CONTINUITY OF PLURISUBHARMONIC ENVELOPES IN NON-ARCHIMEDEAN GEOMETRY AND TEST IDEALS (WITH AN APPENDIX BY JOSÉ IGNACIO BURGOS GIL AND MARTÍN SOMBRA)

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ABSTRACT. Let L be an ample line bundle on a smooth projective variety X over a non-archimedean field K. For a continuous metric on L^{an}, we show in the following two cases that the semipositive envelope is a continuous semipositive metric on L^{an} and that the non-archimedean Monge-Ampère equation has a solution. First, we prove it for curves using results of Thuillier. Second, we show it under the assumption that X is a surface defined geometrically over the function field of a curve over a perfect field k of positive characteristic. The second case holds in higher dimensions if we assume resolution of singularities over k. The proof follows a strategy from Boucksom, Favre and Jonsson, replacing multiplier ideals by test ideals. Finally, the appendix by Burgos and Sombra provides an example of a semipositive metric whose retraction is not semipositive. The example is based on the construction of a toric variety which has two SNC-models which induce the same skeleton but different retraction maps.

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CONTENTS

1. INTRODUCTION

Let L be an ample line bundle on an n-dimensional complex projective variety X and let μ be a smooth volume form on the associated complex manifold X^{an} of total mass $\deg_L(X)$. The *Calabi conjecture* claims that there is a smooth semipositive metric $\|\ \|$ on L^{an}, unique up to positive multiples, solving the *Monge–Ampère equation*

$$(1.1) \qquad\qquad c_1(L, \|\ \|)^{\wedge n} = \mu.$$

This was conjectured by Calabi who proved uniqueness [Cal54, Cal57] and the existence part was solved by Yau [Yau78]. In fact, they proved a more general version in the setting of compact Kähler manifolds, but this will not be relevant for our paper.

The motivation of this paper is the study of the non-archimedean version of this conjecture. We consider a non-archimedean field K with valuation ring K°. Let L be an ample line bundle on an n-dimensional smooth projective variety X over K. The line bundle L induces a line bundle L^{an} on the analytification X^{an} of X as a Berkovich non-archimedean analytic space. In non-archimedean geometry, model metrics on L^{an} play a similar role as smooth metrics on line bundles on complex manifolds. We call a model metric on L^{an} semipositive if it is induced by an nef model. Zhang [Zha95] introduced continuous semipositive metrics on L^{an} as uniform limits of semipositive model metrics. For such metrics, Chambert-Loir [CL06] defined a Monge–Ampère measure $c_1(L, \|\ \|)^{\wedge n}$ on X^{an} which is a positive Radon measure of total mass $\deg_L(X)$. These measures play an important role in arithmetic equidistribution results (see [Yua08]). We refer to Section 2 for details about these notions.

In the *non-archimedean Calabi–Yau problem*, one looks for continuous semipositive metrics on L^{an} solving the Monge–Ampère equation (1.1). Yuan and Zhang [YZ17] proved that such a metric is unique up to constants. The existence of a singular semipositive solution was proven in the case of curves by Thuillier [Thu05, Cor. 3.4.13]. Liu [Liu11] proved existence of a continuous semipositive solution for totally degenerate abelian varieties A if μ is a smooth volume form on the canonical skeleton of A.

Next, we describe the fundamental existence result of Boucksom, Favre and Jonsson [BFJ16a, BFJ15]. We assume that K is a complete discretely valued field with valuation ring K°. We recall that an SNC-model is a regular model of X such that the special fiber has simple normal crossing support. Boucksom, Favre and Jonsson prove in [BFJ15, Thm. A] that the non-archimedean Calabi–Yau problem has a continuous semipositive solution $\|\ \|$ if the following assumptions are satisfied:

(a) The characteristic of the residue field $\tilde{K}$ is zero.
(b) The positive Radon measure μ is supported on the skeleton of a projective SNC-model of X and satisfies $\mu(X^{\mathrm{an}}) = \deg_L(X)$.
(c) The smooth projective variety X is of geometric origin from a 1-dimensional family over $\tilde{K}$.

The last condition will play an important role in this paper. More generally, we say that X is *of geometric origin from a d-dimensional family over the field k* if there is a codimension 1 point b in a normal variety B over k such that K° is the completion of $\mathcal{O}_{B,b}$ and such that X is defined over the function field $k(B)$. In [BGJKM16, Thm. D], we have shown that (c) is not necessary for the existence of a continuous semipositive solution of the non-archimedean Calabi–Yau problem if we assume (a) and (b).

We will later look for a similar result in equicharacteristic $p > 0$. To do so, we have to understand the basic ingredients in the proof of the existence result of Boucksom, Favre and Jonsson. In [BFJ16a], the authors develop a global pluripotential theory on X^{an}

for singular semipositive metrics using the piecewise linear structure on the skeletons of SNC-models. It is here where Assumption (a) enters the first time as resolution of singularities is used to have sufficiently many SNC-models of X at hand. For a continuous metric $\| \|$, we define the *semipositive envelope* $P(\| \|)$ by

$$(1.2) \quad P(\| \|) := \inf\{\| \|' \mid \| \| \leq \| \|' \text{ and } \| \|' \text{ is a semipositive model metric on } L^{\mathrm{an}}\}.$$

It is absolutely crucial for pluripotential theory to prove that $P(\| \|)$ is continuous as this is equivalent to the monotone regularization theorem (see [BFJ16a, Lemma 8.9]). The monotone regularization theorem is the basis in [BFJ15] to introduce the Monge–Ampère measure, capacity and energy for singular semipositive metrics. The proof of continuity of $P(\| \|)$ in [BFJ16a, §8] uses multiplier ideals on regular projective models. In the proof of the required properties of multiplier ideals (see [BFJ16a, Appendix B]), the authors use vanishing results which hold only in characteristic zero, and hence Assumption (a) plays an important role here as well.

A second important result is the *orthogonality property* for $P(\| \|)$ given in [BFJ15, Thm. 7.2]. Multiplier ideals occur again in their proof and it is here where the geometric assumption (c) is used. However, it is shown in [BGJKM16, Thm. 6.3.3] that continuity of $P(\| \|)$ is enough to prove the orthogonality property without assuming (a) or (c). Then the variational method of Boucksom, Favre and Jonsson can be applied to prove existence of a continuous semipositive solution for the non-archimedean Calabi–Yau problem.

This makes it very clear that continuity of the semipositive envelope $P(\| \|)$ plays a crucial role in the non-archimedean Calabi–Yau problem. It is the main object of study in this paper. In Section 2, we will study the basic properties of a slight generalization of $P(\| \|)$ which is called the θ-psh envelope for a closed $(1,1)$-form θ on X. For the sake of simplicity, we will restrict our attention in the introduction to the semipositive envelope $P(\| \|)$, while all the results hold more generally for the θ-psh envelope assuming that the de Rham class of θ is ample.

In Section 3, we will look at continuity of the semipositive envelope in the case of a smooth projective curve X over an arbitrary complete non-archimedean field K. Potential theory on the curve X^{an} was developed in Thuillier's thesis [Thu05]. We will use Thuillier's results and the slope formula of Katz, Rabinoff, and Zureick-Brown [KRZB16, Thm. 2.6] to prove:

Theorem 1.1. *Let L be an ample line bundle on a smooth projective curve X over any non-archimedean field K. Then $P(\| \|)$ is a continuous semipositive metric on L^{an} for any continuous metric $\| \|$ on L^{an}.*

A slightly more general version will be proved in Theorem 3.1. The following is important in the proof: Let L be any line bundle on the smooth projective curve X. We assume that X has a strictly semistable model $\mathscr{X}$ such that L^{an} has a model metric $\| \|_0$ associated to a line bundle on $\mathscr{X}$. For any metric $\| \|$ on L^{an}, we consider the function $\varphi := -\log(\| \|/\| \|_0)$. Let $p_{\mathscr{X}} \colon X^{\mathrm{an}} \to \Delta$ be the canonical retraction to the skeleton Δ associated to $\mathscr{X}$. Then

$$(1.3) \qquad\qquad\qquad \| \|_\Delta := e^{-\varphi \circ p_{\mathscr{X}}} \| \|_0$$

is a metric on L^{an} which does not depend on the choice of $\| \|_0$. The following result is crucial in the proof of Theorem 1.1:

Proposition 1.2. *Using the hypotheses above, we consider a model metric $\| \|$ of L^{an}. Then we have the following properties:*

(i) *The metric $\| \|_\Delta$ is a model metric.*

(ii) *There is an equality of measures* $c_1(L, \| \ \|_\Delta) = (p_{\mathscr{X}})_*(c_1(L, \| \ \|))$.

(iii) *If* $\| \ \|$ *is semipositive, then* $\| \ \|_\Delta$ *is semipositive and* $\| \ \|_\Delta \leq \| \ \|$.

This will be proven in Propositions 3.5 and 3.8. It would make pluripotential theory and the solution of the non-archimedean Calabi–Yau problem much easier if Proposition 1.2 would also hold in higher dimensions as we could work more combinatorically on skeletons. It is still true that $\| \ \|_\Delta$ is a model metric which satisfies $\| \ \|_\Delta \leq \| \ \|$. Burgos and Sombra show in a two dimensional toric counterexample in the Appendix that $\| \ \|_\Delta$ does not have to be semipositive.

We show now that Proposition 1.2 is also crucial for the existence of the solution of the non-archimedean Calabi–Yau problem in the case of curves arguing as in [BFJ16b, §9]. By Thuillier [Thu05, Cor. 3.4.13], there is a semipositive metric $\| \ \|$ solving (1.1), but it might be singular. Here, semipositive means that the metric is an increasing pointwise limit of semipositive model metrics of the ample line bundle L. If we assume that the positive Radon measure μ has support in the skeleton Δ of a strictly semistable model $\mathscr{X}$ of X, then it follows easily from Proposition 1.2 that $\| \ \|_\Delta$ is a continuous semipositive metric solving (1.1). Burgos and Sombra show in their counterexample in the Appendix that this does not hold in higher dimensions either.

To look for solutions of the higher dimensional non-archimedean Monge–Ampère equation in positive characteristic, we will replace the use of multiplier ideals by the use of test ideals. Test ideals were introduced by Hara and Yoshida [HY03] using a generalization of tight closure theory. In Section 4, we will gather the necessary facts about test ideals mainly following [Mus13] and so we work on a smooth variety X over a perfect field k of characteristic $p > 0$. Similarly as in the case of multiplier ideals, one can define an *asymptotic test ideal* $\tau(\lambda\|D\|)$ *of exponent* $\lambda \in \mathbb{R}_{\geq 0}$ for a divisor D on X. Crucial for us is that $\tau(\lambda\|D\|)$ satisfies a subadditivity property and the following *uniform generation property*:

Theorem 1.3. *Let X be a projective scheme over a finitely generated k-algebra R such that X is a smooth n-dimensional variety over k. We assume that H is an ample and basepoint-free divisor, D is a divisor with $h^0(X, \mathcal{O}(mD)) \neq 0$ for some $m \in \mathbb{N}_{>0}$ and E is a divisor such that the $\mathbb{Q}$-divisor $D - \lambda E$ is nef for some $\lambda \in \mathbb{Q}_{\geq 0}$. Then the sheaf $\mathcal{O}_X(K_{X/k} + E + dH) \otimes_{\mathcal{O}_X} \tau(\lambda \cdot \|D\|)$ is globally generated for all $d \geq n + 1$.*

This was proven by Mustaţă if X is projective over k. As we will later work over discrete valuation rings, we need the more general version with X only projective over R. This will be possible in Theorem 4.6 as we can replace the use of Fujita's vanishing theorem in Mustaţă's proof by Keeler's generalization.

Now we come to the *non-archimedean Calabi–Yau problem in equicharacteristic $p > 0$*. For the remaining part of the introduction, we now fix an n-dimensional smooth projective variety X over a complete discretely valued field K of characteristic p. To apply the results on test ideals, we have to require that X is of geometric origin from a d-dimensional family over a perfect field k. We also fix an ample line bundle L on X.

Theorem 1.4. *Under the hypotheses above, we assume that resolution of singularities holds over k in dimension $d + n$. Then the semipositive envelope $P(\| \ \|)$ of a continuous metric $\| \ \|$ on L^{an} is a continuous semipositive metric on L^{an}.*

For the precise definition about resolution of singularities, we refer to Definition 6.1. As resolution of singularities is known in dimension 3 over a perfect field by a result of Cossart and Piltant [CP09, Thm. p. 1839], Theorem 1.4 is unconditional if X is a smooth projective surface of geometric origin from a 1-dimensional family over k.

In Section 7, we will prove Theorem 1.4 in the case when $\| \ \|$ is a model metric associated to a model which is also defined geometrically over k. We will follow the proof of Boucksom, Favre, and Jonsson, replacing multiplier ideals by test ideals. As we use a rather weak notion of resolution of singularities, a rather subtle point in the argument is necessary in the proof of Lemma 7.5 which involves a result of Pépin about semi-factorial models. Theorem 1.4 will be proved in full generality in Section 8 using the dd^c-lemma and basic properties of the semipositive envelope. In fact, we will prove in Theorem 8.2 a slightly more general result.

If we use additionally that embedded resolution of singularities (see Definition 6.2) holds over k in dimension $d+n$, then the family of projective SNC-models will be cofinal in the category of all models of X. We will see in Section 9 that this and Theorem 1.4 allow us to set up the pluripotential theory from [BFJ16a] on X^{an}. By [BFJ15, Thm. 7.2] again, the continuity of $P(\| \ \|)$ yields that the orthogonality property holds for any continuous metric on L^{an}. We will use this in Section 9 to show that the variational method of Boucksom, Favre and Jonsson proves the following result (see Theorem 9.3).

Theorem 1.5. *Let X be an n-dimensional smooth projective variety of geometric origin from a d-dimensional family over a perfect field k of characteristic $p > 0$. We assume that resolution of singularities and embedded resolution of singularities hold over k in dimension $d+n$. Let L be an ample line bundle on X and let μ be a positive Radon measure supported on the skeleton of a projective SNC-model of X with $\mu(X^{\mathrm{an}}) = \deg_L(X)$. Then the non-archimedean Monge–Ampère equation (1.1) has a continuous semipositive metric $\| \ \|$ on L^{an} as a solution.*

Cossart and Piltant [CP08, CP09] have shown resolution of singularities and embedded resolution of singularities in dimension 3 over a perfect field, hence Theorem 1.5 holds unconditionally for a smooth projective surface X of geometric origin from a 1-dimensional family over the perfect field k.

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Notations and conventions

Let X be a scheme. An *ideal* in $\mathcal{O}_X$ is a quasi-coherent ideal sheaf in $\mathcal{O}_X$. A *divisor* on X is always a Cartier divisor on X. Given $m \in \mathbb{N}$ we write $X^{(m)}$ for the set of all $p \in X$ where the local ring $\mathcal{O}_{X,p}$ has Krull dimension m. Let k be a field. A *variety X over k* is an integral k-scheme X which is separated and of finite type. A *curve (resp. surface)* is a variety of dimension one (resp. two).

Throughout this paper $(K, | \ |)$ denotes a complete non-archimedean valued field with valuation ring K° and residue field $\tilde{K}$. Starting in Section 7 we will assume furthermore that the valuation is discrete and that K has positive characteristic $p > 0$. In this case there exists an isomorphism $K^\circ \xrightarrow{\sim} \tilde{K}[[T]]$ [Mat89, Thm. 29.7]. Let X be a K-variety. We denote the analytification of X in the sense of Berkovich [Ber90, Thm. 3.4.1] by X^{an}.

2. Model metrics, semipositive metrics, and envelopes

Let X be a proper variety over a complete non-archimedean valued field $(K, | \ |)$.

2.1. A *model of X* is given by a proper flat scheme $\mathscr{X}$ over $S := \operatorname{Spec} K^\circ$ together with an isomorphism h between X and the generic fiber $\mathscr{X}_\eta$ of the S-scheme $\mathscr{X}$ which we read as an identification. Given a model $\mathscr{X}$ of X there is a canonical surjective *reduction map* $\mathrm{red} \colon X^{\mathrm{an}} \longrightarrow \mathscr{X}_s$ where $\mathscr{X}_s$ denotes the special fiber $\mathscr{X} \otimes_{K^\circ} \tilde{K}$ of $\mathscr{X}$ over S.

Let L be a line bundle on the proper variety X. A *model of* (X, L) or briefly a *model of* L is given by a model $(\mathscr{X}, h)$ of X together with a line bundle $\mathscr{L}$ on $\mathscr{X}$ and an isomorphism h' between L and $h^*(\mathscr{L}|_{\mathscr{X}_\eta})$ which we read as an identification.

Given a model $(\mathscr{X}, \mathscr{L})$ of $(X, L^{\otimes m})$ for some $m \in \mathbb{N}_{>0}$ there is a unique metric $\| \ \|_{\mathscr{L}}$ on L^{an} over X^{an} which satisfies the following: Given an open subset $\mathscr{U}$ of $\mathscr{X}$, a frame t of $\mathscr{L}$ over $\mathscr{U}$, and a section s of L over $U = X \cap \mathscr{U}$ we write $s^{\otimes m} = ht$ for some regular function h on U and get $\|s\| = \sqrt[m]{|h|}$ on $U^{\mathrm{an}} \cap \mathrm{red}^{-1}(\mathscr{U}_s)$. Such a metric on L^{an} is called a *model metric determined on* $\mathscr{X}$.

