HEIGHTS OF COMPLETE INTERSECTIONS IN TORIC VARIETIES

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Abstract. The height of a toric variety and that of its hypersurfaces can be expressed in convex-analytic terms as an adelic sum of mixed integrals of their roof functions and duals of their Ronkin functions.

Here we extend these results to the 2-codimensional situation by presenting a limit formula predicting the typical height of the intersection of two hypersurfaces on a toric variety. More precisely, we prove that the height of the intersection cycle of two effective divisors translated by a strict sequence of torsion points converges to an adelic sum of mixed integrals of roof and duals of Ronkin functions.

This partially confirms a previous conjecture of the authors about the average height of families of complete intersections in toric varieties.

Contents

Introduction 1
Conventions and notations 4
1. Geometric results 5
2. The height of intersection cycles of twists 11
3. An auxiliary function 15
4. A local logarithmic equidistribution 25
5. The adelic vanishing 30
6. Main results 38
Appendix A. Proof of the reductions 42
References 48

Introduction

The study of the Arakelov geometry of toric varieties was inaugurated by Maillot in [Mai00], where among other results he obtained upper bounds for the canonical height of a complete intersection. These can be viewed as an arithmetic analogue of the inequality part in the classical Bernstein–Kushnirenko–Khovanskii theorem [Ber75].

Finding an arithmetic version of the equality part in this theorem is a more challenging problem. A first step towards this goal was taken by Burgos Gil, Philippon and the second author in the monograph [BPS14], where they studied semipositive metrized line bundles on a toric variety which are invariant under the action of the compact torus and gave a convex-analytic formula for the corresponding height of

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this ambient variety. Subsequently, their formula was extended by the first author to compute the height of a hypersurface in similar terms [Gua18b].

For higher-codimensional complete intersections in toric varieties, one can not expect a general convex-analytic formula only depending on the arithmetic size of the defining Laurent polynomials, as shown in [Gua18a, Example 5.1.1] and [GS23, Example 2.1]. However, these two references have suggested that the latter information could be enough to recover the *limit* height of a certain family of complete intersection cycles associated to the Laurent polynomials.

To explain this precisely, let us fix a number field $\mathbb{K}$ with set of places $\mathfrak{M}$, and a split algebraic torus $\mathbb{T}$ over $\mathbb{K}$ of dimension n with character lattice M. For a Laurent polynomial $f = \sum_m \alpha_m \chi^m \in \mathbb{K}[M]$ and a point $t \in \mathbb{T}(\overline{\mathbb{K}})$, we can define the *twist* of f by t as

$$t^* f = \sum_m \alpha_m \chi^m(t)\, \chi^m \in \overline{\mathbb{K}}[M].$$

Now, for $k \leq n$ let

$$\boldsymbol{f} = (f_1, \dots, f_k) \quad \text{and} \quad \boldsymbol{t} = (t_1, \dots, t_k)$$

be a family of nonzero Laurent polynomials in $\mathbb{K}[M]$ and a family of points of $\mathbb{T}(\overline{\mathbb{K}})$, respectively. Whenever the divisors of the twisted Laurent polynomials $t_1^* f_1, \dots, t_k^* f_k$ meet properly on the base extension $\mathbb{T}_{\overline{\mathbb{K}}}$, intersection theory allows to consider the $(n-k)$-dimensional cycle

$$Z_{\mathbb{T}}(\boldsymbol{t}^* \boldsymbol{f}) = \operatorname{div}(t_1^* f_1) \cdots \operatorname{div}(t_k^* f_k) \cdot \mathbb{T}_{\overline{\mathbb{K}}}.$$

Geometrically, this is the cycle obtained by intersecting the translations of the hypersufaces of each f_i by the corresponding point t_i^{-1}.

To be able to consider the height of such cycles we need to fix a compactification of the torus. Let then X be a complete toric variety with torus $\mathbb{T}$ and $\overline{D}_0, \dots, \overline{D}_{n-k}$ a family of semipositive metrized divisors on it. The corresponding height of $Z_{\mathbb{T}}(\boldsymbol{t}^* \boldsymbol{f})$ is defined as the height of the closure of this cycle in $X_{\overline{\mathbb{K}}}$.

With this convention, the following statement is a strengthening of the guesses formulated in [Gua18a, Conjecture 6.4.4] and in [GS23, Conjecture 11.8].

Conjecture A. *For $k \leq n$ let $\boldsymbol{f} = (f_1, \dots, f_k)$ be a family of nonzero Laurent polynomials in $\mathbb{K}[M]$, and X a complete toric variety with torus $\mathbb{T}$ equipped with a family $\overline{D}_0, \dots, \overline{D}_{n-k}$ of semipositive toric metrized divisors. Then for each sequence $(\boldsymbol{\omega}_\ell)_\ell$ of torsion points in $\mathbb{T}(\overline{\mathbb{K}})^k$ whose projection to $(\mathbb{T}^k/\mathbb{T})(\overline{\mathbb{K}})$ via the diagonal action is strict we have*

$$\lim_{\ell \to \infty} \mathrm{h}_{\overline{D}_0, \dots, \overline{D}_{n-k}}(Z_{\mathbb{T}}(\boldsymbol{\omega}_\ell^* \boldsymbol{f})) = \sum_{v \in \mathfrak{M}} n_v \, \mathrm{MI}_M(\vartheta_{\overline{D}_0, v}, \dots, \vartheta_{\overline{D}_{n-k}, v}, \rho_{f_1, v}^\vee, \dots, \rho_{f_k, v}^\vee),$$

where for each $v \in \mathfrak{M}$ we denote by n_v the weight of this place, by $\vartheta_{\overline{D}_i, v}$ the v-adic roof function of $\overline{D}_i$, and by $\rho_{f_j, v}^\vee$ the Legendre–Fenchel dual of the v-adic Ronkin function of f_j.

The precise definitions for the v-adic roof functions, the Legendre–Fenchel dual of the v-adic Ronkin functions and the mixed integral operator MI_M appearing in this conjecture can be found in [BPS14, Definitions 5.1.4 and 2.7.16] and [Gua18b, Definition 2.7], their underlying ideas being briefly recalled in Section 2. The strictness requirement on the sequence $(\boldsymbol{\omega}_\ell)_\ell$ ensures that the cycle $Z_{\mathbb{T}}(\boldsymbol{\omega}_\ell^* \boldsymbol{f})$ is well-defined for

ℓ sufficiently large thanks to Corollary 1.9, and so the limit in the left hand side of the conjectural equality makes sense.

In spirit, Conjecture A affirms that the typical height of the complete intersection defined by a family of Laurent polynomials twisted by torsion points may be predicted in convex-analytic terms from the knowledge of the arithmetic complexity of these Laurent polynomials. As such, it can be considered as an arithmetic analogue of a weak version of the Bernstein–Kushnirenko–Khovanskii theorem computing the typical degree of $Z_{\mathbb{T}}(\boldsymbol{t}^*\boldsymbol{f})$ for $\boldsymbol{t} \in \mathbb{T}(\overline{\mathbb{K}})^k$ (Corollary 1.7).

Till now, the conjecture was only known for $f_1 = f_2 = x_1 + x_2 + 1 \in \mathbb{Q}[x_1^{\pm 1}, x_2^{\pm 1}]$ and the canonical height on the projective plane [GS23, Corollary 5]. In this particular case, the limit height could also be expressed as a quotient of special values of the Riemann zeta function.

Our main result in this paper is the following.

Theorem B. *Conjecture A holds for $k = 2$.*

When specialized to bivariate Fermat polynomials of arbitrary degree, this result confirms [GS23, Conjecture 11.8]. It also implies the validity in the 2-codimensional situation of the preliminary [Gua18a, Conjecture 6.4.4] via the argument explained in [GS23, Section 6].

Independently of us, Destic, Hultberg and Szachniewicz have proven the conjecture in the general case [DHS24]. Their approach is radically different, relying on the application of Yuan's equidistribution theorem and on the theory of globally valued fields from unbounded continuous logic introduced by Ben Yaacov and Hrushovski.

The strategy of the proof of Theorem B is centered on the study of the local contributions to the height. More precisely, applying the arithmetic Bézout theorem we can realize the height of $Z_{\mathbb{T}}(\boldsymbol{\omega}_\ell^*\boldsymbol{f})$ as

$$\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{n-2},\overline{D}_{f_2}^{\mathrm{Ron}}}(Z_{\mathbb{T}}(\omega_{\ell,1}^* f_1)) + \sum_{v\in\mathfrak{M}} \frac{n_v}{\#O(\boldsymbol{\omega}_\ell)_v} \sum_{\boldsymbol{\eta}\in O(\boldsymbol{\omega}_\ell)_v} I_v(\boldsymbol{\eta}), \tag{1}$$

where $\overline{D}_{f_2}^{\mathrm{Ron}}$ is the toric divisor associated to the Newton polytope of f_2 equipped with its Ronkin metric, $O(\boldsymbol{\omega}_\ell)_v$ is the image of the Galois orbit of $\boldsymbol{\omega}_\ell$ in the v-adic analytic torus $\mathbb{T}_v^{\mathrm{an}}$, and $I_v\colon \mathbb{T}(\mathbb{C}_v)^2 \to \mathbb{R}\setminus\{-\infty\}$ is a function defined as an integral over the v-adic analytic toric variety X_v^{an}. As the height of the hypersurface defined by f_1 with respect to the metrized divisors $\overline{D}_0,\dots,\overline{D}_{n-2},\overline{D}_{f_2}^{\mathrm{Ron}}$ is well-understood in convex-analytic terms from [Gua18b], we can reduce the proof of Theorem B to showing that the second summand converges to zero along sequences $(\boldsymbol{\omega}_\ell)_\ell$ with strict image by the diagonal action (Section 2).

The study of the asymptotic behaviour of a function on strict sequences of torsion points immediately calls for the application of the equidistribution results. Unfortunately the equidistribution of roots of unity can not be applied directly to the function I_v. Indeed this function is only defined on the algebraic points of $\mathbb{T}$, and one first needs to extend it to the whole analytic torus. More seriously, this extension does not need to be continuous, and in fact it can take the value $-\infty$. Our focal goal is then to prove that I_v is (the restriction of) a function on the v-adic analytic torus $\mathbb{T}_v^{\mathrm{an}}$ with at most logarithmic singularities. We achieve this by applying Stoll's theorem on the continuity of the fiber integral in the Archimedean case [Sto67] and non-Archimedean analytic and formal geometry otherwise [BGR84, Ber90]. This is done in Section 3.

As a consequence of this property we can apply the local logarithmic equidistribution theorems for torsion points of Dimitrov and Habegger in the Archimedean case [DH24] and of Tate and Voloch in the non-Archimedean case [TV96]. When combined, these two statements show that each v-adic summand in (1) converges to 0 (Section 4).

Still, this local asymptotic vanishing is not enough to conclude the proof of Theorem B. In fact, it can happen that there are infinitely many places for which the corresponding local v-adic contribution does not eventually vanish, and therefore we are obliged to keep track of infinitely many places even for arbitrary large values of ℓ. To ensure that the second summand in (1) asymptotically vanishes we therefore need to prove that there exists a finite subset $\mathfrak{S} \subset \mathfrak{M}$ such that

$$\lim_{\ell\to\infty} \sum_{v\in\mathfrak{M}\setminus\mathfrak{S}} \frac{n_v}{\#O(\boldsymbol{\omega}_\ell)_v} \sum_{\boldsymbol{\eta}\in O(\boldsymbol{\omega}_\ell)_v} I_v(\boldsymbol{\eta}) = 0.$$

This can be done by explicit computation for a specific set $\mathfrak{S}$ of bad places (Section 5).

Because of its local nature, this approach can be used to obtain formulæ for the typical values of the local heights of the cycle $Z_{\mathbb{T}}(\boldsymbol{\omega}^*\boldsymbol{f})$. Moreover, as it relies on equidistribution theorems for which quantitative versions are available [DH24, Sch24], it might yield explicit bounds for the approximation of these local heights to their limit, as already done by Lin for the special case $f_1 = f_2 = 1 + x_1 + x_2$ and the canonical height on the projective plane [Lin24].

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Conventions and notations

A *variety* is an integral separated scheme X of finite type over a field K. A *divisor* on X is a Cartier divisor, unless otherwise said. For an algebraic closure $\overline{K}$ of K, the points of $X(\overline{K})$ are called the *algebraic points* of X.

Set $X_{\overline{K}} = X \times_K \operatorname{Spec}(\overline{K})$ for the base extension. A *cycle* Z of $X_{\overline{K}}$ is a formal finite sum of subvarieties of X with integer coefficients. For an integer r, the cycle Z is an *r-cycle* if these subvarieties can be taken of dimension r. We denote by $|Z|$ the support of Z. For an open subset $U \subset X$ and a cycle Y of U, we set $\overline{Y}$ for its closure in X.

Set $n = \dim(X)$. Then for an integer $0 \leq k \leq n$ and a family of global sections $s_1, \ldots, s_k$ of line bundles on $X_{\overline{K}}$ we denote by $V_X(s_1, \ldots, s_k)$ its zero set. These global sections are said to *meet properly* if for every subset $I \subseteq \{1, \ldots, k\}$ each irreducible component of $V_X(\{s_i\}_{i \in I})$ has codimension $\#I$. When this is the case, intersection theory allows to define the $(n-k)$-cycle of $X_{\overline{K}}$

$$Z_X(s_1, \ldots, s_k) = \mathrm{div}(s_1) \cdots \mathrm{div}(s_k) \cdot X_{\overline{K}}. \tag{2}$$

We have that $|Z_X(s_1, \ldots, s_k)| = V_X(s_1, \ldots, s_k)$.

We denote by $\mathbb{K}$ a number field and by $\mathfrak{M}$ its set of (nontrivial) places. For each $v \in \mathfrak{M}$ we denote by $|\cdot|_v$ the unique absolute value on $\mathbb{K}$ representing v and extending one of the standard absolute values on $\mathbb{Q}$. We set $\mathbb{K}_v$ for the completion of $\mathbb{K}$ with respect to $|\cdot|_v$ and associate to v the positive real weight

$$n_v = \frac{[\mathbb{K}_v : \mathbb{Q}_{v_0}]}{[\mathbb{K} : \mathbb{Q}]}$$

where v_0 is the restriction of this place to the field of rational numbers. We denote by $\mathbb{C}_v$ the completion of an algebraic closure of $\mathbb{K}_v$. The absolute value on $\mathbb{K}_v$ has a unique continuous extension to $\mathbb{C}_v$, that is also denoted by $|\cdot|_v$. When v is non-Archimedean, we denote by $\mathbb{C}_v^\circ$ the corresponding valuation ring, and by $\widetilde{\mathbb{C}}_v$ the associated residue field.

For a variety X over $\mathbb{K}$ and a place $v \in \mathfrak{M}$ we denote by X_v the base change with respect to $\mathbb{C}_v$ and by X_v^{an} the corresponding analytification in the sense of Berkovich [Ber90]. Similarly, for a morphism f of varieties over $\mathbb{K}$ we denote by f_v its base change with respect to $\mathbb{C}_v$ and by f_v^{an} its analytification.

1. Geometric results

Here we recall the basic notions and constructions from toric geometry that we will use throughout, referring to [Ful93] and [BPS14, Chapter 3] for the proofs and more details. In addition, we study the cycles of a toric variety given as the closure of intersection cycles of the torus of generic twists of Laurent polynomials. Our main result here is Theorem 1.5, which realizes them as the intersection cycles of the same twists for the associated global sections. In particular, this allows to express their degree in combinatorial terms (Corollary 1.7).

In this section we denote by K an arbitrary field and by $\mathbb{T}$ a split algebraic torus over K of dimension $n \geq 0$. We set M for the character lattice of $\mathbb{T}$ and $N = M^\vee$ for the dual lattice. Set also $M_\mathbb{R} = M \otimes \mathbb{R}$ and $N_\mathbb{R} = N \otimes \mathbb{R}$ for the associated vector spaces, and let $\langle \cdot, \cdot \rangle$ denote their pairing.

1.A. Geometric toric constructions. Let f be a nonzero Laurent polynomial in the group algebra $K[M]$ and set $Z_\mathbb{T}(f) = \mathrm{div}(f) \cdot \mathbb{T}$ for its associated Weil divisor.

Definition 1.1. Writing $f = \sum_m \alpha_m \chi^m$, the *twist* of f by a point $t \in \mathbb{T}(K)$ is defined as $t^* f = \sum_m \alpha_m \chi^m(t) \chi^m \in K[M]$.

This twist coincides with the pullback of f by the translation-by-t map $\tau_t \colon \mathbb{T} \to \mathbb{T}$ defined by $x \mapsto t \cdot x$. In particular, its Weil divisor agrees with the pushforward of the Weil divisor of f with respect to the inverse translation, that is

$$Z_\mathbb{T}(t^* f) = \tau_{t^{-1},*} Z_\mathbb{T}(f).$$

We denote by $\mathrm{NP}(f) \subset M_{\mathbb{R}}$ the Newton polytope of f and we set $\Psi_{\mathrm{NP}(f)} \colon N_{\mathbb{R}} \to \mathbb{R}$ for its *support function*, defined as

$$\Psi_{\mathrm{NP}(f)}(u) = \inf_{x \in \mathrm{NP}(f)} \langle u, x \rangle \quad \text{for } u \in N_{\mathbb{R}}.$$

Let then Σ be a complete fan on $N_{\mathbb{R}}$ that is *compatible* with $\mathrm{NP}(f)$, in the sense that the support function of this polytope is linear on each of the cones of this fan.

Set X for the toric variety over K associated to Σ. It is an equivariant compactification of the torus $\mathbb{T}$ that is both normal and complete. It admits an open covering by the affine toric varieties corresponding to the cones of the fan:

$$X = \bigcup_{\sigma \in \Sigma} X_{\sigma}. \tag{1.1}$$

The affine toric variety X_0 corresponding to the zero cone is canonically isomorphic to the torus, and is called the *principal open subset* of X.

Since the fan is compatible with the Newton polytope of f, we can associate to this Laurent polynomial a nef toric divisor D_f on X. It is defined as

$$D_f = (X_{\sigma}, \chi^{-m_{\sigma}})_{\sigma \in \Sigma},$$

where for each σ we denote by $m_{\sigma} \in M$ the slope of $\Psi_{\mathrm{NP}(f)}$ on the cone σ. Set also $L_f = \mathcal{O}(D_f)$ for the associated toric line bundle and s_{D_f} for its *canonical rational section*. The first is the subsheaf of the sheaf of rational functions $\mathcal{K}_X$ that is generated on each affine chart X_{σ} by the monomial $\chi^{m_{\sigma}}$, and the second is the rational section of L_f induced by the global section 1 of $\mathcal{K}_X$. We have that $\mathrm{div}(s_{D_f}) = D_f$.

We also set s_f for the rational section of L_f similarly induced by the Laurent polynomial f considered as a global section of $\mathcal{K}_X$. It is related to the canonical rational section by $s_f = f s_{D_f}$. It follows from [Gua18b, Theorem 4.3] that s_f is a global section with Weil divisor equal to the closure in X of the Weil divisor of $\mathbb{T}$ defined by f, that is

$$s_f \in \Gamma(X, L_f) \quad \text{and} \quad Z_X(s_f) = \overline{Z_{\mathbb{T}}(f)}.$$

The action of $\mathbb{T}$ on L_f induces an action on the global sections of this line bundle, which can be described at the level of closed points as follows. For $t \in \mathbb{T}(K)$ we now denote by $\tau_t \colon X \to X$ the translation-by-t map on the toric variety and by $\tau_t^* \colon \mathcal{K}_X \to \mathcal{K}_X$ the corresponding pullback morphism. We have that L_f is invariant with respect to the latter, and so we can pullback global sections by τ_t.

Definition 1.2. For $s \in \Gamma(X, L_f)$ and $t \in \mathbb{T}(K)$, the *twist* of s by t is the pullback global section $\tau_t^* s$. It is denoted as $t^* s$ for convenience.

