vee(x_\infty) = \sum_v n_v \lim_{\ell \to \infty} \vartheta_{v,\ell}(x_\infty) = \lim_{\ell \to \infty} \vartheta_\ell(x_\infty) \geq 0,$$

which proves the statement.

Conversely, suppose now that there exists $x \in \Delta$ with $\vartheta_{\overline{D}}(x) \geq 0$ maximal. In particular, $\Delta \neq \emptyset$. The virtual support function corresponding to the $\mathbb{R}$-divisor $\ell \varphi^* D + A$ is equal to $\ell \Psi + \Phi$. By Lemma A.2(1), $\mathrm{stab}(\ell \Psi + \Phi) \supset \ell \Delta + \Gamma$. In particular, this is a polytope of dimension n. Furthermore, for $v \in \mathfrak{M}_{\mathbb{K}}$, the corresponding function is $\ell \psi_v + \phi_v$. Applying Lemma A.2(2), we deduce that

$$(\ell \psi_v + \phi_v)^\vee (\ell x) \geq \left((\ell \psi_v)^\vee \boxplus \phi_v^\vee\right)(\ell x) = \ell \psi_v^\vee(x) + \phi_v^\vee(0).$$

Hence,

$$\vartheta_{\ell \varphi^* \overline{D} + \overline{A}}(\ell x) = \sum_v n_v (\ell \psi_v + \phi_v)^\vee (\ell x)$$

$$\geq \sum_v n_v (\ell \psi_v^\vee(x) + \phi_v^\vee(0)) \geq \ell \vartheta_{\overline{D}}(x) + n_{v_0} > 0.$$

By (3), this implies that $\ell \varphi^* \overline{D} + \overline{A}$ is big and therefore $\overline{D}$ is pseudo-effective.

We finally prove (5). The metrized $\mathbb{R}$-divisor $\overline{D}$ is effective if and only if D is effective and s_D is small which, by propositions 4.9(3) and 5.1(1), is equivalent to the fact that $0 \in \Delta_D$ and $\vartheta_{\overline{D},v}(0) \geq 0$, for all $v \in \mathfrak{M}_{\mathbb{K}}$. $\qquad\square$

Corollary 6.2. *Let $\overline{D}$ be a semipositive toric metrized $\mathbb{R}$-divisor on X with D big. Then $\mathrm{h}_{\overline{D}}(X) = \widehat{\mathrm{vol}}(X, \overline{D})$ if and only if $\overline{D}$ is nef.*

Proof. This follows easily from Proposition 4.20 and theorems 5.6 and 6.1(2). $\qquad\square$

We also obtain the following arithmetic analogue of the Nakai-Moishezon criterion for toric varieties.

Corollary 6.3. *Let $\overline{D}$ be a semipositive toric metrized $\mathbb{R}$-divisor on X.*

(1) *If D is nef, the following conditions are equivalent:*
   (a) *$\overline{D}$ is nef;*
   (b) *$\mathrm{h}_{\overline{D}}(p) \geq 0$ for every $\mathbb{T}$-invariant point p;*
   (c) *$\mathrm{h}_{\overline{D}}(Y) \geq 0$ for every subvariety Y of $X_{\overline{\mathbb{K}}}$.*
(2) *If D is ample, the following conditions are equivalent:*
   (a) *$\overline{D}$ is ample;*
   (b) *$\mathrm{h}_{\overline{D}}(p) > 0$ for every $\mathbb{T}$-invariant point p;*
   (c) *$\mathrm{h}_{\overline{D}}(Y) > 0$ for every subvariety Y of $X_{\overline{\mathbb{K}}}$.*

Proof. We first prove (1). The equivalence between (1a) and (1b) follows from the proof of Theorem 6.1(2).

It is clear that (1c) implies (1b). For the converse, let Y be a subvariety of X defined over a finite extension $\mathbb{F}$ of $\mathbb{K}$. Using the same argument as in the proof of Theorem 6.1(2), we reduce to points in the principal orbit and we assume that Y meets this principal orbit.

We prove that $\mathrm{h}_{\overline{D}}(Y) \geq 0$ by induction on the dimension of Y. The 0-dimensional case follows from the fact that $\overline{D}$ is nef. Assume now that Y has dimension $d \geq 1$. Let $m \in \Delta$ and s_m the corresponding toric $\mathbb{R}$-section of D. By Bézout formula (3.13),

$$\mathrm{h}_{\overline{D}}(Y) = \mathrm{h}_{\overline{D}}(Y \cdot \mathrm{div}(s_m)) - \sum_{w \in \mathfrak{M}_{\mathbb{F}}} n_w \int_{X_w^{\mathrm{an}}} \log \|s_m\|_w \, \mathrm{c}_1(D, \|\cdot\|_w)^{\wedge d} \wedge \delta_Y.$$

We have that $h_{\overline{D}}(Y \cdot \mathrm{div}\, s_m) \geq 0$ by the inductive hypothesis, and observe that $\|s_m\|_w \leq \|s_m\|_{w,\sup}$. The fact that the metric is semipositive implies that, for each $w \in \mathfrak{M}_{\mathbb{F}}$, the signed measure $\mathrm{c}_1(D, \|\cdot\|_w)^{\wedge d} \wedge \delta_Y$ is nonnegative and of total mass

$\deg_D(Y)$. Using Proposition 5.1(1),

$$\mathrm{h}_{\overline{D}}(Y) \geq - \sum_{w \in \mathfrak{M}_{\mathbb{F}}} n_w \int_{X_w^{\mathrm{an}}} \log \|s_m\|_{w,\sup} \, \mathrm{c}_1(D, \| \cdot \|_w)^{\wedge d} \wedge \delta_Y$$

$$= \sum_{w \in \mathfrak{M}_{\mathbb{F}}} n_w \vartheta_{\overline{D},w}(m) \deg_D(Y) = \vartheta_{\overline{D}}(m) \deg_D(Y).$$

Theorem 6.1(2) implies that $\vartheta_{\overline{D}}(m) \geq 0$, which concludes the proof of (1). The proof of (2) can be done similarly. $\qquad \square$

Proposition 6.4. *Let $\overline{D}, \overline{E}$ be two toric metrized $\mathbb{R}$-divisors on a toric variety X such that $\overline{E}$ is semipositive. The following conditions are equivalent:*

(1) $\overline{D} \geq \overline{E}$;
(2) $\psi_{\overline{D},v} \leq \psi_{\overline{E},v}$ *for all $v \in \mathfrak{M}_{\mathbb{K}}$;*
(3) $\Delta_E \subset \Delta_D$ *and $\vartheta_{\overline{E},v}(x) \leq \vartheta_{\overline{D},v}(x)$ for all $x \in \Delta_E$ and $v \in \mathfrak{M}_{\mathbb{K}}$.*

Proof. (1) $\Leftrightarrow$ (2) We have that $\overline{D} \geq \overline{E}$ if and only if the $\mathbb{R}$-section $s_{\overline{D}-\overline{E}}$ is small. This is equivalent to the fact that, for $v \in \mathfrak{M}_{\mathbb{K}}$, $\psi_{\overline{D}-\overline{E},v}(u) \leq 0$ for all $u \in N_{\mathbb{R}}$, which in turn is equivalent to $\psi_{\overline{E},v}(u) \geq \psi_{\overline{D},v}(u)$ for all u.