2.2. A model metric $\| \ \|$ on $\mathcal{O}_{X^{\mathrm{an}}}$ induces a continuous function $f = -\log\|1\| \colon X^{\mathrm{an}} \to \mathbb{R}$. The space of *model functions*

$$\mathscr{D}(X) = \{f \colon X^{\mathrm{an}} \to \mathbb{R} \mid f = -\log\|1\| \text{ for some model metric } \| \ \| \text{ on } \mathcal{O}_{X^{\mathrm{an}}}\}$$

has a natural structure of a $\mathbb{Q}$-vector space. We say that a model function $f = -\log\|1\|$ is *determined on a model* $\mathscr{X}$ if the model metric $\| \ \|$ is determined on $\mathscr{X}$. A vertical divisor D on $\mathscr{X}$ determines a model $\mathcal{O}(D)$ of $\mathcal{O}_X$ and an associated model function $\varphi_D := -\log\|1\|_{\mathcal{O}(D)}$. Such model functions are called $\mathbb{Z}$-*model functions*. Let $\mathfrak{a}$ denote a vertical ideal of $\mathscr{X}$. Let E denote the exeptional divisor of the blowup of $\mathscr{X}$ in $\mathfrak{a}$. Then $\log|\mathfrak{a}| := \varphi_E$ is called the $\mathbb{Z}$-*model function defined by the vertical ideal* $\mathfrak{a}$.

2.3. Consider a model $\mathscr{X}$ of the proper variety X over K. The rational vector space space $N^1(\mathscr{X}/S)_{\mathbb{Q}}$ is by definition the quotient of $\mathrm{Pic}(\mathscr{X})_{\mathbb{Q}} := \mathrm{Pic}(\mathscr{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ by the subspace generated by classes of line bundles $\mathscr{L}$ such that $\mathscr{L} \cdot C = 0$ for each closed curve C in the special fiber $\mathscr{X}_s$. Note that $N^1(\mathscr{X}/S)_{\mathbb{Q}}$ is finite dimensional by applying [Kle66, Prop. IV.1.4] to $\mathscr{X}_s$. We define $N^1(\mathscr{X}/S) := N^1(\mathscr{X}/S)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$. An element $\alpha \in N^1(\mathscr{X}/S)_{\mathbb{Q}}$ (resp. $\alpha \in N^1(\mathscr{X}/S)$) is called *nef* if $\alpha \cdot C \geq 0$ for all closed curves C in $\mathscr{X}_s$. We call a line bundle $\mathscr{L}$ on $\mathscr{X}$ *nef* if the class of $\mathscr{L}$ in $N^1(\mathscr{X}/S)$ is nef.

2.4. We define $\mathcal{Z}^{1,1}(X)_{\mathbb{Q}}$ as the direct limit

$$(2.1) \qquad\qquad \mathcal{Z}^{1,1}(X)_{\mathbb{Q}} := \varinjlim N^1(\mathscr{X}/S)_{\mathbb{Q}},$$

where $\mathscr{X}$ runs over the isomorphism classes of models of X. The *space of closed* $(1,1)$-*forms on* X is defined as $\mathcal{Z}^{1,1}(X) := \mathcal{Z}^{1,1}(X)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$. Let L be a line bundle on X. Let $\| \ \|$ be a model metric on L^{an} which is determined on $\mathscr{X}$ by a model $\mathscr{L}$ of $L^{\otimes m}$. We multiply the class of $\mathscr{L}$ in $N^1(\mathscr{X}/S)_{\mathbb{Q}}$ by m^{-1} which determines a well defined class $c_1(L, \| \ \|) \in \mathcal{Z}^{1,1}(X)_{\mathbb{Q}} \subseteq \mathcal{Z}^{1,1}(X)$ called the *curvature form* $c_1(L, \| \ \|)$ *of* $(L, \| \ \|)$. We have a natural map $dd^c \colon \mathscr{D}(X) \to \mathcal{Z}^{1,1}(X); \ f \mapsto c_1(\mathcal{O}_X, \| \ \|_{\mathrm{triv}} \cdot e^{-f})$.

A closed $(1,1)$-form θ is called *semipositive* if it is represented by a nef element $\theta_{\mathscr{X}} \in N^1(\mathscr{X}/S)$ for some model $\mathscr{X}$ of X. We say that a model metric $\| \ \|$ on L^{an} for a line bundle L on X is *semipositive* if the same holds for the curvature form $c_1(L, \| \ \|)$.

2.5. For $\theta \in \mathcal{Z}^{1,1}(X)$ we denote by

$$\mathrm{PSH}_{\mathscr{D}}(X, \theta) = \{f \in \mathscr{D}(X) \mid \theta + dd^c f \in \mathcal{Z}^{1,1}(X) \text{ is semipositive}\}$$

the set of θ-*plurisubharmonic* (θ-psh for short) model functions. Recall from [GM16, Prop. 3.12] that the set $\mathrm{PSH}_{\mathscr{D}}(X, \theta)$ is stable under the formation of max.

2.6. If Y is a proper variety over an arbitrary field k, we denote by $N^1(Y)_{\mathbb{Q}}$ the rational vector space $\mathrm{Pic}(Y) \otimes \mathbb{Q}$ modulo numerical equivalence. Similarly, we denote by $N^1(Y) = N^1(Y)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$ the real vector space $\mathrm{Pic}(Y) \otimes \mathbb{R}$ modulo numerical equivalence. A class in $N^1(Y)$ is called *ample* if it is an $\mathbb{R}_{>0}$-linear combination of classes induced by ample line bundles on Y. An element $\alpha \in N^1(Y)_{\mathbb{Q}}$ (resp. $\alpha \in N^1(Y)$) is called *nef* if $\alpha \cdot C \geq 0$ for all closed curves C in Y.

2.7. The restriction maps $N^1(\mathscr{X}/S) \to N^1(X)$, $[\mathscr{L}] \mapsto [\mathscr{L}|_X]$ induce a linear map $\{\ \} \colon \mathcal{Z}^{1,1}(X) \longrightarrow N^1(X)$, $\theta \mapsto \{\theta\}$. We call $\{\theta\}$ the *de Rham class* of θ.

Definition 2.8. Let X be a projective variety over K and $\theta \in \mathcal{Z}^{1,1}(X)$ with de Rham class $\{\theta\} \in N^1(X)$. The θ-*psh envelope* $P_\theta(u)$ of $u \in C^0(X^{\mathrm{an}})$ is the function

$$(2.2) \qquad P_\theta(u) \colon X^{\mathrm{an}} \to \mathbb{R} \cup \{-\infty\}, \ P_\theta(u)(x) = \sup\{\varphi(x) \,|\, \varphi \in \mathrm{PSH}_{\mathscr{D}}(X,\theta) \wedge \varphi \le u\}.$$

Note that $P_\theta(u)$ is a real valued function if and only if there exists a θ-psh model function. For the existence of a θ-psh model function, it is necessary that the de Rham class $\{\theta\}$ is nef (see [GM16, 4.8] and [BFJ16a, Rem. 5.4]). If $\{\theta\}$ is ample, then there exists always a θ-psh model function and hence $P_\theta(u)$ is a real valued function. If there is no θ-psh function, then $P_\theta(u) \equiv -\infty$ by definition.

If the residue characteristic is zero and if the de Rham class $\{\theta\}$ is ample, our definition of $P_\theta(u)$ is by [BFJ16a, Thm. 8.3 and Lemma 8.9] equivalent to the definition of Boucksom, Favre, and Jonsson in [BFJ16a, Def. 8.1] .

The next proposition collects elementary properties of envelopes.

Proposition 2.9. *Let $u, u' \in C^0(X^{\mathrm{an}})$ and $\theta, \theta' \in \mathcal{Z}^{1,1}(X)$.*
  (i) *If $u \le u'$ then $P_\theta(u) \le P_\theta(u')$.*
  (ii) *We have $P_{t\theta+(1-t)\theta'}(tu + (1-t)u') \ge tP_\theta(u) + (1-t)P_{\theta'}(u')$ for all $t \in [0,1]$.*
  (iii) *We have $P_\theta(u) + c = P_\theta(u + c)$ for each $c \in \mathbb{R}$.*
  (iv) *We have $P_\theta(u) - v = P_{\theta+dd^cv}(u - v)$ for each $v \in \mathscr{D}(X)$.*
  (v) *If $P_\theta(u) \not\equiv -\infty$, then we have $\sup_{X^{\mathrm{an}}} |P_\theta(u) - P_\theta(u')| \le \sup_{X^{\mathrm{an}}} |u - u'|$.*
  (vi) *If θ is determined on a model $\mathscr{X}$, if the de Rham class $\{\theta\}$ is ample and if $\theta_m \to \theta$ in $N^1(\mathscr{X}/S)$, then $P_{\theta_m}(u) \to P_\theta(u)$ uniformly on X^{an}.*
  (vii) *We have $P_{t\theta}(tu) = tP_\theta(u)$ for all $t \in \mathbb{R}_{>0}$.*
  (viii) *Assume $P_\theta(u) \not\equiv -\infty$. Then the envelope $P_\theta(u)$ is continuous if and only if it is a uniform limit of θ-psh model functions.*

Proof. The proof of Properties (i)–(vi) in [BFJ16a, Prop. 8.2] works in our setup as well. For (vi) it was used that an ample line bundle extends to an ample line bundle on a sufficiently high model which holds in our more general setting by [GM16, Proposition 4.11]. Property (vii) is obvious for $t \in \mathbb{Q}_{>0}$ and an easy approximation argument then shows (vii) in general. We have seen that θ-psh model functions are closed under max and hence the θ-psh model functions $\varphi \le u$ form a directed family. We conclude that (viii) follows from Dini's Theorem for nets [Kel75, p. 239] and the definition of $P_\theta(u)$. $\square$

For the next proposition, we assume for simplicity that the valuation on K is discrete.

Proposition 2.10. *Let L be an ample line bundle on X, $\mathscr{L}$ an extension to a model $\mathscr{X}$ and $\theta = c_1(L, \|\ \|_{\mathscr{L}}) \in \mathcal{Z}^{1,1}(X)$. For $m > 0$ let*

$$(2.3) \qquad\qquad \mathfrak{a}_m = \mathrm{Im}\left(H^0(\mathscr{X}, \mathscr{L}^{\otimes m}) \otimes_{K^\circ} \mathscr{L}^{\otimes -m} \to \mathcal{O}_{\mathscr{X}}\right)$$

be the m-th base ideal of $\mathscr{L}$ and $\varphi_m := m^{-1}\log|\mathfrak{a}_m|$. Then $\varphi_m \in \mathrm{PSH}_{\mathscr{D}}(X,\theta)$ and

$$(2.4) \qquad\qquad \lim_{m \to \infty} \varphi_m = \sup_{m \in \mathbb{N}} \varphi_m = P_\theta(0)$$

pointwise on X^{an}.

Proof. This is shown as in Step 1 of the proof of [BFJ16a, Thm. 8.5]. Observe that the arguments which show (2.4) in *loc. cit.* on the subset of quasimonomial points give us (2.4) immediately on X^{an} using our different definition of the θ-psh envelope. $\square$

Proposition 2.11. *Let K'/K be a finite normal extension and let $q\colon X' := X \otimes_K K' \to X$ be the natural projection. For $\theta \in \mathcal{Z}^{1,1}(X)$ and $u \in C^0(X^{\mathrm{an}})$, we have*

$$(2.5) \qquad\qquad q^*(P_\theta(u)) = P_{q^*\theta}(q^*(u)).$$

Proof. Splitting the extension K'/K into a purely inseparable part and a Galois part, we can reduce to two cases. In the first case of a purely inseparable extension, the result follows from Lemma 2.13 below. In the second case of a Galois extension, we can apply the argument of [BFJ15, Lemma A.4]. $\qquad\square$

Remark 2.12. (i) We always equip strictly K-analytic spaces with the *G-topology* induced by the strictly K-affinoid domains. We refer to [Ber90, §2.2] for the notion of a strictly K-affinoid domain and to [Ber93, §1.6] for the construction of the G-topology.

(ii) We recall from [GM16, Def. 2.8, 2.11] that a piecewise $\mathbb{Q}$-linear function on a strictly K-analytic space W is a function $f\colon W \to \mathbb{R}$ such that there is a G-covering $\{U_i\}_{i \in I}$ of W by strictly affinoid domains, analytic functions $\gamma_i \in \mathcal{O}(U_i)^\times$ and non-zero $m_i \in \mathbb{N}$ with $m_i f = -\log|\gamma_i|$ on U_i for every $i \in I$. By [GM16, Rem. 2.6, Prop. 2.10], the notions of model functions and $\mathbb{Q}$-linear functions agree.

Lemma 2.13. *Let L be a line bundle on X. Let K'/K be a finite purely inseparable extension and let $q\colon X' := X \otimes_K K' \to X$ be the natural projection. G-topology induced by the strictly K-affinoid domains. Then the map $(X')^{\mathrm{an}} \to X^{\mathrm{an}}$ is a homeomorphism and it also identifies the G-topologies. The map q^* induces a bijection between the set of model metrics on L and the set of model metrics on $q^*(L)$. Moreover this bijection identifies semipositive metrics on L and on $q^*(L)$.*

Proof. As in Remark 2.12 we use the G-topology induced by the strictly K-affinoid domains. We first prove that the map $q\colon (X')^{\mathrm{an}} \to X^{\mathrm{an}}$ is a homeomorphism and that it also identifies the G-topologies. In fact, this follows easily from the following claim:

Step 1: Let V be a strictly affinoid space over K and $V' := V \hat{\otimes}_K K'$. Then the natural projection $q\colon V' \to V$ is a homeomorphism which identifies the G-topologies.

Let $p^e = [K' : K]$ be the degree of the purely inseparable field extension. It is clear that for every $g \in \mathcal{O}(V')$, there is $f \in \mathcal{O}(V)$ with

$$(2.6) \qquad\qquad g^{p^e} = f \circ q.$$

This property easily shows that $q\colon V' \to V$ is a homeomorphism which we read now as an identification. Using that (2.6) holds also for rational functions g on V' and f on V, we see that V and V' have the same strictly rational domains. By the Gerritzen–Grauert theorem [BGR84, Cor. 7.3.5/3], we deduce the Step 1.

Next we prove the bijective correspondence between the model metrics on L and on L'. Since L admits a model metric [GM16, 2.1], it is enough to show that we have a bijective correspondence between model functions on X^{an} and model functions on $(X')^{\mathrm{an}}$. By Remark 2.12 we may check the same correspondence for $\mathbb{Q}$-linear functions. This may then be done G-locally and hence the correspondence follows from the following step.

Step 2: Using the same assumptions as in Step 1, the map $f \mapsto f \circ q$ is an isomorphism from the group of piecewise $\mathbb{Q}$-linear functions on V onto the group of piecewise $\mathbb{Q}$-linear functions on V'.

Using the above definition of piecewise $\mathbb{Q}$-linear functions, Step 1 and (2.6) yield easily Step 2.

To deduce the lemma, it remains to check that the identification between the model metrics on L and L' preserves semipositivity. This is an easy consequence of the projection formula applied to finite morphisms between closed curves in the special fibers of models. $\qquad\square$

3. CONTINUITY OF PLURISUBHARMONIC ENVELOPES ON CURVES

In this section, K is any field endowed with a non-trivial non-archimedean complete valuation $v \colon K \to \mathbb{R}$ with value group $\Gamma \subset \mathbb{R}$. In this section, we consider a smooth projective curve X over K. The goal is to prove the following result:

Theorem 3.1. *If θ is a closed $(1,1)$-form on X^{an} with nef de Rham class $\{\theta\}$ and if $u \in C^0(X^{\mathrm{an}})$, then $P_\theta(u)$ is a uniform limit of θ-psh model functions and thus $P_\theta(u)$ is continuous on X^{an}.*

3.2. As a main tool in the proof, we need strictly semistable models of X and their canonical skeletons. This construction is due to Berkovich in [Ber99]. We recall here only the case of a smooth projective curve X over K for which we can also refer to [Thu05].

A K°-model $\mathscr{X}$ of X as in 2.1 is called *strictly semistable* if there is an open covering of $\mathscr{X}$ by open subsets $\mathcal{U}$ such that there are étale morphisms $\mathcal{U} \to \mathrm{Spec}(K^\circ[x,y]/(xy - \rho_\mathcal{U}))$ for some $\rho_\mathcal{U} \in K^{\circ\circ}$. Applying the construction in [Thu05, §2.2] to the associated formal scheme $\hat{\mathscr{X}}$, we get a canonical *skeleton* $S(\mathscr{X}) \subseteq X^{\mathrm{an}}$ with a proper strong deformation retraction $\tau \colon X^{\mathrm{an}} \to S(\mathscr{X})$. The skeleton $S(\mathscr{X})$ carries a canonical structure of a metrized graph. We note that the generic fiber of the formal scheme $\hat{\mathcal{U}}$ intersects $S(\mathscr{X})$ in an edge of length $v(\rho_\mathcal{U})$. By using the reduction map, the vertices of $S(\mathscr{X})$ correspond to the irreducible components of the special fiber $\mathscr{X}_s$ and the open edges of $S(\mathscr{X})$ correspond to the singular points of $\mathscr{X}_s$.