For the global section s_f we have that

$$t^* s_f = (t^* f) s_{D_f}. \tag{1.2}$$

The next lemma describes the Weil divisor defined by s_f in terms of the orbit decomposition of the toric variety. To state this properly, we recall that to each cone $\sigma \in \Sigma$ we can attach the split algebraic torus

$$O(\sigma) = \mathrm{Spec}(K[\sigma^{\perp} \cap M]),$$

where $\sigma^{\perp} \subset M_{\mathbb{R}}$ denotes the orthogonal linear subspace. On the other hand, the corresponding affine toric variety in (1.1) is $X_{\sigma} = \mathrm{Spec}(K[\sigma^{\vee} \cap M])$ where $\sigma^{\vee} \subset M_{\mathbb{R}}$ denotes the dual cone. In this situation, the inclusion $\sigma^\perp \subset \sigma^\vee$ induces the homomorphism of algebras $K[\sigma^\vee \cap M] \to K[\sigma^\perp \cap M]$ given by

$$\chi^m \longmapsto \begin{cases} \chi^m & \text{if } m \in \sigma^\perp, \\ 0 & \text{otherwise,} \end{cases}$$

and so a closed immersion $\iota_\sigma \colon O(\sigma) \hookrightarrow X_\sigma$.

Varying the cones and identifying the corresponding tori with their images in X gives the different orbits of the action of $\mathbb{T}$ on X, producing a decomposition

$$X = \bigsqcup_{\sigma \in \Sigma} O(\sigma).$$

To each σ we can also associate the face of the Newton polytope of f defined as

$$\mathrm{NP}(f)^\sigma = \{x \in \mathrm{NP}(f) : \langle u, x\rangle = \Psi_{\mathrm{NP}(f)}(u) \text{ for all } u \in \sigma\}.$$

It is contained in $m_\sigma + \sigma^\perp \subset M_\mathbb{R}$ for the slope m_σ of $\Psi_{\mathrm{NP}(f)}$ on this cone, and in fact it is the intersection between this affine subspace and $\mathrm{NP}(f)$. Restricting the monomial expansion of f to this face and translating the exponents we obtain the nonzero Laurent polynomial

$$f(\sigma) = \sum_{m \in \mathrm{NP}(f)^\sigma \cap M} \alpha_m \chi^{m - m_\sigma} \in K[\sigma^\perp \cap M].$$

The inclusion $\sigma^\perp \cap M \subset M$ also induces an inclusion of algebras $K[\sigma^\perp \cap M] \subset K[M]$ and so a homomorphism of tori $\pi_\sigma \colon \mathbb{T} \to O(\sigma)$. These constructions are related to the twisting by

$$(t^* f)(\sigma) = \chi^{m_\sigma}(t) \cdot (\pi_\sigma(t))^*(f(\sigma)). \tag{1.3}$$

Lemma 1.3. *For each $\sigma \in \Sigma$ we have that*

$$V_X(s_f) \cap O(\sigma) = V_{O(\sigma)}(f(\sigma)).$$

In particular, for $t \in \mathbb{T}(K)$ we have that $V_X(t^* s_f) \cap O(\sigma) = V_{O(\sigma)}(\pi_\sigma(t)^* f(\sigma))$.

Proof. On X_σ, the line bundle L_f is generated by χ^{m_σ} and its global section s_f is defined by f. Hence $V_{X_\sigma}(s_f) = V_{X_\sigma}(\chi^{-m_\sigma} f)$, where the latter denotes the zero set of the regular function $\chi^{-m_\sigma} f$ on X_σ. Restricting to the orbit $O(\sigma)$ we get

$$V_X(s_f) \cap O(\sigma) = V_{X_\sigma}(\chi^{-m_\sigma} f) \cap O(\sigma) = V_{O(\sigma)}(\iota_\sigma^*(\chi^{-m_\sigma} f)) = V_{O(\sigma)}(f(\sigma)),$$

proving the first statement. The second statement follows from the first together with the functoriality in (1.3), noting that $\chi^{m_\sigma}(t) \neq 0$. □

Using this result, we show that the zero set of a generic twist of the global section s_f avoids any given finite set of points. Recall that a condition on the algebraic points of $\mathbb{T}$ is *generic* if it holds on a nonempty open subset of $\mathbb{T}_{\overline{K}}$.

Lemma 1.4. *Let $p_1, \dots, p_r \in X_{\overline{K}}$ be a finite family of (not necessarily closed) points. Then for a generic choice of $t \in \mathbb{T}(\overline{K})$ we have that $p_i \notin V_X(t^* s_f)$ for all i.*

Proof. We reduce without loss of generality to a single point $p \in X_{\overline{K}}$, since the general case follows from this one by intersecting the corresponding genericity conditions. Moreover, taking any closed point in the closure $\overline{\{p\}}$ we can also suppose that $p \in X(\overline{K})$.

Take $\sigma \in \Sigma$ such that $p \in O(\sigma)$. By Lemma 1.3, for $t \in \mathbb{T}(\overline{K})$ we have that $p \notin V_X(t^* s_f)$ if and only if $p \notin V_{O(\sigma)}(\pi_\sigma(t)^* f(\sigma))$. This is equivalent to the fact that

$$f(\sigma)(\pi_\sigma(t) \cdot p) \neq 0,$$

which in turn is equivalent to $t \notin \pi_\sigma^{-1}(V_{O(\sigma)}(p^* f(\sigma)))$. Since the Laurent polynomial $f(\sigma)$ is nonzero, this latter condition is open and nonempty, and so generic. $\square$

For $0 \leq r \leq n$, the *degree* of an r-cycle Z of $\mathbb{T}_{\overline{K}}$ with respect to a family $D_1, \dots, D_r$ of divisors on X is defined by taking its closure, that is

$$\deg_{D_1,\dots,D_r}(Z) \coloneqq \deg_{D_1,\dots,D_r}(\overline{Z}),$$

where the latter degree is defined by considering the base change of the divisors with respect to $\overline{K}$.

Recall that to each toric divisor D on X we can associate a lattice polytope $\Delta_{D_i} \subset M_\mathbb{R}$. For a family $D_1, \dots, D_n$ of nef toric divisors on X we have that

$$\deg_{D_1,\dots,D_n}(\mathbb{T}_{\overline{K}}) = \mathrm{MV}_M(\Delta_{D_1}, \dots, \Delta_{D_n}), \tag{1.4}$$

where MV_M denotes the mixed volume function for the Haar measure vol_M on $M_\mathbb{R}$ that gives covolume 1 to the lattice M and acts on families of n convex bodies of this vector space [BPS14, Definition 2.7.14].

1.B. The dimension of intersection cycles of twists. Here we study the intersection cycles defined by the twists of several Laurent polynomials. We start by considering an integer $0 \leq k \leq n$ and a family of nonzero Laurent polynomials in $K[M]$

$$\boldsymbol{f} = (f_1, \dots, f_k).$$

Fix a complete fan Σ on $N_\mathbb{R}$ which is compatible with the Newton polytopes of these Laurent polynomials and set X for the associated complete toric variety. This compatibility allows to consider the family $s_{\boldsymbol{f}} = (s_{f_1}, \dots, s_{f_k})$ where s_{f_i} is the global section of the nef toric line bundle L_{f_i} on X associated to f_i.

Let $\mathbb{T}^k$ be the product of k-many copies of the torus $\mathbb{T}$, and for $\boldsymbol{t} \in \mathbb{T}(\overline{K})^k$ set

$$\boldsymbol{t}^* \boldsymbol{f} = (t_1^* f_1, \dots, t_k^* f_k) \quad \text{and} \quad \boldsymbol{t}^* s_{\boldsymbol{f}} = (t_1^* s_{f_1}, \dots, t_k^* s_{f_k})$$

for the corresponding families of twists over $\overline{K}$ (Definitions 1.1 and 1.2).

The following is the main result of this section, realizing the closure of the intersection cycle of the torus defined by a generic twist of $\boldsymbol{f}$ as the intersection cycle of the toric variety defined by the same twist of $s_{\boldsymbol{f}}$.

Theorem 1.5. *There is a proper closed subset $H \subset \mathbb{T}^k$ such that for $\boldsymbol{t} \in (\mathbb{T}^k \setminus H)(\overline{K})$ we have that $t_1^* s_{f_1}, \dots, t_k^* s_{f_k}$ meet properly and that $\overline{Z_\mathbb{T}(\boldsymbol{t}^* \boldsymbol{f})} = Z_X(\boldsymbol{t}^* s_{\boldsymbol{f}})$.*

We first prove the next auxiliary result.

Lemma 1.6. *There is $\boldsymbol{t} \in \mathbb{T}(\overline{K})^k$ such that every irreducible component of the zero set $V_X(\boldsymbol{t}^* s_{\boldsymbol{f}})$ has codimension k and is not contained in $X \setminus X_0$.*

Proof. We proceed by induction on k. The case $k = 0$ is tautological, and so we suppose that $k \geq 1$. The inductive step follows by applying Lemma 1.4 to the generic point of each of the components of $V_X(t_1^* s_{f_1}, \dots, t_{k-1}^* s_{f_{k-1}})$ and of the intersection of this zero set with $X \setminus X_0$. As a consequence, there is $t_k \in \mathbb{T}(\overline{K})$ such that $V_X(t_k^* s_{f_k})$

does not contain any of the components of $V_X(t_1^* s_{f_1}, \dots, t_{k-1}^* s_{f_{k-1}})$. Setting $\boldsymbol{t} = (t_1, \dots, t_{k-1}, t_k)$ we have that every component of

$$V_X(\boldsymbol{t}^* s_{\boldsymbol{f}}) = V_X(t_1^* s_{f_1}, \dots, t_{k-1}^* s_{f_{k-1}}) \cap V_X(t_k^* s_{f_k})$$

has codimension k, as stated.

Moreover, by the inductive hypothesis all the components of the intersection of $V_X(t_1^* s_{f_1}, \dots, t_{k-1}^* s_{f_{k-1}})$ with $X \setminus X_0$ have codimension k. By construction of t_k, none of them is contained in $V_X(\boldsymbol{t}^* s_{\boldsymbol{f}})$, and so this zero set has no component contained in $X \setminus X_0$. □

Proof of Theorem 1.5. We first show that the conditions in Lemma 1.6 are generic. To this end, for each i denote by $\mu_i \colon \mathbb{T}^k \times X \to X$ the i-th multiplication map defined by

$$(t_1, \dots, t_k, p) \longmapsto t_i \cdot p.$$

The pullback $\mu_i^* s_{f_i}$ is a global section of the line bundle $\mu_i^* L_{f_i}$ on $\mathbb{T}^k \times X$, and we put

$$\Omega = V_{\mathbb{T}^k \times X}(\mu_1^* s_{f_1}, \dots, \mu_k^* s_{f_k}).$$

This a closed set of $\mathbb{T}^k \times X$ whose algebraic points are the pairs $(\boldsymbol{t}, p) \in (\mathbb{T}^k \times X)(\overline{K}) = \mathbb{T}(\overline{K})^k \times X(\overline{K})$ such that

$$s_{f_1}(t_1 \cdot p) = \dots = s_{f_k}(t_k \cdot p) = 0.$$

Hence Ω is the incidence closed subset of the global sections s_{f_i}, $i = 1, \dots, k$, with respect to the action of the torus on the toric variety.

For $\boldsymbol{t} \in \mathbb{T}(\overline{K})^k$ the fiber of the projection map $\pi \colon \mathbb{T}^k \times X \to \mathbb{T}^k$ restricted to this closed subset identifies with the zero set of the family of twisted global sections $\boldsymbol{t}^* s_{\boldsymbol{f}}$, that is

$$(\pi|_\Omega)^{-1}(\boldsymbol{t}) = \{\boldsymbol{t}\} \times V_X(\boldsymbol{t}^* s_{\boldsymbol{f}}).$$

Let W be an irreducible component of Ω. We have that the map π is closed because X is complete, and so the image $\pi(W) \subset \mathbb{T}^k$ is a closed subset.

On the one hand, if $\pi(W) \neq \mathbb{T}^k$ then $\pi(W) \subset \mathbb{T}^k$ is a proper closed subset and so $(\pi|_W)^{-1}(\boldsymbol{t}) = \emptyset$ for $\boldsymbol{t}$ generic. On the other hand, if $\pi(W) = \mathbb{T}^k$ then the restriction

$$\pi|_W \colon W \longrightarrow \mathbb{T}^k \tag{1.5}$$

is surjective. As Ω is defined by k equations, dimension theory implies that the codimension of W cannot exceed k. Moreover, by the first part of Lemma 1.6 the map in (1.5) has a fiber of dimension $n-k$, and by the theorem of dimension of fibers [Har77, Exercise 3.22 at page 95] this quantity bounds above the relative dimension of W over $\mathbb{T}^k$. Altogether this implies that

$$n - k \leq \dim(W) - \dim(\mathbb{T}^k) \leq n - k.$$

It follows that this relative dimension is equal to $n-k$, and so the theorem of dimension of fibers also gives that $\dim((\pi|_W)^{-1}(\boldsymbol{t})) = n - k$ for $\boldsymbol{t}$ generic.

Note that

$$(\pi|_\Omega)^{-1}(\boldsymbol{t}) = \bigcup_W (\pi|_W)^{-1}(\boldsymbol{t}),$$

the union being over the irreducible components of Ω. Hence intersecting the genericity conditions corresponding to each W we get that $V_X(\boldsymbol{t}^* s_{\boldsymbol{f}})$ is either empty or has pure codimension k for $\boldsymbol{t}$ generic.

Applying the same reasoning to the intersection $\Omega \cap (\mathbb{T}^k \times (X \setminus X_0))$ and using the second part of Lemma 1.6 we deduce that $V_X(\boldsymbol{t}^* s_{\boldsymbol{f}}) \cap (X \setminus X_0)$ is either empty or has pure codimension $k+1$ in X for a generic choice of $\boldsymbol{t}$. We obtain that $V_X(\boldsymbol{t}^* s_{\boldsymbol{f}})$ has no irreducible component contained in $X \setminus X_0$ for this choice of $\boldsymbol{t}$.

This proves that the conditions in Lemma 1.6 applied to the family $s_{f_1}, \dots, s_{f_k}$ hold for $\boldsymbol{t}$ generic. Applying this to the global sections corresponding to each index subset $I \subset \{1, \dots, k\}$ and intersecting the resulting genericity conditions we get that a generic twist of the global sections meets properly, as stated.

Summing up, for a generic choice of $\boldsymbol{t}$ the intersection cycle $Z_X(\boldsymbol{t}^* s_{\boldsymbol{f}})$ can be defined and has no irreducible component contained in $X \setminus X_0$. In particular, it coincides with the closure of $Z_{\mathbb{T}}(\boldsymbol{t}^* \boldsymbol{f})$ in X, as stated. □

We next use this result to compute the degree of the intersection cycle of the torus defined by a generic twist of $\boldsymbol{f}$.

Corollary 1.7. *Let $D_1, \dots, D_{n-k}$ be nef toric divisors on X and $H \subset \mathbb{T}^k$ the proper closed subset in Theorem 1.5. Then for $\boldsymbol{t} \in (\mathbb{T}^k \setminus H)(\overline{K})$ we have that*

$$\deg_{D_1,\dots,D_{n-k}}(Z_{\mathbb{T}}(\boldsymbol{t}^* \boldsymbol{f})) = \mathrm{MV}_M(\Delta_{D_1}, \dots, \Delta_{D_{n-k}}, \mathrm{NP}(f_1), \dots, \mathrm{NP}(f_k)).$$

Proof. By Theorem 1.5, for a generic choice of $\boldsymbol{t} \in \mathbb{T}(\overline{K})^k$ we have that

$$\deg_{D_1,\dots,D_{n-k}}(Z_{\mathbb{T}}(\boldsymbol{t}^* \boldsymbol{f})) = \deg_{D_1,\dots,D_{n-k}}(\mathrm{div}(t_1^* s_{f_1}) \cdots \mathrm{div}(t_{n-k}^* s_{f_{n-k}}) \cdot X).$$

By Bézout's formula, this latter degree coincides with $\deg_{D_1,\dots,D_{n-k},D_{f_1},\dots,D_{f_k}}(\mathbb{T}_{\overline{K}})$ where D_{f_i} is the nef toric divisor on X associated to f_i for each i. The polytope of D_{f_i} coincides with $\mathrm{NP}(f_i)$ and so the statement follows from (1.4). □

Following [Bil97], a sequence of algebraic points of a torus is said to be *strict* if it eventually avoids any fixed proper algebraic subgroup. In this subsection we slightly relax this notion to one that suits better to twists of systems of equations.

Consider the diagonal embedding $\mathbb{T} \hookrightarrow \mathbb{T}^k$ defined by $t \mapsto (t, \dots, t)$. The quotient $\mathbb{T}^k/\mathbb{T}$ is isomorphic to the torus $\mathbb{T}^{k-1}$ through the map

$$[(t_1, t_2, \dots, t_k)] \longmapsto (t_1^{-1} t_2, \dots, t_1^{-1} t_k). \tag{1.6}$$

Set $\varpi \colon \mathbb{T}^k \to \mathbb{T}^k/\mathbb{T}$ for the quotient homomorphism.

Definition 1.8. A sequence $(\boldsymbol{t}_\ell)_\ell$ in $\mathbb{T}(\overline{K})^k$ is *quasi-strict* if for every proper algebraic subgroup $G \subset \mathbb{T}^k/\mathbb{T}$ there is $\ell_0 \in \mathbb{N}$ such that $\varpi(\boldsymbol{t}_\ell) \notin G(\overline{K})$ for all $\ell \geq \ell_0$.

Notice that any strict sequence in $\mathbb{T}(\overline{K})^k$ is also quasi-strict. In fact, a sequence $(\boldsymbol{t}_\ell)_\ell$ in $\mathbb{T}(\overline{K})^k$ is quasi-strict if and only if the sequence $((t_{\ell,1}^{-1} t_{\ell,2}, \dots, t_{\ell,1}^{-1} t_{\ell,k}))_\ell$ in $\mathbb{T}(\overline{K})^{k-1}$ is strict.

The next result is a direct consequence of Theorem 1.5 and Corollary 1.7 together with Laurent's theorem proving the toric Manin-Mumford conjecture.

Corollary 1.9. *Let $(\boldsymbol{\omega}_\ell)_\ell$ be a quasi-strict sequence of torsion points of $\mathbb{T}(\overline{K})^k$. Then for ℓ sufficiently large the global sections $\boldsymbol{\omega}_{\ell,1}^* s_{f_1}, \dots, \boldsymbol{\omega}_{\ell,k}^* s_{f_k}$ meet properly and*

$$\overline{Z_{\mathbb{T}}(\boldsymbol{\omega}_\ell^* \boldsymbol{f})} = Z_X(\boldsymbol{\omega}_\ell^* s_{\boldsymbol{f}}).$$

In particular, for a family $D_1, \dots, D_{n-k}$ of nef toric divisors on X we have that

$$\deg_{D_1,\dots,D_{n-k}}(Z_{\mathbb{T}}(\boldsymbol{\omega}_\ell^* \boldsymbol{f})) = \mathrm{MV}_M(\Delta_{D_1}, \dots, \Delta_{D_{n-k}}, \mathrm{NP}(f_1), \dots, \mathrm{NP}(f_k))$$

for ℓ sufficiently large.

Proof. Let $H \subset \mathbb{T}^k$ be the minimal closed subset satisfying the conditions in Theorem 1.5. One can check that these conditions are invariant under the diagonal action of $\mathbb{T}(\overline{K})$. This implies that $\varpi(H) \subset \mathbb{T}^k/\mathbb{T}$ is a proper closed subset.

The sequence $(\boldsymbol{\omega}_\ell)_\ell$ projects to a strict sequence of torsion points of $(\mathbb{T}^k/\mathbb{T})(\overline{K})$. By Laurent's theorem [BG06, Theorem 7.4.7], this strict sequence eventually avoids $\varpi(H)$, and so $(\boldsymbol{\omega}_\ell)_\ell$ eventually avoids H. The statement then follows readily from Theorem 1.5 and Corollary 1.7. □

2. The height of intersection cycles of twists

In this section we move to the arithmetic setting, approaching the main content of the paper. After recalling from [BPS14, Gua18b] the notions and objects playing a role in the statement of Conjecture A we present a strategy for its proof, which reduces it to a local logarithmic equidistribution statement and a global adelic one. We also provide several reduction steps that can be employed for the proof, relegating the burdensome details to Appendix A.