(2) $\Rightarrow$ (3) By considering the corresponding recession functions, we deduce that $\Psi_E(u) \geq \Psi_D(u)$ for all u, which implies that

$$\Delta_E = \{x \in M_{\mathbb{R}} \mid \langle x, u \rangle \geq \Psi_E(u) \text{ for all } u \in N_{\mathbb{R}}\}$$
$$\subset \{x \in M_{\mathbb{R}} \mid \langle x, u \rangle \geq \Psi_D(u) \text{ for all } u \in N_{\mathbb{R}}\} = \Delta_D.$$

Similarly, for $x \in \Delta_E$,

$$\vartheta_{\overline{E},v}(x) = \inf_{u \in N_{\mathbb{R}}} \left(\langle x, u \rangle - \psi_{\overline{E},v}(u) \right) \leq \inf_{u \in \mathbb{N}_{\mathbb{R}}} \left(\langle x, u \rangle - \psi_{\overline{D},v}(u) \right) = \vartheta_{\overline{D},v}(x).$$

(3) $\Rightarrow$ (2) Let $v \in \mathfrak{M}_{\mathbb{K}}$. Since $\overline{E}$ is semipositive, the function $\psi_{\overline{E},v}$ is concave and so $\psi_{\overline{E},v} = \vartheta_{\overline{E},v}^{\vee}$. The fact that $\vartheta_{\overline{E},v} \leq \vartheta_{\overline{D},v}$ on Δ_E implies that $\vartheta_{\overline{E},v}^{\vee} \geq \vartheta_{\overline{D},v}^{\vee}$ on $N_{\mathbb{R}}$. Moreover,

$$\vartheta_{\overline{D},v}^{\vee} = \mathrm{conc}(\psi_{\overline{D},v}) \geq \psi_{\overline{D},v},$$

where $\mathrm{conc}(\psi_{\overline{D},v})$ denotes the concave envelope of the function $\psi_{\overline{D},v}$ (Appendix A). Hence $\psi_{\overline{E},v} \geq \psi_{\overline{D},v}$, which concludes the proof. $\qquad \square$

Example 6.5. This example is due to Moriwaki [Mor11]. Let $X = \mathbb{P}^n_{\mathbb{Q}}$ with homogeneous coordinates $(z_0 : \cdots : z_n)$ and $D = \mathrm{div}(z_0)$. Then X is a toric variety and D is a toric divisor corresponding to the support function $\Psi \colon \mathbb{R}^n \to \mathbb{R}$ given by

$$\Psi(u_1, \ldots, u_n) = \min(0, u_1, \ldots, u_n).$$

Let $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_n)$ be a collection of positive real numbers. Consider the toric semipositive metric on D given, under the correspondence in Proposition 4.16(2), by the family of functions

$$\psi_v(u_1, \ldots, u_n) = \begin{cases} \frac{-1}{2} \log(\alpha_0 + \alpha_1 \, \mathrm{e}^{-2u_1} + \cdots + \alpha_n \, \mathrm{e}^{-2u_n}) & \text{if } v = \infty, \\ \Psi & \text{otherwise.} \end{cases}$$

For $v = \infty$, this metric agrees with the weighted Fubini-Study metric given by

$$\|s(z_0 : \cdots : z_n)\|_{\infty}^2 = \frac{z_0 \overline{z}_0}{\alpha_0 z_0 \overline{z}_0 + \cdots + \alpha_n z_n \overline{z}_n}$$

for the toric section s corresponding to the linear form z_0. For the non-Archimedean places, it agrees with the canonical metric and is induced by the canonical model $\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1)$. We denote $\overline{D}_{\boldsymbol{\alpha}}$ this toric metrized divisor.

Let $\{e_1, \ldots, e_n\}$ be the standard basis of $M_{\mathbb{R}} = (\mathbb{R}^n)^{\vee}$ and $(x_1, \ldots, x_n)$ the coordinates of a point $x \in M_{\mathbb{R}}$ with respect to this basis. Set also $e_0 = 0$ and $x_0 = 1 - \sum_i x_i$. The polytope $\Delta = \operatorname{stab}(\Psi)$ is the standard simplex $\operatorname{conv}(e_0, \ldots, e_n)$ of $\mathbb{R}^n$. Computing as in [BPS14, Example 2.4.3], one sees that the local roof functions are given, for $(x_1, \ldots, x_n) \in \Delta$, by

$$\vartheta_{\overline{D}_{\boldsymbol{\alpha}}, v}(x_1, \ldots, x_n) = \begin{cases} \frac{-1}{2} \sum_{i=0}^n x_i \log(x_i/\alpha_i) & \text{if } v = \infty, \\ 0 & \text{otherwise.} \end{cases}$$

This is an adelic family of continuous concave functions on Δ. By *loc. cit.*, Theorem 4.8.1(2), the metrized divisor $\overline{D}_{\boldsymbol{\alpha}}$ is semipositive. Its roof function is

$$\vartheta_{\overline{D}_{\boldsymbol{\alpha}}}(x_1, \ldots, x_n) = \frac{-1}{2} \sum_{i=0}^n x_i \log(x_i/\alpha_i).$$

The values of the roof function at the vertices of Δ are given by $\vartheta_{\overline{D}_{\boldsymbol{\alpha}}}(e_i) = (1/2) \log(\alpha_i)$, $i = 0, \ldots, n$. Moreover

$$\sup_{x \in \Delta} \vartheta_{\overline{D}_{\boldsymbol{\alpha}}}(x) = \frac{1}{2} \log(\alpha_0 + \cdots + \alpha_n).$$

Write $\Theta_{\boldsymbol{\alpha}} = \{x \in \Delta \mid \vartheta_{\overline{D}_{\boldsymbol{\alpha}}}(x) \geq 0\}$. From Theorem 6.1, we deduce part of the main result of [Mor11]:

(1) $\overline{D}_{\boldsymbol{\alpha}}$ is ample if and only if $\alpha_i > 1$, $i = 0, \ldots, n$;
(2) $\overline{D}_{\boldsymbol{\alpha}}$ is nef if and only if $\alpha_i \geq 1$, $i = 0, \ldots, n$;
(3) $\overline{D}_{\boldsymbol{\alpha}}$ is big if and only if $\sum_i \alpha_i > 1$;
(4) $\overline{D}_{\boldsymbol{\alpha}}$ is pseudo-effective if and only if $\sum_i \alpha_i \geq 1$;
(5) $\overline{D}_{\boldsymbol{\alpha}}$ is effective if and only if $\alpha_0 \geq 1$.

By [BPS14, Theorem 5.2.5] and Theorem 5.6,

$$\mathrm{h}_{\overline{D}_{\boldsymbol{\alpha}}}(X) = \widehat{\mathrm{vol}}_{\chi}(X, \overline{D}_{\boldsymbol{\alpha}}) = (n+1)! \int_{\Delta} \vartheta_{\overline{D}_{\boldsymbol{\alpha}}} \, \mathrm{dvol} = \frac{1}{2} \sum_{i=0}^n \left(\log(\alpha_i) + \sum_{j=1}^i \frac{1}{j} \right).$$

Apparently, there is no such a simple formula for the arithmetic volume

$$\widehat{\mathrm{vol}}(X, \overline{D}_{\boldsymbol{\alpha}}) = (n+1)! \int_{\Theta_{\boldsymbol{\alpha}}} \vartheta_{\overline{D}_{\boldsymbol{\alpha}}} \, \mathrm{dvol}.$$

7. Toric versions of Dirichlet's unit theorem, Zariski decomposition and Fujita approximation

In this section, we give a complete answer to questions 3.24 and 3.29 in the toric case and a partial answer to Question 3.28. Namely, we give a criterion for when a *toric* Zariski decomposition or a strong *toric* Zariski decomposition exists. We will show in §8 that, for big toric arithmetic $\mathbb{R}$-divisors and under some restrictions, the existence of a non-necessarily toric Zariski decomposition implies the existence of a toric one.

Definition 7.1. Let X be a proper toric variety over $\mathbb{K}$ and $\overline{D}$ a toric metrized $\mathbb{R}$-divisor on X. A *toric Zariski decomposition* of $\overline{D}$ is a Zariski decomposition $\varphi^* \overline{D} = \overline{P} + \overline{E}$ such that φ is a birational toric map of proper toric varieties and $\overline{P}$ (hence $\overline{E}$) is a toric metrized $\mathbb{R}$-divisor.