Remark 3.3. By definition, a strictly semistable model $\mathscr{X}$ of X is proper over K°. Using that X is a curve, we will deduce that $\mathscr{X}$ is projective over K°. Indeed, the special fiber $\mathscr{X}_s$ is a proper curve over the residue field and hence projective. It is easy to construct an effective Cartier divisor D on $\mathscr{X}$ whose support intersects any irreducible component of $\mathscr{X}_s$ in a single closed point. By [Liu06, Exercise 7.5.3], the restriction of D to $\mathscr{X}_s$ is ample. It follows from [EGAIV, Cor. 9.6.4] that D is ample and hence $\mathscr{X}$ is projective.

Similarly, we can define strictly semistable formal models of X^{an}. Using that X is a smooth projective curve, the algebraization theorem of Grothendieck [EGAIII, Thm. 5.4.5] and its generalizations to the non-noetherian setting [Abb11, Cor. 2.13.9], [FK88, Prop. I.10.3.2] show that formal completion induces an equivalence of categories between strictly semistable algebraic models of X and strictly semistable formal models of X^{an}. Here, we need a similar argument as above to construct an effective formal Cartier divisor which restricts to an ample Cartier divisor on the special fiber.

Definition 3.4. A function $f \colon S(\mathscr{X}) \to \mathbb{R}$ is called *piecewise linear* if there is a subdivision of $S(\mathscr{X})$ such that the restriction of f to each edge of the subdivison is affine. We call such an f *integral Γ-affine* if there is a subdivision such that each edge e has length in Γ, such that $f|_e$ has integer slopes, and such that $f(v) \in \Gamma$ for each vertex v of the subdivision.

Proposition 3.5. *Let $\mathscr{X}$ be a strictly semistable model of X and let $f \colon X^{\mathrm{an}} \to \mathbb{R}$ be a function. Then the following properties hold:*

(a) *If f is a $\mathbb{Z}$-model function, then $f|_{S(\mathscr{X})}$ is a piecewise linear function which is integral Γ-affine.*

(b) *The function f is a $\mathbb{Z}$-model function determined on $\mathscr{X}$ if and only if $f = F \circ \tau$ for some function $F \colon S(\mathscr{X}) \to \mathbb{R}$ which is affine on each edge of $S(\mathscr{X})$ with integer slopes and with $f(v) \in \Gamma$ for each vertex v of $S(\mathscr{X})$.*

(c) *If G is a piecewise linear function on $S(\mathscr{X})$ which is integral Γ-affine, then $G \circ \tau$ is a $\mathbb{Z}$-model function.*

Proof. In the G-topology on X^{an} induced by the strictly K-affinoid domains, a $\mathbb{Z}$-model function is given locally by $-\log|\gamma|$ for a rational function γ on X. Hence (a) follows from [GRW16, Prop. 5.6]. Property (b) was proven in [GH15, Prop. B.7] for any dimension.

To prove (c), we choose a subdivision of $S(\mathscr{X})$ as in Definition 3.4 for G. As in [BPR13, §3], this subdivison is the skeleton of a strictly semistable model $\mathscr{X}'$ dominating $\mathscr{X}$ and with the same retraction τ. Then (c) follows from (b). Note that the quoted papers have the standing assumption that K is algebraically closed, but it is straightforward to verify that this was not used for the quoted results. $\square$

3.6. Now we consider a model function f on X^{an}. Using the setting of Proposition 3.5 and (b), we see that $f = F \circ \tau$ for a piecewise linear function F on $S(\mathscr{X})$ such that mF is integral Γ-affine for some non-zero $m \in \mathbb{N}$. We also assume that θ is a closed $(1,1)$-form on X^{an} which is determined on our given strictly semistable model $\mathscr{X}$.

We have the following useful characterization for f to be θ-psh in terms of slopes:

Proposition 3.7. *Under the hypotheses from 3.6, the model function f is θ-psh if and only if F satisfies for all $x \in \Delta := S(\mathscr{X})$*

$$(3.1) \qquad \sum_{\nu \in T_x(\Delta)} w_x(\nu)\lambda_{x,\nu}(F) + \deg(\theta|_{\mathcal{C}_x}) \geq 0,$$

where ν ranges over the set $T_x(\Delta)$ of outgoing tangent directions at x. Here, $\lambda_{x,\nu}(F)$ denotes the slope of F at x along ν and we have the weight $w_x(\nu) := [\tilde{K}(p_\nu) : \tilde{K}]$ for the singularity p_ν of $\mathscr{X}_s$ corresponding to the edge of $S(\mathscr{X})$ at x in the direction of ν. Moreover, if x is a vertex of $S(\mathscr{X})$, then $\mathcal{C}_x$ denotes the corresponding irreducible component $\mathcal{C}_x$ of $\mathscr{X}_s$ and if x is not a vertex, then $\deg(\theta|_{\mathcal{C}_x}) := 0$.

Proof. If we pass to the completion $\mathbb{C}_K$ of an algebraic closure of K, there is a strictly semistable model $\mathscr{X}'$ dominating $\mathscr{X}$ such that f is determined on $\mathscr{X}'$. This is proven in [BL85, §7]. We note that the property θ-psh holds if and only if the corresponding property holds after base change to $\mathbb{C}_K$. This is a consequence of the projection formula in algebraic intersection theory. Since the degree is invariant under base change, it follows from [Thu05, Prop. 2.2.21] that the left hand side of (3.1) is invariant under base change as well. We conclude that we may assume that K is algebraically closed and that $\mathscr{X} = \mathscr{X}'$, i.e. f is determined on $\mathscr{X}$. Then (3.1) follows from the slope formula of Katz, Rabinoff, and Zureick-Brown [KRZB16, Thm. 2.6]. $\square$

The next result is crucial for the proof of Theorem 3.1. It is well-known to the experts, but in our quite general setting we could not find a proof in the literature (a special case was proven in [GH15, B.16]). The result is related to the fact that the retraction from a graph to a subgraph preserves subharmonicity of functions (see for example [Jon15, Sect. 2.5.1] for the case of trees).

Proposition 3.8. *Let $\mathscr{X}$ be a strictly semistable model of X with canonical retraction $\tau \colon X^{\mathrm{an}} \to S(\mathscr{X})$. Let $\theta \in \mathcal{Z}^{1,1}(X)$ be determined on $\mathscr{X}$ and let $\varphi \colon X^{\mathrm{an}} \to \mathbb{R}$ be an arbitrary θ-psh model function. Then $\varphi \circ \tau \colon X^{\mathrm{an}} \to \mathbb{R}$ is a θ-psh model function with $\varphi \leq \varphi \circ \tau$.*

Proof. It follows from Proposition 3.5 that $\varphi \circ \tau$ is a model function. To check that $\varphi \circ \tau$ is θ-psh, we may assume K algebraically closed as we have seen in the proof of Proposition 3.7. Moreover, we have seen that there is a strictly semistable model $\mathscr{X}'$ of X dominating $\mathscr{X}$ such that f is determined on $\mathscr{X}'$. Then

$$\Delta := S(\mathscr{X}) \subset \Delta' := S(\mathscr{X}')$$

and $\varphi \circ \tau$ is constant along edges of Δ' which are not contained in Δ. By Proposition 3.5, there is a piecewise linear function F' on Δ' with $\varphi = F' \circ \tau'$ for the canonical retraction $\tau' \colon X^{\mathrm{an}} \to \Delta'$ such that mF' is integral Γ-affine for a non-zero $m \in \mathbb{N}$. Moreover, the function F' is affine on the edges of Δ'. Let F be the restriction of F' to Δ. The same arguments as in [BFJ16a, Prop. 5.7] show that the θ-psh function φ is a uniform limit of functions of the form $\frac{1}{m} \log |\mathfrak{a}|$ with non-zero $m \in \mathbb{N}$ and with a vertical fractional ideal sheaf $\mathfrak{a}$ on $\mathscr{X}$. By [Ber99, Thm. 5.2(ii)], we deduce that $\varphi \leq \varphi \circ \tau$. Using the terminology introduced in Proposition 3.7, for all $x \in \Delta$ and $v \in T_x(\Delta')$ we obtain $\lambda_{x,\nu}(F') = \lambda_{x,\nu}(F)$ if $v \in T_x(\Delta')$ and $\lambda_{x,\nu}(F') \leq 0$ if $\nu \in T_x(\Delta') \setminus T_x(\Delta)$. This implies

$$0 \leq \sum_{\nu \in T_x(\Delta')} w_x(\nu)\lambda_{x,\nu}(F') + \deg(\theta|_{\mathcal{C}_x}) \leq \sum_{\nu \in T_x(\Delta)} w_x(\nu)\lambda_{x,\nu}(F) + \deg(\theta|_{\mathcal{C}_x})$$

Here this first inequality comes from Proposition 3.7, since φ is θ-psh. Applying Proposition 3.7 again, we conclude that $\varphi \circ \tau = F \circ \tau$ is θ-psh. $\qquad\square$

Remark 3.9. Proposition 3.8 does not hold for higher dimensional varieties. We refer to the Appendix for a toric counterexample in dimension two by José Burgos and Martín Sombra.

The following special case of model functions is crucial for the proof of Theorem 3.1. Especially for model functions, we can say much more about the envelope.

Proposition 3.10. *Let θ be a closed $(1,1)$-form with nef de Rham class $\{\theta\}$ on the smooth projective curve X over K and let $f \colon X^{\mathrm{an}} \to \mathbb{R}$ be a model function. We assume that θ and f are determined on the strictly semistable model $\mathscr{X}$ of X. Let $\tau \colon X^{\mathrm{an}} \to S(\mathscr{X})$ be the canonical retraction to the skeleton. Then the following properties hold:*

- *(i) There is $F \colon S(\mathscr{X}) \to \mathbb{R}$ which is affine on each edge and with $P_\theta(f) = F \circ \tau$.*
- *(ii) If $\Gamma \subset \mathbb{Q}$ and if $\theta \in \mathcal{Z}^{1,1}(X)_{\mathbb{Q}}$, then $P_\theta(f)$ is a θ-psh model function which is determined on $\mathscr{X}$.*

Proof. Step 1. By Proposition 2.9(iv), we have

$$P_\theta(f) = P_{\theta + dd^c f}(0) + f.$$

Hence replacing θ by $\theta + dd^c f$ and f by 0, we can assume that $f = 0$ by Proposition 3.5(b).

Step 2. Let $\Delta := S(\mathscr{X})$ denote the skeleton of $\mathscr{X}$. By Propositions 3.5 and 3.8, we get that

$$(3.2) \qquad\qquad\qquad P_\theta(0) = \sup_{F \in \mathcal{A}} F \circ \tau$$

for the set $\mathcal{A}$ of non-positive piecewise linear functions F on Δ such that mF is integral Γ-affine for some $m \in \mathbb{N}_{>0}$ and such that $F \circ \tau$ is θ-psh. Note that the piecewise linear functions are not assumed to be affine on the edges of Δ. Since X is a smooth projective curve and the de Rham class $\{\theta\}$ is nef, it is clear that $\mathcal{A}$ is non-empty. We introduce the function $F_0 \colon \Delta \to \mathbb{R}$ defined by

$$F_0 := \sup_{F \in \mathcal{A}} F.$$

By (3.2), we get that $P_\theta(0) = F_0 \circ \tau$. Hence we can reduce (i) to prove that F_0 is affine on each edge of Δ.

Step 3. For $F \in \mathcal{A}$, let $L(F) \colon \Delta \to \mathbb{R}$ be the function which is affine on the edges of Δ and which agrees with F on the set V of vertices of Δ. As $F \leq 0$, we deduce

immediately $L(F) \leq 0$. Since $F \circ \tau$ is θ-psh, Proposition 3.7 shows that F is convex on each edge of Δ and hence $F \leq L(F)$.

By passing from F to $L(F)$, the slopes do not decrease in the vertices and using that $F \circ \tau$ is θ-psh, it follows from Proposition 3.7 that $L(F) \circ \tau$ is θ-psh as well.

The slopes of the function $L(F)$ might be non-rational. However, we can approximate the slopes of $L(F)$ in a rational way at any vertex and thus for any $\varepsilon > 0$ we find a piecewise linear function $L_\varepsilon(F)$ on Δ such that

   (i) $L_\varepsilon(F)$ agrees with F on V.
   (ii) $L_\varepsilon(F)$ has rational slopes.
   (iii) $L_\varepsilon(F)$ is convex on the edges of Δ.
   (iv) $L_\varepsilon(F) \geq F$.
   (v) $\sup |L_\varepsilon(F) - L(F)| < \varepsilon$

We claim that $L_\varepsilon(F) \in \mathcal{A}$. It follows from (ii) that $m L_\varepsilon(F)$ is integral Γ-affine for some $m \in \mathbb{N}_{>0}$. Since F and $L(F)$ agree on V, it is clear from (i) and (iii) that $L_\varepsilon(F) \leq L(F) \leq 0$. To show that $L_\varepsilon(F) \circ \tau$ is θ-psh, we use the slope criterion from Proposition 3.7. Note that (3.1) is fulfilled in the interior of each edge of Δ by (iii). In a vertex of Δ, the inequality (3.1) is satisfied by using the corresponding inequality for F, (i) and (iv). This proves $L_\varepsilon(F) \in \mathcal{A}$. As a consequence we find

$$(3.3) \qquad F_0 = \sup_{F \in \mathcal{A}} F = \sup_{F \in \mathcal{A}} L_\varepsilon(F) = \sup_{F \in \mathcal{A}} L(F).$$

Step 4. We claim $F_0 = L(F_0)$. First note that since the max of convex functions in convex, F_0 is convex on each edge of Δ, thus $F_0 \leq L(F_0)$.

We pick an $\varepsilon > 0$. For any $v \in V$, there is $f_v \in \mathcal{A}$ with $f_v(v) > L(F_0)(v) - \varepsilon$. It follows from [GM16, Prop. 3.12] that the maximum of two θ-psh model functions is again a θ-psh model function. Using 2.5, we conclude that $\mathcal{A}$ is closed under the operation max. Thus $L(\max\{f_v \mid v \in V\}) \in \mathcal{A}$ is in ε-distance to $L(F_0)$ at every vertex of Δ and hence at every point of Δ. As $\varepsilon > 0$ can be chosen arbitrarily small, (3.3) yields $L(F_0) \leq F_0$ and hence we get Step 4. Note that Step 4 proves (i).

Step 5. In the case $\Gamma \subset \mathbb{Q}$, we have $L(F) \in \mathcal{A}$ for $F \in \mathcal{A}$. Indeed, we note that in this special case the edges have rational lengths and F takes rational values at V. We deduce from Proposition 3.5 that $L(F) \circ \tau$ is a model function. Let $\mathcal{B}$ be the set of $F \in \mathcal{A}$ such that F is affine on every edge of Δ. For $F \in \mathcal{A}$ we thus have $L(F) \in \mathcal{B}$ which shows via (3.3) that we might restrict the sup to $\mathcal{B}$ in the definition of F_0. Recall that V is the set of vertices of Δ. Since the edge lengths of Δ are rational, the map $\Psi \colon \mathcal{B} \to \mathbb{R}^V$ defined by $\Psi(F) = (F(v))_{v \in V}$ identifies $\mathcal{B}$ with the rational points of a rational polyhedron in $\mathbb{R}^{\check V}$ defined by the linear inequalities of Proposition 3.7. Note that by affineness on the edges, we need to check the slope inequalities only at the vertices. If a rational linear form φ is bounded from above on a rational polyhedron P, then $\varphi|_P$ achieves its maximum in a rational point. Hence there exists $G \in \mathcal{B}$ such that

$$(3.4) \qquad \sum_{v \in V} G(v) = \max_{F \in \mathcal{B}} \left(\sum_{v \in V} F(v) \right).$$

We claim that $G = F_0$. Considering $F \in \mathcal{B}$, we get $\max(G, F) \in \mathcal{A}$ by Step 4 and hence $H' := L(\max(G, F)) \in \mathcal{B}$. Hence $H' \geq G$, $H' \geq F$ and $H' \in \mathcal{B}$. But by (3.4) we deduce that for $v \in V$ we have $H'(v) = G(v)$. Since functions in $\mathcal{B}$ are determined by their values on V, we have $H' = G$. Hence $G \geq F$, whence $G = F_0$. It follows that $P_\theta(0) = F_0 \circ \tau = G \circ \tau$ is a θ-psh function proving (ii). □

Proof of Theorem 3.1. By Proposition 2.9 (viii) it is enough to prove the continuity of $P_\theta(u)$. By the semistable reduction theorem [BL85, §7], there is a finite field extension K'/K such that $X' := X \otimes_K K'$ has a strictly semistable model $\mathscr{X}'$ with $\theta' := q^*\theta$ determined on $\mathscr{X}'$, where $q \colon X' \to X$ is the canonical map. It follows from Proposition 3.10 that $P_{\theta'}(u \circ q)$ is continuous. We know from Lemma 2.11 that

$$P_{\theta'}(u \circ q) = q^*(P_\theta(u)).$$

By [Ber90, Prop. 1.3.5], the topological space of X^{an} is the quotient of $(X')^{\mathrm{an}}$ by the automorphism group of K'/K. We conclude that $P_\theta(u)$ is continuous. □

In the following, we consider an ample line bundle L on the projective smooth curve X over K. Recall that we have defined the semipositive envelope $P(\| \|)$ of a continuous metric $\| \|$ on L^{an} in (1.2).