2.A. Arithmetic toric constructions. Throughout this section we denote by $\mathbb{T}$ a split algebraic torus over $\mathbb{K}$ of dimension $n \geq 0$ and X a complete toric variety compactifying $\mathbb{T}$. Recall that for each $v \in \mathfrak{M}$ we respectively denote by $\mathbb{T}_v$ and X_v the base change of the torus and the toric variety with respect to the complete and algebraically closed field $\mathbb{C}_v$, and we set $\mathbb{T}_v^{\mathrm{an}}$ and X_v^{an} for their analytification. The *v-adic compact torus* is the subset of $\mathbb{T}_v^{\mathrm{an}}$ defined as

$$\mathbb{S}_v = \big\{t \in \mathbb{T}_v^{\mathrm{an}} : |\chi^m(t)|_v = 1 \text{ for all } m \in M\big\}. \tag{2.1}$$

It is an analytic group acting on X_v^{an}.

We also denote by $\mathrm{val}_v \colon \mathbb{T}_v^{\mathrm{an}} \to N_{\mathbb{R}}$ the *v-adic valuation map* [BPS14, Section 4.1]. Choosing a splitting $\mathbb{T} \simeq \mathbb{G}_{\mathrm{m}}^n$ identifies $N_{\mathbb{R}}$ with $\mathbb{R}^n$ and the set of rigid points of $\mathbb{T}_v^{\mathrm{an}}$ with $(\mathbb{C}_v^{\times})^n$. In these coordinates, the v-adic valuation map writes down at the level of these rigid points as

$$\mathrm{val}_v(t_1, \dots, t_n) = (-\log|t_1|_v, \dots, -\log|t_n|_v).$$

Let $\overline{D}$ be a *semipositive toric (adelically) metrized divisor* on X. It is the datum

$$\overline{D} = \big(D, (\|\cdot\|_v)_{v\in\mathfrak{M}}\big),$$

where D is a toric divisor on X and each $\|\cdot\|_v$ is a semipositive $\mathbb{S}_v$-invariant and continuous metric on the analytic line bundle $\mathcal{O}(D)_v^{\mathrm{an}}$ that is induced by a single integral model of the pair $(X, O(D))$ for all but finitely many $v \in \mathfrak{M}$ [BPS14, Definition 4.9.1].

Let s_D be the canonical rational section of the line bundle $\mathcal{O}(D)$ associated to D. For each v the function $\log\|s_D\|_v \colon \mathbb{T}_v^{\mathrm{an}} \to \mathbb{R}$ factors through the valuation map. In fact,

$$\log\|s_D\|_v = \psi_{\overline{D},v} \circ \mathrm{val}_v$$

for a concave function $\psi_{\overline{D},v} \colon N_{\mathbb{R}} \to \mathbb{R}$, called the *$v$-adic metric function* of $\overline{D}$. The Legendre–Fenchel dual of $\psi_{\overline{D},v}$ is a continuous concave function

$$\vartheta_{\overline{D},v} \colon \Delta_D \longrightarrow \mathbb{R},$$

where $\Delta_D \subset M_{\mathbb{R}}$ is the polytope associated to D. It is called the *v-adic roof function* of $\overline{D}$. The adelicity of $\overline{D}$ implies that $\vartheta_{\overline{D},v}$ is the zero function on Δ_D for all except finitely many $v \in \mathfrak{M}$.

For $0 \leq r \leq n$, the *height* of an r-cycle Z of $\mathbb{T}_{\mathbb{K}}$ with respect to a family of semipositive metrized divisors $\overline{D}_0, \dots, \overline{D}_r$ on X is defined through its closure as

$$\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_r}(Z) = \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_r}(\overline{Z}).$$

The latter is defined by considering the base change of these metrized divisors to any finite extension of $\mathbb{K}$ over which $\overline{Z}$ is defined, and applying the recursive definition of the height in [BPS14, Chapter 1].

The following result due to Burgos, Philippon and the second author expresses the height of the torus with respect to a family of semipositive toric metrized divisors in convex-analytic terms. It involves the mixed integral operator MI_M with respect to the Haar measure vol_M on $M_{\mathbb{R}}$ acting on families of $n+1$ concave functions on convex bodies of $M_{\mathbb{R}}$, see for instance [BPS14, Definition 2.7.16].

Theorem 2.1 ([BPS14, Theorem 5.2.5])**.** *Let $\overline{D}_0, \dots, \overline{D}_n$ be semipositive toric metrized divisors on X. Then*

$$\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_n}(\mathbb{T}_{\mathbb{K}}) = \sum_{v \in \mathfrak{M}} n_v \, \mathrm{MI}_M(\vartheta_{\overline{D}_0,v}, \dots, \vartheta_{\overline{D}_n,v}).$$

Let now $f \in \mathbb{K}[M]$ be a nonzero Laurent polynomial. When investigating the height of its Weil divisor it is necessary to consider an adelic family of functions measuring the arithmetic complexity of this Laurent polynomial. To this end, for each $v \in \mathfrak{M}$ we consider the *v-adic Ronkin function* of f, which is a concave function

$$\rho_{f,v} \colon N_{\mathbb{R}} \longrightarrow \mathbb{R}$$

defined as the average of $\log|f|_v \colon \mathbb{T}_v^{\mathrm{an}} \to \mathbb{R} \cup \{-\infty\}$ on the fibers of the v-adic valuation map [Gua18b, Section 2]. Its Legendre–Fenchel dual $\rho_{f,v}^{\vee}$ is a continuous concave function on the Newton polytope of f.

Theorem 2.2 ([Gua18b, Theorem 5.12])**.** *Let $f \in \mathbb{K}[M]$ be a nonzero Laurent polynomial and $\overline{D}_0, \dots, \overline{D}_{n-1}$ semipositive toric metrized divisors on X. Then*

$$\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{n-1}}(Z_{\mathbb{T}}(f)) = \sum_{v \in \mathfrak{M}} n_v \, \mathrm{MI}_M(\vartheta_{\overline{D}_0,v}, \dots, \vartheta_{\overline{D}_{n-1},v}, \rho_{f,v}^{\vee}).$$

2.B. An approach to Conjecture A. Our main concern in this article is the height of higher codimensional cycles in X. To this end, for an integer $0 \leq k \leq n$ fix a family $\boldsymbol{f} = (f_1, \dots, f_{n-k})$ of nonzero Laurent polynomials in $\mathbb{K}[M]$ and a family $\overline{D}_0, \dots, \overline{D}_{n-k}$ of semipositive toric metrized divisors on X. Conjecture A in the introduction predicts that

$$\lim_{\ell \to \infty} \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{n-k}}(Z_{\mathbb{T}}(\boldsymbol{\omega}_\ell^* \boldsymbol{f})) = \sum_{v \in \mathfrak{M}} n_v \, \mathrm{MI}_M(\vartheta_{\overline{D}_0,v}, \dots, \vartheta_{\overline{D}_{n-k},v}, \rho_{f_1,v}^{\vee}, \dots, \rho_{f_k,v}^{\vee})$$

for any quasi-strict sequence $(\boldsymbol{\omega}_\ell)_\ell$ of torsion points of $\mathbb{T}(\overline{\mathbb{K}})^k$.

Remark 2.3. For $k = 0$ this claim boils down to Theorem 2.1, while for $k = 1$ it coincides with Theorem 2.2 noticing that

$$\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{n-1}}(Z_{\mathbb{T}}(\omega^* f)) = \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{n-1}}(Z_{\mathbb{T}}(f))$$

for any torsion point $\omega \in \mathbb{T}(\overline{\mathbb{K}})$, which follows from the arithmetic projection formula and the definition of toric metrics.

We now outline a strategy for proving Conjecture A, arguing by induction on the number k of Laurent polynomials.

As the conjecture holds true for $k=0$ and $k=1$ (Remark 2.3), we just need to prove the inductive step. Let $k\geq 2$ and assume for convenience that the fan of X is compatible with the Newton polytopes of $f_1,\dots,f_k$. As we will later see, this can always be done without loss of generality (Proposition 2.6). Under this compatibility condition, for each Laurent polynomial f_i we consider the nef toric divisor D_{f_i} on X and the global section s_{f_i} of the toric line bundle $\mathcal{O}(D_{f_i})$. Following [Gua18b, Section 5], we can enrich the divisor D_{f_i} with its adelic family of v-adic Ronkin metrics. The obtained pair

$$\overline{D}_{f_i}^{\mathrm{Ron}}=(D_{f_i},(\|\cdot\|_{\mathrm{Ron},v})_{v\in\mathfrak{M}}) \tag{2.2}$$

is a semipositive toric metrized divisor on X whose v-adic metric function coincides with the Ronkin function $\rho_{f_i,v}$.

Let $H\subset\mathbb{T}^k$ be the proper closed subset given by Theorem 1.5 and take a family of torsion points $\boldsymbol{\omega}=(\omega_1,\dots,\omega_k)\in\mathbb{T}(\overline{\mathbb{K}})^k$ not lying in it. By this result, the twisted global sections $\omega_1^*s_{f_1},\dots,\omega_k^*s_{f_k}$ meet properly and

$$\overline{Z_{\mathbb{T}}(\boldsymbol{\omega}^*\boldsymbol{f})}=Z_X(\boldsymbol{\omega}^*s_{\boldsymbol{f}})=\mathrm{div}(\omega_k^*s_{f_k})\cdot Z_X\big(\widehat{\boldsymbol{\omega}}^*s_{\widehat{\boldsymbol{f}}}\big)$$

with $\widehat{\boldsymbol{f}}=(f_1,\dots,f_{k-1})$ and $\widehat{\boldsymbol{\omega}}=(\omega_1,\dots,\omega_{k-1})$. Applying Theorem 1.5 to the family $\widehat{\boldsymbol{f}}$, up to possibly enlarging H we also have that $\overline{Z_{\mathbb{T}}(\widehat{\boldsymbol{\omega}}^*\widehat{\boldsymbol{f}})}=Z_X(\widehat{\boldsymbol{\omega}}^*s_{\widehat{\boldsymbol{f}}})$. Hence

$$\overline{Z_{\mathbb{T}}(\boldsymbol{\omega}^*\boldsymbol{f})}=\mathrm{div}(\omega_k^*s_{f_k})\cdot\overline{Z_{\mathbb{T}}(\widehat{\boldsymbol{\omega}}^*\widehat{\boldsymbol{f}})}.$$

This expression is particularly favourable to compute the height of this cycle of $X_{\mathbb{K}}$ using the recursive definition of the height. To do so, let us consider the finite extension $\mathbb{K}(\boldsymbol{\omega})/\mathbb{K}$ and denote by $\mathfrak{M}(\boldsymbol{\omega})$ its set of places. Each $w\in\mathfrak{M}(\boldsymbol{\omega})$ restricts to a place $v\in\mathfrak{M}$, and its weight is

$$n_w=\frac{[\mathbb{K}(\boldsymbol{\omega})_w:\mathbb{K}_v]}{[\mathbb{K}(\boldsymbol{\omega}):\mathbb{K}]}\,n_v \tag{2.3}$$

by the multiplicativity of degrees of finite extensions. Applying the arithmetic Bézout formula over the number field $\mathbb{K}(\boldsymbol{\omega})$ yields

$$\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{n-k}}(Z_{\mathbb{T}}(\boldsymbol{\omega}^*\boldsymbol{f}))=\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{n-k},\overline{D}_{f_k}^{\mathrm{Ron}}}(Z_{\mathbb{T}}(\widehat{\boldsymbol{\omega}}^*\widehat{\boldsymbol{f}}))\;+\sum_{w\in\mathfrak{M}(\boldsymbol{\omega})}n_wJ_w(\boldsymbol{\omega}), \tag{2.4}$$

where for each $w\in\mathfrak{M}(\boldsymbol{\omega})$ we have set

$$J_w(\boldsymbol{\omega})=\int_{X_w^{\mathrm{an}}}\log\|\omega_k^*s_{f_k}\|_{\mathrm{Ron},w}\;\mathrm{c}_1(\overline{D}_{0,w})\wedge\ldots\wedge\mathrm{c}_1(\overline{D}_{n-k,w})\wedge\delta_{Z_X(\widehat{\boldsymbol{\omega}}^*s_{\widehat{\boldsymbol{f}}})_w^{\mathrm{an}}}.$$

Here the integral is with respect to the w-adic mixed Monge–Ampère measure of the extension to $\mathbb{K}(\boldsymbol{\omega})$ of the metrized divisors, restricted to the analytic $(n-k+1)$-cycle defined by the first $k-1$ twisted global sections.

We can write (2.4) in a friendlier way by grouping together the local w-adic terms according to their restriction to $\mathbb{K}$ and reformulating the resulting sums in term of Galois orbits.

For $t\in\mathbb{T}(\overline{\mathbb{K}})$ we denote by $O(t)\subset\mathbb{T}(\overline{\mathbb{K}})$ its *Galois orbit*, that is the orbit of this algebraic point under the action of the absolute Galois group of $\mathbb{K}$. To pass from the algebraic to the analytic setting, we need to choose a $\mathbb{K}$-embedding

$$\iota_v \colon \overline{\mathbb{K}} \longhookrightarrow \mathbb{C}_v.$$

This choice induces a composite injective map $\mathbb{T}(\overline{\mathbb{K}}) \to \mathbb{T}_v(\mathbb{C}_v) \to \mathbb{T}_v^{\mathrm{an}}$ that we also denote by ι_v. The *v-adic Galois orbit* of $t \in \mathbb{T}(\overline{\mathbb{K}})$ is now defined as the finite set

$$O(t)_v = \iota_v(O(t)) \subset \mathbb{T}_v^{\mathrm{an}}.$$

Since the different $\mathbb{K}$-embeddings of $\overline{\mathbb{K}}$ into $\mathbb{C}_v$ differ by an element of $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$, the set $O(t)_v$ does not depend on the choice of the embedding ι_v. Moreover, its cardinality coincides with that of $O(t)$.

We also consider the function $I_v \colon (\mathbb{T}^k \setminus H)(\mathbb{C}_v) \to \mathbb{R}$ defined for $\boldsymbol{t} = (t_1, \dots, t_k)$ as

$$I_v(\boldsymbol{t}) = \int_{X_v^{\mathrm{an}}} \log \|t_k^* s_{f_k}\|_{\mathrm{Ron},v} \ \mathrm{c}_1(\overline{D}_{0,v}) \wedge \dots \wedge \mathrm{c}_1(\overline{D}_{n-k,v}) \wedge \delta_{Z_X(\widehat{\boldsymbol{t}}^* s_{\widehat{\boldsymbol{f}}})_v^{\mathrm{an}}}$$

with $\widehat{\boldsymbol{t}} = (t_1, \dots, t_{k-1})$.

Claim 2.4. *We have that* $\displaystyle\sum_{w \in \mathfrak{M}(\boldsymbol{\omega})} n_w J_w(\boldsymbol{\omega}) = \sum_{v \in \mathfrak{M}} \frac{n_v}{\# O(\boldsymbol{\omega})} \sum_{\boldsymbol{\eta} \in O(\boldsymbol{\omega})_v} I_v(\boldsymbol{\eta}).$

Proof. Let $v \in \mathfrak{M}$ and denote by $\mathfrak{M}(\boldsymbol{\omega})_v$ the set of places of $\mathbb{K}(\boldsymbol{\omega})$ over v. The finite extension $\mathbb{K}(\boldsymbol{\omega})/\mathbb{K}$ is Galois and we denote by $G_{\boldsymbol{\omega}}$ its Galois group. This finite group acts by composition on the set $\mathfrak{M}(\boldsymbol{\omega})_v$, and by [Neu99, Chapter II, Proposition 9.1] this action is transitive.

In particular, the degree $[\mathbb{K}(\boldsymbol{\omega})_w : \mathbb{K}_v]$ is the same for all $w \in \mathfrak{M}(\boldsymbol{\omega})_v$, and by [Neu99, Chapter II, Corollary 8.4] these degrees sum up to $[\mathbb{K}(\boldsymbol{\omega}) : \mathbb{K}]$. Combining this with the orbit-stabilizer theorem we obtain that

$$\frac{[\mathbb{K}(\boldsymbol{\omega})_w : \mathbb{K}_v]}{[\mathbb{K}(\boldsymbol{\omega}) : \mathbb{K}]} = \frac{1}{\# \mathfrak{M}(\boldsymbol{\omega})_v} = \frac{\# \operatorname{stab}(w)}{\# G_{\boldsymbol{\omega}}} \quad \text{for any } w \in \mathfrak{M}(\boldsymbol{\omega})_v, \tag{2.5}$$

where $\operatorname{stab}(w)$ denotes the stabilizer of w with respect to the action of $G_{\boldsymbol{\omega}}$.

Choose the place $w_v \in \mathfrak{M}(\boldsymbol{\omega})$ induced by the embedding $\iota_v \colon \overline{\mathbb{K}} \hookrightarrow \mathbb{C}_v$. Then (2.5) together with the transitivity of the action gives that

$$\sum_{w \in \mathfrak{M}(\boldsymbol{\omega})_v} \frac{[\mathbb{K}(\boldsymbol{\omega})_w : \mathbb{K}_v]}{[\mathbb{K}(\boldsymbol{\omega}) : \mathbb{K}]} J_w(\boldsymbol{\omega}) = \frac{1}{\# G_{\boldsymbol{\omega}}} \sum_{\sigma \in G_{\boldsymbol{\omega}}} J_{\sigma(w_v)}(\boldsymbol{\omega}).$$

For each $\sigma \in G_{\boldsymbol{\omega}}$ we have that $J_{\sigma(w_v)}(\boldsymbol{\omega}) = I_v(\iota_v \circ \sigma(\boldsymbol{\omega}))$, as it can be checked using the isomorphism $X_{\sigma(w_v)}^{\mathrm{an}} \simeq X_v^{\mathrm{an}}$ and the change of variables formula.

On the other hand, the finite group $G_{\boldsymbol{\omega}}$ acts freely and transitively on the Galois orbit of $\boldsymbol{\omega}$. Hence the assignment $\sigma \mapsto \iota_v \circ \sigma(\boldsymbol{w})$ is a bijection between $G_{\boldsymbol{\omega}}$ and the v-adic Galois orbit of this point. Altogether we deduce that

$$\sum_{w \in \mathfrak{M}(\boldsymbol{\omega})_v} \frac{[\mathbb{K}(\boldsymbol{\omega})_w : \mathbb{K}_v]}{[\mathbb{K}(\boldsymbol{\omega}) : \mathbb{K}]} J_w(\boldsymbol{\omega}) = \frac{1}{\# O(\boldsymbol{\omega})} \sum_{\boldsymbol{\eta} \in O(\boldsymbol{\omega})_v} I_v(\boldsymbol{\eta}).$$

The claim follows multiplying this equality by n_v and summing over all $v \in \mathfrak{M}$, together with the relation between the weights of $\mathbb{K}$ and those of $\mathbb{K}(\boldsymbol{\omega})$ in (2.3). $\square$

Combining this claim with (2.4) we deduce the following recursive formula.

Proposition 2.5. *In this setting, for any torsion point* $\boldsymbol{\omega} \in \mathbb{T}(\overline{\mathbb{K}})^k \setminus H(\overline{\mathbb{K}})$ *we have that*

$$\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{n-k}}(Z_{\mathbb{T}}(\boldsymbol{\omega}^*\boldsymbol{f})) = \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{n-k},\overline{D}^{\mathrm{Ron}}_{f_k}}(Z_{\mathbb{T}}(\widehat{\boldsymbol{\omega}}^*\widehat{\boldsymbol{f}})) + \sum_{v\in\mathfrak{M}} \frac{n_v}{\#O(\boldsymbol{\omega})} \sum_{\boldsymbol{\eta}\in O(\boldsymbol{\omega})_v} I_v(\boldsymbol{\eta}).$$

Let us now see how this formula could lead to the proof of the inductive step of Conjecture A. Let $(\boldsymbol{\omega}_\ell)_\ell$ be a quasi-strict sequence of torsion points of $\mathbb{T}(\overline{\mathbb{K}})^k$. Setting $\widehat{\boldsymbol{\omega}}_\ell = (\omega_{\ell,1},\dots,\omega_{\ell,k-1})$ for each ℓ, the sequence $(\widehat{\boldsymbol{\omega}}_\ell)_\ell$ of torsion points of $\mathbb{T}(\overline{\mathbb{K}})^{k-1}$ is also quasi-strict. Using the inductive hypothesis, the first summand in the right-hand side of the formula in Proposition 2.5 converges to the desired limit as $\ell \to \infty$. Thus the proof of the conjecture is reduced to showing that the second summand in this formula asymptotically vanishes, namely

$$\lim_{\ell\to\infty} \sum_{v\in\mathfrak{M}} \frac{n_v}{\#O(\boldsymbol{\omega}_\ell)} \sum_{\boldsymbol{\eta}\in O(\boldsymbol{\omega}_\ell)_v} I_v(\boldsymbol{\eta}) = 0. \tag{2.6}$$

When approaching this problem, the following reduction steps might be useful.