We denote by $\Upsilon_{\mathbb{T}}(\overline{D}) \subset \Upsilon(\overline{D})$ the subset formed by the elements $(\varphi, \overline{P})$, with φ a proper birational toric map of proper toric varieties and $\overline{P}$ a toric metrized $\mathbb{R}$-divisor. The equivalence relation $\sim$ in Definition 3.27 induces an equivalence relation on $\Upsilon_{\mathbb{T}}(\overline{D})$ and the order relation on $\Upsilon(\overline{D})/\sim$ induces an order relation on $\Upsilon_{\mathbb{T}}(\overline{D})/\sim$. A *toric strong Zariski decomposition* is the greatest element of $\Upsilon_{\mathbb{T}}(\overline{D})/\sim$, if it exists.

Theorem 7.2. *Let X be a proper toric variety over $\mathbb{K}$ and $\overline{D}$ a toric metrized $\mathbb{R}$-divisor on X.*

(1) *Assume that $\mathbb{K}$ is an A-field. Then, for every $a \in \Theta_{\overline{D}}$ there exists an $\alpha \in \mathbb{K}^{\times} \otimes \mathbb{R}$ such that*

$$(7.3) \qquad\qquad \overline{D} + \widehat{\mathrm{div}}(\alpha\chi^a) \geq 0.$$

Therefore, if $\overline{D}$ is pseudo-effective, then there exists $a \in \Delta_D$ and $\alpha \in \mathbb{K}^{\times} \otimes \mathbb{R}$ satisfying (7.3).

(2) *If $\overline{D}$ is pseudo-effective, then there exists a toric strong Zariski decomposition of $\overline{D}$ if and only if $\Theta_{\overline{D}}$ is a quasi-rational polytope.*

(3) *If $\overline{D}$ is big, the following statements are equivalent:*
   (a) *there exists a toric Zariski decomposition of $\overline{D}$;*
   (b) *there exists a strong toric Zariski decomposition of $\overline{D}$;*
   (c) *$\Theta_{\overline{D}}$ is a quasi-rational polytope.*

(4) *Assume that $\overline{D}$ is big. Then, for every $\varepsilon > 0$, there exists a birational toric map $\varphi \colon X' \to X$ of proper toric varieties and toric metrized $\mathbb{R}$-divisors $\overline{A}$, $\overline{E}$ on X' such that $\overline{A}$ is ample, $\overline{E}$ is effective,*

$$\varphi^*\overline{D} = \overline{A} + \overline{E} \quad and \quad \widehat{\mathrm{vol}}(X',\overline{A}) \geq \widehat{\mathrm{vol}}(X,\overline{D}) - \varepsilon.$$

Proof. (1) Let S be a finite subset of $\mathfrak{M}_{\mathbb{K}}$ containing the Archimedean places and those places such that $\vartheta_{\overline{D},v}(a) \neq 0$. Let $\gamma_v \in \mathbb{R}$, $v \in \mathfrak{M}_{\mathbb{K}}$, such that $\gamma_v \leq \vartheta_{\overline{D},v}(a)$ for all $v \in S$, $\gamma_v = 0$ for $v \notin S$, and $\sum_v n_v \gamma_v = 0$. Dirichlet's unit theorem for A-fields [Wei74, Chapter IV, §4, Theorem 9] implies that there exists $\alpha \in \mathbb{K}^{\times} \otimes \mathbb{R}$ such that $\log|\alpha|_v = \gamma_v$ for all v. Set $\overline{D}' = \overline{D} + \widehat{\mathrm{div}}(\alpha\chi^a)$. Then, for all $v \in \mathfrak{M}_{\mathbb{K}}$ and $x \in M_{\mathbb{R}}$,

$$\vartheta_{\overline{D}',v}(x) = \vartheta_{\overline{D},v}(x+a) - \gamma_v.$$

In particular, $\vartheta_{\overline{D}',v}(0) = \vartheta_{\overline{D},v}(a) - \gamma_v \geq 0$ for all v and so $\overline{D}' \geq 0$, as stated.

If $\overline{D}$ is pseudo-effective, then $\Theta_{\overline{D}} \neq \emptyset$ by Theorem 6.1(4), which proves the second statement.

(2) Let $\varphi \colon X' \to X$ be a birational toric map and $\overline{P}$ a nef toric metrized $\mathbb{R}$-divisor on X' such that $\overline{P} \leq \varphi^*\overline{D}$. In particular, $\overline{P}$ is semipositive. By Proposition 6.4, $\Delta_P \subset \Delta_{\varphi^*D} = \Delta_D$ and $\vartheta_{\overline{P},v}(x) \leq \vartheta_{\varphi^*\overline{D},v} = \vartheta_{\overline{D},v}(x)$ for all $v \in \mathfrak{M}_{\mathbb{K}}$ and $x \in \Delta_P$. Furthermore, by Theorem 6.1(2), $\vartheta_{\overline{P}} \geq 0$ on Δ_P. Hence, for $x \in \Delta_P$,

$$(7.4) \qquad\qquad \vartheta_{\overline{D}}(x) = \sum_v n_v \vartheta_{\overline{D},v}(x) \geq \vartheta_{\overline{P}}(x) \geq 0$$

and $\Delta_P \subset \Theta_{\overline{D}}$.

Assume that $(\varphi, \overline{P})$ is maximal. Suppose that $\Theta_{\overline{D}}$ is not a quasi-rational polytope. Then $\Delta_P \neq \Theta_{\overline{D}}$ and so there is a quasi-rational polytope $\Delta' \supsetneq \Delta_P$ contained in $\Theta_{\overline{D}}$. Let Σ' be a common refinement of Σ and $\Sigma_{\Delta'}$. Set $X' = X_{\Sigma'}$ and let $\varphi \colon X' \to X$ be the associate birational toric map. Let P' be the toric $\mathbb{R}$-divisor of X' determined by Δ' and $\overline{P}'$ the toric semipositive metrized $\mathbb{R}$-divisor associated to the restriction to Δ' of the family of concave functions $\{\vartheta_{\overline{D},v}\}_v$ under the correspondence in Proposition 4.19(2). We have that $\overline{P}'$ is nef (by Theorem 6.1(2)), $\overline{P} \leq \varphi^*\overline{P}'$ (by Proposition 6.4) and $\overline{P} \neq \overline{P}'$ since the associated polytopes are different. Hence, $\overline{P}$ is not maximal contradicting the hypothesis. We conclude that $\Theta_{\overline{D}} = \Delta_P$ is a quasi-rational polytope.

Conversely, assume that $\Theta_{\overline{D}}$ is a quasi-rational polytope. Let Σ' be a common refinement of Σ and $\Sigma_{\Theta_{\overline{D}}}$. Set $X' = X_{\Sigma'}$ and let $\varphi \colon X' \to X$ be the associated birational toric map.

Let P be the toric $\mathbb{R}$-divisor of X' determined by $\Theta_{\overline{D}}$ and $\overline{P}$ the toric semipositive metrized $\mathbb{R}$-divisor associated to the restriction to $\Theta_{\overline{D}}$ of the family of concave functions $\{\vartheta_{\overline{D},v}\}_v$ under the correspondence in Proposition 4.19(2). Hence, $\overline{P}$ is nef (by Theorem 6.1(2)) and $\overline{P} \le \varphi^* \overline{D}$ (by Proposition 6.4).

Now we show the maximality of the class of $(\varphi, \overline{P})$. Let Σ_1 be a refinement of Σ, $\varphi_1 \colon X_1 \to X$ the corresponding birational toric map and $\overline{P}_1$ a nef toric metrized $\mathbb{R}$-divisor on X_1 with $\varphi_1^* \overline{D} \ge \overline{P}_1$. By Proposition 6.4, $\Delta_{P_1} \subset \Delta_D$ and $\vartheta_{\overline{P}_1,v}(x) \le \vartheta_{\overline{D},v}(x)$ for all $v \in \mathfrak{M}_{\mathbb{K}}$ and $x \in \Delta_{P_1}$. Since $\overline{P}_1$ is nef, Theorem 6.1(2) implies that $\vartheta_{\overline{P}_1}(x) \ge 0$ for all $x \in \Delta_{P_1}$. Hence, $\Delta_{P_1} \subset \Theta_{\overline{D}}$. By construction, $\Delta_{P_1} \subset \Delta_P$ and $\vartheta_{\overline{P}_1}(x) \le \vartheta_{\overline{P}}(x)$ for $x \in \Delta_{P_1}$.