Corollary 3.11. *Assume that $\Gamma \subset \mathbb{Q}$. Let $\| \|$ be a model metric on L. Then $P(\| \|)$ is a semipositive model metric on L.*

Proof. By definition, we have $P(\| \|) = \| \| e^{-P_\theta(0)}$ for $\theta := c_1(L, \| \|)$, hence the claim follows from Proposition 3.10. □

From now on, we assume that K is discretely valued. The goal is to prove some rationality results for the non-archimedean volumes on the line bundle L of the smooth projective curve X over K. Non-archimedean volumes $\mathrm{vol}(L, \| \|_1, \| \|_2)$ with respect to continuous metrics $\| \|_1, \| \|_2$ on L^{an} are analogues of volumes $\mathrm{vol}(L)$ in algebraic geometry. We refer to [BGJKM16, Def. 4.1.2] for the precise definition. By the Riemann–Roch theorem, $\mathrm{vol}(L) \in \mathbb{Q}$ in the special case of curves. We will show a similar result about non-archimedean volumes.

Corollary 3.12. *Let K be a field endowed with a complete discrete valuation with value group $\Gamma \subset \mathbb{Q}$. Let $\| \|_1$ and $\| \|_2$ be two model metrics on the line bundle L of the smooth projective curve X over K. Then $\mathrm{vol}(L, \| \|_1, \| \|_2) \in \mathbb{Q}$.*

Proof. If $\deg(L) \leq 0$, then it is clear from the definition that $\mathrm{vol}(L, \| \|_1, \| \|_2) = 0$. So we may assume that L is ample. We need the energy $E(L, \| \|_1, \| \|_2)$ with respect to continuous semipositive metrics $\| \|_1, \| \|_2$ on L^{an} introduced in [BGJKM16, Def. 2.4.4]. By Corollary 3.11, the envelopes $P(\| \|_1)$ and $P(\| \|_2)$ are semipositive model metrics on L^{an}. In particular, they are continuous and hence it follows from [BGJKM16, Cor. 6.2.2] that

$$\mathrm{vol}(L, \| \|_1, \| \|_2) = E(L, P(\| \|_1), P(\| \|_2)).$$

In the case of semipositive model metrics associated to line bundles $\mathscr{L}_1, \mathscr{L}_2$ on a K°-model $\mathscr{X}$, our assumption $\Gamma \subset \mathbb{Q}$ yields that the energy is defined as a $\mathbb{Q}$-linear combination of intersection numbers of the line bundles $\mathscr{L}_1, \mathscr{L}_2$ on $\mathscr{X}$ proving the claim. □

Remark 3.13. When $\dim(X) \geq 3$, there are varieties with line bundles L such that $\mathrm{vol}(L)$ is irrational (see [ELMN05, Example 2.2] or [Laz04, Example 2.3.8]). Hence with our definition and normalization of non-archimedean volumes, we get for a model metric $\| \|$ that $\mathrm{vol}(L, \| \|, e^\lambda \| \|) = \mathrm{vol}(L)\lambda$ which produces irrational non-archimedean volumes. The following natural questions remain open:

(i) What happens if X is a variety of dimension two? Are non-archimedean volumes rational? Note that by Zariski decomposition, $\mathrm{vol}(L)$ is rational then (see for instance [Laz04, Cor. 2.3.22]).

(ii) If we normalize our non-archimedean volumes by $\mathrm{vol}(L)$, can we find an example of some model metrics $\|\ \|$ and $\|\ \|'$ with irrational $\mathrm{vol}(L, \|\ \|, \|\ \|')$? The idea is to avoid the trivial example above.

4. Asymptotic test ideals

We recall definitions and some basic properties from the theory of generalized and asymptotic test ideals developed in [BMS08, Sect. 2] and [Mus13, Sect. 3]. We refer to [ST12] for a more comprehensive overview of the theory of test ideals. Let X be a smooth variety over a perfect field k of characteristic $p > 0$. Let $F\colon X \to X$ denote the Frobenius morphism which is induced by the p-th power ring morphism on affine subsets. Write

$$\omega_{X/k} = \det \Omega_{X/k} = \mathcal{O}_X(K_{X/k})$$

for some canonical divisor $K_{X/k}$ on X.

Let $\mathfrak{a}$ be an ideal in $\mathcal{O}_X$ and $e \in \mathbb{N}_{>0}$. There is a unique ideal $\mathfrak{a}^{[p^e]}$ in $\mathcal{O}_X$ such that for every open affine U in X the ideal $\mathfrak{a}^{[p^e]}(U)$ in $\mathcal{O}_X(U)$ is generated by $\{u^{p^e} \mid u \in \mathfrak{a}(U)\}$. We have [BMS08, bottom of p. 44]

$$(4.1) \qquad\qquad \mathfrak{a}(U) = \{a \in \mathcal{O}_X(U) \mid a^{p^e} \in \mathfrak{a}^{[p^e]}(U)\}$$

We recall the following facts from [Mus13, p. 540]: There is a canonical trace map $\mathrm{Tr}\colon F_*(\omega_{X/k}) \to \omega_{X/k}$ whose construction can be based on the Cartier isomorphism [Kat70, Thm. (7.2) and Eq. (7.2.3)]. Mustaţă gives an explicit description of the trace map [Mus13, top of p. 540]. Given $e \in \mathbb{N}_{>0}$ there is an iterated trace map $\mathrm{Tr}^e\colon F^e_*(\omega_{X/k}) \to \omega_{X/k}$. For an ideal $\mathfrak{a}$ in $\mathcal{O}_X$ there exists a unique ideal $\mathfrak{a}^{[1/p^e]}$ in $\mathcal{O}_X$ with

$$(4.2) \qquad\qquad \mathrm{Tr}^e(F^e_*(\mathfrak{a} \cdot \omega_{X/k})) = \mathfrak{a}^{[1/p^e]} \cdot \omega_{X/k}.$$

This definition of $\mathfrak{a}^{[1/p^e]}$ is compatible with [BMS08, Def. 2.2]. Hence we have

$$(4.3) \qquad\qquad \left(\mathfrak{a}^{[p^e]}\right)^{[1/p^e]} = \mathfrak{a} \subseteq \left(\mathfrak{a}^{[1/p^e]}\right)^{[p^e]}$$

by [BMS08, Lemma 2.4(iv)].

Definition 4.1. [BMS08, Def. 2.9] Given an ideal $\mathfrak{a}$ in $\mathcal{O}_X$ and $\lambda \in \mathbb{R}_{\geq 0}$ one defines the *test ideal of $\mathfrak{a}$ of exponent λ* to be

$$\tau(\mathfrak{a}^\lambda) := \bigcup_{e \in \mathbb{N}_{>0}} \left(\mathfrak{a}^{\lceil \lambda p^e \rceil}\right)^{[1/p^e]}$$

where given $r \in \mathbb{R}$ we write $\lceil r \rceil$ for the smallest integer $\geq r$.

Remark 4.2.   (i) Observe that we have $\tau(\mathfrak{a}^\lambda) = (\mathfrak{a}^{\lceil \lambda p^e \rceil})^{[1/p^e]}$ for large $e \in \mathbb{N}$ as X is noetherian. The equality

$$(4.4) \qquad\qquad \tau((\mathfrak{a}^m)^\lambda) = \tau(\mathfrak{a}^{\lambda m})$$

for $m \in \mathbb{N}$ shows that the notation in Definition 4.1 is compatible with taking powers of ideals [BMS08, Cor. 2.15]. We have $\tau(\mathfrak{a}^\lambda) \subseteq \tau(\mathfrak{b}^\lambda)$ for ideals $\mathfrak{a} \subseteq \mathfrak{b}$ in $\mathcal{O}_X$ [BMS08, Prop. 2.11(i)].

(ii) Choose e such that $\tau(\mathfrak{a}) = (\mathfrak{a}^{[p^e]})^{[1/p^e]}$. For any ideal $\mathfrak{b}$ in $\mathcal{O}_X$ such that $\mathfrak{a}^{[p^e]} \subseteq \mathfrak{b}^{[p^e]}$ we get $\mathfrak{a} \subseteq \mathfrak{b}$ from (4.1). Hence (4.3) implies

$$(4.5) \qquad\qquad \mathfrak{a} \subseteq \tau(\mathfrak{a}).$$

Let $\mathfrak{a}_\bullet$ be *graded sequence of ideals in $\mathcal{O}_X$*, i.e. a family $(\mathfrak{a}_m)_{m \in \mathbb{N}_{>0}}$ of ideals in $\mathcal{O}_X$ such that $\mathfrak{a}_m \cdot \mathfrak{a}_n \subseteq \mathfrak{a}_{m+n}$ for all $m, n \in \mathbb{N}_{>0}$ and $\mathfrak{a}_m \neq (0)$ for some $m > 0$.

Definition 4.3. [Mus13, p. 541] Choose $\lambda \in \mathbb{R}_{\geq 0}$. Define the *asymptotic test ideal of exponent* λ as

$$\tau(\mathfrak{a}_\bullet^\lambda) := \bigcup_{m \in \mathbb{N}} \tau(\mathfrak{a}_m^{\lambda/m}).$$

Remark 4.4.   (i) We have $\tau(\mathfrak{a}_\bullet^\lambda) = \tau(\mathfrak{a}_m^{\lambda/m})$ for suitable $m \in \mathbb{N}$ which are divisible enough [Mus13, p. 541].

 (ii) For all $m \in \mathbb{N}$ we have [Mus13, p. 541, l. 4]

(4.6) $$\tau(\mathfrak{a}_m) \subseteq \tau(\mathfrak{a}_\bullet^m).$$

 (iii) For all $m \in \mathbb{N}$ we have the *Subadditivity Property* [Mus13, Prop. 3.1(ii)]

(4.7) $$\tau(\mathfrak{a}_\bullet^{m\lambda}) \subseteq \tau(\mathfrak{a}_\bullet^\lambda)^m.$$

Definition 4.5. Let D be a divisor on X with $h^0(X, \mathcal{O}_X(mD)) \neq 0$ for some $m > 0$. Define the *asymptotic test ideal of exponent* $\lambda \in \mathbb{R}_{\geq 0}$ *associated with* X *and* D as

$$\tau(\lambda \cdot \|D\|) := \tau(\mathfrak{a}_\bullet^\lambda)$$

where $\mathfrak{a}_\bullet$ denotes the graded sequence of base ideals for D, i.e. $\mathfrak{a}_m$ is the image of the natural map

$$H^0(X, \mathcal{O}(mD)) \otimes_k \mathcal{O}_X(-mD) \to \mathcal{O}_X.$$

If D is a $\mathbb{Q}$-divisor such that $h^0(X, \mathcal{O}_X(mD)) \neq 0$ for some positive integer m such that mD is a usual divisor then we put $\tau(\lambda \cdot \|D\|) := \tau(\lambda/r \cdot \|rD\|)$ for some $r \in \mathbb{N}$ such that rD has integral coefficients.

   We finish with a slight generalization of Mustaţă's uniform generation property [Mus13, Thm. 4.1]. Observe that in *loc. cit.* it is required that the variety X is projective over the ground field k.

Theorem 4.6. *Let R be a k-algebra of finite type over a perfect field k of characteristic $p > 0$. Let X be an integral scheme of dimension n which is projective over the spectrum of R and smooth over k. Let D, E, and H be divisors on X and $\lambda \in \mathbb{Q}_{\geq 0}$ such that*

   (i) $\mathcal{O}_X(H)$ is an ample, globally generated line bundle,
   (ii) $h^0(X, \mathcal{O}_X(mD)) > 0$ for some $m > 0$, and
   (iii) the $\mathbb{Q}$-divisor $E - \lambda D$ is nef.

Then the sheaf $\mathcal{O}_X(K_{X/k} + E + dH) \otimes_{\mathcal{O}_X} \tau(\lambda \cdot \|D\|)$ is globally generated for all $d \geq n+1$.

Proof. We literally follow Mustaţă's proof with two modifications. The proof requires Mumford's theorem on Castelnuovo-Mumford regularity for the projective scheme X over R which holds also in this more general setting [BS13, 20.4.13]. Furthermore we replace the use of Fujita's vanishing theorem to the sheaves $\mathscr{F}_j := \mathcal{O}_{\mathscr{X}}(K_{X/k} + T_j)$, $j = 1, \dots, r$ and the ample divisor $(d-i)H$ by an application of Keeler's generalization [Kee03, Thm. 1.5]. □

5. Descent for model functions

   Let K denote a complete discretely valued field with valuation ring K°. Let R be a discrete valuation subring of K° whose completion is K°. Then K is the completion of the field of fractions F of R. In this section we show that all model functions on analytifications of varieties over K are already defined over R.

   An R-model of a projective variety over F is defined completely analogously to the complete case treated in 2.1.

Definition 5.1. Let X be a projective variety over K. We say that a *model function* $\varphi \colon X^{\mathrm{an}} \to \mathbb{R}$ *is defined over R* if there exists a projective variety Y over F, with an isomorphism $Y \otimes_F K \simeq X$, an R-model $\mathscr{Y}$ of Y, a vertical divisor D_0 on $\mathscr{Y}$ such that $\varphi = \frac{1}{m}\varphi_D$ where $m \in \mathbb{N}_{>0}$ and D is the vertical divisor on $\mathscr{Y} \otimes_R K^\circ$ obtained by pullback from D_0. Likewise we define the notion of a *vertical ideal sheaf defined over R*.

Here is the announced descent result.

Proposition 5.2. *Let Y be a projective variety over F and let $X := Y \otimes_F K$.*

(a) *Any K°-model of X is dominated by the base change of a projective R-model of Y to K°.*

(b) *If a projective K°-model $\mathscr{X}$ of X dominates $\mathscr{Y} \otimes_R K^\circ$ for a projective R-model $\mathscr{Y}$ of Y, then $\mathscr{X} \simeq \mathscr{Y}' \otimes_R K^\circ$ for a projective R-model $\mathscr{Y}'$ of Y dominating $\mathscr{Y}$.*

(c) *Every model function on X^{an} is defined over R.*

Proof. To prove (a), we pick any projective R-model $\mathscr{Y}$ of Y. By [Lü93, Lemma 2.2], there is a blowing up $\pi \colon \mathscr{X}' \to \mathscr{Y} \otimes_R K^\circ$ such that $\mathscr{X}'$ dominates $\mathscr{X}$. Since π is a projective morphism, $\mathscr{X}'$ is a projective K°-model dominating $\mathscr{X}$. Hence (a) follows from (b).

To prove (b), we note that the morphism $\mathscr{X} \to \mathscr{Y} \otimes_R K^\circ$ is a blowing up morphism along a vertical closed subscheme Z of $\mathscr{Y} \otimes_R K^\circ$ (see [Liu06, Thm. 8.1.24]). Since the ideal sheaf of Z contains a power of the uniformizer of R, we may define it over R and hence the same is true for the blowing up morphism and for $\mathscr{X}$ proving (b).

To prove (c), we may assume that the model function is associated to a vertical Cartier divisor D. Replacing D by $D + \mathrm{div}(\lambda)$ for a suitable non-zero $\lambda \in R$ and using (a) and (b), we may assume that D is an effective Cartier divisor on a projective R-model $\mathscr{Y}$ of Y. As in (b), we see that the ideal sheaf of D is defined by the ideal sheaf of a Cartier divisor D_0 defined over R proving (c). $\qquad\square$

6. Resolution of singularities

For our applications, we need that regular projective models are cofinal in the categroy of models which makes it necessary to assume resolution of singularities in a certain dimension.

Definition 6.1. Let k be a field. We say that *resolution of singularities holds over k in dimension n* if for every quasi-projective variety Y over k of dimension n there exists a regular variety $\tilde{Y}$ over k and a projective morphism $\tilde{Y} \to Y$ which is an isomorphism over the regular locus of Y.

To transfer results from [BFJ16a, BFJ15] to our context, it is essential to show that projective models are dominated by SNC-models. In order to this we are going to use the following assumption.

Definition 6.2. We say that *embedded resolution of singularities in dimension m* holds over a field k if for every quasi-projective regular variety Y over k of dimension m and every proper closed subset Z of Y, there is a projective morphism $\pi : Y' \to Y$ of quasi-projective regular varieties over k such that the set $\pi^{-1}(Z)$ is the support of a normal crossing divisor and such that π is an isomorphism over $Y \setminus Z$.

Hironaka has shown that resolution of singularities and embedded resolution of singularities holds over a field of characteristic zero in any dimension. Resolution of singularities holds over arbitrary fields in dimension one (Dedekind, M. Noether, Riemann)

and in dimension two (Abhyankar, Lipman). Cossart and Piltant have proven that resolution of singularities and embedded resolution of singularities hold in dimension three over perfect fields.

Theorem 6.3 (Cossart-Piltant). *Resolution of singularities and embedded resolution of singularities hold in dimension three over any perfect field.*

Proof. This is shown in [CP09, Thm. on p. 1839] and [CP08, Prop. 4.1]. □

7. Uniform convergence to the envelope of the zero function

Let K be a complete discretely valued field of positive characteristic $p > 0$. Let X be a smooth projective variety over K, L an ample line bundle on X, and $(\mathscr{X}, \mathscr{L})$ a model of (X, L) over K°. For $m \in \mathbb{N}_{>0}$ let $\mathfrak{a}_m$ denote the m-th base ideal of $\mathscr{L}$ as in (2.3). For a scheme B, we recall our convention $B^{(1)} = \{p \in B \mid \dim \mathcal{O}_{B,p} = 1\}$.