Proposition 2.6. *When proving Conjecture A for a given* $0 \le k \le n$ *it is enough to suppose that:*

(1) the Laurent polynomials $f_1,\dots,f_k$ *have coefficients in the ring of integers of* $\mathbb{K}$*, their supports contain the lattice point* $0 \in M$*, and (up to possibly replacing* $\mathbb{K}$ *by a finite extension) they are absolutely irreducible, i.e. they are irreducible as elements of* $\overline{\mathbb{K}}[M]$

(2) the toric variety X *is smooth and projective, and its fan is compatible with the Newton polytopes of the Laurent polynomials*

(3) the toric divisors $D_0,\dots,D_{n-k}$ *are very ample*

(4) the semipositive toric metrics of $\overline{D}_0,\dots,\overline{D}_{n-k}$ *are smooth at Archimedean places and algebraic at non-Archimedean ones*

(5) the torsion points in the sequence are of the form $\boldsymbol{\omega}_\ell = (1,\omega_{\ell,2},\dots,\omega_{\ell,k})$ *with* 1 *the neutral element of* $\mathbb{T}(\overline{\mathbb{K}})$ *and* $((\omega_{\ell,2},\dots,\omega_{\ell,k}))_\ell$ *a strict sequence in* $\mathbb{T}(\overline{\mathbb{K}})^{k-1}$*.*

Proving Proposition 2.6 would result weighty here, and therefore we postpone this to Appendix A.

3. An auxiliary function

With the aim of proving Theorem B and in view of the reduction steps in Proposition 2.6 we now place ourselves in the following setting. Let $f, g \in \mathbb{K}[M]$ be two absolutely irreducible Laurent polynomials such that both of their supports contain the lattice point $0 \in M$. We write them as

$$f = \sum_m \alpha_m \chi^m \quad \text{and} \quad g = \sum_m \beta_m \chi^m$$

with $\alpha_m, \beta_m \in \mathbb{K}$ that are zero for all but finitely many m. Let also X be a smooth projective toric variety compactifying $\mathbb{T}$ with fan compatible with the Newton polytopes of f and g. Furthermore we denote by $\overline{D}_0,\dots,\overline{D}_{n-2}$ a family of semipositive toric metrized divisors on X with very ample underlying divisors, and with smooth metrics at the Archimedean places and algebraic metrics at the non-Archimedean ones.

Recall from Section **1.A** that D_f denotes the nef toric divisor on X associated to the Newton polytope of f and s_f the global section of $\mathcal{O}(D_f)$ associated to this Laurent polynomial. We set $Z = Z_X(s_f)$ for the corresponding hypersurface of X.

Fix $v \in \mathfrak{M}$. To treat the v-adic summand in (2.6) in the present 2-codimensional case it will be convenient to consider the function $F_v\colon \mathbb{T}(\mathbb{C}_v) \to \mathbb{R} \cup \{-\infty\}$ defined by

$$F_v(t) = \int_{X_v^{\mathrm{an}}} \log |t^*g|_v \ \mathrm{c}_1(\overline{D}_{0,v}) \wedge \ldots \wedge \mathrm{c}_1(\overline{D}_{n-2,v}) \wedge \delta_{Z_v^{\mathrm{an}}}. \tag{3.1}$$

This integral is computed with respect to the v-adic mixed Monge–Ampère measure of these metrized divisors on the analytification of the hypersurface defined by f. We devote this section to the study of the regularity of this auxiliary function.

3.A. The goal of this section. We first introduce a class of functions on $\mathbb{T}_v^{\mathrm{an}}$ with controlled behaviour along closed algebraic subsets, similarly as those considered by Chambert-Loir and Thuillier along Weil divisors [CT09].

Definition 3.1. Let $H \subset \mathbb{T}_v$ be a proper closed subset and $h_1, \ldots, h_s \in \mathbb{C}_v[M]$ a system of Laurent polynomials defining it. A function $\varphi\colon \mathbb{T}_v^{\mathrm{an}} \to \mathbb{R} \cup \{-\infty\}$ is said to have *at most logarithmic singularities along* H^{an} if the two following conditions are met:

(1) the restriction of φ to $\mathbb{T}_v^{\mathrm{an}} \setminus H^{\mathrm{an}}$ is a continuous function with real values,
(2) for each point of H^{an} there exist an open neighbourhood $U \subset \mathbb{T}_v^{\mathrm{an}}$ and positive real numbers c_1, c_2 such that

$$\varphi \geq c_1 \log \max_{j=1,\ldots,s} |h_j|_v - c_2 \quad \text{on } U.$$

Remark 3.2. Let $H' \subset \mathbb{T}_v$ be a closed subset containing H, and $g_1, \ldots, g_r$ a system of Laurent polynomials defining it. By Hilbert's Nullstellensatz, there is an integer $\kappa_1 \geq 1$ such that $g_i^{\kappa_1} \in (h_1, \ldots, h_s)$ for every i. Therefore for any open subset $U \subset \mathbb{T}_v^{\mathrm{an}}$ with compact closure there is a positive real number κ_2 such that

$$\log \max_{j=1,\ldots,s} |h_j|_v \geq \kappa_1 \log \max_{i=1,\ldots,r} |g_i|_v - \kappa_2 \quad \text{on } U.$$

Hence if φ has at most logarithmic singularities along H^{an} then it also has at most logarithmic singularities along $(H')^{\mathrm{an}}$.

Applying this observation to the case when $H' = H$, it implies moreover that Definition 3.1 does not depend on the choice of the defining system of Laurent polynomials.

To study the singularities of the auxiliary function we introduce the proper closed subset of the torus defined as

$$\Upsilon = V_{\mathbb{T}}(\{\alpha_m \beta_{m'+m_0} \chi^{m'+m_0} - \alpha_{m'} \beta_{m+m_0} \chi^{m+m_0}\}_{m,m' \in M}) \tag{3.2}$$

if there is $m_0 \in M$ such that $\mathrm{supp}(g) = \mathrm{supp}(f) + m_0$, and as $\Upsilon = \emptyset$ otherwise.

Lemma 3.3. *For $t \in \mathbb{T}(\mathbb{C}_v)$ we have that $F_v(t) = -\infty$ if and only if $t \in \Upsilon(\mathbb{C}_v)$.*

Proof. First suppose that there is $m_0 \in M$ such that $\mathrm{supp}(g) = \mathrm{supp}(f) + m_0$. Then for $t \in \mathbb{T}(\mathbb{C}_v)$ we have that $t \in \Upsilon(\mathbb{C}_v)$ if and only if there is $\lambda \in \mathbb{C}_v^\times$ such that $\beta_{m+m_0} \chi^{m+m_0}(t) = \lambda \alpha_m$ for all m, or equivalently that

$$t^*g = \lambda \chi^{m_0} f.$$

Otherwise f and t^*g are coprime for all $t \in \mathbb{T}(\mathbb{C}_v)$ because both Laurent polynomials are absolutely irreducible and their supports do not coincide modulo a translation.

We conclude that $t \in \Upsilon(\mathbb{C}_v)$ if and only if f and t^*g coincide up to a monomial factor. When this condition holds we have that $|t^*g\,(x)|_v = 0$ for all $x \in Z_v^{\mathrm{an}}$ and so $F_v(t) = -\infty$. Otherwise t^*g is a nonzero rational function on the hypersurface Z

and therefore $F_v(t) \in \mathbb{R}$ because the v-adic mixed Monge–Ampère measure integrates functions with at most logarithmic singularities. □

The following is our main result here. Its proof is rather long and technical, and will occupy us for the rest of the section.

Theorem 3.4. *The function F_v extends uniquely to a function $\mathbb{T}_v^{\mathrm{an}} \to \mathbb{R} \cup \{-\infty\}$ with at most logarithmic singularities along Υ_v^{an} and taking the value $-\infty$ on this analytic subvariety.*

3.B. Archimedean continuity. We start by showing that when v is Archimedean, the auxiliary function is continuous outside its singularity locus. The proof is an application of Stoll's theorem on the continuity of the fiber integral [Sto67].

Proposition 3.5. *Let v be an Archimedean place of $\mathbb{K}$. Then the restriction of F_v to $\mathbb{T}_v^{\mathrm{an}} \setminus \Upsilon_v^{\mathrm{an}}$ is a continuous function with real values.*

Proof. After fixing an isometry we identify $\mathbb{C}_v$ with the field of complex numbers and consequently we drop the index v from the notation throughout. In particular, the v-adic analytifications of $\mathbb{T}_v$ and Υ_v are respectively identified with $\mathbb{T}(\mathbb{C})$ and $\Upsilon(\mathbb{C})$. Thus we can write the auxiliary function as

$$F(t) = \int_{Z(\mathbb{C})} \log |g(t \cdot x)| \, d\lambda \quad \text{for } t \in \mathbb{T}(\mathbb{C}) \tag{3.3}$$

where λ denotes the $(n-1, n-1)$-form $\bigwedge_{i=0}^{n-2} c_1(\overline{D}_i|_Z)$. Up to desingularizing we can assume that Z is a smooth complete variety over $\mathbb{C}$.

Consider the finite open covering of X by the affine toric varieties X_σ, $\sigma \in \Sigma$, as in (1.1). Choose a family of compact semialgebraic subsets $K_\sigma \subset (X_\sigma \cap Z)(\mathbb{C})$, $\sigma \in \Sigma$, such that

$$\bigcup_{\sigma \in \Sigma} K_\sigma = Z(\mathbb{C}).$$

Such a covering can be obtained, for instance, as the restriction to $Z(\mathbb{C})$ of the Batyrev-Tschinkel decomposition of $X(\mathbb{C})$ [Mai00, Section 3.2]. By the inclusion-exclusion formula we have

$$F(t) = \sum_{P \subset \Sigma} (-1)^{\#P-1} \int_{K_P} \log |g(t \cdot x)| \, d\lambda \tag{3.4}$$

with $K_P = \bigcap_{\sigma \in P} K_\sigma$ for $P \subset \Sigma$.

Fix P. Choose then any $\sigma \in P$ and write the restriction of the rational function g to the corresponding affine toric variety as a quotient of nonzero regular functions

$$g|_{X_\sigma} = \frac{h_1}{h_2}$$

that are coprime as elements of $\mathbb{C}[M]$. By [Sto67, Theorem 4.9], for each j the function

$$t \longmapsto \int_{K_P} \log |h_j(t \cdot x)| \, d\lambda \tag{3.5}$$

is continuous at any point $t_0 \in \mathbb{T}(\mathbb{C}) \setminus \Upsilon(\mathbb{C})$. We now check this claim by placing ourselves in the notation and terminology of *loc. cit.*. To this end consider first the smooth complex manifold

$$M = (X_\sigma \cap Z)(\mathbb{C}) \times \mathbb{T}(\mathbb{C})$$

equipped with the differential form of bidegree $(n-1,n-1)$ obtained as the pullback of λ with respect to the projection onto the first factor $M\to (X_\sigma\cap Z)(\mathbb{C})$. Consider also the $(n-1)$-fibering obtained as the projection onto the second factor $M\to \mathbb{T}(\mathbb{C})$. Furthermore choose an open subset $B\subset \mathbb{T}(\mathbb{C})$ with compact closure containing the point t_0 and set

$$G=\operatorname{int}(K_P)\times B\subset M,$$

where $\operatorname{int}(K_P)$ denotes the interior of this compact semialgebraic subset.

Notice that the boundary of G restricted to the fiber of t_0 is of measure zero with respect to the chosen $(n-1,n-1)$-form. Moreover the holomorphic function $M\to\mathbb{C}$ defined by $(x,t)\mapsto h_j(t\cdot x)$ is not identically zero on the fiber over t_0 because $t_0\notin \Upsilon(\mathbb{C})$. Then the hypotheses of Stoll's theorem are satisfied, and the continuity at t_0 of the integral in (3.5) follows from this result together with the computation of the multiplicities of the fibering given by [Sto66, Lemma 5.2].

Finally the continuity on $\mathbb{T}(\mathbb{C})\setminus\Upsilon(\mathbb{C})$ in (3.5) implies that of the P-term in the inclusion-exclusion formula in (3.4), and in turn that of F. □

3.C. Non-Archimedean continuity. When v is non-Archimedean, the function F_v is only defined at the rigid points of $\mathbb{T}_v^{\mathrm{an}}$. Here we show that it can be uniquely extended to the whole of the v-adic analytic torus with singularities along Υ_v^{an}. Its proof relies on a number of technical results and constructions from non-Archimedean formal and analytic geometry.

Proposition 3.6. *Let v be a non-Archimedean place of $\mathbb{K}$. Then F_v extends uniquely to a function $\mathbb{T}_v^{\mathrm{an}}\to\mathbb{R}\cup\{-\infty\}$ which is continuous with real values on $\mathbb{T}_v^{\mathrm{an}}\setminus\Upsilon_v^{\mathrm{an}}$ and takes the value $-\infty$ on Υ_v^{an}.*

Proof. If the claimed extension exists then it is unique, by the density of $\mathbb{T}(\mathbb{C}_v)$ in $\mathbb{T}_v^{\mathrm{an}}\setminus\Upsilon_v^{\mathrm{an}}$. We thus focus on the proof of its existence.

We start by providing a more explicit expression for the function F_v from which the definition of the extension will be more natural. We denote by $\iota\colon Z\hookrightarrow X$ the inclusion of the hypersurface Z into the toric variety X. Note that we can consider the pullback with respect to ι of any toric divisor on X because Z is not contained in the boundary $X\setminus X_0$.

Since the metrics in the v-adic metrized divisors $\overline{D}_{0,v},\dots,\overline{D}_{n-2,v}$ are algebraic, by passing to the formal completion along the special fiber they are also formal in the sense of [Gub98, Section 7]. Thus, the metrics of the pullbacks $\iota_v^*\overline{D}_{0,v},\dots,\iota_v^*\overline{D}_{n-2,v}$ are formal. It then follows from [Gub07, Proposition 3.11] and [Gub98, Proposition 1.11] that there exist a distinguished formal analytic variety $\mathfrak{Z}$ over $\mathbb{C}_v$ with generic fiber Z_v^{an} and reduced special fiber $\widetilde{\mathfrak{Z}}$, and for each $i=0,\dots,n-2$ a pair $(\mathfrak{L}_i,e_i)$ consisting of a formal analytic line bundle on $\mathfrak{Z}$ and a positive integer inducing the metric of $\iota_v^*\overline{D}_{i,v}$. In this setting the v-adic Monge–Ampère measure in the definition of F_v is the discrete measure on X_v^{an} given by

$$\text{(3.6)}\quad \mathrm{c}_1(\overline{D}_{0,v})\wedge\dots\wedge \mathrm{c}_1(\overline{D}_{n-2,v})\wedge\delta_{Z_v^{\mathrm{an}}}=\iota_{v,*}^{\mathrm{an}}(\mathrm{c}_1(\iota_v^*\overline{D}_{0,v})\wedge\dots\wedge\mathrm{c}_1(\iota_v^*\overline{D}_{n-2,v}))$$
$$=\sum_{V\in\widetilde{\mathfrak{Z}}^{(0)}}\frac{\deg_{\widetilde{\mathfrak{L}}_0,\dots,\widetilde{\mathfrak{L}}_{n-2}}(V)}{e_0\cdots e_{n-2}}\,\delta_{\iota_v^{\mathrm{an}}(\xi_V)},$$

where $\widetilde{\mathfrak{L}}_i$ is the line bundle on $\widetilde{\mathfrak{Z}}$ induced by $\mathfrak{L}_i$, the index set $\widetilde{\mathfrak{Z}}^{(0)}$ is the collection of irreducible components of $\widetilde{\mathfrak{Z}}$, and for each $V\in\widetilde{\mathfrak{Z}}^{(0)}$ we denote by ξ_V the unique point of Z_v^{an} which is sent to the generic point η_V of V by the reduction map. Hence the auxiliary function takes the form

$$F_v(t) = \sum_{V \in \widetilde{\mathfrak{Z}}^{(0)}} \frac{\deg_{\widetilde{\mathfrak{L}}_0,\dots,\widetilde{\mathfrak{L}}_{n-2}}(V)}{e_0 \cdots e_{n-2}} \log |t^*g\,(\xi_V)|_v \quad \text{for } t \in \mathbb{T}(\mathbb{C}_v), \tag{3.7}$$

where the Laurent polynomial t^*g is seen as an element of $\mathbb{C}_v[M]/(f)$.

For what follows we need a more explicit description of the analytic points in the support of the measure. Since $\mathfrak{Z}$ is a distinguished formal analytic variety with reduced special fiber, by definition it is locally isomorphic to the formal spectrum of a distinguished $\mathbb{C}_v$-affinoid algebra $\mathscr{A}$ whose reduction $\widetilde{\mathscr{A}}$ is a reduced $\widetilde{\mathbb{C}}_v$-algebra. Therefore for each $V \in \widetilde{\mathfrak{Z}}^{(0)}$ there is a distinguished $\mathbb{C}_v$-affinoid algebra $\mathscr{A}_V$ such that $\eta_V \in \operatorname{Spec}(\widetilde{\mathscr{A}_V})$. Following the proof of [Ber90, Proposition 2.4.4], up to localizing we can assume that $\widetilde{\mathscr{A}_V}$ is a domain, in which case ξ_V can be described as the point of the Berkovich spectrum $\mathcal{M}(\mathscr{A}_V) \subset Z_v^{\mathrm{an}}$ corresponding to the sup-seminorm on $\mathscr{A}_V$. Precisely, this point is the multiplicative seminorm on $\mathscr{A}_V$ defined as

$$|a(\xi_V)|_v = \sup_{x \in \mathcal{M}(\mathscr{A}_V)} |a(x)|_v \quad \text{for } a \in \mathscr{A}_V. \tag{3.8}$$

We want to apply this norm to the Laurent polynomial t^*g for $t \in \mathbb{T}(\mathbb{C}_v)$. By [BGR84, 6.4.3/Theorem 1 and 6.2.1/Proposition 4], for each V the corresponding $\mathbb{C}_v$-affinoid algebra $\mathscr{A}_V$ is reduced and therefore the seminorm in (3.8) is actually a norm. This latter fact implies that ξ_V avoids every proper analytic subset of $\mathcal{M}(\mathscr{A}_V)$, and in particular the analytification of the boundary $Z \setminus X_0$. Hence up to localizing again we can assume without loss of generality that $\mathbb{C}_v[M]/(f) \subset \mathscr{A}_V$, and so

$$|t^*g\,(\xi_V)|_v = \sup_{x \in \mathcal{M}(\mathscr{A}_V)} |t^*g\,(x)|_v. \tag{3.9}$$

Let now $t \in \mathbb{T}_v^{\mathrm{an}}$ be an arbitrary analytic point and $\mathscr{H}(t)$ its complete residue field, which is a complete valued field extension of $\mathbb{C}_v$. For each $V \in \widetilde{\mathfrak{Z}}^{(0)}$ we plan to define $|t^*g\,(\xi_V)|_v$ extending the expression in (3.9).