Taking a common refinement Σ'' of Σ' and Σ_1, we consider the corresponding birational toric maps $\nu \colon X'' \to X'$ and $\nu_1 \colon X'' \to X_1$. Proposition 6.4 then implies that $\nu^* \overline{P} \ge \nu_1^* \overline{P}_1$, which proves the statement.

(3) Since a big metrized divisor is pseudo-effective the equivalence of (3b) and (3c) follows from (2). Furthermore (3b) and (3c) imply (3a), it remains to prove that (3a) implies (3c). Assume that $\varphi^* \overline{D} = \overline{P} + \overline{E}$ is a toric Zariski decomposition. As before, $\Delta_P \subset \Theta_{\overline{D}}$ and $\vartheta_{\overline{P}} \le \vartheta_{\overline{D}}$. The equality of arithmetic volumes implies

$$(7.5) \qquad \int_{\Delta_P} \vartheta_{\overline{P}} \, \mathrm{dvol}_M = \int_{\Theta_{\overline{D}}} \vartheta_{\overline{D}} \, \mathrm{dvol}_M \, .$$

Since $\overline{D}$ is big, the interior of $\Theta_{\overline{D}}$ is nonempty and the function $\vartheta_{\overline{D}}$ is strictly positive on it. Then, equation (7.5) implies that $\Theta_{\overline{D}}$ is equal to Δ_P, which is a quasi-rational polytope.

(4) For each $\varepsilon > 0$ we pick a quasi-rational polytope Δ' contained in the interior of $\Theta_{\overline{D}}$ and such that

$$(n+1)! \int_{\Theta_{\overline{D}} \setminus \Delta'} \vartheta_{\overline{D}} \, \mathrm{dvol}_M \le \varepsilon.$$

Let Σ' be a common refinement of Σ and $\Sigma_{\Delta'}$. We consider the corresponding toric variety $X' = X_{\Sigma'}$ and the birational toric map $\varphi \colon X' \to X$. We set $\overline{A}$ for the toric $\mathbb{R}$-divisor corresponding to Δ' together with the metrics induced by the restriction to this polytope of the family of concave functions $\{\vartheta_{\overline{D},v}\}_v$.

By concavity, $\vartheta_{\overline{A}}$ is strictly positive on Δ'. Theorem 6.1(1) then implies that $\overline{A}$ is ample. By Proposition 6.4, $\overline{A} \le \varphi^* \overline{D}$ and, by construction,

$$\widehat{\mathrm{vol}}(X', \overline{A}) = (n+1)! \int_{\Delta'} \vartheta_{\overline{D}} \, \mathrm{dvol}_M \ge (n+1)! \int_{\Theta_{\overline{D}}} \vartheta_{\overline{D}} \, \mathrm{dvol}_M - \varepsilon = \widehat{\mathrm{vol}}(X, \overline{D}) - \varepsilon,$$

which concludes the proof. $\qquad \square$

Remark 7.6. If $\overline{D}$ as in the previous theorem is pseudo-effective but not big, then a toric Zariski decomposition always exists. Take any $a \in \Theta_{\overline{D}}$. Since the set $\{a\}$ is a quasi-rational polytope, using the construction of Theorem 7.2(2) we can associate to it a nef toric metrized divisor on X such that $\overline{P} \le \overline{D}$ and $\widehat{\mathrm{vol}}(X, \overline{P}) = 0 = \widehat{\mathrm{vol}}(X, \overline{D})$. Clearly, in this case, the decomposition may be non-unique.

Example 7.7. Consider again the toric metrized divisor $\overline{D}_{\boldsymbol{\alpha}}$ on $\mathbb{P}^n_{\mathbb{Q}}$ in Example 6.5. We also use the notation and results therein.

First suppose that $\sum_i \alpha_i \ge 1$ or, equivalently, that $\overline{D}_{\boldsymbol{\alpha}}$ is pseudo-effective. By Theorem 7.2(1), Dirichlet's unit theorem holds true in this case, as it does for any pseudo-effective metrized toric $\mathbb{R}$-divisor over an A-field.

If $\sum_i \alpha_i = 1$, then the set $\Theta_{\boldsymbol{\alpha}}$ is a point. Otherwise, this is a compact subset of Δ of dimension n with smooth boundary. In this case, $\Theta_{\boldsymbol{\alpha}}$ is not a polytope, unless

$n = 0, 1$. Hence, by Theorem 7.2(2), $\overline{D}$ admits a strong toric Zariski decomposition if and only if either $\sum_i \alpha_i = 1$, or $\sum_i \alpha_i > 1$ and $n = 0, 1$.

Now suppose that $\sum_i \alpha_i > 1$ or, equivalently, that $\overline{D}$ is big. Then, by Theorem 7.2(3), $\overline{D}$ admits a toric Zariski decomposition (and a strong Zariski decomposition) if and only if $n = 0, 1$.

Finally, Theorem 7.2(4) shows that a Fujita approximation of $\overline{D}_{\boldsymbol{\alpha}}$ always exists, as it does for any big metrized toric $\mathbb{R}$-divisor over a global field.

8. Zariski decomposition on toric varieties

In the previous section, we gave a characterization for when a toric Zariski decomposition exists. Now, we will study when the existence of a not necessarily toric Zariski decomposition implies the existence of a toric one. For technical reasons, we will restrict this study to algebraic metrized $\mathbb{R}$-divisors arising from arithmetic $\mathbb{R}$-divisors as in Example 3.19 and assume $\mathbb{K} = \mathbb{Q}$.

Definition 8.1. Let X be a smooth projective toric variety over $\mathbb{Q}$, $\mathcal{X}$ a model of X over $\mathbb{Z}$ and $\overline{\mathcal{D}}$ an arithmetic $\mathbb{R}$-divisor on $\mathcal{X}$ of C^0-type. Let D be the restriction of $\mathcal{D}$ to X and $\overline{D}$ the algebraic metrized $\mathbb{R}$-divisor on X associated to $\overline{\mathcal{D}}$. We say that $\overline{\mathcal{D}}$ is a *toric arithmetic $\mathbb{R}$-divisor* if $\overline{D}$ is a toric metrized $\mathbb{R}$-divisor. In particular, D is a toric $\mathbb{R}$-divisor on X.

The following is the main result of this section.

Theorem 8.2. *Let X be a smooth projective toric variety over $\mathbb{Q}$, $\overline{\mathcal{D}}$ a big toric arithmetic $\mathbb{R}$-divisor and $\overline{D}$ the associated toric metrized $\mathbb{R}$-divisor. Then, the following conditions are equivalent:*

(1) *there is a birational map $\sigma\colon \mathcal{Y} \to \mathcal{X}$ of flat normal generically smooth projective schemes over $\mathbb{Z}$ and a decomposition $\sigma^*\overline{\mathcal{D}} = \overline{\mathcal{P}} + \overline{\mathcal{E}}$ with $\overline{\mathcal{P}}$ and $\overline{\mathcal{E}}$ arithmetic $\mathbb{R}$-divisors on $\mathcal{Y}_{\mathbb{Q}}$ such that the corresponding metrized $\mathbb{R}$-divisors give a Zariski decomposition of $\overline{D}$;*

(2) *there is a toric Zariski decomposition of $\overline{D}$;*

(3) $\Theta_{\overline{D}}$ *is a quasi-rational polytope.*

Before proving the theorem we will need some preliminaries.

Let X be a proper variety over a field K. A *divisorial valuation* of X is a valuation ν on $\mathrm{K}(X)$ given, for $f \in \mathrm{K}(X)$ by

$$\nu(f) = \mathrm{ord}_H(\sigma^* f),$$

where $\sigma\colon Y \to X$ is a birational map of normal proper varieties over K and $H \in \mathrm{Div}(Y)$ is a prime Weil divisor. We denote by $\mathrm{DV}(X)$ the set of all divisorial valuations of X.