Assumption 7.1. There exist a normal affine variety B over a perfect field k, a codimension one point $b \in B^{(1)}$, a projective regular integral scheme $\mathscr{X}_B$ over B, and line bundles $\mathscr{L}_B$ and $\mathcal{A}_B$ over $\mathscr{X}_B$ such that there exist

(i) a flat morphism $h\colon \operatorname{Spec} K^\circ \to \operatorname{Spec} \mathcal{O}_{B,b} \to B$,

(ii) an isomorphism $\mathscr{X}_B \otimes_B \operatorname{Spec} K^\circ \xrightarrow{\sim} \mathscr{X}$,

(iii) an isomorphism $h^* \mathscr{L}_B \xrightarrow{\sim} \mathscr{L}$ over the isomorphism in (ii),

(iv) and an isomorphism $\mathcal{A}_B|_{\mathscr{X}_{B,\eta}} \xrightarrow{\sim} \mathscr{L}_B|_{\mathscr{X}_{B,\eta}}$ where η is the generic point of B and the line bundle $\mathcal{A}_B$ on $\mathscr{X}_B$ is ample.

Usually we read all the isomorphisms above as identifications.

Note that all relevant information in Assumption 7.1 is over the discrete valuation ring $\mathcal{O}_{B,b}$. The next remark makes this statement precise and gives an equivalent local way to formulate this assumption.

Remark 7.2. Suppose that $\mathscr{X}$ and $\mathscr{L}$ are defined over a subring R of K° by a line bundle $\mathscr{L}_R$ on a projective regular integral scheme $\mathscr{X}_R$ over R. We assume furthermore that R is a discrete valuation ring which is *defined geometrically by a d-dimensional normal variety B over a field k*, i.e. there exist $b \in B^{(1)}$ and an isomorphism $h\colon R \xrightarrow{\sim} \mathcal{O}_{B,b}$. We read the isomorphism h as an identification. Then Assumption 7.1 is equivalent to the existence of data $(R, k, B, b, h, \mathscr{X}_R, \mathscr{L}_R)$ as above assuming furthermore that the field k is perfect and the restriction of $\mathscr{L}_R$ to the generic fiber $\mathscr{X}_{R,\eta}$ over R extends to an ample line bundle $\mathcal{A}_R$ on $\mathscr{X}_R$.

One direction of the equivalence is clear by base change from B to $\operatorname{Spec} \mathcal{O}_{B,b}$. On the other hand, replacing B by an open affine neighbourhood of b, it is clear by [EGAIV, Cor. 9.6.4] that $\mathcal{A}_R$ extends to an ample line bundle $\mathcal{A}_B$ on a projective integral scheme $\mathscr{X}_B$ over B and that $\mathscr{L}_R$ extends to a line bundle $\mathscr{L}_B$ on $\mathscr{X}_B$. Since the regular locus of $\mathscr{X}_B$ is open [GW10, Cor. 12.52] and since the fiber of $\mathscr{X}_B$ over b is contained in the regular locus, we may assume that $\mathscr{X}_B$ is also regular by shrinking B again.

Theorem 7.3. *Let θ be defined by the line bundle $\mathscr{L}$. If the pair $(\mathscr{X}, \mathscr{L})$ satisfies Assumption 7.1, then $(m^{-1} \log |\mathfrak{a}_m|)_{m \in \mathbb{N}_{>0}}$ is a sequence of θ-psh model functions which converges uniformly on X^{an} to $P_\theta(0)$.*

If the field K has equicharacteristic zero, this result was proven by Boucksom, Favre, and Jonsson [BFJ16a, Thm. 8.5] without Assumption 7.1. We will follow their strategy of proof replacing the use of multiplier ideals by the use of test ideals. The required results about test ideals are gathered in Section 4.

Proof. We start with the observation that we have

(7.1) $$\Gamma(\mathscr{X}_B, \mathscr{L}_B^{\otimes m}) \neq 0$$

for some $m > 0$. In fact we have

$$\Gamma(\mathscr{X}_B, \mathscr{L}_B^{\otimes m}) \otimes_R K \xrightarrow{\sim} \Gamma(X, L^{\otimes m}) \neq 0$$

by flat base change and the ampleness of L for some $m > 0$.

We have a cartesian diagram

$$
\begin{array}{ccc}
\mathscr{X} & \xrightarrow{\ g\ } & \mathscr{X}_B \\
\downarrow & & \downarrow \searrow \\
\operatorname{Spec} K^\circ & \xrightarrow{\ h\ } & B \longrightarrow \operatorname{Spec} k.
\end{array}
$$

We observe that $\mathscr{X}_B$ is a smooth variety over the perfect field k and write

$$\mathfrak{a}_{B,m} = \operatorname{Im}\big(H^0(\mathscr{X}_B, \mathscr{L}_B^{\otimes m}) \otimes_k \mathscr{L}_B^{\otimes -m} \to \mathcal{O}_{\mathscr{X}_B}\big)$$

for the m-th base ideal of $\mathscr{L}_B$. Consider the ideal $g^{-1}(\mathfrak{a}_{B,m}) \cdot \mathcal{O}_{\mathscr{X}}$ in $\mathcal{O}_{\mathscr{X}}$ generated by $g^{-1}(\mathfrak{a}_{B,m})$. We have $g^{-1}(\mathfrak{a}_{B,m}) \cdot \mathcal{O}_{\mathscr{X}} = g^* \mathfrak{a}_{B,m}$ as g is flat. Sections of $\mathfrak{a}_m$ are locally of the form $s \cdot t^{-1}$ where $s \in \Gamma(\mathscr{X}, \mathscr{L}^{\otimes m})$ is a global section and t is a local section of $\mathscr{L}^{\otimes m}$. Flat base change [Har77, Prop. III.9.3] gives

$$H^0(\mathscr{X}, \mathscr{L}^{\otimes m}) = H^0(\mathscr{X}_B, \mathscr{L}_B^{\otimes m}) \otimes_R K^\circ.$$

Hence the formation of base ideals is compatible with base change, i.e. we have

(7.2) $$\mathfrak{a}_m = g^{-1}(\mathfrak{a}_{B,m}) \cdot \mathcal{O}_{\mathscr{X}} = g^* \mathfrak{a}_{B,m}$$

for all $m \in \mathbb{N}_{>0}$.

The family $\mathfrak{a}_{B,\bullet} = (\mathfrak{a}_{B,m})_{m>0}$ defines a graded sequence of ideals in the sense of Section 4. Let $\mathfrak{b}_{B,m} := \tau(\mathfrak{a}_{B,\bullet}^m)$ denote the associated asymptotic test ideal of exponent m. Motivated by (7.2) we define

$$\mathfrak{b}_m := g^{-1}\mathfrak{b}_{B,m} \cdot \mathcal{O}_{\mathscr{X}} = g^* \mathfrak{b}_{B,m}$$

as the ideal in $\mathcal{O}_{\mathscr{X}}$ generated by $\mathfrak{b}_{B,m}$. These ideals have the following properties:

(a) We have $\mathfrak{a}_m \subset \mathfrak{b}_m$ for all $m \in \mathbb{N}_{>0}$.
(b) We have $\mathfrak{b}_{ml} \subset \mathfrak{b}_m^l$ for all $l, m \in \mathbb{N}_{>0}$.
(c) There is $m_0 \geq 0$ such that $\mathcal{A}^{\otimes m_0} \otimes \mathscr{L}^{\otimes m} \otimes \mathfrak{b}_m$ is globally generated for all $m > 0$.

Properties (a) and (b) follow from the corresponding properties of $\mathfrak{a}_{B,m}$ and $\mathfrak{b}_{B,m}$ mentioned in (4.5), (4.6), and (4.7) if we observe (7.2).

Property (c) is a consequence of the generalization of Mustaţă's uniform generation property given in Theorem 4.6. Write $\mathscr{L}_B = \mathcal{O}(D)$ for some divisor D on $\mathscr{X}_B$ and choose a divisor H on $\mathscr{X}_B$ such that $\mathcal{O}(H)$ is ample and globally generated. Fix $d > \dim \mathscr{X}_B$ and a canonical divisor $K_{\mathscr{X}_B/k}$ on the smooth k-variety $\mathscr{X}_B$. As $\mathcal{A}_B$ is ample we find some $m_0 \in \mathbb{N}$ such that $\mathcal{A}_B^{\otimes m_0} \otimes \mathcal{O}(-K_{X/k} - dH)$ is globally generated. Given $m \in \mathbb{N}_{>0}$ we put $E := mD$. Since $\mathscr{L}_B$ satisfies (7.1), for any $m \in \mathbb{N}_{>0}$ we may use $E := mD$ and $\lambda := m$ in Theorem 4.6 to see that the sheaf

$$\mathcal{O}(K_{\mathscr{X}_B/k} + dH) \otimes \mathscr{L}_B^{\otimes m} \otimes \mathfrak{b}_{B,m}$$

is globally generated. As a consequence, our choice of m_0 implies that $\mathcal{A}_B^{\otimes m_0} \otimes \mathscr{L}_B^{\otimes m} \otimes \mathfrak{b}_{B,m}$ is globally generated. Base change to K° proves (c).

Now we follow the proof of [BFJ16a, Thm. 8.5]. Step 1 of *loc. cit.* holds not only on quasi-monomial points of X^{an}, but pointwise on the whole X^{an} using Proposition

2.10 and our different definition of $P_\theta(0)$. Then Step 2 of *loc. cit.* works in our setting using properties (a), (b), and (c) above. The only difference is that all inequalities hold immediately on X^{an} and not only on the quasi-monomial points of X^{an}. $\square$

Corollary 7.4. *Let X be a smooth n-dimensional projective variety over K with a closed $(1,1)$-form θ. Let $\mathscr{L}$ be a line bundle on a K°-model $\mathscr{X}$ of X defining θ and with $L = \mathscr{L}|_X$ ample. We assume that $(\mathscr{X}, \mathscr{L})$ is the base change of $(\mathscr{X}_R, \mathscr{L}_R)$ for a line bundle $\mathscr{L}_R$ of a projective integral scheme $\mathscr{X}_R$ over a subring R of K° and that R is a discrete valuation ring defined geometrically by a d-dimensional normal variety B over a perfect field k (as in Remark 7.2). If resolution of singularities holds over k in dimension $d+n$, then $P_\theta(0)$ is a uniform limit of θ-psh model functions and hence $P_\theta(0)$ is continuous on X^{an}.*

Proof. It follows from our assumptions that X is base change of the generic fiber $\mathscr{X}_{R,\eta}$ of $\mathscr{X}_R/R$ to K. Since X is smooth, we conclude that $\mathscr{X}_{R,\eta}$ is smooth as well [EGAIV, Cor. 17.7.3]. By Proposition 2.9(vii), it is enough to prove the claim for any positive multiple of θ. Using this and Lemma 7.5 below, we see that by passing to dominant models, we may assume that $\mathscr{X}_R$ is *regular* and that the restriction of $\mathscr{L}_R$ to $\mathscr{X}_{R,\eta}$ extends to an ample line bundle on $\mathscr{X}_R$. By Remark 7.2, these conditions are equivalent to Assumption 7.1 and hence the claim follows from Theorem 7.3. $\square$

Lemma 7.5. *Let R be a discrete valuation ring which is defined geometrically by a d-dimensional normal variety B over the field k (as in Remark 7.2). Let $\mathscr{X}_R$ be a projective integral scheme over R with n-dimensional regular generic fiber $X' := \mathscr{X}_{R,\eta}$. We assume that resolution of singularities holds over k in dimension $d + n$. Then for any ample line bundle L' on X', there exists $m \in \mathbb{N}_{>0}$ and an ample extension $\mathscr{L}_R'$ of $(L')^{\otimes m}$ to a regular R-model $\mathscr{X}_R'$ of X' with a projective morphism $\mathscr{X}_R' \to \mathscr{X}_R$ over R extending the identity on X'.*

Proof. The proof proceeds in three steps. First, we use a result of Lütkebohmert about vertical blowing ups to show that L' may be assumed to extend to an ample line bundle $\mathscr{H}$ on $\mathscr{X}_R$. In a second step, we show that $\mathscr{X}_R$ may be also assumed to be semi-factorial by a theorem of Pépin. In a third step, we use resolution of singularities to construct our desired regular model $\mathscr{X}_R'$.

Step 1: Replacing L' by a positive tensor power, we may assume that L' has an ample extension $\mathscr{H}_R$ to a projective R-model $\mathscr{Y}_R$. There is a blow up $\pi : \mathscr{Z}_R \to \mathscr{Y}_R$ in an ideal sheaf $\mathcal{J}$ supported in the special fiber of $\mathscr{Y}_R$ such that the identity on X' extends to a morphism $\mathscr{Z}_R \to \mathscr{X}_R$ [Lü93, Lemma 2.2]. Then $\pi^{-1}(\mathcal{J}) = \mathcal{O}_{\mathscr{Z}_R/\mathscr{Y}_R}(1)$ and hence there is $\ell \in \mathbb{N}_{>0}$ such that $\pi^*(\mathscr{H}^{\otimes \ell}) \otimes \mathcal{O}_{\mathscr{Z}_R/\mathscr{Y}_R}(1)$ is ample [Har77, Prop. II.7.10]. We conclude that by replacing $\mathscr{X}_R$ by $\mathscr{Z}_R$ and by passing to a positive tensor power of L', we may assume that L' has an ample extension $\mathscr{H}_R$ to $\mathscr{X}_R$. This completes the first step.

Step 2: By a result of Pépin [Pé13, Thm. 3.1], there is a a blowing-up morphism $\pi' : \mathscr{X}_R' \to \mathscr{X}_R$ centered in the special fiber of $\mathscr{X}_R$ such that $\mathscr{X}_R'$ is semi-factorial. The latter means that every line bundle on the generic fiber $\mathscr{X}_{R,\eta}'$ of $\mathscr{X}_R'$ over R extends to a line bundle on $\mathscr{X}_R'$. Similarly as in the first step, we may assume that a positive tensor power of L' extends to an ample line bundle on $\mathscr{X}_R'$. Replacing $\mathscr{X}_R$ by $\mathscr{X}_R'$ and L' by this positive tensor power, we get the second step.

Step 3: We may assume that B is affine. Using $R = \mathcal{O}_{B,b}$ for some $b \in B^{(1)}$, it is clear that $\mathscr{X}_R$ extends to a projective integral scheme $\mathscr{X}_B$ over B. By using resolution of singularities over k in dimension $d + n$, there is a regular integral scheme $\mathscr{X}_B'$ and a projective morphism $\varphi_B : \mathscr{X}_B' \to \mathscr{X}_B$ which is an isomorphism over the regular locus of

$\mathscr{X}_B$. Since X' is contained in the regular locus of $\mathscr{X}_B$, we conclude that φ_B maps the generic fiber $\mathscr{X}'_{B,\eta}$ of $\mathscr{X}'_B$ over B isomorphically onto $X' = \mathscr{X}_{B,\eta}$. As usual, we read this isomorphism as an identification. Then we get an induced projective morphism

$$\varphi_R : \mathscr{X}'_R \coloneqq \mathscr{X}'_B \times_B \operatorname{Spec}(R) \longrightarrow \mathscr{X}_R$$

extending the identity on X'. The same argument as in the first step gives $m \in \mathbb{N}_{>0}$ such that

$$\mathscr{L}'_R \coloneqq \varphi_R^*(\mathscr{H}_R^{\otimes m}) \otimes \mathcal{O}_{\mathscr{X}'_R / \mathscr{X}_R}(1)$$

is an ample line bundle on $\mathscr{X}'_R$. Let F be the restriction of $\mathcal{O}_{\mathscr{X}'_R / \mathscr{X}_R}(1)$ to $\mathscr{X}'_{R,\eta} = \mathscr{X}_{R,\eta} = X'$. Then $\mathscr{L}'_R$ is a model of $(L')^{\otimes m} \otimes F$. To prove the lemma, we have to ensure that F may be assumed to be $\mathcal{O}_{X'}$. To do so, we use that $\mathscr{X}_R$ is semi-factorial to extend F to a line bundle $\mathscr{F}_B$ on $\mathscr{X}_R$. Then we may replace $\mathcal{O}_{\mathscr{X}'_R / \mathscr{X}_R}(1)$ by $\mathcal{O}_{\mathscr{X}'_R / \mathscr{X}_R}(1) \otimes \varphi_R^*(\mathscr{F}^{-1})$ to deduce the claim. □

8. CONTINUITY OF THE ENVELOPE

We present consequences of our application of test ideals in the last subsection under the assumption that we have resolution of singularities. As in Section 7 let K be a complete discretely valued field of positive characteristic $p > 0$. Let X be a smooth projective variety over K of dimension n. Consider $\theta \in \mathcal{Z}^{1,1}(X)$ with ample de Rham class $\{\theta\} \in N^1(X)$.

Definition 8.1. We say that X is *of geometric origin from a d-dimensional family over a field k* if there exist a normal d-dimensional variety B over k, a point $b \in B^{(1)}$ of codimension one and a projective variety Y over $K' = k(B)$ such that

(i) there exists an isomorphism $\widehat{\mathcal{O}}_{B,b} \xrightarrow{\sim} K^\circ$ of rings where $\widehat{\mathcal{O}}_{B,b}$ denotes the completion of the discrete valuation ring $R \coloneqq \mathcal{O}_{B,b}$,

(ii) an isomorphism $Y \otimes_{K'} K \simeq X$ over K with $K' \to K$ induced by (i).

Usually, we read these isomorphisms as identifications. Moreover, if L is a line bundle (resp. if θ is a closed $(1,1)$-form) on X, we say that (X, L) (resp. (X, θ)) is *of geometric origin from a d-dimensional family over a field k* if the above conditions are satisfied and if we can also find a line bundle L' on Y inducing L by the base change K/K' (resp. a line bundle $\mathscr{L}_R$ on an R-model $\mathscr{X}_R$ of Y inducing θ by the base change K°/R).