To this end, recall that $\mathscr{A}_V$ is a distinguished $\mathbb{C}_v$-affinoid algebra with integral reduction $\widetilde{\mathscr{A}_V}$. Since $\widetilde{\mathbb{C}}_v$ is algebraically closed, the tensor product $\widetilde{\mathscr{A}_V} \otimes_{\widetilde{\mathbb{C}}_v} \widetilde{\mathscr{H}(t)}$ is an integral $\widetilde{\mathscr{H}(t)}$-algebra [Bou81, V.17.5/Corollaire 2]. Hence thanks to [Bos69, Satz 6.4] the completed tensor product $\mathscr{A}_V \widehat{\otimes}_{\mathbb{C}_v} \mathscr{H}(t)$ is a distinguished $\mathscr{H}(t)$-algebra. Set then

$$\xi_{V,t} \in \mathcal{M}(\mathscr{A}_V \widehat{\otimes}_{\mathbb{C}_v} \mathscr{H}(t))$$

for the sup-seminorm on this completed tensor product. As before, the algebra $\mathscr{A}_V \widehat{\otimes}_{\mathbb{C}_v} \mathscr{H}(t)$ is reduced and so this multiplicative seminorm is actually a norm. Moreover it coincides with the completed tensor product norm, as shown in the proof of [Ber90, Proposition 5.2.5].

We have that $t^*g \in \mathscr{H}(t)[M]/(f) \simeq \mathbb{C}_v[M]/(f) \otimes_{\mathbb{C}_v} \mathscr{H}(t) \subset \mathscr{A}_V \widehat{\otimes}_{\mathbb{C}_v} \mathscr{H}(t)$, and so we can consider the norm for this twist. By the previous discussion it can be expressed as

$$|t^*g\,(\xi_{V,t})|_v = \sup_x |t^*g\,(x)|_v = |t^*g|_{\mathscr{A}_V \widehat{\otimes}_{\mathbb{C}_v} \mathscr{H}(t)}, \tag{3.10}$$

where x ranges over the Berkovich spectrum $\mathcal{M}(\mathscr{A}_V \widehat{\otimes}_{\mathbb{C}_v} \mathscr{H}(t))$.

Finally, the sought extension for F_v is defined as

$$F_v(t) = \sum_{V\in\widetilde{\mathfrak{Z}}^{(0)}} \frac{\deg_{\widetilde{\mathfrak{L}}_0,\dots,\widetilde{\mathfrak{L}}_{n-2}}(V)}{e_0\cdots e_{n-2}} \log|t^*g\,(\xi_{V,t})|. \tag{3.11}$$

When $t\in\mathbb{T}(\mathbb{C}_v)$ we have that $\mathscr{H}(t)=\mathbb{C}_v$ and so the above expression agrees with that for the auxiliary function in (3.7).

Moreover the defined extension takes the value $-\infty$ if and only if there is at least one V for which $|t^*g\,(\xi_{V,t})|_v=0$. Since $\xi_{V,t}$ is a norm, this is equivalent to the fact that $t^*g=0\in\mathscr{H}(t)[M]/(f)$. As both f and g are absolutely irreducible, this vanishing occurs if and only if t^*g and f coincide up to a nonzero scalar in $\mathscr{H}(t)$, which translates into the condition that

$$\alpha_m\beta_{m'}\chi^{m'}(t)-\alpha_{m'}\beta_m\chi^m(t)=0\in\mathscr{H}(t) \quad \text{for all } m,m'\in M.$$

This is equivalent to the fact that the ideal of Υ_v is contained in the kernel of the multiplicative seminorm corresponding to the analytic point t, and in turn to the fact that $t\in\Upsilon_v^{\mathrm{an}}$.

To conclude, we are left to show that the function F_v is continuous on $\mathbb{T}^{\mathrm{an}}\setminus\Upsilon_v^{\mathrm{an}}$. Let $t_0\in\mathbb{T}_v^{\mathrm{an}}$ and choose a $\mathbb{C}_v$-affinoid algebra $\mathscr{B}$ such that the Berkovich spectrum $\mathcal{M}(\mathscr{B})$ is a neighbourhood of t_0. Then [Ber90, Lemma 5.2.6] applied to the element $\sum_m\beta_m\chi^m\otimes\chi^m\in\mathscr{A}_V\widehat{\otimes}_{\mathbb{C}_v}\mathscr{B}$ gives that the function on $\mathcal{M}(\mathscr{B})$ defined as

$$t\longmapsto |t^*g|_{\mathscr{A}_V\widehat{\otimes}_{\mathbb{C}_v}\mathscr{H}(t)}$$

is continuous. Because of (3.10) and the fact that $|t^*g\,(\xi_{V,t})|_v>0$ for $t\in\mathbb{T}_v^{\mathrm{an}}\setminus\Upsilon_v^{\mathrm{an}}$ for all V this implies that F_v is continuous with real values on this set. □

Remark 3.7. The extension of the auxiliary function can be also written as

$$F_v(t)=\int_{X_v^{\mathrm{an}}}\log|g(t*x)|\ \mathrm{c}_1(\overline{D}_{0,v})\wedge\ldots\wedge\mathrm{c}_1(\overline{D}_{n-2,v})\wedge\delta_{Z_v^{\mathrm{an}}},$$

where $t*x\in X^{\mathrm{an}}$ is the *peaked product* between $t\in\mathbb{T}^{\mathrm{an}}$ and $x\in X^{\mathrm{an}}$ as defined in [Ber90, §5.2], see also [BPS14, Definition 4.2.2]. This follows from the previous proof, precisely from (3.10) and (3.11). This is the non-Archimedean analogue of the expression for F_v in (3.3).

3.D. A logarithmic lower bound. Here we prove the following lower bound for the v-adic auxiliary function.

Proposition 3.8. *Let $v\in\mathfrak{M}$. There is a system of Laurent polynomials $h_1,\dots,h_s\in\mathbb{C}_v[M]$ defining Υ_v and a positive real number c such that*

$$F_v\geq\log\max_{j=1,\dots,s}|h_j|_v-c \quad \text{on } \mathbb{T}(\mathbb{C}_v).$$

The main idea of the proof is to compare the value of F_v at t with the valuation of t^*g at the 0-cycle in Z_v obtained by cutting the latter with suitably generic global sections. The Poisson formula for the sparse resultant from [DS15] will then allow to see that such a valuation depends polynomially on t in a way that is related with the vanishing of the Laurent polynomials defining Υ_v.

To turn this strategy into practice we need a series of side results. The first gives a simple criterion for the proper intersection of a family of global sections in the ample case.

Lemma 3.9. *Let Y be a complete variety over $\mathbb{C}_v$ of dimension e, and for $i = 1, \dots, k$ with $1 \le k \le e+1$ let s_i be a global section of an ample line bundle L_i on Y. Then*

$$\dim(V_Y(s_1, \dots, s_k)) = e - k \tag{3.12}$$

if and only if $s_1, \dots, s_k$ meet properly.

Proof. If $s_1, \dots, s_k$ meet properly then the condition (3.12) clearly holds.

Conversely assume that this condition is satisfied. If $s_1, \dots, s_k$ do not meet properly then there is $I \subset \{1, \dots, k\}$ such that

$$\dim(V_Y(\{s_i\}_{i\in I}) > e - \#I.$$

Suppose that I is the maximal index subset satisfying this inequality and take an irreducible component C of this zero set with $\dim(C) > e - \#I$. Take also an index $i_0 \in \{1, \dots, k\} \setminus I$, which is possible because (3.12) ensures that $I \neq \{1, \dots, k\}$.

This irreducible component has positive dimension because $\#I \le k - 1 \le e$, and cannot be contained in the hypersurface $V_Y(s_{i_0})$ because this would contradict the maximality of I. Hence this hypersurface cuts C, and so $C\cdot\mathrm{div}(s_{i_0})$ is a cycle of positive degree with respect to the ample line bundle L_{i_0}. In particular $C \cap V_Y(s_{i_0}) \neq \emptyset$, and by Krull's Hauptidealsatz this intersection has dimension $\dim(C) - 1$. Hence

$$\dim(V_Y(\{s_i\}_{i\in I\cup\{i_0\}})) \ge \dim(V_Y(\{s_i\}_{i\in I})) - 1 > e - \#(I \cup \{i_0\}),$$

contradicting again the maximality of I and thus proving the statement. □

We also need the next formula from [BE21, Lemma 8.17(ii)]. It can be seen as a higher dimensional metric version of the Weil reciprocity law as explained in [GS23, Remark 9.7], and a self-contained proof for complex smooth projective curves can be found in [GS23, Proposition 9.6].

Proposition 3.10. *Let Y be a complete variety over $\mathbb{C}_v$ of dimension e and $\overline{L}_0 = (L_0, \|\cdot\|_0), \dots, \overline{L}_e = (L_e, \|\cdot\|_e)$ semipositive metrized line bundles on Y. Let $s_{e-1} \in \Gamma(Y, L_{e-1})$ and $s_e \in \Gamma(Y, L_e)$ be global sections that meet properly. Then*

$$\int_{Y^{\mathrm{an}}} \log\|s_{e-1}\|_{e-1} \big(\mathrm{c}_1(\overline{L}_0)\wedge\dots\wedge\mathrm{c}_1(\overline{L}_{e-2})\wedge\mathrm{c}_1(\overline{L}_e) - \mathrm{c}_1(\overline{L}_0)\wedge\dots\wedge\mathrm{c}_1(\overline{L}_{e-2})\wedge\delta_{Z_Y(s_e)^{\mathrm{an}}}\big)$$
$$= \int_{Y^{\mathrm{an}}} \log\|s_e\|_e \big(\mathrm{c}_1(\overline{L}_0)\wedge\dots\wedge\mathrm{c}_1(\overline{L}_{e-2})\wedge\mathrm{c}_1(\overline{L}_{e-1}) - \mathrm{c}_1(\overline{L}_0)\wedge\dots\wedge\mathrm{c}_1(\overline{L}_{e-2})\wedge\delta_{Z_Y(s_{e-1})^{\mathrm{an}}}\big).$$

We next give a lower bound for the auxiliary function at a point in terms of the evaluation of the corresponding twist of g at a certain 0-cycle of X_v. The evaluation of a Laurent polynomial $q \in \mathbb{C}_v[M]$ at a 0-cycle $W = \sum_{i\in I} k_i x_i$ of X_v is the scalar defined as

$$q(W) = \prod_{i\in I} q(x_i)^{k_i} \in \mathbb{C}_v.$$

Recall from Section **1.A** that D_f and D_g are the nef toric divisors associated to the Newton polytopes of f and g respectively, and $s_f \in \Gamma(X, \mathcal{O}(D_f))$ and $s_g \in \Gamma(X, \mathcal{O}(D_g))$ the global sections induced by these Laurent polynomials. Moreover s_{D_g} denotes the canonical rational section of $\mathcal{O}(D_g)$, which in our current setting is also a global section because of the assumption that $0 \in \mathrm{supp}(g)$.

Lemma 3.11. *Let $\sigma_i \in \Gamma(X_v, \mathcal{O}(D_i))$, $i = 0, \dots, n-2$, such that the global sections $\sigma_0, \dots, \sigma_{n-2}, s_f$ meet properly and their common zero set is contained in the principal open subset $X_{v,0}$. Then there is $c > 0$ such that*

$$F_v(t) \geq \log |t^* g\,(Z_{X_v}(\sigma_0, \dots, \sigma_{n-2}, s_f))|_v - c \quad \text{for } t \in \mathbb{T}(\mathbb{C}_v).$$

Proof. Set $W = Z_{X_v}(\sigma_0, \dots, \sigma_{n-2}, s_f)$ for short. This is an effective 0-cycle of X_v whose support $|W|$ is contained in $Z_v \cap X_{v,0}$. Its degree coincides with $\deg_{D_0,\dots,D_{n-2}}(Z)$, and so it is positive because these divisors are ample. In particular $W \neq 0$.

Let $t \in \mathbb{T}(\mathbb{C}_v)$. We suppose without loss of generality that $t^* g\,(W) \neq 0$ because otherwise the right-hand side of the sought inequality equals $-\infty$ and so it is trivially verified. Then the twist $t^* s_g = t^* g\, s_{D_g}$ has no zeros on $|W|$ because of this assumption and the fact that the zero set of s_{D_g} is contained in the boundary $X_v \setminus X_{v,0}$, that is

$$|W| \cap V_{X_v}(t^* s_g) = V_{X_v}(\sigma_0, \dots, \sigma_{n-2}, s_f, t^* s_g) = \emptyset. \tag{3.13}$$

This implies that $t \notin \Upsilon(\mathbb{C}_v)$ because the support $|W|$ is contained in Z_v and nonempty. Hence the global sections s_f and $t^* s_g$ meet properly. We can then apply Lemma 3.9 to each irreducible component C of $V_{X_v}(s_f, t^* s_g)$ and the ample line bundles $\mathcal{O}(D_0)|_C, \dots, \mathcal{O}(D_{n-2})|_C$ equipped with global sections $\sigma_0|_C, \dots, \sigma_{n-2}|_C$, to deduce that $\sigma_0, \dots, \sigma_{n-2}$ meet properly on this zero set. In turn, this implies that for $i = 0, \dots, n-2$ the global sections σ_i and $t^* s_g$ meet properly on $V_{X_v}(\sigma_{i+1}, \dots, \sigma_{n-2}, s_f)$.

Similarly, the fact that $V_{X_v}(\sigma_0, \dots, \sigma_{n-2}, s_f, s_{D_g}) = \emptyset$ implies that for each i the global sections σ_i and s_{D_g} meet properly on $V_{X_v}(\sigma_{i+1}, \dots, \sigma_{n-2}, s_f)$.

Now take any semipositive metric $\|\cdot\|$ on the analytic line bundle $\mathcal{O}(D_g)_v^{\mathrm{an}}$ and consider the metrized divisor $\overline{D}_{g,v} = (D_{g,v}, \|\cdot\|)$ on X_v. Since $t^* s_g = t^* g\, s_{D_g}$ we have that $\|t^* s_g(x)\| = |t^* g(x)|_v \, \|s_{D_g}(x)\|$ for all $x \in X_{v,0}^{\mathrm{an}}$ and so

$$F_v(t) = \int_{X_v^{\mathrm{an}}} \log \|t^* s_g\| \ \mathrm{c}_1(\overline{D}_{0,v}) \wedge \ldots \wedge \mathrm{c}_1(\overline{D}_{n-2,v}) \wedge \delta_{Z_v^{\mathrm{an}}} - \int_{X_v^{\mathrm{an}}} \log \|s_{D_g}\| \ \mathrm{c}_1(\overline{D}_{0,v}) \wedge \ldots \wedge \mathrm{c}_1(\overline{D}_{n-2,v}) \wedge \delta_{Z_v^{\mathrm{an}}}. \tag{3.14}$$

By the previous discussion we can apply twice Proposition 3.10 to the restriction to the hypersurface $Z_v = Z_{X_v}(s_f)$ of the metrized line bundles of $\overline{D}_{0,v}, \dots, \overline{D}_{n-2,v}, \overline{D}_{g,v}$, first to the global sections σ_{n-2} and $t^* s_g$, and secondly to σ_{n-2} and s_{D_g}. Subtracting the resulting formulae and taking into account (3.14) we obtain

$$\begin{aligned} F_v(t) = & \int_{X_v^{\mathrm{an}}} \log |t^* g|_v \ \mathrm{c}_1(\overline{D}_{0,v}) \wedge \ldots \wedge \mathrm{c}_1(\overline{D}_{n-3,v}) \wedge \delta_{Z_{X_v}(\sigma_{n-2}, s_f)^{\mathrm{an}}} \\ & + \int_{X_v^{\mathrm{an}}} \log \|\sigma_{n-2}\|_{n-2} \ \mathrm{c}_1(\overline{D}_{0,v}) \wedge \ldots \wedge \mathrm{c}_1(\overline{D}_{n-3,v}) \wedge \delta_{Z_{X_v}(s_f, s_{D_g})^{\mathrm{an}}} \\ & - \int_{X_v^{\mathrm{an}}} \log \|\sigma_{n-2}\|_{n-2} \ \mathrm{c}_1(\overline{D}_{0,v}) \wedge \ldots \wedge \mathrm{c}_1(\overline{D}_{n-3,v}) \wedge \delta_{Z_{X_v}(s_f, t^* s_g)^{\mathrm{an}}}. \end{aligned}$$

Iterating this procedure we obtain

$$F_v(t) = \int_{X_v^{\mathrm{an}}} \log |t^* g|_v \ \delta_{W^{\mathrm{an}}} + \sum_{i=0}^{n-2} (S_i + T_i) = \log |t^* g\,(W)|_v + \sum_{i=0}^{n-2} (S_i + T_i) \tag{3.15}$$

with

$$S_i = \int_{X_v^{\mathrm{an}}} \log \|\sigma_i\|_i \ \mathrm{c}_1(\overline{D}_{0,v}) \wedge \ldots \wedge \mathrm{c}_1(\overline{D}_{i-1,v}) \wedge \delta_{Z_{X_v}(\sigma_{i+1},\ldots,\sigma_{n-2},s_f,s_{D_g})^{\mathrm{an}}},$$

$$T_i = -\int_{X_v^{\mathrm{an}}} \log \|\sigma_i\|_i \ \mathrm{c}_1(\overline{D}_{0,v}) \wedge \ldots \wedge \mathrm{c}_1(\overline{D}_{i-1,v}) \wedge \delta_{Z_{X_v}(\sigma_{i+1},\ldots,\sigma_{n-2},s_f,t^*s_g)^{\mathrm{an}}}.$$

The first of these quantities is independent of t whereas the second can be bounded below in terms of the sup-norm of the global section σ_i as

$$\begin{aligned} T_i &\geq -\log \|\sigma_i\|_{i,\sup} \int_{X_v^{\mathrm{an}}} \mathrm{c}_1(\overline{D}_{0,v}) \wedge \ldots \wedge \mathrm{c}_1(\overline{D}_{i-1,v}) \wedge \delta_{Z_{X_v}(\sigma_{i+1},\ldots,\sigma_{n-2},s_f,t^*s_g)^{\mathrm{an}}} \\ &= -\log \|\sigma_i\|_{i,\sup} (D_0 \cdots D_{i-1} \cdot D_{i+1} \cdots D_{n-2} \cdot D_f \cdot D_g \cdot X). \end{aligned}$$

The statement follows then from (3.15) together with this lower bound. □

We also need the following upper bound for the coefficients of a Laurent polynomial.

Lemma 3.12. *For $r \geq 1$ let $\mathcal{A} \subset \mathbb{Z}^r$ be a finite subset and $U \subset \mathbb{G}_{\mathrm{m}}^r$ open and nonempty. Then there is a finite subset $F \subset U(\mathbb{C}_v)$ such that for every Laurent polynomial $p \in \mathbb{C}_v[\mathbb{Z}^r]$ of the form $p = \sum_{a\in\mathcal{A}} p_a \chi^a$ we have*

$$\max_{a\in\mathcal{A}} |p_a|_v \leq \max_{u\in F} |p(u)|_v.$$

Proof. Let k be a positive integer. Then for each $m \in \mathbb{Z}^r$ we have

$$\frac{1}{k^r} \sum_{\zeta\in\mu_k^r} \chi^m(\zeta) = \begin{cases} 1 & \text{if } m \in k\mathbb{Z}^r, \\ 0 & \text{otherwise,} \end{cases}$$

where μ_k denotes the set of k-roots of unity of $\mathbb{C}_v$.

Take k large enough so that the reduction map $\mathbb{Z}^r \to \mathbb{Z}^r/k\mathbb{Z}^r$ is injective on $\mathcal{A}$. Then the latter observation implies that for any $\eta \in \mathbb{G}_{\mathrm{m}}^r(\mathbb{C}_v)$ and each $a \in \mathcal{A}$ we have

$$p_a = \frac{1}{k^r} \sum_{\zeta\in\mu_k^r} \chi^{-a}(\eta\cdot\zeta)\, p(\eta\cdot\zeta).$$

Choosing $\eta = (\eta_1, \ldots, \eta_r)$ with $|\eta_1|_v = \ldots = |\eta_r|_v = 1$ and assuming furthermore that $|k|_v = 1$ if v is non-Archimedean, we deduce from this identity that

$$\max_{a\in\mathcal{A}} |p_a|_v \leq \max_{\zeta\in\mu_k^r} |p(\eta\cdot\zeta)|_v.$$

The result follows then by taking $F = \eta \cdot \mu_k^r$ for any choice of η lying both in the nonempty open subset $\bigcap_{\zeta\in\mu_k^r} \zeta \cdot U(\mathbb{C}_v)$ and in the dense $\{\eta \in \mathbb{G}_{\mathrm{m}}^r(\mathbb{C}_v) : |\eta_1|_v = \cdots = |\eta_r|_v = 1\}$. □

Proof of Proposition 3.8. For $i = 0, \ldots, n-2$ denote by $\Delta_i \subset M_{\mathbb{R}}$ the lattice polytope of the toric divisor D_i and set

$$p_i = \sum_{m\in\Delta_i\cap M} \chi^m \in \mathbb{C}_v[M],$$

which is a Laurent polynomial whose Newton polytope coincides with Δ_i. Hence the corresponding nef toric divisor on X_v agrees with the base change $D_{i,v}$ and we can consider the global section $s_{p_i} \in \Gamma(X_v, \mathcal{O}(D_{i,v}))$.