A divisorial valuation ν defines a map $\mathrm{mult}_\nu\colon \mathrm{Car}(X) \to \mathbb{Z}$ given, for $D \in \mathrm{Car}(X)$, by $\mathrm{mult}_\nu(D) = \nu(f_D)$ where $f_D \in \mathrm{K}(X)$ is a local equation of D on a neighbourhood of the point $\sigma(H)$. This map is a group morphism and so it extends to a map

$$\mathrm{mult}_\nu\colon \mathrm{Car}(X)_{\mathbb{R}} \longrightarrow \mathbb{R}.$$

Let $\rho\colon Z \to X$ be a birational map of normal proper varieties. Since $\mathrm{DV}(X) = \mathrm{DV}(Z)$, for $\nu \in \mathrm{DV}(X)$ the map mult_ν extends to a map $\mathrm{Car}(Z)_{\mathbb{R}} \to \mathbb{R}$.

We will consider the following notion of arithmetic multiplicity for metrized $\mathbb{R}$-divisors over a global field $\mathbb{K}$.

Definition 8.3. Let X be a proper variety over $\mathbb{K}$ and $\overline{D}$ a metrized $\mathbb{R}$-divisor on X. The *arithmetic multiplicity* of $\overline{D}$ is the function defined, for $\nu \in \mathrm{DV}(X)$, by

$$\mu_{\overline{D}}(\nu) = \inf \left\{ \frac{1}{\ell}\, \mathrm{mult}_\nu(\mathrm{div}(s)) \,\Big|\, \ell \geq 1, s \in \widehat{\Gamma}(X, \ell\overline{D})^{\times} \right\}.$$

Let $\rho\colon Y \to X$ be a birational map from a normal proper variety Y over $\mathbb{K}$ and $E \in \mathrm{Car}(Y)_{\mathbb{R}}$. We say that the arithmetic multiplicity of $\overline{D}$ is *represented* by E when $\mu_{\overline{D}}(\nu) = \mathrm{mult}_\nu(E)$ for all $\nu \in \mathrm{DV}(X)$.

The *arithmetic multiplicity* of an arithmetic $\mathbb{R}$-divisor is defined as the arithmetic multiplicity of its associated metrized $\mathbb{R}$-divisor.

We will show that, for a toric metrized $\mathbb{R}$-divisor $\overline{D}$, the convex set $\Theta_{\overline{D}}$ can be expressed as the intersection of a family of halfspaces defined by the arithmetic multiplicity of $\overline{D}$. When this arithmetic multiplicity is representable, this convex set can be expressed as the intersection of a *finite* sub-family of these halfspaces which implies that, in this case, $\Theta_{\overline{D}}$ is a polytope.

Proposition 8.4. *Let X be a proper toric variety over $\mathbb{K}$ and $\overline{D}$ a big toric metrized $\mathbb{R}$-divisor on X. Then*

(1) *for all $\nu \in \mathrm{DV}(X)$,*
$$\mu_{\overline{D}}(\nu) = \inf\{\mathrm{mult}_\nu(\mathrm{div}(s_a)) \mid a \in \Theta_{\overline{D}} \cap M_{\mathbb{Q}}\};$$

(2) $\Theta_{\overline{D}} = \{a \in M_{\mathbb{R}} \mid \mu_{\overline{D}}(\nu) \leq \mathrm{mult}_\nu(\mathrm{div}(s_a))$ *for all $\nu \in \mathrm{DV}(X)\}$;*

(3) *assume that there is a birational map $\sigma\colon Y \to X$ from a normal proper variety Y over $\mathbb{K}$ and an $\mathbb{R}$-divisor E on Y that represents $\mu_{\overline{D}}$. Then, there are prime Weil divisors $H_i \in \mathrm{Div}(Y)$, $i = 1, \ldots, l$, such that*
$$\Theta_{\overline{D}} = \{a \in M_{\mathbb{R}} \mid \mu_{\overline{D}}(\nu_i) \leq \mathrm{mult}_{\nu_i}(\mathrm{div}(s_a)), \ i = 1, \ldots, l\},$$
where ν_i is the divisorial valuation defined by H_i. In particular, $\Theta_{\overline{D}}$ is a quasi-rational polytope.

Proof. (1) Let $\ell \geq 1$ and $m \in \Theta_{\ell\overline{D}} \cap M$. Proposition 5.1(1) implies that $s_m = (\chi^m, \ell D) \in \widehat{\Gamma}(X, \ell\overline{D})^\times$. Since $\mathrm{mult}_\nu(\mathrm{div}(s_m)) = \ell\,\mathrm{mult}_\nu(\mathrm{div}(s_a))$ for $a = m/\ell$, it follows that
$$\mu_{\overline{D}}(\nu) \leq \inf\{\mathrm{mult}_\nu(\mathrm{div}(s_a)) \mid a \in \Theta_{\overline{D}} \cap M_{\mathbb{Q}}\}.$$

For the reverse inequality, let $s \in \widehat{\Gamma}(X, \ell\overline{D})^\times$. Write $s = (f, \ell D)$ with $f = \sum_{m \in \ell\Delta_D} c_m \chi^m$. By Corollary 5.5, $c_m \chi^m \in \widehat{\Gamma}(X, \ell\overline{D})^\times$ for all m such that $c_m \neq 0$. For all such m, Proposition 5.1(1) together with the product formula imply that $m \in \ell\Theta_{\overline{D}}$. Hence,
$$\mathrm{mult}_\nu(\mathrm{div}(s)) = \ell\,\mathrm{mult}_\nu(D) + \nu\left(\sum_{m \in \ell\Delta_D} c_m \chi^m\right) \geq \ell\,\mathrm{mult}_\nu(D) + \min_{m \in \ell\Theta_{\overline{D}} \cap M} \nu(\chi^m).$$

This implies that $\mu_{\overline{D}}(\nu) \geq \inf\{\mathrm{mult}_\nu(\mathrm{div}(s_a)) \mid a \in \Theta_{\overline{D}} \cap M_{\mathbb{Q}}\}$, which proves the formula.

(2) Write $\Omega = \{a \in M_{\mathbb{R}} \mid \mu_{\overline{D}}(\nu) \leq \mathrm{mult}_\nu(\mathrm{div}(s_a)) \ \forall \nu \in \mathrm{DV}(X)\}$ for short. By (1), $\Theta_{\overline{D}} \cap M_{\mathbb{Q}} \subset \Omega$. Since $\overline{D}$ is big, $\Theta_{\overline{D}} \cap M_{\mathbb{Q}}$ is dense in $\Theta_{\overline{D}}$ and so $\Theta_{\overline{D}} \subset \Omega$.

For the reverse inclusion, let $b \in M_{\mathbb{R}} \setminus \Theta_{\overline{D}}$. Since $\Theta_{\overline{D}}$ is convex and closed, there is a real number $\varepsilon > 0$ and a primitive element $u \in N$ such that $\langle b, u \rangle < \langle x, u \rangle - \varepsilon$ for all $x \in \Theta_{\overline{D}}$. Let Σ' be a complete unimodular regular refinement of Σ containing the ray $\mathbb{R}_{\geq 0} u$. Let H be the prime Weil divisor of $X_{\Sigma'}$ corresponding to this ray and ν_H the associated divisorial valuation of X. Then, for any $a = m/\ell \in \Theta_{\overline{D}} \cap M_{\mathbb{Q}}$,
$$\nu_H(\chi^b) = \langle b, u \rangle < \langle a, u \rangle - \varepsilon = \nu_H(\chi^m)/\ell - \varepsilon.$$

By (1), $\mathrm{mult}_{\nu_H}(\mathrm{div}(s_b)) < \mu_{\overline{D}}(\nu_H)$ and so $b \notin \Omega$, which proves the statement.