We can now formulate our main result about the continuity of the envelope:

Theorem 8.2. *Let X be a smooth n-dimensional projective variety over K of geometric origin from a d-dimensional family over a perfect field k. Assume that resolution of singularities holds over k in dimension $d + n$. If θ is a closed $(1,1)$-form on X with ample de Rham class $\{\theta\}$ and if $u \in C^0(X^{\mathrm{an}})$, then $P_\theta(u)$ is a uniform limit of θ-psh model functions and thus $P_\theta(u)$ is continuous on X^{an}.*

Using resolution of singularities in dimension three over a perfect field proven by Cossart–Piltant (see Theorem 6.3), we get the following application:

Corollary 8.3. *Let X be a smooth projective surface over K of geometric origin from a 1-dimensional family over a perfect field k. Then the conclusion of Theorem 8.2 holds unconditionally.*

We will prove Theorem 8.2 in several steps. First, we prove it in a completely geometric situation:

Lemma 8.4. *If we assume additionally that (X, θ) is of geometric origin from a d-dimensional family over a perfect field k, then Theorem 8.2 holds.*

Proof. Recall that the space of model functions $\mathscr{D}(X)$ is dense in $C^0(X^{\mathrm{an}})$ for the topology of uniform convergence [Gub98, Thm. 7.12]. Hence we may assume that $u \in \mathscr{D}(X)$ by Proposition 2.9(v). Observe that by Proposition 2.9(vii), we may replace (θ, u) by a suitable multiple. Hence we may assume without loss of generality that the model function u is defined by a vertical divisor on a K°-model $\mathscr{X}'$. It is clear that we can choose $\mathscr{X}'$ dominating the geometric model $\mathscr{X} = \mathscr{X}_R \otimes_R K^\circ$ of θ from Definition 8.1. It follows from Proposition 5.2(a) that we may assume $\mathscr{X} = \mathscr{X}'$. By Proposition 2.9(iv) we get

$$(8.1) \qquad\qquad P_\theta(u) = P_{\theta + dd^c u}(0) + u.$$

By construction, the class $\theta + dd^c u$ is induced by a line bundle on $\mathscr{X}_R$ and hence Corollary 7.4 yields that $P_{\theta + dd^c u}(0)$ is a uniform limit of $(\theta + dd^c u)$-psh model functions φ_i. Then $P_\theta(u)$ is the uniform limit of the sequence of θ-psh functions $\varphi_i + u$ by (8.1). $\qquad\square$

In the lemma above we have proven Theorem 8.2 under the additional assumption that the $(1,1)$-form θ is defined geometrically. In the next lemma, we relax this assumption a bit only assuming that the de Rham class of θ is defined geometrically.

Lemma 8.5. *If we assume additionally that the de Rham class $\{\theta\}$ is induced by an ample line bundle L on X such that (X, L) is of geometric origin from a d-dimensional family over the perfect field k, then Theorem 8.2 holds.*

Proof. By Proposition 2.9(vi), we may assume that $\theta \in \mathcal{Z}^{1,1}(X)_{\mathbb{Q}}$. By Proposition 2.9(vii), we may replace L by a positive tensor power and θ by the corresponding multiple, and so we may assume that L is very ample. In the notation of Definition 8.1, the assumption that (X, L) is of geometric origin means that L is the pull-back of a line bundle L' on the projective variety Y over $K' = k(B)$. It follows easily from [Har77, Prop. III.9.3] and [EGAIV, Prop. 2.7.1(xii)] that L' is very ample. Then L' extends to a very ample line bundle on a projective R-model of Y for the discrete valuation ring $R = \mathcal{O}_{B,b}$ from Definition 8.1. By base change to K°, we conclude that there is a closed $(1,1)$-form θ' on X^{an} with de Rham class $\{\theta'\} = \{\theta\}$ such that (X, θ') is of geometric origin from a d-dimensional family over k.

By the dd^c-lemma in [BFJ16a, Thm. 4.3] (see also the second author's thesis [Jel16, Thm. 4.2.7] for generalizations) and using the rationality assumption on θ from the beginning of the proof, there is $v \in \mathscr{D}(X)$ such that $\theta' = \theta + dd^c v$. It follows from Proposition 2.9(iv) that

$$P_\theta(u) - v = P_{\theta + dd^c v}(u - v) = P_{\theta'}(u - v).$$

By Lemma 8.4, the function $P_{\theta'}(u - v)$ is a uniform limit of θ'-psh functions. Adding v, we get the claim for $P_\theta(u)$. $\qquad\square$

To prove Theorem 8.2 in full generality, the idea is to reduce to the above geometric situation by a similar trick as in [BFJ15, Appendix A].

Proof of Theorem 8.2. We note first that by Proposition 2.9 (viii) the property that $P_\theta(u)$ is a uniform limit of θ-psh model functions is equivalent to the property that it is a continuous function. Let K'/K be a finite normal extension and denote by $q \colon X' := X \otimes_K K' \to X$ the natural projection. Let $\theta' = q^* \theta \in \mathcal{Z}^{1,1}(X')$. Then by Lemma 2.11 we have that

$$q^* P_\theta(u) = P_{\theta'}(q^* u).$$

It follows from [Ber90, Prop. 1.3.5] that X^{an} is as a topological space equal to the quotient of $(X')^{\mathrm{an}}$ by the automorphism group of K'/K. We conclude that $P_\theta(u)$ is continuous if and only if $P_{\theta'}(q^*u)$ is continuous.

Hence we can replace K by a finite normal extension. Adapting the same argument as in [BFJ15, Lemma A.7] to characteristic p, there exists a finite normal extension K'/K and a function field F of transcendence degree d over k with K' as completion as in Definiton 8.1 such that $X' := X \otimes_K K'$ is the base change of a projective variety Y over F with $N^1(Y/F)_{\mathbb{Q}} \to N^1(X'/K')_{\mathbb{Q}}$ surjective. Replacing K' by K, we can assume that there is Y as above with a surjective map

$$(8.2) \qquad\qquad N^1(Y/F)_{\mathbb{Q}} \to N^1(X/K)_{\mathbb{Q}}$$

induced by the natural projection $X \to Y$. To prove continuity of $P_\theta(u)$, we may assume that the de Rham class $\{\theta\}$ is in $N^1(X)_{\mathbb{Q}}$ by using an approximation argument based on Proposition 2.9(vi). We conclude from surjectivity in (8.2) that there is a non-zero $m \in \mathbb{N}$ such that $\{m\theta\}$ is induced by a line bundle L with (X, L) of geometric origin from a d-dimensional family over k (in fact from Y). By Proposition 2.9(vii), we have $P_\theta(u) = \frac{1}{m} P_{m\theta}(mu)$ and hence continuity follows from Lemma 8.5. $\square$

9. The Monge–Ampère equation

Let K be a field endowed with a complete discrete absolute value. Boucksom, Favre, and Jonsson have shown in [BFJ16a, BFJ15] that the Monge–Ampère equation for a Radon measure supported on the skeleton of a smooth projective variety over K has a solution if the variety is of geometric origin from a one-dimensional family over a field of characteristic zero. In this section, we will explain that the same is true in characteristic $p > 0$ if we assume resolution of singularities (see Section 6 for precise definitions).

In the following, we work under the following assumptions:

(A1) The n-dimensional smooth projective variety X over K is of geometric origin from a d-dimensional family over a perfect field k of characteristic $p > 0$.

(A2) Resolution of singularities holds over k in dimension $d + n$.

(A3) Embedded resolution of singularities holds over k in dimension $d + n$.

Note that assumptions (A2) and (A3) are unconditional for $n = 2$ and $d = 1$ by Theorem 6.3 of Cossart and Piltant. For the following, it is crucial to have in mind that models of X can be defined geometrically which follows from Proposition 5.2.

To transfer the results from [BFJ16a, BFJ15], it is essential to note that every projective K°-model of X is dominated by a projective SNC-model of X. To see this, we note first that we may assume that the given K°-model is of geometric origin over the perfect field k by Proposition 5.2. Using resolution of singularities in dimension $d + n$ similarly as in the third step of the proof of Lemma 7.5, we deduce that there is a regular projective scheme $\mathscr{X}_R$ over R as in Definition 8.1 dominating the given model. Applying in the same way embedded resolution of singularities in dimension $d + n$ to the non-smooth fibers of $\mathscr{X}_R$ over R, we may assume that the singular fibers of $\mathscr{X}_R$ have the same support as a strict normal crossing divisor. Then base change to K° yields the claim as base change of the discrete valuation ring $\mathcal{O}_{B,b}$ to its completion K° preserves regularity [Sta17, Tag 0BG4] and strict normal crossing support.

Having sufficiently many projective SNC-models of X at hand, the density results of skeletons in X^{an} given in [BFJ16a, Sect. 3] also hold in our case of equicharacteristic p. Given a closed $(1,1)$-form θ with ample de Rham class $\{\theta\}$, the notion of θ-psh functions on X^{an} introduced in [BFJ16a, Sect. 7] keeps the same properties in our case. In fact, the results of [BFJ16a, Sect. 1–7] and their proofs carry over to our setting.

Note that we have already proven the continuity of the θ-psh envelope in Theorem 8.2, which is the analogue of [BFJ16a, Thm. 8.3], by using test ideals instead of multiplier ideals. For $u \in C^0(X^{\mathrm{an}})$, we recall from 2.8 that we have used a different definition of the θ-psh envelope $P_\theta(u)$ than in [BFJ16a, Def. 8.1]. Both definitions agree in the equicharacteristic zero situation by [BFJ16a, Thm. 8.3 and Lemma 8.9]. If the characteristic of K is positive and the Assumptions (A1)–(A3) hold then we have explained above how to define θ-psh functions. We claim now that in this case both definitions of the envelope agree as well. Indeed, it follows from [BFJ16a, Lemma 8.4] that the definitions agree on quasi-monomial points of X^{an}. For any $x \in X^{\mathrm{an}}$, we consider the net $p_{\mathscr{X}}(x)$ with $\mathscr{X}$ ranging over all SNC models of X. By [BFJ16a, Cor. 3.9], this net of quasi-monomial points converges to x. It follows from continuity that the net $P_\theta(u)(p_{\mathscr{X}}(x))$ converges to $P_\theta(u)(x)$ for our definition of the envelope. By [BFJ16a, Thm. 7.11, Prop. 8.2(i)], the same convergence holds for their envelope and hence both definitions agree.

This yields now in the same way as in [BFJ16a, Thm. 8.7] that the following *monotone regularization* holds:

Corollary 9.1. *Under the assumptions (A1)–(A3), let θ be a closed $(1,1)$-form on X^{an} with ample de Rham class. Then every θ-psh function on X^{an} is the pointwise limit of a decreasing net of θ-psh model functions on X^{an}.*

In [BFJ15, Sect. 3], the monotone regularization is the basic ingredient to generalize the Monge–Ampère operator from θ-psh model functions to bounded θ-psh functions and hence it applies also to our setting leading to the same results as in [BFJ15, Sect. 3]. For a bounded θ-psh function φ, we denote by $\mathrm{MA}_\theta(\varphi)$ the associated Monge-Ampère measure on X^{an}. The definitions, results and arguments from [BFJ15, Sect. 4–6] carry over without change. In particular, we may choose a decreasing sequence of θ-psh model functions on X^{an} in the monotone regularization from Corollary 9.1 similarly as in [BFJ15, Prop. 4.7].

A crucial step is now to prove the following *orthogonality property*:

Theorem 9.2. *Under the assumptions (A1)–(A3), let θ be a closed $(1,1)$-form on X^{an} with ample de Rham class. Then for every continuous function f on X^{an} with θ-psh envelope $P_\theta(f)$, we have the orthogonality property*

$$\int_{X^{\mathrm{an}}} (f - P_\theta(f))\mathrm{MA}_\theta(P_\theta(f)) = 0.$$

Proof. By Proposition 2.9(vi) and the continuity of the Monge–Ampère measure given in [BFJ15, Thm. 3.1], we may assume that $\theta \in \mathcal{Z}^{1,1}(X)_{\mathbb{Q}}$. Using Proposition 2.9(vii), we may assume that θ is induced by a line bundle of a model of X. Then the claim follows from [BGJKM16, Thm. 6.3.2] as $P_\theta(f)$ is continuous by Theorem 8.2. $\square$

As a consequence of the orthogonality property, we get differentiability of $E \circ P_\theta$ as in [BFJ15, Thm. 7.2] where E is the energy from [BFJ15, Sect. 6]. We have now all ingredients available to solve the *following Monge–Ampère equation.*

Theorem 9.3. *Under the assumptions (A1)–(A3), let θ be a closed $(1,1)$-form on X^{an} with ample de Rham class $\{\theta\}$ and let μ be a positive Radon measure on X^{an} of mass $\{\theta\}^n$ supported on the skeleton of a projective SNC-model. Then there is a continuous θ-psh function φ on X^{an} such that $\mathrm{MA}_\theta(\varphi) = \mu$ and φ is unique up to additive constants.*

Proof. It was shown in [BFJ15, §8.1] that uniqueness follows from a result of Yuan and Zhang in [YZ17]. To prove existence of a θ-psh solution φ, we use the variational method of Boucksom, Favre, and Jonsson. The basic tools needed here are upper semicontinuity

of the energy [BFJ15, Prop. 6.2], the compactness theorem [BFJ16a, Thm. 7.10], and the differentiability of $E \circ P_\theta$. As explained above, all these results are available in our setting. It remains to see that φ is continuous and this is done by estimates in the spirit of Kolodziej as in [BFJ15, §8.3].   □

APPENDIX A. THE SKELETON AND THE RETRACTION IN THE TORIC CASE
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In this appendix we give a combinatorial description of the skeleton associated to a toric model of a toric variety, and of the corresponding retraction. We will use this description to show an example of two models of the same variety that have the same skeleton but different retractions. In turn this will give a counterexample to a higher dimensional extension of Proposition 3.8.

Let $(K, |\ |)$ be a complete non-archimedean discretely valued field, K° the valuation ring, k the residue field, and $S = \mathrm{Spec}(K^\circ)$. Let ϖ be a uniformizer of K° and write

$$\lambda_K = -\log|\varpi|.$$

Let X be a smooth projective variety over K of dimension n and X^{an} the associated Berkovich analytic space. Let $\mathscr{X}$ be an SNC model of X over S, that is an SNC projective scheme $\mathscr{X}$ over S with generic fiber X such that the special fiber, which is not assumed to be reduced, agrees as a closed subset with a simple normal crossing divisor D of $\mathscr{X}$. To the model $\mathscr{X}$ we can associate a skeleton $\Delta_{\mathscr{X}} \subset X^{\mathrm{an}}$ and a retraction $p_{\mathscr{X}} : X^{\mathrm{an}} \to \Delta_{\mathscr{X}}$, see [BFJ16a, §3] for details.

Let L be an ample line bundle on X and $\mathscr{L}$ a nef model of L on $\mathscr{X}$. Let θ be the semipositive $(1,1)$-form in the class of L corresponding to the model $\mathscr{L}$. Let μ be a positive Radon measure on X^{an} with support in $\Delta_{\mathscr{X}}$ such that $\mu(X^{\mathrm{an}}) = \deg_L(X)$. The Monge-Ampère equation looks for a θ-psh function φ on X^{an} such that

$$(A.1) \qquad\qquad\qquad (dd^c\varphi + \theta)^{\wedge n} = \mu.$$

With the generality we are discussing in this paragraph, there is not yet a definition of the class of θ-psh functions with all the properties of classical pluripotential theory, but every good definition of this class should include the class of θ-psh model functions as introduced in 2.5.

The following question is natural and in case of being true would be of great help to solve the Monge-Ampère equation in positive and mixed characteristic.

Question 1. *With the previous hypotheses, is it true that any solution φ to the Monge-Ampère equation* (A.1) *satisfies*

$$(A.2) \qquad\qquad\qquad\qquad \varphi = \varphi \circ p_{\mathscr{X}}?$$

We will see that this question has a negative answer by exhibiting a counterexample in the context of toric varieties. In fact, that this question has a negative answer is related with Proposition 3.8 not being true in higher dimension. To this aim, we will consider a smooth projective variety X of dimension 2 and two models $\mathscr{X}$ and $\mathscr{X}'$ that have the same skeleton

$$\Delta = \Delta_{\mathscr{X}} = \Delta_{\mathscr{X}'}$$

but with different retractions $p_{\mathscr{X}} \neq p_{\mathscr{X}'}$. We will fix a semipositive $(1,1)$-form θ that is realized in both models and construct two model functions φ and φ' on X^{an} satisfying

$$(A.3) \qquad\qquad\qquad \varphi \neq \varphi', \qquad \varphi\mid_\Delta = \varphi'\mid_\Delta,$$
$$(A.4) \qquad\qquad\qquad \varphi = \varphi \circ p_{\mathscr{X}}, \quad \varphi' = \varphi' \circ p_{\mathscr{X}'}.$$

As a consequence of these properties, we deduce that $\varphi = \varphi' \circ p_{\mathscr{X}}$ and that $\varphi' = \varphi \circ p_{\mathscr{X}'}$. Moreover φ' will be a θ-psh model function while the model function φ will not be θ-psh. Let $\mu := (dd^c\varphi' + \theta)^{\wedge 2}$. This is a positive measure with support on Δ and φ' is a solution of the corresponding Monge-Ampère equation. Since

$$\varphi' \neq \varphi = \varphi' \circ p_{\mathscr{X}},$$

we see that φ' is a counterexample to Question 1 for the model $\mathscr{X}$. Moreover, if Proposition 3.8 were true in dimension 2, then $\varphi = \varphi' \circ p_{\mathscr{X}}$ would be a θ-psh model function, but it is not.