By Theorem 1.5 there exists a nonempty open subset $U \subset \mathbb{T}^{n-1}$ such that for all $(u_0, \ldots, u_{n-2}) \in U(\mathbb{C}_v)$ the global sections $u_0^* s_{p_0}, \ldots, u_{n-2}^* s_{p_{n-2}}, s_f$ meet properly and their zero set is contained in $X_{0,v}$. Indeed, the direct application of the theorem ensures these properties for a generic twist of $s_{p_0}, \dots, s_{p_{n-2}}, s_f$, but they also hold without twisting s_f because of their invariance under the diagonal action of $\mathbb{T}$ on $\mathbb{T}^n$.

In this situation, the zero set in X_v of $u_0^* s_{p_0}, \dots, u_{n-2}^* s_{p_{n-2}}, s_f$ agrees with the zero set in $\mathbb{T}_v$ of the Laurent polynomials $u_0^* p_0, \dots, u_{n-2}^* p_{n-2}, f$ and so

$$Z_{X_v}(u_0^* s_{p_0}, \dots, u_{n-2}^* s_{p_{n-2}}, s_f) = Z_{\mathbb{T}_v}(u_0^* p_0, \dots, u_{n-2}^* p_{n-2}, f).$$

Then for $t \in \mathbb{T}(\mathbb{C}_v)$ the Poisson formula for the sparse resultant [DS15, Theorem 1.1] shows that the evaluation of $t^* g$ at this effective 0-cycle can be written as the quotient of the resultant of $u_0^* p_0, \dots, u_{n-2}^* p_{n-2}, f, t^* g$ by the product of the resultants of a finite number of initial parts of $u_0^* p_0, \dots, u_{n-2}^* p_{n-2}, f$. Since all these resultants are polynomials in the coefficients of the corresponding family of Laurent polynomials, this implies that there are $P \in \mathbb{C}_v[M \oplus M^{\oplus(n-1)}]$ and $Q \in \mathbb{C}_v[M^{\oplus(n-1)}] \setminus \{0\}$ with

$$t^* g\left(Z_{X_v}(u_0^* s_{p_0}, \dots, u_{n-2}^* s_{p_{n-2}}, s_f)\right) = \frac{P(t, u_0, \dots, u_{n-2})}{Q(u_0, \dots, u_{n-2})}. \tag{3.16}$$

Combining with Lemma 3.11, this implies that for each $(u_0, \dots, u_{n-2}) \in U(\mathbb{C}_v)$ there is $c > 0$ such that

$$F_v(t) \geq \log|P(t, u_0, \dots, u_{n-2})|_v - c \quad \text{for all } t \in \mathbb{T}(\mathbb{C}_v). \tag{3.17}$$

Now consider P as a Laurent polynomial in $\mathbb{C}_v[M][M^{\oplus(n-1)}]$ and write it as

$$P = \sum_{a \in \mathcal{A}} P_a \, \chi^a$$

for a finite subset $\mathcal{A} \subset M^{\oplus(n-1)}$ and $P_a \in \mathbb{C}_v[M]$ for each $a \in \mathcal{A}$.

We claim that the family $P_a \in \mathbb{C}_v[M]$, $a \in \mathcal{A}$, defines the closed subset $\Upsilon_v \subset \mathbb{T}_v$. Indeed it is enough to show this for the $\mathbb{C}_v$-points of the torus. For $t \in \mathbb{T}(\mathbb{C}_v)$ we have that $P_a(t) = 0$ for all a if and only if $P(t, u_0, \dots, u_{n-2}) = 0$ for all $(u_0, \dots, u_{n-2}) \in U(\mathbb{C}_v)$. By the identity in (3.16) this is equivalent to the fact that $t^* g$ vanishes on the support of the 0-cycle therein for all such $(u_0, \dots, u_{n-2})$, which is the same as

$$V_X(s_f, t^* s_g) \cap V_X(u_0^* s_{p_0}, \dots, u_{n-2}^* s_{p_{n-2}}) \neq \emptyset. \tag{3.18}$$

On the other hand $t \in \Upsilon(\mathbb{C}_v)$ if and only if $\operatorname{codim}(V_X(s_f, t^* s_g)) \leq 1$. Since $u_i^* s_{p_i}$, $i = 0, \dots, n-2$, are $n-1$ generic global sections of ample line bundles, this latter condition is equivalent to that in (3.18). Hence $P_a(t) = 0$ for all a if and only if $t \in \Upsilon(\mathbb{C}_v)$, proving this claim.

Finally, by Lemma 3.12 there is a finite subset $F \subset U(\mathbb{C}_v)$ such that

$$\max_{(u_0, \dots, u_{n-2}) \in F} |P(t, u_0, \dots, u_{n-2})|_v \geq \max_{a \in \mathcal{A}} |P_a(t)|_v \quad \text{for all } t \in \mathbb{T}(\mathbb{C}_v).$$

The statement then follows by taking the maximum of the lower bounds in (3.17) for $(u_0, \dots, u_{n-2}) \in F$. □

3.E. Proof of Theorem 3.4. When v is Archimedean we have that $\mathbb{T}_v^{\mathrm{an}} = \mathbb{T}_v(\mathbb{C}_v)$, and so the statement is an immediate consequence of Propositions 3.5 and 3.8.

When v is non-Archimedean, Proposition 3.6 shows that there is a unique extension $\mathbb{T}_v^{\mathrm{an}} \to \mathbb{R} \cup \{-\infty\}$ that is continuous with real values outside Υ_v^{an} and takes the value $-\infty$ therein. The result then follows from Proposition 3.8 together with the density of $\mathbb{T}_v(\mathbb{C}_v)$ in $\mathbb{T}_v^{\mathrm{an}}$.

4. A local logarithmic equidistribution

We now begin building up the proof of Theorem B following the strategy proposed in Section **2.B**. Our main result here shows that in the two-codimensional situation each v-adic summand in (2.6) vanishes asymptotically (Theorem 4.4).

Throughout this section we denote by $\mathbb{T}$ a split algebraic torus over the number field $\mathbb{K}$, of dimension n and with character lattice M.

4.A. Logarithmic equidistribution of torsion points. In this subsection we recall the results concerning the distribution of Galois orbits of torsion points in analytic tori that will play a key role for our present local treatment.

When v is Archimedean, the analytic torus $\mathbb{T}_v^{\mathrm{an}}$ identifies with $\mathbb{T}_v(\mathbb{C}_v)$. We have that $\mathbb{S}_v$ is a compact subgroup of this topological group, and we denote by ν_v its Haar probability measure.

The following result is a direct consequence of a theorem of Dimitrov and Habegger [DH24]. For a closed subset $H \subset \mathbb{T}_v$, its analytification H^{an} identifies with $H(\mathbb{C}_v)$. Following *loc. cit.*, we then say that H is *essentially atoral* if the closure of $H^{\mathrm{an}} \cap \mathbb{S}_v$ in $\mathbb{T}_v$ is a finite union of subvarieties of codimension at least 2 and proper torsion cosets, that is translates of subtori by torsion points.

Theorem 4.1 (Archimedean logarithmic equidistribution of torsion points)**.** *Let v be an Archimedean place of $\mathbb{K}$. Let $H \subset \mathbb{T}_v$ be an essentially atoral closed algebraic subset and $\varphi\colon \mathbb{T}_v(\mathbb{C}_v) \to \mathbb{R} \cup \{-\infty\}$ a function with at most logarithmic singularities along $H(\mathbb{C}_v)$. Then for any strict sequence $(\omega_\ell)_\ell$ of torsion points of $\mathbb{T}(\overline{\mathbb{K}})$ we have that*

$$\lim_{\ell \to +\infty} \frac{1}{\#O(\omega_\ell)} \sum_{y \in O(\omega_\ell)_v} \varphi(y) = \int_{\mathbb{S}_v} \varphi \, d\nu_v.$$

Proof. After fixing an isometry between $(\mathbb{C}_v, |\cdot|_v)$ and $(\mathbb{C}, |\cdot|)$ and an isomorphism between $\mathbb{T}$ and $\mathbb{G}_{\mathrm{m}}^n$ we can identify $\mathbb{T}_v(\mathbb{C}_v)$ with $(\mathbb{C}^\times)^n$. Under this identification, the compact torus $\mathbb{S}_v$ corresponds to the polycircle

$$(S^1)^n = \{(z_1, \dots, z_n) \in (\mathbb{C}^\times)^n : |z_1| = \dots = |z_n| = 1\}$$

and ν_v to its Haar probability measure.

Let $h_1, \dots, h_s \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a system of Laurent polynomials defining H and consider the Laurent polynomial defined by

$$h(x_1, \dots, x_n) = \sum_{i=1}^{s} \overline{h_i}(x_1^{-1}, \dots, x_n^{-1}) \, h_i(x_1, \dots, x_n),$$

where $\overline{h_i}$ denotes the complex conjugate of h_i. For all $z = (z_1, \dots, z_n) \in (S^1)^n$ we have that $z_i \overline{z_i} = 1$ for each i, and so

$$h(z) = \sum_{i=1}^{s} \overline{h_i}(\overline{z}) \, h_i(z) = \sum_{i=1}^{s} |h_i(z)|^2.$$

This readily implies that $H(\mathbb{C}) \cap (S^1)^n = V_{\mathbb{G}_{\mathrm{m}}^n}(h)(\mathbb{C}) \cap (S^1)^n$. As the closed subset H is essentially atoral, the Laurent polynomial h is essentially atoral in the sense of [DH24].

Since $V_{\mathbb{G}_{\mathrm{m}}^n}(h)$ contains H, by Remark 3.2 we have that φ has at most logarithmic singularities along the analytification of this hypersurface, and so there are positive real numbers c_1, c_2 such that

$$\varphi(z) \geq c_1 \log|h(z)| - c_2 \quad \text{for all } z \in (S^1)^n$$

thanks to the compactness of $(S^1)^n$. Applying the measure-theoretical lemma [CT09, Lemma 6.3] together with Bilu's equidistribution theorem [Bil97] we can now reduce the proof of the theorem to the case $\varphi = \log|h|$. This boils down to a particular case of [DH24, Corollary 8.9], as we next explain.

To do so, fix an embedding $\iota\colon \overline{\mathbb{K}} \hookrightarrow \mathbb{C}$ and for each ℓ consider the Galois group $G_\ell = \mathrm{Gal}(\mathbb{K}(\omega_\ell)/\mathbb{K})$. In the present situation, we have to prove that

$$\lim_{\ell \to \infty} \frac{1}{\# G_\ell} \sum_{\sigma \in G_\ell} \log|h(\iota \circ \sigma(\omega_\ell))| = \mathrm{m}(\iota(h)), \tag{4.1}$$

where $\mathrm{m}(\iota(h))$ denotes the Mahler measure of the complex Laurent polynomial $\iota(h)$.

Set $\Gamma_b = (\mathbb{Z}/b\mathbb{Z})^\times$ for $b \in \mathbb{N}$. With this notation, for each ℓ the group $\Gamma_{\mathrm{ord}(\omega_\ell)}$ agrees with the Galois group of the cyclotomic extension $\mathbb{Q}(\omega_\ell)/\mathbb{Q}$. Notice that G_ℓ is isomorphic to the Galois group of $\mathbb{Q}(\omega_\ell)/(\mathbb{K} \cap \mathbb{Q}(\omega_\ell))$ [Lan02, Chapter VI, Theorem 1.12], and hence G_ℓ can be identified with a subgroup of $\Gamma_{\mathrm{ord}(\omega_\ell)}$. On the other hand, the conductor $\mathfrak{f}_{G_\ell}$ is the minimal positive integer $b \,|\, \mathrm{ord}(\omega_\ell)$ such that G_ℓ contains the kernel of the reduction map $\Gamma_{\mathrm{ord}(\omega_\ell)} \to \Gamma_b$.

To apply [DH24, Corollary 8.9] we need to show that both $[\Gamma_{\mathrm{ord}(\omega_\ell)} : G_\ell]$ and $\mathfrak{f}_{G_\ell}$ are uniformly bounded above. When this is case, this result ensures that the discrepancy between the Mahler measure $\mathrm{m}(\iota(h))$ and its ℓ-th approximant in (4.1) is bounded above by

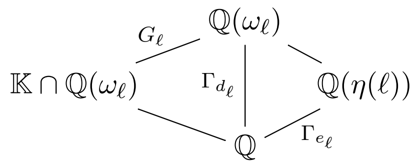
$$c\, \delta(\omega_\ell)^{-\kappa} \tag{4.2}$$

for all ℓ such that the strictness degree $\delta(\omega_\ell)$ si sufficiently large, where both c, κ are positive real numbers depending only on h.

As $\mathbb{K}$ is a number field, it has finitely many subfields. Therefore there is a finite subset $S \subset \overline{\mathbb{K}}$ of roots of unity such that for every ℓ there is $\eta(\ell) \in S$ with

$$\mathbb{K} \cap \mathbb{Q}(\omega_\ell) = \mathbb{K} \cap \mathbb{Q}(\eta(\ell)) \quad \text{and} \quad \mathbb{Q}(\eta(\ell)) \subseteq \mathbb{Q}(\omega_\ell).$$

Setting for short $d_\ell = \mathrm{ord}(\omega_\ell)$ and $e_\ell = \mathrm{ord}(\eta(\ell))$, there is a diagram



for each ℓ. This readily implies that

$$[\Gamma_{d_\ell} : G_\ell] = [\mathbb{K} \cap \mathbb{Q}(\omega_\ell) : \mathbb{Q}] \leq [\mathbb{K} : \mathbb{Q}].$$

On the other hand, the Galois group of the extension $\mathbb{Q}(\omega_\ell)/\mathbb{Q}(\eta(\ell))$ is isomorphic to the kernel of the reduction map $\Gamma_{d_\ell} \to \Gamma_{e_\ell}$ and it is a subgroup of G_ℓ, implying that $\mathfrak{f}_{G_\ell} \leq e_\ell$. Therefore

$$\mathfrak{f}_{G_\ell} \leq \max_{\eta \in S} \mathrm{ord}(\eta) \quad \text{for all } \ell,$$

and so it is also uniformly bounded above.

As the sequence $(\omega_\ell)_\ell$ is strict, its strictness degree diverges and the required convergence follows from (4.2). □

Remark 4.2. It is expected that the technical condition on H in Theorem 4.1 is not necessary for the conclusion of the theorem, as discussed in [DH24, Conjecture 1.3].

When v is non-Archimedean, recall that the *Gauss point* ζ_v of $\mathbb{T}_v^{\mathrm{an}}$ is the point of this Berkovich analytic space corresponding to the multiplicative seminorm on $\mathbb{C}_v[M]$ defined by

$$\|f\|_{\zeta_v} = \max_m |c_m|_v \quad \text{for } f = \sum_m c_m \chi^m \in \mathbb{C}_v[M].$$

It is a point of $\mathbb{S}_v$, and the Dirac delta measure at it is the non-Archimedean analogue of the Haar probability measure on the compact torus from the Archimedean case.

The following result is a direct consequence of the theorem of Tate and Voloch on linear forms in p-adic roots of unity [TV96].

Theorem 4.3 (non-Archimedean logarithmic equidistribution of torsion points)**.** *Let v be a non-Archimedean place of $\mathbb{K}$ and $\varphi\colon \mathbb{T}_v^{\mathrm{an}} \to \mathbb{R}\cup\{-\infty\}$ a function with at most logarithmic singularities along H^{an} for a closed subset $H \subset \mathbb{T}_v$. Then for any strict sequence $(\omega_\ell)_\ell$ of torsion points of $\mathbb{T}(\overline{\mathbb{K}})$ we have that*

$$\lim_{\ell\to+\infty} \frac{1}{\#O(\omega_\ell)} \sum_{y\in O(\omega_\ell)_v} \varphi(y) = \varphi(\zeta_v).$$

Proof. As in the proof of Theorem 4.1 we can assume that $\mathbb{T} = \mathbb{G}_{\mathrm{m}}^n$. Consider a nonzero Laurent polynomial $h \in \mathbb{C}_v[x_1^{\pm1},\dots,x_n^{\pm1}]$ vanishing on H. Then φ has logarithmic singularities along $V_{\mathbb{G}_{\mathrm{m}}^n}(h)^{\mathrm{an}}$ by Remark 3.2, and so it is sufficient to show the result for $\varphi = \log|h|_v$, similarly as for the proof of Theorem 4.1. This amounts to show that

$$\lim_{\ell\to+\infty} \frac{1}{\#O(\omega_\ell)} \sum_{y\in O(\omega_\ell)_v} \log|h(y)|_v = \log\|h\|_{\zeta_v}. \tag{4.3}$$

By Tate–Voloch's theorem [TV96, Theorem 2], there is a constant $c>0$ such that for any family $\eta_1,\dots,\eta_n \in \mathbb{C}_v$ of roots of unity with $h(\eta_1,\dots,\eta_n)\neq 0$ we have that $|h(\eta_1,\dots,\eta_n)|_v \geq c$. In particular, for such a family of roots of unity we have that

$$c \leq |h(\eta_1,\dots,\eta_n)|_v \leq \|h\|_{\zeta_v}. \tag{4.4}$$

Let $\rho\colon \mathbb{T}_v^{\mathrm{an}} \to \mathbb{R}$ be a continuous function with compact support and value 1 on $\mathbb{S}_v$. Its existence is ensured by [CD12, Corollaire (3.3.4)] since $\mathbb{S}_v$ is compact. We now consider the function $\psi\colon \mathbb{T}_v^{\mathrm{an}} \to \mathbb{R}$ defined as

$$\psi(x) = \rho(x)\log\max(|h|_x, c).$$

It is a continuous and compactly supported function, and so it can be extended to a continuous function on the analytic projective space. Applying Chambert-Loir's non-Archimedean equidistribution theorem as in [Cha06, Exemple 3.2] we thus obtain

$$\lim_{\ell\to+\infty} \frac{1}{\#O(\omega_\ell)} \sum_{y\in O(\omega_\ell)_v} \psi(y) = \psi(\zeta_v).$$

Since the sequence $(\omega_\ell)_\ell$ is strict, for ℓ large enough the Galois orbit $O(\omega_\ell)$ eventually avoids the zero set of h and so for each $y \in O(\omega_\ell)_v$ we have that $\psi(y) = \log|h(y)|_v$. Finally $\psi(\zeta_v) = \log\|h\|_{\zeta_v}$ because of (4.4), proving (4.3). □

4.B. The local vanishing. We now come back to the situation of the present paper, and address the main objective of this section. We show that, in the 2-codimensional situation and under suitable hypotheses, each of the local error terms in the recursive expression for the height in Proposition 2.5 converges to zero for strict sequences of torsion points.

We set ourselves in the same hypotheses and notations of Section 3. In particular, $f, g \in \mathbb{K}[M]$ are two absolutely irreducible Laurent polynomials whose supports contain the origin lattice point, X is a smooth projective toric variety compactifying $\mathbb{T}$ whose fan is compatible with the Newton polytopes of f and g, and $\overline{D}_0, \dots, \overline{D}_{n-2}$ is a family of semipositive toric metrized divisors on X with very ample underlying divisors and with smooth metrics at the Archimedean places and algebraic metrics at the non-Archimedean ones.

For all $v \in \mathfrak{M}$, similarly as in Section **2.B**, we consider the function $I_v \colon \mathbb{T}(\mathbb{C}_v) \to \mathbb{R} \cup \{-\infty\}$ defined by

$$I_v(t) = \int_{X_v^{\mathrm{an}}} \log \|t^* s_g\|_{\mathrm{Ron},v} \ \mathrm{c}_1(\overline{D}_{0,v}) \wedge \dots \wedge \mathrm{c}_1(\overline{D}_{n-2,v}) \wedge \delta_{Z_X(s_f)_v^{\mathrm{an}}},$$

where $\| \cdot \|_{\mathrm{Ron},v}$ stands for the v-adic Ronkin metric on D_g as in (2.2). This function takes the value $-\infty$ precisely on the $\mathbb{C}_v$-points of the proper closed subset Υ of $\mathbb{T}$ defined in (3.2).