(3) Let $\{H_1, \ldots, H_l\}$ be the set of prime Weil divisors on Y containing all the components of E and of $\sigma^{-1}(X \setminus X_{\Sigma,0})$. For short, write
$$\Omega' = \{a \in M_{\mathbb{R}} \mid \mu_{\overline{D}}(\nu_i) \leq \mathrm{mult}_{\nu_i}(\mathrm{div}(s_a)), \ i = 1, \ldots, l\}.$$

By (2), $\Theta_{\overline{D}} \subset \Omega'$. For the reverse inclusion, let $b \in \Omega'$. Then, for all $\nu \in \mathrm{DV}(X)$,

$$\mu_{\overline{D}}(\nu) = \mathrm{mult}_\nu(E) = \sum_{i=1}^{l} \mathrm{mult}_{\nu_i}(E) \, \mathrm{mult}_\nu(H_i) = \sum_{i=1}^{l} \mu_{\overline{D}}(\nu_i) \, \mathrm{mult}_\nu(H_i)$$

$$\leq \sum_{i=1}^{l} \mathrm{mult}_{\nu_i}(\mathrm{div}(s_b)) \, \mathrm{mult}_\nu(H_i) = \mathrm{mult}_\nu(\mathrm{div}(s_b)).$$

By (2), this implies that $b \in \Theta_{\overline{D}}$ and so we obtain the first statement. The quasi-rationality of $\Theta_{\overline{D}}$ follows from the fact that

$$\mu_{\overline{D}}(\nu_i) = \mathrm{mult}_{\nu_i}(\mathrm{div}\, s_a)$$

is an affine equation in a with integral slope. $\qquad\qquad\qquad\qquad\square$

The relationship between the arithmetic multiplicity and the Zariski decomposition is given by the following result of Moriwaki [Mor12a, Theorems 2.5 and 4.1.1].

Theorem 8.5. *Let $\sigma\colon \mathcal{Y} \to \mathcal{X}$ be a birational morphism of generically smooth normal projective varieties over $\mathbb{Z}$. Denote by X and Y the generic fibre of $\mathcal{X}$ and $\mathcal{Y}$, respectively. Let $\overline{D}$ be a big arithmetic $\mathbb{R}$-divisor on $\mathcal{X}$. If $\overline{D}$ admits a Zariski decomposition $\sigma^*\overline{D} = \overline{P} + \overline{\mathcal{E}}$ with $\overline{P}, \overline{\mathcal{E}}$ arithmetic $\mathbb{R}$-divisors on $\mathcal{Y}$, then the arithmetic multiplicity of $\overline{D}$ is represented by the $\mathbb{R}$-divisor $E = \mathcal{E}\mid_Y \in \mathrm{Car}(Y)_{\mathbb{R}}$.*

Proof of Theorem 8.2. By Theorem 7.2(3), we know that (2) is equivalent to (3). Theorem 8.5 and Proposition 8.4(3) show that (1) implies (3).

It only remains to prove that (3) implies (1). The only difficulty is to show that the metrized $\mathbb{R}$-divisors appearing in the Zariski decomposition $\overline{D} = \overline{P} + \overline{E}$ obtained by Theorem 7.2(3) come from arithmetic $\mathbb{R}$-divisors. Clearly, it is enough to show that this is the case for the nef part $\overline{P}$.

Suppose that $\Theta_{\overline{D}}$ is a quasi-rational polytope. Let Σ' be a complete unimodular regular refinement of Σ and $\Sigma_{\Theta_{\overline{D}}}$. Set $X' = X_{\Sigma'}$ and let $\varphi\colon X' \to X$ be the associated birational toric map. Recall the construction of $\overline{P}$ in the proof of Theorem 7.2(3): P is the toric $\mathbb{R}$-divisor on X' determined by $\Theta_{\overline{D}}$ and $\overline{P}$ is the toric semipositive metrized $\mathbb{R}$-divisor associated to the restriction to $\Theta_{\overline{D}}$ of the family of concave functions $\{\vartheta_{\overline{D},v}\}_v$ under the correspondence in Proposition 4.19(2). For $v \neq \infty$, the function $\psi_{\overline{D},v}$ is piecewise affine, by [BPS14, Proposition 4.5.10(1)], and so is each local roof function $\vartheta_{\overline{D},v}$.

For each $v \in \mathfrak{M}_{\mathbb{Q}}$, consider the functions

$$\zeta_v = \vartheta_v\big|_{\Theta_{\overline{D}}} \qquad \text{and} \qquad \varphi_v = \zeta_v^\vee$$

and the finite set of places $S = \{v \in \mathfrak{M}_{\mathbb{Q}} \setminus \{\infty\} \mid \zeta_v \not\equiv 0\}$. For each $v \in S$, let $p_v \in \mathbb{Z}$ be the corresponding prime number. Choose a subdivision Π_v of $N_{\mathbb{R}}$ so that φ_v is piecewise affine on Π_v and $\mathrm{rec}(\Pi_v) = \Sigma'$. This can be done as follows. Let Ψ' be a strictly concave function on Σ', which exists because of the condition that Σ' is a regular fan. Then, Π_v can be constructed as the subdivision determined by the concave function $\varphi_v + \Psi'$ as in [BPS14, Definition 2.2.5].

For each finite subset $S' \subset S$ we denote $p_{S'} = \prod_{v \in S'} p_v$ and $\mathbb{Z}_{S'} = \mathbb{Z}[1/p_{S'}]$. Consider the toric scheme $\mathcal{X}_{\Pi_v}$ over $\mathbb{Z}_{S \setminus \{v\}}$ obtained by the construction in [BPS14, §3.5] using $\mathbb{Z}_{S \setminus \{v\}}$ and p_v in place of K° and ϖ. The function φ_v defines a Cartier divisor $\mathcal{P}_{\varphi_v}$ on $\mathcal{X}_{\Pi_v}$ as in the case of toric varieties over a field. The restriction of $\mathcal{P}_{\varphi_v}$ to the generic fibre X' agrees with P.

For $v, w \in S$, the restriction of the models $\mathcal{X}_{\Pi_v}$ and $\mathcal{X}_{\Pi_w}$ to $\mathbb{Z}_{S \setminus \{v,w\}}$ can be identified by using the element p_v/p_w. Under this identification, the Cartier divisors $\mathcal{P}_{\varphi_v}$ and $\mathcal{P}_{\varphi_w}$ correspond to each other. Since these identifications satisfy the

cocycle condition, we can glue together the schemes $\mathcal{X}_{\Pi_v}$, $v \in S$, into a model $\mathcal{X}'$ over $\mathbb{Z}$ of X', and the divisors $\mathcal{P}_{\varphi_v}$ into a model $\mathcal{P}$ of P. Then

$$g(x) = -2\varphi_\infty(\mathrm{val}_\infty(x))$$

is a Green function of C^0 and PSH-type for P. Hence $\overline{\mathcal{P}} = (\mathcal{P}, g)$ is a nef arithmetic $\mathbb{R}$-divisor and, by construction, $\overline{P}$ is its associated metrized $\mathbb{R}$-divisor, which concludes the proof of the theorem. $\qquad\square$

Example 8.6. Consider again the toric metrized divisor $\overline{D}_{\boldsymbol\alpha}$ on $\mathbb{P}_{\mathbb{Q}}^n$ in Example 6.5 and let $\overline{\mathcal{D}}_{\boldsymbol\alpha}$ be the associated toric arithmetic divisor.

With the notation therein, suppose that $n \geq 2$, and that $\sum_i \alpha_i > 1$ or, equivalently, that $\overline{D}_{\boldsymbol\alpha}$ is big. As noted in Example 7.7, the convex set $\Theta_{\boldsymbol\alpha}$ is not a polytope in this case. Hence, Theorem 8.2 shows that $\overline{\mathcal{D}}_{\boldsymbol\alpha}$ does not admit a Zariski decomposition into arithmetic $\mathbb{R}$-divisors.