We place ourselves in the framework and notation of [BPS14]. The results below will make explicit the skeleton and the retraction associated to a toric SNC model of a toric variety.

Let $\mathbb{T} \simeq \mathbb{G}_m^n$ be a split torus over K. We denote by

$$M = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m), \quad N = \mathrm{Hom}(\mathbb{G}_m, \mathbb{T}),$$

the lattices of characters and one-parameter subgroups of $\mathbb{T}$. Then $M = N^\vee$. We also denote $N_{\mathbb{R}} = N \otimes \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes \mathbb{R}$. The pairing between $u \in N_{\mathbb{R}}$ and $x \in M_{\mathbb{R}}$ is denoted by $\langle x, u \rangle$.

Let now X be a proper toric variety over K and $\mathscr{X}$ a proper toric model of X over S. Then X is described by a complete fan Σ in $N_{\mathbb{R}}$ and $\mathscr{X}$ is described by a complete SCR-polyhedral complex Π in $N_{\mathbb{R}}$ whose recession fan satisfies $\mathrm{rec}(\Pi) = \Sigma$ [BPS14, Thm. 3.5.4].

There is a map $\zeta_K \colon N_{\mathbb{R}} \to \mathbb{T}^{\mathrm{an}}$ that sends $u \in N_{\mathbb{R}}$ to the seminorm on $K[M]$ given by

$$\left| \sum_{m \in M} \alpha_m \chi^m \right| = \max_m |\alpha_m| e^{-\lambda_K \langle m, u \rangle}.$$

This is a particular case of the map denoted by θ_σ in [BPS14, Prop.-Def. 4.2.12] composed with the homothety of ratio λ_K.

There is also a map $\mathrm{val}_K \colon \mathbb{T}^{\mathrm{an}} \to N_{\mathbb{R}}$ that sends a point $p \in \mathbb{T}^{\mathrm{an}}$ to the point $\mathrm{val}_K(p) \in N_{\mathbb{R}}$ determined by

$$\langle m, \mathrm{val}_K(p) \rangle = -\frac{1}{\lambda_K} \log |\chi^m(p)|,$$

see [BPS14, Sec. 4.1]. From the definition, it follows that $\mathrm{val}_K \circ \zeta_K = \mathrm{Id}_{N_{\mathbb{R}}}$.

To each polyhedron $\Lambda \in \Pi$ there is associated an orbit $O(\Lambda)$ for the action of $\mathbb{T}_k$ on the special fiber $\mathscr{X}_s$ [BPS14, §3.5]. We denote by ξ_Λ the generic point of $O(\Lambda)$.

The relation of val_K with the reduction map is given by [BPS14, Cor. 4.5.2]: a point $p \in \mathbb{T}^{\mathrm{an}}$ satisfies $\mathrm{red}(p) \in O(\Lambda)$ if and only if $\mathrm{val}_K(p) \in \mathrm{relint}(\Lambda)$. The relation of ζ_K with the reduction map is given by the next result.

Lemma A.1. *Let $\Lambda \in \Pi$. If u lies in the relative interior of Λ, then $\mathrm{red}(\zeta_K(u)) = \xi_\Lambda$.*

Proof. We use the notation of [BPS14, §3.5]. In particular, we write $\widetilde{N} = N \oplus \mathbb{Z}$ and $\widetilde{M} = M \oplus \mathbb{Z}$. Let σ be the cone of $\widetilde{N}$ generated by the set $\{(x, 1) \mid x \in \Lambda\}$ and write $\widetilde{M}_\Lambda = \widetilde{M} \cap \sigma^\vee$, where $\sigma^\vee$ is the dual cone of σ. Let $\mathscr{X}_\Lambda$ be the affine toric scheme associated to Λ. The ring of functions of $\mathscr{X}_\Lambda$ is

$$K^\circ[\mathscr{X}_\Lambda] = K^\circ[\widetilde{M}_\Lambda]/(\chi^{(0,1)} - \varpi).$$

The orbit $O(\Lambda)$ is a closed subscheme of $\mathscr{X}_\Lambda$. If $u \in \mathrm{relint}(\Lambda)$, the ideal of $O(\Lambda)$ is the ideal generated by the monomials $\chi^{(m,l)}$ with $(m, l) \in \widetilde{M}_\Lambda$ and $\langle m, u \rangle + l > 0$.

The generic fiber of $\mathscr{X}_\Lambda$ is the affine toric variety $X_{\mathrm{rec}(\Lambda)} = \mathrm{Spec}(K[M_{\mathrm{rec}(\Lambda)}])$. The natural inclusion $K^\circ[\mathscr{X}_\Lambda] \subset K[M_{\mathrm{rec}(\Lambda)}]$ is given by $\chi^{(m,l)} \mapsto \varpi^l \chi^m$. Any point $p \in X^{\mathrm{an}}_{\mathrm{rec}(\Lambda)}$ determines a seminorm on $K^\circ[\mathscr{X}_\Lambda]$. The set of points of $X^{\mathrm{an}}_{\mathrm{rec}(\Lambda)}$ whose reduction belongs to $\mathscr{X}_\Lambda$ is

$$C = \{p \in X^{\mathrm{an}}_{\mathrm{rec}(\Lambda)} \mid |f(p)| \le 1, \ \forall f \in K^\circ[\mathscr{X}_\Lambda]\}.$$

Given a point $p \in C$, then $\mathrm{red}(p)$ is the point corresponding to the prime ideal

$$\mathfrak{q}_p = \{f \in K^\circ[\mathscr{X}_\Lambda] \mid |f(p)| < 1\}.$$

Every $f \in K^\circ[\mathscr{X}_\Lambda]$ can be written as a sum

$$f = \sum_{(m,l) \in \widetilde{M}_\Lambda} \alpha_{(m,l)} \chi^{(m,l)},$$

with $|\alpha_{(m,l)}| = 0, 1$ and only a finite number of coefficients $\alpha_{(m,l)}$ different from zero.

By the definition of ζ_K,

$$\mathfrak{q}_{\zeta_K(u)} = \left\{\sum \alpha_{(m,l)} \chi^{(m,l)} \in K^\circ[\mathscr{X}_\Lambda] \,\Big|\, |\varpi|^l e^{-\lambda_K \langle m, u \rangle} < 1\right\}$$
$$= \left\{\sum \alpha_{(m,l)} \chi^{(m,l)} \in K^\circ[\mathscr{X}_\Lambda] \,\Big|\, \langle m, u \rangle + l > 0\right\}.$$

Since $u \in \mathrm{relint}(\Lambda)$, we deduce that $\mathfrak{q}_{\zeta_K(u)}$ is the ideal of $O(\Lambda)$ and therefore $\mathrm{red}(\zeta_K(u)) = \xi_\Lambda$. $\qquad\square$

We add now to X the condition of being regular, which is equivalent to Σ being unimodular, and to $\mathscr{X}$ the condition of being an SNC model. By [KKMS73, Chap. IV, §3.I item d)], $\mathscr{X}$ is regular if and only if the rational fan in $N_\mathbb{R} \times \mathbb{R}_{\ge 0}$ generated by $\Pi \times \{1\}$ is unimodular. In this case, the model is always an SNC model. On the other hand, by [BMPS16, Example 3.6.11] the model will be strictly semistable (SNC with reduced special fiber) if, in addition, all the vertices are lattice points.

Since a unimodular fan is necessarily simplicial, for each polyhedron $\Lambda \in \Pi$, of dimension t, we can write

$$(\mathrm{A.5}) \qquad\qquad \Lambda = \mathrm{conv}(p_0, \ldots, p_s) + \mathrm{cone}(v_{s+1}, \ldots, v_t),$$

where $s \le t$, p_i are points of $N_\mathbb{R}$ and v_i are vectors on the tangent space to $N_\mathbb{R}$ at a point, that we identify with $N_\mathbb{R}$.

We define the *combinatorial skeleton* as

$$\Delta_\Pi = \bigcup_{\substack{\Lambda \in \Pi \\ \Lambda \text{ bounded}}} \Lambda.$$

There is a *combinatorial retraction* $p_\Pi \colon N_\mathbb{R} \to \Delta_\Pi$ defined as follows. Let $u \in N_\mathbb{R}$ and let $\Lambda \in \Pi$ be a polyhedron of dimension t with $u \in \Lambda$. Write Λ as in equation (A.5). Therefore u can be written uniquely as

$$(\mathrm{A.6}) \qquad\qquad u = \sum_{i=0}^{s} a_i p_i + \sum_{j=s+1}^{t} \lambda_j v_j,$$

with $a_i, \lambda_j \ge 0$, and $\sum a_i = 1$. Then

$$p_\Pi(u) = \sum_{i=0}^{s} a_i p_i.$$

Remark A.2. The fan Σ determines a compactification N_Σ of $N_\mathbb{R}$ as in [BPS14, §4.1] such that the retraction p_Π can be extended to a continuous map $N_\Sigma \to \Delta_\Pi$.

The following result explicites the skeleton and retraction associated to the model $\mathscr{X}$.

Theorem A.3. *With the previous hypotheses, the skeleton $\Delta_{\mathscr{X}} \subset X^{\mathrm{an}}$ is given by*

$$\Delta_{\mathscr{X}} = \zeta_K(\Delta_\Pi).$$

The restriction to $\mathbb{T}^{\mathrm{an}}$ of the retraction $p_{\mathscr{X}}$ is the composition

$$p_{\mathscr{X}} \mid_{\mathbb{T}^{\mathrm{an}}} = \zeta_K \circ p_\Pi \circ \mathrm{val}_K .$$

Proof. We start by recalling the construction of $\Delta_{\mathscr{X}}$ and $p_{\mathscr{X}}$ from [BFJ16a]. Note that, in loc. cit. the residue field k is of characteristic zero, but once we assume that the model $\mathscr{X}$ is an SNC model, using the results of [MN15, § 3.1] it is possible to extend the presentation of [BFJ16a] to the case of positive and mixed characteristic.

Let $\mathrm{Div}_0(\mathscr{X})$ be the group of vertical Cartier divisors on $\mathscr{X}$. Denote $\mathrm{Div}_0(\mathscr{X})_{\mathbb{R}} = \mathrm{Div}_0(\mathscr{X}) \otimes \mathbb{R}$ and let $\mathrm{Div}_0(\mathscr{X})^*_{\mathbb{R}}$ be the dual. As explained in 2.2, each $D \in \mathrm{Div}_0(\mathscr{X})$ determines a model function φ_D. The map $D \mapsto \varphi_D$ is linear in D and can be extended by linearity to a map $\mathrm{Div}_0(\mathscr{X})_{\mathbb{R}} \to C^0(X^{\mathrm{an}})$.

There is a map $\mathrm{ev}_{\mathscr{X}} \colon X^{\mathrm{an}} \to \mathrm{Div}_0(\mathscr{X})^*_{\mathbb{R}}$ determined by

(A.7) $$\langle D, \mathrm{ev}_{\mathscr{X}}(x) \rangle = \varphi_D(x).$$

Let $D_1, \ldots, D_\ell$ be the components of the special fiber $\mathscr{X}_s$. Each D_i, $i = 1, \ldots, \ell$, determines a divisorial point $x_i \in X^{\mathrm{an}}$ and we denote by $e_i = \mathrm{ev}_{\mathscr{X}}(x_i)$. For each $J \subset \{1, \ldots, \ell\}$ we write $D_J = \bigcap_{j \in J} D_j$ and $\sigma_J = \mathrm{conv}(e_j, j \in J)$. Then the abstract skeleton of $\mathscr{X}$ is

$$\Delta_{\mathscr{X}}^{\mathrm{abs}} = \bigcup_{\substack{J \subset \{1, \ldots, \ell\} \\ D_J \neq \emptyset}} \sigma_J \subset \mathrm{Div}_0(\mathscr{X})^*_{\mathbb{R}}.$$

By [BFJ16a, Thm. 3.1], the image of $\mathrm{ev}_{\mathscr{X}}$ is $\Delta_{\mathscr{X}}^{\mathrm{abs}}$ and there exists a unique function $\mathrm{emb}_{\mathscr{X}} \colon \Delta_{\mathscr{X}}^{\mathrm{abs}} \to X^{\mathrm{an}}$ such that

(i) $\mathrm{ev}_{\mathscr{X}} \circ \mathrm{emb}_{\mathscr{X}} = \mathrm{Id}_{\Delta_{\mathscr{X}}^{\mathrm{abs}}}$;

(ii) for each $s \in \Delta_{\mathscr{X}}^{\mathrm{abs}}$, if $s \in \mathrm{relint}(\sigma_J)$, then $\mathrm{red}(\mathrm{emb}_{\mathscr{X}}(s)) = \xi_{D_J}$, where ξ_{D_J} is the generic point of D_J.

Then the skeleton and the retraction are given by

$$\Delta_{\mathscr{X}} = \mathrm{emb}_{\mathscr{X}}(\Delta_{\mathscr{X}}^{\mathrm{abs}}) \quad \text{and} \quad p_{\mathscr{X}} = \mathrm{emb}_{\mathscr{X}} \circ \mathrm{ev}_{\mathscr{X}} .$$

We now go back to the regular toric case. In particular, X is a toric smooth projective variety over K and $\mathscr{X}$ is a toric projective SNC model. Then all the divisors of $\mathrm{Div}_0(\mathscr{X})$ are toric divisors. Therefore, for $D \in \mathrm{Div}_0(\mathscr{X})_{\mathbb{R}}$, the function φ_D is invariant under the action of the compact torus $\mathbb{S} = \mathrm{val}_K^{-1}(0)$. The restriction of φ_D to $\mathbb{T}^{\mathrm{an}}$ factorizes as

(A.8) $$\varphi_D \mid_{\mathbb{T}^{\mathrm{an}}} = -\phi_D \circ \mathrm{val}_K,$$

where ϕ_D is the function from [BPS14, Def. 4.3.6] corresponding to the trivial line bundle $\mathcal{O}_X$ with the metric determined by D and the section 1.

We now define $\mathrm{ev}_\Pi \colon N_{\mathbb{R}} \to \mathrm{Div}_0(\mathscr{X})^*$ by

$$\langle D, \mathrm{ev}_\Pi(u) \rangle = -\phi_D(u).$$

By construction, the restriction of ev_Π to each polyhedron $\Lambda \in \Pi$ is affine. Moreover, using (A.7) and (A.8) we deduce that

(A.9) $$\mathrm{ev}_{\mathscr{X}} \mid_{\mathbb{T}^{\mathrm{an}}} = \mathrm{ev}_\Pi \circ \mathrm{val}_K .$$

As before let $D_1, \ldots, D_\ell$ be the components of the special fiber $\mathscr{X}_s$ and x_i the divisorial point determined by D_i. Then the set of vertices of Π is $\Pi^0 = \{u_1, \ldots, u_\ell\}$, where

$u_i = \mathrm{val}_K(x_i)$. Therefore $\mathrm{ev}_\Pi(u_i) = e_i$. Since ev_Π is affine in each polyhedron of Π we deduce that the image of ev_Π is $\Delta_{\mathscr{X}}^{\mathrm{abs}}$ and that ev_Π determines a homeomorphism $\Delta_\Pi \to \Delta_{\mathscr{X}}^{\mathrm{abs}}$. We define $\mathrm{emb}_\Pi \colon \Delta_\Pi^{\mathrm{abs}} \to N_\mathbb{R}$ as the composition of the inverse of this homeomorphism with the inclusion $\Delta_\Pi \hookrightarrow N_\mathbb{R}$. Using equation (A.9) and Lemma A.1 one can check that $\zeta_K \circ \mathrm{emb}_\Pi$ satisfies the conditions (1) and (2) that characterize $\mathrm{emb}_{\mathscr{X}}$. Therefore

$$\text{(A.10)} \qquad\qquad \mathrm{emb}_{\mathscr{X}} = \zeta_K \circ \mathrm{emb}_\Pi \,.$$

We next claim that $p_\Pi = \mathrm{emb}_\Pi \circ \mathrm{ev}_\Pi$. Indeed, for every $D \in \mathrm{Div}_0(\mathscr{X})$, since D is a model of the trivial vector bundle, we know that $\mathrm{rec}(\phi_D)$ is the zero function. Therefore, writing any $u \in \Lambda \in \Pi$ is as in (A.6), one can show that

$$\phi_D = \phi_D \circ p_\Pi.$$

This implies that $\mathrm{ev}_\Pi = \mathrm{ev}_\Pi \circ p_\Pi$. By construction $\mathrm{emb}_\Pi \circ \mathrm{ev}_\Pi$ is the identity in the image of p_Π. Therefore

$$\text{(A.11)} \qquad\qquad \mathrm{emb}_\Pi \circ \mathrm{ev}_\Pi = \mathrm{emb}_\Pi \circ \mathrm{ev}_\Pi \circ p_\Pi = p_\Pi.$$

Using equations (A.11) (A.10) and (A.9) we deduce that

$$\Delta_{\mathscr{X}} = \mathrm{emb}_{\mathscr{X}}(\Delta_{\mathscr{X}}^{\mathrm{abs}}) = \zeta_K(\mathrm{emb}_\Pi(\Delta_{\mathscr{X}}^{\mathrm{abs}})) = \zeta_K(\Delta_\Pi)$$

and

$$p_{\mathscr{X}}|_{\mathbb{T}^{\mathrm{an}}} = \mathrm{emb}_{\mathscr{X}} \circ \mathrm{ev}_{\mathscr{X}}|_{\mathbb{T}^{\mathrm{an}}} = \zeta_K \circ \mathrm{emb}_\Pi \circ \mathrm{ev}_\Pi \circ \mathrm{val}_K = \zeta_K \circ p_\Pi \circ \mathrm{val}_K$$

concluding the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$
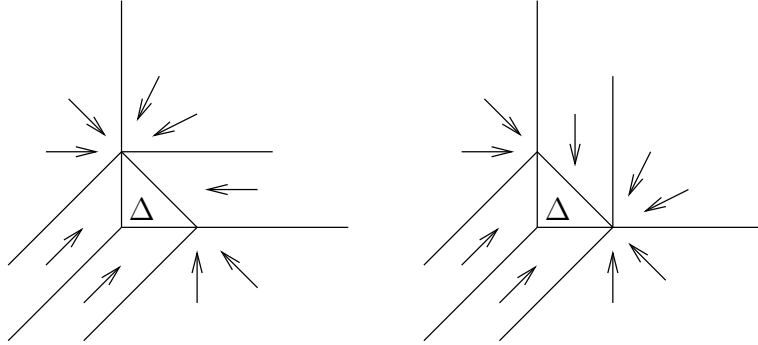

FIGURE 1. Subdivisions corresponding to the toric models $\mathscr{X}$ and $\mathscr{X}'$.