By combining the local logarithmic equidistribution theorems for torsion points from Section **4.A**, the definition of the Ronkin function and Theorem 3.4, we obtain the following asymptotic vanishing for I_v.

Theorem 4.4. *In the previous hypotheses and notations, assume moreover that if v is Archimedean the closed subset $\Upsilon_v \subset \mathbb{T}_v$ is essentially atoral. Then, for all strict sequence $(\omega_\ell)_\ell$ of torsion points of $\mathbb{T}(\overline{\mathbb{K}})$ we have*

$$\lim_{\ell \to \infty} \frac{1}{\# O(\omega_\ell)} \sum_{\eta \in O(\omega_\ell)_v} I_v(\eta) = 0.$$

Proof. Recall from (1.2) that for all $t \in \mathbb{T}(\mathbb{C}_v)$ we have the equality $t^* s_g = (t^* g)\, s_{D_g}$ of rational sections of the toric line bundle $\mathcal{O}(D_g)$. Together with the definition of the v-adic Ronkin metric on D_g this relation yields

$$\log \|t^* s_g\|_{\mathrm{Ron},v} = \log |t^* g| + \rho_{g,v} \circ \mathrm{val}_v \quad \text{on } X_{0,v}^{\mathrm{an}}, \tag{4.5}$$

where $\rho_{g,v}$ is the v-adic Ronkin function of g. The measure

$$\mathrm{c}_1(\overline{D}_{0,v}) \wedge \dots \wedge \mathrm{c}_1(\overline{D}_{n-2,v}) \wedge \delta_{Z_X(s_f)_v^{\mathrm{an}}}$$

has zero mass on the analytic boundary $X_v^{\mathrm{an}} \setminus X_{0,v}^{\mathrm{an}}$, and to lighten the notation we write μ for its restriction to $X_{0,v}^{\mathrm{an}}$. It follows then from (4.5) that

$$I_v(t) = F_v(t) + \int_{X_{0,v}^{\mathrm{an}}} (\rho_{g,v} \circ \mathrm{val}_v)\, d\mu \tag{4.6}$$

for all $t \in \mathbb{T}(\mathbb{C}_v)$, with F_v being the v-adic auxiliary function from (3.1). In particular, for a strict sequence $(\omega_\ell)_\ell$ of torsion points of $\mathbb{T}(\overline{\mathbb{K}})$ we obtain that

$$\lim_{\ell \to \infty} \frac{1}{\# O(\omega_\ell)} \sum_{\eta \in O(\omega_\ell)_v} I_v(\eta) = \lim_{\ell \to \infty} \frac{1}{\# O(\omega_\ell)} \sum_{\eta \in O(\omega_\ell)_v} F_v(\eta) + \int_{X_{0,v}^{\mathrm{an}}} (\rho_{g,v} \circ \mathrm{val}_v)\, d\mu. \tag{4.7}$$

To conclude the proof it is hence enough to show that the right hand side of (4.7) vanishes. We do this by distinguishing the Archimedean and non-Archimedean cases.

Suppose first that v is Archimedean. In this case, the valuation map $\mathrm{val}_v \colon X_{0,v}^{\mathrm{an}} \to N_{\mathbb{R}}$ is a group homomorphism, and for each $x \in X_{0,v}^{\mathrm{an}}$ the fiber of $\mathrm{val}_v(x)$ agrees with the coset of x under the preimage of 0, that is with the set $\mathbb{S}_v \cdot x$. Thus, denoting by ν_v the Haar probability measure on $\mathbb{S}_v$, the definition of the Ronkin function and Fubini–Tonelli's theorem yield

$$\begin{aligned}\int_{X_{0,v}^{\mathrm{an}}} (\rho_{g,v} \circ \mathrm{val}_v)\, d\mu &= \int_{X_{0,v}^{\mathrm{an}}} \int_{\mathrm{val}_v^{-1}(\mathrm{val}_v(x))} -\log|g|_v \, d\nu_v \, d\mu(x) \\ &= \int_{X_{0,v}^{\mathrm{an}}} \int_{\mathbb{S}_v} -\log|g(\theta \cdot x)|_v \, d\nu_v(\theta)\, d\mu(x) \\ &= \int_{\mathbb{S}_v} \int_{X_{0,v}^{\mathrm{an}}} -\log|(\theta^* g)(x)|_v \, d\mu(x)\, d\nu_v(\theta) \\ &= -\int_{\mathbb{S}_v} F_v \, d\nu_v. \end{aligned} \tag{4.8}$$

On the other hand, thanks to Theorem 3.4 the auxiliary function F_v is a function on $\mathbb{T}_v^{\mathrm{an}}$ with at most logarithmic singularities along $\Upsilon_v(\mathbb{C}_v)$, and Υ_v is an essentially atoral closed algebraic subset of $\mathbb{T}_v$ because of the hypotheses. Therefore, Theorem 4.1 together with (4.7) and (4.8) implies that

$$\lim_{\ell \to \infty} \frac{1}{\# O(\omega_\ell)} \sum_{\eta \in O(\omega_\ell)_v} I_v(\eta) = 0.$$

Let us now suppose that v is non-Archimedean. Again by Theorem 3.4, the auxiliary function extends to a function F_v on $\mathbb{T}_v^{\mathrm{an}}$ with at most logarithmic singularities along Υ_v^{an}. Therefore, applying Theorem 4.3 to (4.7) yields

$$\lim_{\ell \to \infty} \frac{1}{\# O(\omega_\ell)} \sum_{\eta \in O(\omega_\ell)_v} I_v(\eta) = F_v(\zeta_v) + \int_{X_{0,v}^{\mathrm{an}}} (\rho_{g,v} \circ \mathrm{val}_v)\, d\mu, \tag{4.9}$$

where ζ_v is the Gauss point of $\mathbb{T}_v^{\mathrm{an}}$. To compute this sum, we recall from the proof of Proposition 3.6 the expression for the non-Archimedean Monge–Ampère measure μ in (3.6) and the definition of the extended auxiliary function in (3.11). Using those, we obtain that the right hand side in (4.9) is equal to a weighted sum of

$$\log|\zeta_v^* g\,(\xi_{V,\zeta_v}))|_v + (\rho_{g,v} \circ \mathrm{val}_v)(\xi_V),$$

where the sum ranges over the set of irreducible components of the special fiber of a suitable distinguished formal analytic model of $Z_X(s_f)_v$, with ξ_V and ξ_{V,ζ_v} as in the cited proof.

We claim that each of the previous terms is zero, which will be enough to conclude. On the one hand, let $\mathscr{A}_V$ be as in the proof of Proposition 3.6 and $\mathscr{B}$ be a completion of $\mathbb{C}_v[M]$ with respect to the Gauss norm. They are $\mathbb{C}_v$-affinoid algebras satisfying $\xi_V \in \mathcal{M}(\mathscr{A}_V)$ and $\zeta_v \in \mathcal{M}(\mathscr{B})$ respectively. Because of [Poi13, Lemme 3.1], both canonical morphisms $\mathscr{A}_V \widehat{\otimes}_{\mathbb{C}_v} \mathscr{B} \to \mathscr{A}_V \widehat{\otimes}_{\mathbb{C}_v} \mathscr{H}(\zeta_v)$ and $\mathscr{A}_V \widehat{\otimes}_{\mathbb{C}_v} \mathscr{B} \to \mathscr{H}(\xi_V) \widehat{\otimes}_{\mathbb{C}_v} \mathscr{B}$ are isometries. Therefore, writing $g = \sum_m \beta_m \chi^m$ and using the definition in (3.10), we have that

$$|\zeta_v^* g\,(\xi_{V,\zeta_v})|_v = \Big|\sum_m \beta_m \chi^m \otimes \chi^m(\zeta_v)\Big|_{\mathscr{A}_V \widehat{\otimes}_{\mathbb{C}_v} \mathscr{H}(\zeta_v)} = \Big|\sum_m \beta_m \chi^m \otimes \chi^m\Big|_{\mathscr{A}_V \widehat{\otimes}_{\mathbb{C}_v} \mathscr{B}}$$
$$= \Big|\sum_m \beta_m \chi^m(\xi_V) \otimes \chi^m\Big|_{\mathscr{H}(\xi_V) \widehat{\otimes}_{\mathbb{C}_v} \mathscr{B}}.$$

As ζ_v is the Gauss norm on $\mathscr{B}$, the completed tensor product norm on $\mathscr{H}(\xi_V)\widehat{\otimes}_{\mathbb{C}_v}\mathscr{B}$ agrees with the Gauss norm on the corresponding algebra, see for instance [Bos14, Appendix B/Proposition 7], so that we get

$$|\zeta_v^* g\,(\xi_{V,\zeta_v})|_v = \max_m |\beta_m|_v |\chi^m(\xi_V)|_v. \tag{4.10}$$

On the other hand, the expression for the non-Archimedean Ronkin function from [Gua18b, Remark 2.8] implies that

$$\begin{aligned}(\rho_{g,v} \circ \mathrm{val}_v)(\xi_V) &= \min_m(\langle m, \mathrm{val}_v(\xi_V)\rangle - \log|\beta_m|_v) \\ &= \min_m(-\log|\chi^m(\xi_V)|_v - \log|\beta_m|_v) = -\log\max_m(|\beta_m|_v|\chi^m(\xi_V)|_v).\end{aligned} \tag{4.11}$$

Thus, putting together (4.10) and (4.11) concludes the proof. □

5. The adelic vanishing

We continue in the 2-codimensional situation and keep the setting at the beginning of Section 3. Our aim here is to show that for a strict sequence of torsion points the sum of the correcting integrals over all places outside a specific finite subset converges to zero. To this end, we consider the following conditions on a place v of $\mathbb{K}$:

(1) the restriction of v to $\mathbb{Q}$ is a p-adic absolute value for a prime number $p \neq 2$,
(2) all nonzero coefficients of f and g have v-adic absolute value equal to 1,
(3) for $i = 0, \dots, n-2$ the v-adic metric of $\overline{D}_i$ is canonical.

Whenever these conditions are met, v is a non-Archimedean place and $f, g \in \mathbb{C}_v^\circ[M]$. This allows to consider the following additional conditions on v:

(4) the value group of v coincides with that of its restriction to $\mathbb{Q}$, namely $|\mathbb{K}^\times|_v = p^{\mathbb{Z}}$,
(5) the reductions $\widetilde{f}, \widetilde{g} \in \widetilde{\mathbb{C}}_v[M]$ are irreducible Laurent polynomials.

We denote by $\mathfrak{S} \subset \mathfrak{M}$ the subset of places for which at least one of the conditions (1)-(5) fails to hold.

Lemma 5.1. *The set $\mathfrak{S}$ is finite.*

Proof. This is equivalent to the fact that each of the above conditions fails for at most finitely many places. This is clear for (1), (2) and (3) because the number field $\mathbb{K}$ has a finite number of places above each place of $\mathbb{Q}$, every $\alpha \in \mathbb{K}^\times$ has unitary v-adic absolute value for almost all v, and for each i the non-Archimedean metrics of $\overline{D}_i$ are defined by the canonical integral model of the line bundle $\mathcal{O}(D_i)$ for almost all v.

For the remaining we assume that the first three conditions are met. To the non-Archimedean place v it corresponds a prime ideal $\mathfrak{p}$ of the ring of integers $\mathcal{O}_\mathbb{K}$ with ramification equal to the index of the value group $|\mathbb{Q}^\times|_v$ inside $|\mathbb{K}^\times|_v$. Thus the finiteness for (4) follows from the well-known fact that only finitely many rational primes ramify in a given number field [Neu99, Proposition I.8.4]. Furthermore the algebraic closure of the residue field $\mathcal{O}_\mathbb{K}/\mathfrak{p}$ coincides with $\widetilde{\mathbb{C}}_v$. Hence the finiteness for (5) is given by the classical result of Ostrowski stating that the reductions of an absolutely irreducible polynomial over $\mathbb{K}$ are absolutely irreducible for all but a finite number of prime ideals of $\mathcal{O}_{\mathbb{K}}$ [Ost19, Hilfssatz at the end of page 296]. □

As in Section **4.B**, for each $v \in \mathfrak{M}$ we denote by $I_v \colon \mathbb{T}(\mathbb{C}_v) \to \mathbb{R} \cup \{-\infty\}$ the function defined by

$$I_v(t) = \int_{X_v^{\mathrm{an}}} \log \|t^* s_g\|_{\mathrm{Ron},v} \ \mathrm{c}_1(\overline{D}_{0,v}) \wedge \ldots \wedge \mathrm{c}_1(\overline{D}_{n-2,v}) \wedge \delta_{Z_v^{\mathrm{an}}}$$

with $\| \cdot \|_{\mathrm{Ron},v}$ the v-adic Ronkin metric on the divisor D_g. We devote the rest of this section to the proof of the following result.

Theorem 5.2. *For all strict sequence $(\omega_\ell)_\ell$ of torsion points of $\mathbb{T}(\overline{\mathbb{K}})$ we have*

$$\lim_{\ell \to \infty} \sum_{v \in \mathfrak{M} \setminus \mathfrak{S}} \frac{n_v}{\# O(\omega_\ell)} \sum_{\eta \in O(\omega_\ell)_v} I_v(\eta) = 0. \tag{5.1}$$

5.A. Computing the integral. We start by giving an explicit expression for the function I_v for $v \in \mathfrak{M} \setminus \mathfrak{S}$ in terms of the coefficients of the involved Laurent polynomials. To this end, write again

$$f = \sum_m \alpha_m \chi^m \quad \text{and} \quad g = \sum_m \beta_m \chi^m$$

with $\alpha_m, \beta_m \in \mathbb{K}$ that are zero except for finitely many m. Recall that $\mathbb{S}_v \subset \mathbb{T}_v^{\mathrm{an}}$ is the v-adic compact torus of $\mathbb{T}$ as in (2.1).

Proposition 5.3. *Let $v \in \mathfrak{M} \setminus \mathfrak{S}$ and $t \in \mathbb{T}(\mathbb{C}_v) \cap \mathbb{S}_v$. If there is $m_0 \in M$ such that* $\mathrm{supp}(g) = \mathrm{supp}(f) + m_0$ *then*

$$I_v(t) = \deg_{D_0,\dots,D_{n-2}}(Z) \ \log \max_{m,m' \in M} |\alpha_{m-m_0} \beta_{m'} \chi^{m'}(t) - \alpha_{m'-m_0} \beta_m \chi^m(t)|_v.$$

Otherwise $I_v(t) = 0$.

To prove this statement we need the following construction for a fixed $v \in \mathfrak{M} \setminus \mathfrak{S}$. By the condition (1) this place is non-Archimedean, and we can consider the associated $\mathbb{C}_v$-affinoid algebra $\mathbb{C}_v\langle M \rangle$ of *strictly convergent Laurent series* [BGR84, Section 6.1.4]. As a set, it consists of the Laurent series $\sum_m \gamma_m \chi^m$ such that for every $\varepsilon > 0$ we have that $|\gamma_m|_v < \varepsilon$ for all but a finite number of $m \in M$. We equip this set with the standard addition and multiplication, and with the corresponding Gauss norm $\| \cdot \|_{\zeta_v}$.

We then consider the distinguished $\mathbb{C}_v$-affinoid algebra

$$\mathscr{A} = \mathbb{C}_v\langle M \rangle / (f) \tag{5.2}$$

and denote by $\| \cdot \|$ its sup-seminorm. This latter is the quotient of the Gauss norm, and so it is defined as

$$\|q + (f)\| = \inf_{h \in \mathbb{C}_v\langle M \rangle} \|q - h\, f\|_{\zeta_v} \quad \text{for } q \in \mathbb{C}_v\langle M \rangle. \tag{5.3}$$

The reduction of $\mathscr{A}$ agrees with $\widetilde{\mathscr{A}} = \widetilde{\mathbb{C}}_v[M]/(\widetilde{f})$, and so it is an integral $\widetilde{\mathbb{C}}_v$-algebra because of the condition (5).

Lemma 5.4. *Let $q = \sum_m \kappa_m \chi^m \in \mathbb{C}_v[M]$ with $|\kappa_m|_v = 1$ for all $m \in \mathrm{supp}(q)$ such that its reduction is either invertible or irreducible. If there is $m_0 \in M$ such that* $\mathrm{supp}(q) = \mathrm{supp}(f) + m_0$ *then*

$$\|q + (f)\| = \max_{m,m' \in M} |\alpha_{m-m_0} \kappa_{m'} - \alpha_{m'-m_0} \kappa_m|_v. \tag{5.4}$$

Otherwise $\|q+(f)\|=1$.

Proof. By the hypotheses and the condition (2) we have that $\|f\|_{\zeta_v}=\|q\|_{\zeta_v}=1$ and so $\|q+(f)\|\leq\|q\|_{\zeta_v}=1$. On the other hand, let $h=\sum_m\gamma_m\chi^m\in\mathbb{C}_v\langle M\rangle$ and suppose that $\|q-hf\|_{\zeta_v}<1$. The ultrametric inequality implies that $\|h\|_{\zeta_v}=1$, and so we can consider the reduction of $q-hf$ to obtain that

$$\widetilde{q}=\widetilde{h}\widetilde{f}\in\widetilde{\mathbb{C}}_v[M]. \tag{5.5}$$

When $\widetilde{q}$ is invertible this equality cannot hold because $\widetilde{f}$ is irreducible, and so in this case $\|q+(f)\|=1$.

Hence for the rest of this proof we can restrict to the case when $\widetilde{q}$ is irreducible. Then (5.5) implies that $\widetilde{h}$ is a monomial or equivalently that $h=\gamma_{m_0}\chi^{m_0}+h'$ with $|\gamma_{m_0}|_v=1$ and $h'=\sum_{m\neq m_0}\gamma_m\chi^m$ satisfying $\|h'\|_{\zeta_v}<1$. We deduce that $\operatorname{supp}(q)=\operatorname{supp}(\widetilde{q})=\operatorname{supp}(\widetilde{f})+m_0=\operatorname{supp}(f)+m_0$, completing the proof of the second statement.

For the first suppose that $\operatorname{supp}(q)=\operatorname{supp}(f)+m_0$ for some m_0. We have that $\|q+(f)\|=\|\chi^{-m_0}q+(f)\|$ and so we can also suppose without loss of generality that $m_0=0$. With this normalization consider the finite subset

$$\mathcal{C}=\operatorname{supp}(f)=\operatorname{supp}(q)\subset M.$$

It verifies that $0\in\mathcal{C}$ and that $\mathcal{C}\setminus\{0\}\neq\emptyset$ because f is absolutely irreducible. Then the right hand side of (5.4) can be written as

$$\begin{aligned}\max_{m,m'\in M}|\alpha_m\kappa_{m'}-\alpha_{m'}\kappa_m|_v&=\max_{m,m'\in\mathcal{C}}|\alpha_m\kappa_{m'}-\alpha_{m'}\kappa_m|_v\\&=\max_{m,m'\in\mathcal{C}}\left|\frac{\kappa_m}{\alpha_m}-\frac{\kappa_{m'}}{\alpha_{m'}}\right|_v=\max_{m\in\mathcal{C}\setminus\{0\}}\left|\frac{\kappa_m}{\alpha_m}-\frac{\kappa_0}{\alpha_0}\right|_v,\end{aligned} \tag{5.6}$$

with the last equality coming from the ultrametric property. Denoting this quantity by ρ we then have

$$\left\|q-\frac{\kappa_0}{\alpha_0}f\right\|_{\zeta_v}=\rho,$$

and so the left hand side of (5.4) is less or equal to the right hand side.

If $\rho=0$ the proof is complete, so that we assume now that $\rho>0$. To prove the equality in this case, suppose that there exists $h\in\mathbb{C}_v\langle M\rangle$ such that $\|q-hf\|_{\zeta_v}<\rho$. Since $\rho\leq1$ we can consider again the reduction of the Laurent series $q-hf$ as in (5.5), and using the fact that the supports of $\widetilde{f}$ and $\widetilde{q}$ coincide we deduce that $h=\gamma_0+h'$ with $|\gamma_0|_v=1$ and h' a Laurent series with no constant term satisfying $\|h'\|_{\zeta_v}<1$.