Remark 8.7. In principle, Theorem 8.2 leaves open the possibility that the existence of a general Zariski decomposition of $\overline{D}$ does not imply the existence of a toric one. It would be interesting to settle this question and extend Theorem 8.2 to arbitary metrized $\mathbb{R}$-divisors. Since the main reason why we restrict ourselves to arithmetic $\mathbb{R}$-divisors is the use of Theorem 8.5, one step in this direction would be to extend this last result to metrized $\mathbb{R}$-divisors.

Appendix A. Convex analysis of asymptotically conic functions

In this appendix we extend some definitions and constructions from [BPS14, Chapter 2] to functions which are not necessarily concave. We will freely use the notations and conventions in *loc. cit.*.

Definition A.1. Let $f\colon N_{\mathbb{R}} \to \mathbb{R}$ be a function. The *stability set* of f is the subset of $M_{\mathbb{R}}$ given by

$$\mathrm{stab}(f) = \{x \in M_{\mathbb{R}} \mid x - f \text{ is bounded below}\}.$$

When f is a concave function, this coincides with the definition of stability set in convex analysis [Roc70]. If $\mathrm{stab}(f) \neq \emptyset$, this is a convex set. The function f determines a concave function $f^\vee\colon \mathrm{stab}(f) \to \mathbb{R}$ defined as

$$f^\vee(x) = \inf_{u \in N_{\mathbb{R}}} \langle x, u \rangle - f(u).$$

When f is concave, the function $f^\vee$ is the Legendre-Fenchel dual of f.

Lemma A.2. *Let $f, g\colon N_{\mathbb{R}} \to \mathbb{R}$ be two functions. Then*
   (1) $\mathrm{stab}(f + g) \supset \mathrm{stab}(f) + \mathrm{stab}(g)$.
   (2) $(f + g)^\vee(x) \geq (f^\vee \boxplus g^\vee)(x)$ *for all* $x \in \mathrm{stab}(f) + \mathrm{stab}(g)$.

Proof. (1) Let $x \in \mathrm{stab}(f) + \mathrm{stab}(g)$ and take $y \in \mathrm{stab}(f)$ and $z \in \mathrm{stab}(g)$ such that $x = y + z$. Then $y - f$ and $z - g$ are bounded below. Therefore $(y + z) - (f + g)$ is also bounded below, and so $x \in \mathrm{stab}(f + g)$.

(2) For $x \in \mathrm{stab}(f) + \mathrm{stab}(g)$ we have that

$$(f^\vee \boxplus g^\vee)(x) = \sup_{y+z=x} (f^\vee(y) + g^\vee(z)), \quad (f + g)^\vee(x) = \inf_u (\langle x, u \rangle - f(u) - g(u))$$

and

$$\sup_{y+z=x} (f^\vee(y) + g^\vee(z)) = \sup_{y+z=x} \left(\inf_u(\langle y, u \rangle - f(u)) + \inf_v(\langle z, v \rangle - g(v)) \right)$$

$$\leq \sup_{y+z=x} \inf_u(\langle y, u \rangle + \langle z, u \rangle - f(u) - g(u)) = \inf_u(\langle x, u \rangle - f(u) - g(u)),$$

from where we deduce the statement. $\qquad\square$

A conic function on $N_{\mathbb{R}}$ is a function $\Psi\colon N_{\mathbb{R}} \to \mathbb{R}$ such that $\Psi(\gamma u) = \gamma\Psi(u)$ for all $u \in N_{\mathbb{R}}$ and $\gamma \geq 0$.

Definition A.3. Let $f\colon N_{\mathbb{R}} \to \mathbb{R}$ be a function. We say that f is *asymptotically conic* if there is a conic function Ψ on $N_{\mathbb{R}}$ such that $|f - \Psi|$ is bounded. Such a conic function Ψ is necessarily unique. We call it the *recession function* of f and we denote it by $\mathrm{rec}(f)$.

Remark A.4. Let $f\colon N_{\mathbb{R}} \to \mathbb{R}$ be an asymptotically conic function. Then

$$\mathrm{rec}(f)(u) = \lim_{\gamma \to \infty} \frac{f(\gamma u)}{\gamma}.$$

Hence, for a concave asymptotically conic function, the notion of recession function coincides with the usual one in convex analysis, see for instance [Roc70, Theorem 8.5].

The stability set of an asymptotically conic function $f\colon N_{\mathbb{R}} \to \mathbb{R}$ agrees with that of its recession function. Hence,

$$\mathrm{stab}(f) = \{x \in M_{\mathbb{R}} \mid \langle x, u \rangle \geq \mathrm{rec}(f)(u) \text{ for all } u \in N_{\mathbb{R}}\}.$$

If f is an asymptotically conic function $f\colon N_{\mathbb{R}} \to \mathbb{R}$ with $\mathrm{stab}(f) \neq \emptyset$, the *concave envelope* of f, denoted $\mathrm{conc}(f)$, is defined as the smallest concave function $h\colon N_{\mathbb{R}} \to \mathbb{R}$ such that $h \geq f$. We have that $f^{\vee} = \mathrm{conc}(f)^{\vee}$ and $f^{\vee\vee} = \mathrm{conc}(f)$.

Lemma A.5. *Let $f, g\colon N_{\mathbb{R}} \to \mathbb{R}$ be two asymptotically conic functions such that $0 \in \mathrm{stab}(g)$. Then*

(1) $\mathrm{stab}(f + \varepsilon'g) \subset \mathrm{stab}(f + \varepsilon g)$ *for* $0 \leq \varepsilon' \leq \varepsilon$ *and*

$$\bigcap_{\varepsilon > 0} \mathrm{stab}(f + \varepsilon g) = \mathrm{stab}(f).$$

(2) *Assume that* $\mathrm{stab}(f) \neq \emptyset$. *Then, for* $0 \leq \varepsilon' \leq \varepsilon$,

$$(f + \varepsilon g)^{\vee}\big|_{\mathrm{stab}(f+\varepsilon'g)} \geq (f + \varepsilon'g)^{\vee} + (\varepsilon - \varepsilon')g^{\vee}(0)$$

and, for $x \in \mathrm{stab}(f)$,

$$\lim_{\varepsilon \to 0} (f + \varepsilon g)^{\vee}(x) = f^{\vee}(x).$$

Proof. (1) Let $0 \leq \varepsilon' \leq \varepsilon$. By Lemma A.2(1),

$$\mathrm{stab}(f + \varepsilon g) \supset \mathrm{stab}(f + \varepsilon'g) + \mathrm{stab}((\varepsilon - \varepsilon')g) \supset \mathrm{stab}(f + \varepsilon'g)$$

because $\varepsilon - \varepsilon' \geq 0$ and so $0 \in (\varepsilon - \varepsilon')\,\mathrm{stab}(g) = \mathrm{stab}((\varepsilon - \varepsilon')g)$. Now let $x \in \mathrm{stab}(f + \varepsilon g)$ for all $\varepsilon > 0$. We have that $\mathrm{stab}(f + \varepsilon g) = \mathrm{stab}(\mathrm{rec}(f) + \varepsilon\,\mathrm{rec}(g))$ and so, for all $u \in N_{\mathbb{R}}$,

$$\langle x, u \rangle \geq \mathrm{rec}(f)(u) + \varepsilon\,\mathrm{rec}(g)(u).$$

Letting $\varepsilon \to 0$, we obtain $\langle x, u \rangle \geq \mathrm{rec}(f)(u)$. Since this holds for all $u \in N_{\mathbb{R}}$, we deduce that $x \in \mathrm{stab}(\mathrm{rec}(f)) = \mathrm{stab}(f)$. Hence, $\bigcap_{\varepsilon > 0} \mathrm{stab}(f + \varepsilon g) \subset \mathrm{stab}(f)$. The reverse inclusion follows from the argument above with $\varepsilon' = 0$.