Consider the toric variety $X = \mathbb{P}^2_K$. Let $\mathbb{P}^2_S$ be the projective space over S, and let $(x_0 : x_1 : x_2)$ be homogeneous coordinates of the special fiber $\mathbb{P}^2_k$. Consider the model $\mathscr{X}$ of X obtained by blowing up $\mathbb{P}^2_S$ at the line $x_2 = 0$ inside the special fiber and then blowing up the strict transform of the line $x_1 = 0$. The SCR-polyhedral subdivision Π associated to this model is depicted in the left side of figure 1. Consider also the model $\mathscr{X}'$ of X obtained as before, switching x_1 and x_2. The SCR-polyhedral subdivision Π' associated to this new model is depicted in the right side of figure 1. Both toric schemes $\mathscr{X}$ and $\mathscr{X}'$ are SNC models (even more, they are strictly semistable models) of $\mathbb{P}^2_K$.

The skeleton associated to both models is the simplex

$$\Delta = \mathrm{conv}((0,0), (1,0), (0,1))$$

and both retractions p_Π and $p_{\Pi'}$ are also depicted on the same figure. For instance the retraction p_Π sends every point of σ_2 to the point $(1,0)$, while the same retraction

restricted to the polyhedron σ_1 is the horizontal projection onto the segment $\overline{(0,1)(1,0)}$ along the direction $(-1,0)$. By contrast, the retraction $p_{\Pi'}$ sends both cones σ_1' and σ_2' to the point $(1,0)$.

Consider the divisor D of $\mathbb{P}_K^2$ given by the line at infinity and the divisor $\mathcal{D}$ of $\mathbb{P}_S^2$ given by the closure of D. Let $L = \mathcal{O}_{\mathbb{P}_K^2}(D)$ and $\mathscr{L} = \mathcal{O}_{\mathbb{P}_S^2}(\mathcal{D})$. Then $\mathscr{L}$ is a model of L in $\mathbb{P}_S^2$ and can be pulled back to both $\mathscr{X}$ and $\mathscr{X}'$. Let θ be the closed $(1,1)$-form defined by this model.

Let $\Psi\colon N_{\mathbb{R}} \to \mathbb{R}$ be the function

$$\Psi(u,v) = \min(u,v,0).$$

This is the function that determines the toric divisor D.

By [BPS14, Thm. 4.8.1], the space of all continuous θ-psh functions on X^{an} that are invariant under the action of the compact torus $\mathbb{S}$ can be identified with the set of all bounded functions $f\colon N_{\mathbb{R}} \to \mathbb{R}$ such that $\Psi + f$ is concave. This identification sends $f\colon N_{\mathbb{R}} \to \mathbb{R}$ to the unique continuous function $\varphi\colon X^{\mathrm{an}} \to \mathbb{R}$ such that $\varphi\mid_{\mathbb{T}^{\mathrm{an}}} = -f \circ \mathrm{val}_K$.

Let $g\colon \Delta \to \mathbb{R}$ the affine function that has the value 1 at the point $(1,0)$ and the value 0 at the points $(0,0)$ and $(0,1)$ and put

$$f = g \circ p_{\Pi}, \qquad f' = g \circ p_{\Pi'}.$$

One easily verifies that

$$\Psi + f' = \min(1, 1 + v, u)$$

which is concave. On the other hand, the restriction of $\Psi + f$ to σ_3 is 0 while its restriction to σ_1 is $1 - v$, hence $\Psi + f$ is not concave.

Let φ' be the continuous function on X^{an} whose restriction to $\mathbb{T}^{\mathrm{an}}$ is $-f' \circ \mathrm{val}_K$. It is a model θ-psh function. The function $-f \circ \mathrm{val}_K$ also extends to a model function φ on X^{an} but it is not θ-psh because f is not concave [BPS14, Thm. 3.7.1 (2)].

We now write

$$\mu = (dd^c\varphi' + \theta)^{\wedge 2}.$$

By [BPS14, Thm. 4.7.4] the measure μ is the atomic measure with support in $\zeta_K((1,0))$ with total mass one. Hence its support is contained in $\Delta = \Delta_{\mathscr{X}}$.

Summing up, μ is a measure with support in $\Delta = \Delta_{\mathscr{X}}$, the θ-psh function φ' is a solution of the corresponding Monge-Ampère equation but $\varphi' \neq \varphi' \circ p_{\mathscr{X}}$ showing that the answer to Question 1 is negative. Moreover $\varphi = \varphi' \circ p_{\mathscr{X}}$ is not θ-psh, showing that Proposition 3.8 does not extend to dimension ≥ 2.

Consider now a common refinement of the subdivisions corresponding to the toric models $\mathscr{X}$ and $\mathscr{X}'$ and such that the corresponding toric model $\mathscr{X}''$ is SNC. Then the skeleton $\Delta_{\mathscr{X}''}$ will be strictly bigger than Δ. In particular $\Delta_{\mathscr{X}''}$ contains the unit square $0 \leq u,v \leq 1$. It can be shown that $\varphi' = \varphi' \circ p_{\mathscr{X}''}$. Thus even if the solutions of the Monge-Ampère equation do not factor through the retraction corresponding to the model $\mathscr{X}$, they factor through the retraction associated to a refined model. Mattias Jonsson asked us if one can hope this phenomenon to hold in general. More concretely, one can ask the following question.

Question 2. *With the same hypotheses as in Question 1, is it true that there exists a morphism of models $\mathscr{X}' \to \mathscr{X}$ such that any solution φ of the Monge-Ampère equation* (A.1) *satisfies*

(A.12) $$\varphi = \varphi \circ p_{\mathscr{X}'}?$$

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References

[Abb11]   Ahmed Abbes. *Éléments de géométrie rigide. Vol. I. Construction et étude géométrique des espaces rigides.* Progress in Mathematics 286. Basel: Birkhäuser. xv, 477 p., 2011.

[BPR13]   Matthew Baker, Sam Payne, and Joseph Rabinoff. On the structure of non-Archimedean analytic curves. In *Tropical and non-Archimedean geometry*, volume 605 of *Contemp. Math.*, pages 93–121. Amer. Math. Soc., Providence, RI, 2013.

[Ber90]   Vladimir G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990.

[Ber93]   Vladimir G. Berkovich. Etale cohomology for non-Archimedean analytic spaces. *Publications Mathématiques de l'IHÉS*, 78(1):5–161, 1993.

[Ber99]   Vladimir G. Berkovich. Smooth p-adic analytic spaces are locally contractible. *Invent. Math.*, 137(1):1–84, 1999.

[BMS08]   Manuel Blickle, Mircea Mustaţă, and Karen E. Smith. Discreteness and rationality of F-thresholds. *Michigan Math. J.*, 57:43–61, 2008. Special volume in honor of Melvin Hochster.

[BGR84]   Siegfried Bosch, Ulrich Güntzer, and Reinhold Remmert. *Non-Archimedean analysis*, volume 261 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1984.

[BL85]   Siegfried Bosch and Werner Lütkebohmert. Stable reduction and uniformization of abelian varieties. I. *Math. Ann.*, 270(3):349–379, 1985.

[BFJ15]   Sébastien Boucksom, Charles Favre, and Mattias Jonsson. Solution to a non-Archimedean Monge-Ampère equation. *J. Amer. Math. Soc.*, 28(3):617–667, 2015.

[BFJ16a]   Sébastien Boucksom, Charles Favre, and Mattias Jonsson. Singular semipositive metrics in non-Archimedean geometry. *J. Algebraic Geom.*, 25(1):77–139, 2016.

[BFJ16b]   Sébastien Boucksom, Charles Favre, and Mattias Jonsson. The non-Archimedean Monge-Ampère equation. In *Nonarchimedean and tropical geometry. Based on two Simons symposia, Island of St. John, March 31 – April 6, 2013 and Puerto Rico, February 1–7, 2015*, pages 31–49. Cham: Springer, 2016.

[BS13]   Markus P. Brodmann and Rodney Y. Sharp. *Local cohomology*, volume 136 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2013.

[BGJKM16] José Ignacio Burgos Gil, Walter Gubler, Philipp Jell, Klaus Künnemann, and Florent Martin. Differentiability of non-archimedean volumes and non-archimedean Monge-Ampère equations (with an appendix by Robert Lazarsfeld). `arXiv:1608.01919`, 2016.

[BMPS16]   José Ignacio Burgos Gil, Atsushi Moriwaki, Patrice Philippon, and Martín Sombra. Arithmetic positivity on toric varieties. *J. Algebraic Geom.*, 25(2):201–272, 2016.

[BPS14]   José Ignacio Burgos Gil, Patrice Philippon, and Martin Sombra. Arithmetic geometry of toric varieties. Metrics, measures and heights. *Astérisque*, 360, 2014.

[Cal54]   Eugenio Calabi. The space of Kähler metrics. In *Proceedings of the International Congress of Mathematicians, Amsterdam, 1954. Vol. 2*, pages 206–207. Erven P. Noordhoff N. V., Groningen; North-Holland Publishing Co., Amsterdam, 1954.

[Cal57]   Eugenio Calabi. On Kähler manifolds with vanishing canonical class. In *Algebraic geometry and topology. A symposium in honor of S. Lefschetz*, pages 78–89. Princeton University Press, Princeton, N. J., 1957.

[CL06]   Antoine Chambert-Loir. Mesures et équidistribution sur les espaces de Berkovich. *J. Reine Angew. Math.*, 595:215–235, 2006.

[CP08]   Vincent Cossart and Olivier Piltant. Resolution of singularities of threefolds in positive characteristic. I. *J. Algebra*, 320(3):1051–1082, 2008.

[CP09]   Vincent Cossart and Olivier Piltant. Resolution of singularities of threefolds in positive
             characteristic. II. *J. Algebra*, 321(7):1836–1976, 2009.
[ELMN05]   Lawrence Ein, Robert Lazarsfeld, Mircea Mustaţă, Michael Nakamaye, and Mihnea Popa.
             Asymptotic invariants of line bundles. *Pure Appl. Math. Q.*, 1(2, Special Issue: In memory
             of Armand Borel. Part 1):379–403, 2005.
[FK88]   Eberhard Freitag and Reinhardt Kiehl. *Étale cohomology and the Weil conjecture*, vol-
             ume 13 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin,
             1988.
[GW10]   Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry I*. Advanced Lectures in Mathe-
             matics. Vieweg + Teubner, Wiesbaden, 2010.
[EGAIII]   Alexander Grothendieck and Jean Dieudonné. Éléments de géométrie algébrique. III. Étude
             cohomologique des faisceaux cohérents. *Inst. Hautes Études Sci. Publ. Math.*, 11, 17, 1961-
             63.
[EGAIV]   Alexander Grothendieck and Jean Dieudonné. Éléments de géométrie algébrique. IV. Étude
             locale des schémas et des morphismes de schémas. *Inst. Hautes Études Sci. Publ. Math.*,
             20, 24, 28, 32, 1964–67.
[Gub98]   Walter Gubler. Local heights of subvarieties over non-archimedean fields. *J. Reine Angew.
             Math.*, 498:61–113, 1998.
[GH15]   Walter Gubler and Julius Hertel. Local heights of toric varieties over non-archimedean
             fields. `arXiv:1512.06574`, 2015.
[GM16]   Walter Gubler and Florent Martin. On Zhang's semipositive metrics. `arXiv:1608.08030`,
             2016.
[GRW16]   Walter Gubler, Joseph Rabinoff, and Annette Werner. Skeletons and tropicalizations. *Adv.
             Math.*, 294:150–215, 2016.
[HY03]   Nobuo Hara and Ken-Ichi Yoshida. A generalization of tight closure and multiplier ideals.
             Trans. Amer. Math. Soc., 355(8):3143–3174, 2003.
[Har77]   Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Grad-
             uate Texts in Mathematics, No. 52.
[Jel16]   Philipp Jell. *Differential forms on Berkovich analytic spaces and their cohomology*. PhD
             thesis, Universität Regensburg, 2016. `urn:nbn:de:bvb:355-epub-347884`.
[Jon15]   Mattias Jonsson. Dynamics of Berkovich spaces in low dimensions. In *Berkovich spaces
             and applications*, volume 2119 of *Lecture Notes in Math.*, pages 205–366. Springer, Cham,
             2015.
[KRZB16]   Eric Katz, Joseph Rabinoff, and David Zureick-Brown. Uniform bounds for the number of
             rational points on curves of small Mordell-Weil rank. *Duke Math. J.*, 165(16):3189–3240,
             2016.
[Kat70]   Nicholas M. Katz. Nilpotent connections and the monodromy theorem: Applications of a
             result of Turrittin. *Inst. Hautes Études Sci. Publ. Math.*, 39:175–232, 1970.
[Kee03]   Dennis S. Keeler. Ample filters of invertible sheaves. *J. Algebra*, 259(1):243–283, 2003.
[Kel75]   John L. Kelley. *General topology. 2nd ed.* Springer-Verlag, New York-Heidelberg, 1975.
             Graduate Texts in Mathematics, No. 27.
[KKMS73]   George Kempf, Finn Faye Knudsen, David Mumford, and Bernard Saint-Donat. *Toroidal
             embeddings. I.* Lecture Notes in Mathematics, Vol. 339. Springer-Verlag, Berlin, 1973.
[Kle66]   Steven L. Kleiman. Toward a numerical theory of ampleness. *Ann. of Math. (2)*, 84:293–
             344, 1966.
[Laz04]   Robert Lazarsfeld. *Positivity in algebraic geometry. I. Classical setting: line bundles and
             linear series*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*.
             Springer-Verlag, Berlin, 2004.
[Liu06]   Qing Liu. *Algebraic geometry and arithmetic curves*. Oxford University Press, 2006.
[Liu11]   Yifeng Liu. A non-archimedean analogue of the Calabi-Yau theorem for totally degenerate
             abelian varieties. *J. Differ. Geom.*, 89(1):87–110, 2011.
[Lü93]   Werner Lütkebohmert. On compactification of schemes. *Manuscripta Math.*, 80(1):95–111,
             1993.
[Mat89]   Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Ad-
             vanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989.
[Mus13]   Mircea Mustaţă. The non-nef locus in positive characteristic. In *A celebration of algebraic
             geometry*, volume 18 of *Clay Math. Proc.*, pages 535–551. Amer. Math. Soc., Providence,
             RI, 2013.

[MN15]   Mircea Mustaţă and Johannes Nicaise. Weight functions on non-Archimedean analytic spaces and the Kontsevich-Soibelman skeleton. *Algebr. Geom.*, 2(3):365–404, 2015.

[Pé13]   Cédric Pépin. Modèles semi-factoriels et modèles de Néron. *Math. Ann.*, 355(1):147–185, 2013.

[ST12]   Karl Schwede and Kevin Tucker. A survey of test ideals. In *Progress in commutative algebra 2*, pages 39–99. Walter de Gruyter, Berlin, 2012.

[Sta17]   The Stacks Project Authors. *Stacks Project*. `http://stacks.math.columbia.edu`, 2017.

[Thu05]   Amaury Thuillier. *Théorie du potentiel sur les courbes en géométrie analytique non-archimédienne. Applications à la théorie d'Arakelov.* PhD thesis, Université de Rennes I, 2005.

[Yau78]   Shing Tung Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. *Comm. Pure Appl. Math.*, 31:399–411, 1978.

[Yua08]   Xinyi Yuan. Big line bundles over arithmetic varieties. *Invent. Math.*, 173(3):603–649, 2008.

[YZ17]   Xinyi Yuan and Shou-Wu Zhang. The arithmetic Hodge index theorem for adelic line bundles. *Math. Ann.*, 367(3-4):1123–1171, 2017.

[Zha95]   Shouwu Zhang. Small points and adelic metrics. *J. Algebraic Geom.*, 4(2):281–300, 1995.

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