(2) Let $0 \leq \varepsilon' \leq \varepsilon$. By Lemma A.2(2), for $x \in \mathrm{stab}(f + \varepsilon'g)$,

$$\big(f + \varepsilon g\big)^{\vee}(x) \geq \big((f + \varepsilon'g)^{\vee} \boxplus \big((\varepsilon - \varepsilon')g\big)^{\vee}\big)(x) \geq (f + \varepsilon'g)^{\vee}(x) + \big(\varepsilon - \varepsilon'\big)g^{\vee}(0),$$

which proves the first assertion.

Letting $\varepsilon' = 0$, we deduce that $\liminf_{\varepsilon \to 0} \big(f + \varepsilon g\big)^{\vee}(x) \geq f^{\vee}(x)$ for $x \in \mathrm{stab}(f)$. For the reverse inequality, given $\delta > 0$ let $u_0 \in N_{\mathbb{R}}$ such that $f^{\vee}(x) \geq \langle x, u_0 \rangle - f(u_0) - \delta$. Then

$$(f + \varepsilon g)^{\vee}(x) \leq \langle x, u_0 \rangle - f(u_0) - \varepsilon g(u_0) \leq f^{\vee}(x) - \varepsilon g(u_0) + \delta.$$

Letting $\varepsilon \to 0$, we deduce that

$$\limsup_{\varepsilon \to 0}(f + \varepsilon g)^{\vee}(x) \leq f^{\vee}(x) + \delta.$$

Since this holds for all $\delta > 0$, we obtain $\limsup_{\varepsilon \to 0}(f + \varepsilon g)^{\vee}(x) \leq f^{\vee}(x)$. Therefore $\lim_{\varepsilon \to 0}(f + \varepsilon g)^{\vee}(x)$ exists and is equal to $f^{\vee}(x)$, which concludes the proof. $\qquad \square$

References

[BC11]   S. Boucksom and H. Chen, *Okounkov bodies of filtered linear series*, Compos. Math. **147** (2011), 1205–1229.

[BFJ11]   S. Boucksom, C. Favre, and M. Jonsson, *Solution to a non-Archimedean Monge-Ampère equation*, J. Amer. Math. Soc. **28** (2015), 617–667.

[Bil97]   Y. Bilu, *Limit distribution of small points on algebraic tori*, Duke Math. J. **89** (1997), 465–476.

[BPS14]   J. I. Burgos Gil, P. Philippon, and M. Sombra, *Arithmetic geometry of toric varieties. Metrics, measures and heights*, Astérisque, vol. 360, Soc. Math. France, 2014.

[BR06]   M. H. Baker and R. Rumely, *Equidistribution of small points, rational dynamics, and potential theory*, Ann. Inst. Fourier (Grenoble) **56** (2006), 625–688.

[Bur97]   J. I. Burgos Gil, *Arithmetic Chow rings and Deligne-Beilinson cohomology*, J. Alg. Geom. **6** (1997), 335–377.

[Cha06]   A. Chambert-Loir, *Mesures et équidistribution sur les espaces de Berkovich*, J. Reine Angew. Math. **595** (2006), 215–235.

[Che10]   H. Chen, *Arithmetic Fujita approximation*, Ann. Sci. Éc. Norm. Supér. (4) **43** (2010), 555–578.

[CLS11]   D. A. Cox, J. B. Little, and H. K. Schenck, *Toric varieties*, Grad. Stud. Math., vol. 124, Amer. Math. Soc., 2011.

[EGA]   A. Grothendieck and J. Dieudonné, *Éléments de Géométrie Algébrique. I*, Grunlehren der Math. Wissenschaften **166** (1971); *II*, Publ. Math. I.H.É.S. **8** (1961); *III*, ibidem **11** (1961); *IV*, ibidem **20** (1964), **24** (1965), **28** (1966), **32** (1967).

[FR06]   C. Favre and J. Rivera-Letelier, *Équidistribution quantitative des points de petite hauteur sur la droite projective*, Math. Ann. **335** (2006), 311–361.

[Ful93]   W. Fulton, *Introduction to toric varieties*, Ann. of Math. Stud., vol. 131, Princeton Univ. Press, 1993.

[Gau09]   É. Gaudron, *Géométrie des nombres adélique et lemmes de Siegel généralisés*, Manuscripta Math. **130** (2009), 159–182.

[GS88]   H. Gillet and C. Soulé, *Amplitude arithmétique*, C. R. Acad. Sci. Paris Sér. I Math. **307** (1988), 887–890.

[GS90]   _____, *Arithmetic intersection theory*, Publ. Math. Inst. Hautes Études Sci. **72** (1990), 94–174.

[Lan83]   S. Lang, *Fundamentals of Diphantine Geometry*, Springer-Verlag, 1983.

[Lan94]   _____, *Algebraic number theory*, 2nd ed., Grad. Texts in Math., vol. 110, Springer-Verlag, 1994.

[Laz04]   R. Lazarsfeld, *Positivity in algebraic geometry. I*, Ergeb. Math. Grenzgeb. (3), Springer-Verlag, 2004.

[Liu02]   Q. Liu, *Algebraic geometry and arithmetic curves*, Oxf. Grad. Texts Math., vol. 6, Oxford Univ. Press, 2002.

[Mor09]   A. Moriwaki, *Continuity of volumes on arithmetic varieties*, J. Algebraic Geom. **18** (2009), 407–457.

[Mor11]   _____, *Big arithmetic divisors on the projective spaces over* $\mathbb{Z}$, Kyoto J. Math. **51** (2011), 503–534.

[Mor12a] _____, *Arithmetic linear series with base conditions*, Math. Z. **272** (2012), 1383–1401.

[Mor12b] _____, *Numerical characterization of nef arithmetic divisors on arithmetic surfaces*, Ann. Fac. Sci. Toulouse Math. (6) **23** (2014), 717–753.

[Mor12c] _____, *Zariski decompositions on arithmetic surfaces*, Publ. Res. Inst. Math. Sci. **48** (2012), 799–-898.

[Mor13]   _____, *Toward Dirichlet's unit theorem on arithmetic varieties*, Kyoto J. Math. **53** (2013), 197–259.

[Roc70]   R. T. Rockafellar, *Convex analysis*, Princeton Math. Series, vol. 28, Princeton Univ. Press, 1970.

[SUZ97]   L. Szpiro, E. Ullmo, and S.-W. Zhang, *Équirépartition des petits points*, Invent. Math. **127** (1997), 337–347.

[Wei74]   A. Weil, *Basic number theory*, third ed., Die Grundlehren der Mathematischen Wissenschaften, Band 144, Springer-Verlag, 1974.

[Yua08]   X. Yuan, *Big line bundles over arithmetic varieties*, Invent. Math. **173** (2008), 603–649.

[Yua09a] _____, *On volumes of arithmetic line bundles*, Compos. Math. **145** (2009), 1447–1464.

[Yua09b] _____, *On volumes of arithmetic line bundles. II*, e-print arXiv:0909.3680v1, 2009.

[Zha95a] S.-W. Zhang, *Positive line bundles on arithmetic varieties*, J. Amer. Math. Soc. **8** (1995), 187–221.

[Zha95b] _____, *Small points and adelic metrics*, J. Algebraic Geom. **4** (1995), 281–300.

Instituto de Ciencias Matemáticas (CSIC-UAM-UCM-UCM3). Calle Nicolás Cabrera 15, Campus UAB, Cantoblanco, 28049 Madrid, Spain

Email address: burgos@icmat.es

URL: http://www.icmat.es/miembros/burgos

Department of Mathematics, Faculty of Science, Kyoto University. 606-8502 Kyoto, Japan

Email address: moriwaki@math.kyoto-u.ac.jp

Institut de Mathématiques de Jussieu – U.M.R. 7586 du CNRS, Équipe de Théorie des Nombres. Case 247, 4 place Jussieu, 75252 Paris cedex 05, France

Email address: patrice.philippon@imj-prg.fr

URL: http://www.math.jussieu.fr/~pph

ICREA & Universitat de Barcelona, Departament d'Àlgebra i Geometria. Gran Via 585, 08007 Barcelona, Spain

Email address: sombra@ub.edu

URL: http://atlas.mat.ub.es/personals/sombra