It follows from (5.6) that $\|q-\gamma_0f\|_{\zeta_v}\geq\rho$, and since

$$\|q-\gamma_0f-h'f\|_{\zeta_v}=\|q-hf\|_{\zeta_v}<\rho \tag{5.7}$$

this implies that $\|h'\|_{\zeta_v}=\|h'f\|_{\zeta_v}=\|q-\gamma_0f\|_{\zeta_v}$. Choose then $\tau\in\mathbb{C}_v^\times$ with $|\tau|_v=\|h'\|_{\zeta_v}$, so that setting $h''=\tau^{-1}h'$ and $q'=\tau^{-1}(q-\gamma_0f)$ we have that $\|h''\|_{\zeta_v}=\|q'\|_{\zeta_v}=1$. By (5.7) we have $\|q'-h''f\|_{\zeta_v}<1$ and so the reductions of these Laurent polynomials satisfy

$$\widetilde{q'}=\widetilde{h''}\widetilde{f}.$$

Since $\operatorname{supp}(\widetilde{q'})\subset\operatorname{supp}(q')\subset\operatorname{supp}(f)=\operatorname{supp}(\widetilde{f})$ this implies that $\widetilde{h''}$ is a nonzero constant, which is not possible because h' has no constant term. ☐

Proof of Proposition 5.3. Let $\mathcal{X}$ be the canonical model of the toric variety X and $\mathcal{Z} \subset \mathcal{X}$ the closure of the hypersurface Z. By the condition (3), for each i the metric of $\overline{D}_{i,v}$ is the algebraic metric induced by the canonical model of the line bundle $\mathcal{O}(D_{i,v})$ [BPS14, Definition 3.6.3]. This is a line bundle on $\mathcal{X}$, and we denote by $\mathcal{L}_i$ its pullback to $\mathcal{Z}$.

These integral models induce a distinguished formal analytic variety $\mathfrak{Z}$ over $\mathbb{C}_v$ with generic fiber Z_v^{an} and special fiber $\widetilde{\mathfrak{Z}} = \widetilde{\mathcal{Z}}$. The latter is an integral scheme over $\widetilde{\mathbb{C}}_v$ by the condition (5). For each i we also obtain a formal analytic line bundle $\mathfrak{L}_i$ on $\mathfrak{Z}$. Its reduction is a line bundle on $\widetilde{\mathfrak{Z}}$ which coincides with the reduction $\widetilde{\mathcal{L}}_i$ of $\mathcal{L}_i$.

Now let $t \in \mathbb{T}(\mathbb{C}_v) \cap \mathbb{S}_v$ and recall from (4.6) that

$$I_v(t) = F_v(t) + \int_{X_{0,v}^{\mathrm{an}}} (\rho_{g,v} \circ \mathrm{val}_v)\, \mathrm{c}_1(\overline{D}_{0,v}) \wedge \ldots \wedge \mathrm{c}_1(\overline{D}_{n-2,v}) \wedge \delta_{Z_v^{\mathrm{an}}} \tag{5.8}$$

where F_v is the auxiliary function in (3.1), $\rho_{g,v}$ the v-adic Ronkin function of g, and val_v the v-adic valuation map. As in (3.6), the above Monge-Ampère measure is equal to $\deg_{\widetilde{\mathfrak{L}}_0,\ldots,\widetilde{\mathfrak{L}}_{n-2}}(\widetilde{\mathfrak{Z}})\, \delta_{\xi_{\widetilde{\mathfrak{Z}}}}$, where $\xi_{\widetilde{\mathfrak{Z}}}$ is the sup-seminorm of the the distinguished $\mathbb{C}_v$-affinoid algebra $\mathscr{A}$ in (5.2). In our setting

$$\deg_{\widetilde{\mathfrak{L}}_0,\ldots,\widetilde{\mathfrak{L}}_{n-2}}(\widetilde{\mathfrak{Z}}) = \deg_{\widetilde{\mathcal{L}}_0,\ldots,\widetilde{\mathcal{L}}_{n-2}}(\widetilde{\mathcal{Z}}) = \deg_{D_0,\ldots,D_{n-2}}(Z)$$

because of our previous considerations and the fact that the degree of the fibers of a proper map is constant [Ful98, Example 20.3.3].

Thus by the formula in (4.11) and Lemma 5.4, the second term in (5.8) is equal to

$$-\deg_{D_0,\ldots,D_{n-2}}(Z) \log \max_m |\chi^m(\xi_{\widetilde{\mathfrak{Z}}})|_v = -\deg_{D_0,\ldots,D_{n-2}}(Z) \log \max_m \|\chi^m + (f)\| = 0.$$

On the other hand, by (3.7) we also have

$$F_v(t) = \deg_{D_0,\ldots,D_{n-2}}(Z)\ \log |t^*g\,(\xi_{\widetilde{\mathfrak{Z}}})|_v = \deg_{D_0,\ldots,D_{n-2}}(Z)\ \log \|t^*g + (f)\|.$$

The statement then follows from Lemma 5.4. □

5.B. Bounds for the integral. Fix $v \in \mathfrak{M} \setminus \mathfrak{S}$ and let $\mathcal{A} \subset M$ be a nonempty finite subset together with a family of scalars $\gamma_m \in \mathbb{K}$ with $|\gamma_m|_v = 1$, $m \in \mathcal{A}$. For a torsion point $\omega \in \mathbb{T}(\overline{\mathbb{K}})$ we set

$$K_v(\omega) = \frac{1}{\#O(\omega)} \sum_{\eta \in O(\omega)_v} \log \max_{m \in \mathcal{A}} |\chi^m(\eta) - \gamma_m|_v. \tag{5.9}$$

This quantity arises naturally when computing the mean of the function I_v over the v-adic Galois orbit of ω, as we will see in Section **5.C**. In this subsection we give the bounds needed for the proof of our adelic vanishing theorem.

To this end, we denote by $H\colon \mathbb{Z}^{\mathcal{A}} \to M$ the linear map defined by

$$H(a) = \sum_{m \in \mathcal{A}} a_m m, \tag{5.10}$$

and by $\phi\colon \mathbb{T} \to \mathbb{G}_{\mathrm{m}}^{\mathcal{A}}$ the corresponding monomial map, defined by

$$\phi(t) = (\chi^m(t))_{m \in \mathcal{A}}.$$

For $a = (a_m)_{m \in \mathcal{A}} \in \mathbb{Z}^{\mathcal{A}}$ and $u \in \mathbb{G}_{\mathrm{m}}^{\mathcal{A}}(\overline{\mathbb{K}}) = (\overline{\mathbb{K}}^{\times})^{\mathcal{A}}$ we set $u^a = \prod_{m \in \mathcal{A}} u_m^{a_m} \in \overline{\mathbb{K}}^{\times}$. With this notation we have that

$$\chi^{H(a)}(t) = \phi(t)^a.$$

Denote by p_v the unique prime number such that the restriction of v to $\mathbb{Q}$ is the p_v-adic place. We consider separately the case when this prime number divides the order of $\phi(\omega)$ and when it does not.

Proposition 5.5. *If* $p_v \mid \mathrm{ord}(\phi(\omega))$ *then* $\dfrac{-\log(p_v)}{\#O(\phi(\omega))} \leq K_v(\omega) \leq 0$.

Proposition 5.6. *Suppose that* $p_v \nmid \mathrm{ord}(\phi(\omega))$ *and let* $a \in \mathbb{Z}^{\mathcal{A}}$.
(1) If $\chi^{H(a)}(\omega) = 1$ *then*

$$\frac{\log|\gamma^a - 1|_v}{\#O(\phi(\omega))} \leq K_v(\omega) \leq 0.$$

(2) If $\chi^{H(a)}(\omega) \neq 1$ *and* $\gamma^a = 1$ *then* $K_v(\omega) = 0$.

The proofs of these results rely on the following reduction properties of roots of unity. For a prime number p, let $\mathbb{C}_p$ be the algebraically closed field of p-adic numbers, equipped with the standard p-adic absolute value $|\cdot|_p$. Its residue field is identified with an algebraic closure of the finite field of p elements, and we denote by

$$\mathrm{red}\colon \mathbb{C}_p^\circ \to \overline{\mathbb{F}}_p$$

the corresponding reduction map. Also, we write φ for the Euler totient function.

Lemma 5.7. *Let* ξ *be a root of unity in* $\mathbb{C}_p$ *of order* $d = p^e\, b$*, with* $e \geq 0$ *and* $b \in \mathbb{Z}$ *such that* $p \nmid b$.
(1) The element $\mathrm{red}(\xi)$ *is a root of unity in* $\overline{\mathbb{F}}_p$ *of order* b*, and the reduction map is* $\varphi(p^e)$*-to-1 on the set of roots of unity of order* d *in* $\mathbb{C}_p$.
(2) For each root of unity $\rho \in \mathbb{C}_p$ *of order* b *we have that*

$$|\xi - \rho|_p = \begin{cases} p^{-1/\varphi(p^e)} & \textit{if } \mathrm{red}(\xi) = \mathrm{red}(\rho), \\ 1 & \textit{otherwise.} \end{cases}$$

Proof. For (1), consider first the case in which $e = 0$. In this situation, the polynomial $x^b - 1$ over $\mathbb{F}_p$ is separable since $p \nmid b$, and then $\mathrm{red}(\xi)$ is a root of a single factor in

$$x^b - 1 = \prod_{c \mid b} \mathrm{red}(\Phi_c) \in \mathbb{F}_p[x],$$

where $\mathrm{red}(\Phi_c)$ denotes the reduction to $\mathbb{F}_p$ of the cyclotomic polynomial $\Phi_c \in \mathbb{Z}_p[x]$. Since $\Phi_b(\xi) = 0$ we deduce that $\mathrm{red}(\xi)$ is not a root of $\mathrm{red}(\Phi_c)$ for all $c \neq b$ and so $\mathrm{ord}(\mathrm{red}(\xi)) = b$, as stated. Moreover, the separable polynomials Φ_b and $\mathrm{red}(\Phi_b)$ have as many roots, in $\mathbb{C}_p$ and in $\overline{\mathbb{F}}_p$ respectively, as their common degree. Since by Hensel's lemma the reduction map between the sets of their zeros is surjective, we conclude that the map is 1-to-1 in this case.

When $e \geq 1$ the Frobenius homomorphism $\mathrm{Frob}\colon \overline{\mathbb{F}}_p \to \overline{\mathbb{F}}_p$ mapping x to x^{p^e} is an automorphism, and so the above implies that

$$\mathrm{ord}(\mathrm{red}(\xi)) = \mathrm{ord}(\mathrm{Frob}(\mathrm{red}(\xi))) = \mathrm{ord}(\mathrm{red}(\xi^{p^e})) = b.$$

The statement about the degree of the reduction map follows from a cardinality count and from the surjectivity of the morphism $x \mapsto x^{p^e}$ between the set of roots of unity of order d and the ones of order b in $\mathbb{C}_p$.

For (2), the case in which $b = 1$ is given by [GS23, Lemma 2.5], that is

$$|\xi - 1|_p = p^{-1/\varphi(p^e)}.$$

When b is arbitrary, we then have that

$$\prod_{\rho^b=1} |\xi - \rho|_p = |\xi^b - 1|_p = p^{-1/\varphi(p^e)}.$$

Since $|\xi - \rho|_p = 1$ whenever $\mathrm{red}(\xi) \neq \mathrm{red}(\rho)$, and since the equality $\mathrm{red}(\rho) = \mathrm{red}(\xi)$ is realized for a single root of unity ρ of order b in $\mathbb{C}_p$ because of (1), the claim follows. □

Let us now focus on the proof of Propositions 5.5 and 5.6. Notice that, for each $\eta \in O(\omega)_v$ and $a \in \mathbb{Z}^{\mathcal{A}}$, simple algebraic manipulations allow to write

$$\chi^{H(a)}(\eta) - \gamma^a = \phi(\eta)^a - \gamma^a = \sum_{m\in\mathcal{A}} \varepsilon_m \, (\chi^m(\eta) - \gamma_m),$$

with each ε_m being the product between a monomial in $(\chi^m(\eta))_m$ and one in $(\gamma_m)_m$. In particular, we have that $|\varepsilon_m|_v = 1$ for all $m \in \mathcal{A}$ and so

$$|\chi^{H(a)}(\eta) - \gamma^a|_v \leq \max_{m\in\mathcal{A}} |\chi^m(\eta) - \gamma_m|_v, \tag{5.11}$$

which turns out to be useful to lower bound the quantity $K_v(\omega)$ in (5.9).

Proof of Proposition 5.5. The second inequality is clear from the fact that $|\chi^m(\eta)|_v = |\gamma_m|_v = 1$ for all m, and so we focus on the proof of the first one. To this end, let $d = \mathrm{ord}(\phi(\omega))$; since p_v divides d by hypothesis, we can write $d = p_v^e b$ with $e \geq 1$ and b an integer such that $p_v \nmid b$. Choose also $a \in \mathbb{Z}^{\mathcal{A}}$ such that the root of unity $\chi^{H(a)}(\omega) = \phi(\omega)^a$ has order equal to d; this is possible by writing

$$\phi(\omega) = (\zeta^{c_1}, \ldots, \zeta^{c_r})$$

for a root of unity ζ of order d in $\overline{\mathbb{K}}$ and for $0 \leq c_i \leq d-1$ with $\gcd(c_1, \ldots, c_r, d) = 1$.

Identifying $(\mathbb{C}_v, |\cdot|_v)$ with $(\mathbb{C}_{p_v}, |\cdot|_{p_v})$, the character $\chi^{H(a)}$ thus induces a surjective map from the v-adic Galois orbit $O(\omega)_v$ to the set μ_d° of roots of unity in $\mathbb{C}_{p_v}$ of order d. This map is $\#O(\omega)/\varphi(d)$-to-1 and so, together with (5.11), it yields

$$K_v(\omega) \geq \frac{1}{\#O(\omega)} \sum_{\eta\in O(\omega)_v} \log|\chi^{H(a)}(\eta) - \gamma^a|_v = \frac{1}{\varphi(d)} \sum_{\xi\in\mu_d^\circ} \log|\xi - \gamma^a|_{p_v}. \tag{5.12}$$

We can now suppose that there is some $\xi_0 \in \mu_d^\circ$ for which $|\xi_0 - \gamma^a|_{p_v} < 1$, because otherwise the statement holds trivially. Under this assumption $\mathrm{red}(\gamma^a) = \mathrm{red}(\xi_0)$ and so Lemma 5.7(1) ensures that γ^a has the same reduction as a root of unity ρ of order b in $\mathbb{C}_{p_v}$. It follows for instance from [Ser79, Proposition 16 at page 77] that ρ lives in a unramified extension of $\mathbb{K}_v$. Combined with the assumption (4) on v, this yields

$$|\rho - \gamma^a|_{p_v} \leq p_v^{-1} < p_v^{-1/\varphi(p_v^e)} < 1,$$

where the second inequality follows from the fact that $e \geq 1$ and that $p_v > 2$ by the assumption (1) on v. Therefore, using Lemma 5.7(2) we obtain that

$$|\xi - \gamma^a|_{p_v} = |\xi - \rho + \rho - \gamma^a|_{p_v} = \begin{cases} p_v^{-1/\varphi(p_v^e)} & \text{if } \mathrm{red}(\xi) = \mathrm{red}(\rho), \\ 1 & \text{otherwise.} \end{cases} \tag{5.13}$$

Thanks to Lemma 5.7(1) there are exactly $\varphi(p^e)$ roots of unity $\xi \in \mu_d^\circ$ satisfying $\mathrm{red}(\xi) = \mathrm{red}(\rho)$. Hence, we conclude from (5.12) and (5.13) that

$$K_v(\omega) \geq \frac{\varphi(p_v^e)}{\varphi(d)} \log p_v^{-1/\varphi(p_v^e)} = \frac{-\log(p_v)}{\varphi(d)} = \frac{-\log(p_v)}{\#O(\phi(\omega))},$$

as desired. □

Proof of Proposition 5.6. Again, the inequality $K_v(\omega) \leq 0$ is clear from the fact that $|\chi^m(\eta)|_v = |\gamma_m|_v = 1$ for all m, and so we focus on proving lower bounds for $K_v(\omega)$. In doing this, we call $d = \mathrm{ord}(\phi(\omega))$ and we identify $(\mathbb{C}_v, |\cdot|_v)$ with $(\mathbb{C}_{p_v}, |\cdot|_{p_v})$.

To prove (1) we start by remarking that for each $\eta \in O(\omega)_v$ the inequality

$$\max_{m\in\mathcal{A}} |\chi^m(\eta) - \gamma_m|_{p_v} \neq 1 \tag{5.14}$$

holds if and only if $\mathrm{red}(\chi^m(\eta)) = \mathrm{red}(\gamma_m)$ for all $m \in \mathcal{A}$. We can suppose that (5.14) is realized for a certain η_0, otherwise the sought bound is obviously true. We claim that in this case (5.14) happens for exactly $\#O(\omega)/\#O(\phi(\omega))$ choices of $\eta \in O(\omega)_v$. Indeed, if η satisfies (5.14) it must also fulfil

$$\mathrm{red}(\chi^m(\eta)) = \mathrm{red}(\gamma_m) = \mathrm{red}(\chi^m(\eta_0)) \quad \text{for all } m \in \mathcal{A}.$$

But each $\chi^m(\eta)$ is a coordinate of $\phi(\eta)$ and hence it is a root of unity in $\mathbb{C}_{p_v}$ of order dividing d. In particular, its order is coprime with p_v by hypothesis and hence it must happen that $\chi^m(\eta) = \chi^m(\eta_0)$ because of Lemma 5.7(1); thus, $\phi(\eta) = \phi(\eta_0)$. We deduce that η and η_0 are conjugate by an element of the Galois group of the extension $\mathbb{K}(\omega)/\mathbb{K}(\phi(\omega))$, and there are precisely $\#O(\omega)/\#O(\phi(\omega))$ such conjugates.

For all of these $\eta \in O(\omega)_v$ we have by (5.11) and by hypothesis on a that

$$|1 - \gamma^a|_v \leq \max_{m\in\mathcal{A}} |\chi^m(\eta) - \gamma_m|_v,$$

which implies the desired lower bound on $K_v(\omega)$.

To prove (2), we have by the equality $\gamma^a = 1$ and by (5.11) that

$$K_v(\omega) \geq \frac{1}{\#O(\omega)} \sum_{\eta\in O(\omega)_v} \log |\chi^{H(a)}(\eta) - 1|_{p_v}.$$

For any given $\eta \in O(\omega)_v$, by hypotheses $\chi^{H(a)}(\eta) = \phi(\eta)^a$ is a root of unity in $\mathbb{C}_{p_v}$ different from 1, and with order coprime with p_v. Therefore, by Lemma 5.7(1) its reduction has the same order, and in particular $\mathrm{red}(\chi^{H(a)}(\eta)) \neq \mathrm{red}(1)$. It follows that $|\chi^{H(a)}(\eta) - 1|_{p_v} = 1$, proving the statement. □

We will also need to control the size of a relation of multiplicative dependence for a torsion point of a torus.

Proposition 5.8. *There is $a \in \mathbb{Z}^{\mathcal{A}}$ with $H(a) \neq 0$ such that*

$$\chi^{H(a)}(\omega) = 1 \quad \text{and} \quad \max_{m\in\mathcal{A}} |a_m| \leq c \, \mathrm{ord}(\phi(\omega))^{1/\mathrm{rank}(H(\mathbb{Z}^{\mathcal{A}}))}$$

for a constant $c > 0$ independent of ω.

Proof. Set $d = \mathrm{ord}(\phi(\omega))$. For every $a \in \mathbb{Z}^{\mathcal{A}}$ we have that $\chi^{H(a)}(\omega) = \phi(\omega)^a$ is a d-th root of unity. Hence we can consider the homomorphism $\mathbb{Z}^{\mathcal{A}}/\mathrm{Ker}(H) \to \mu_d$ defined as

$$a \longmapsto \chi^{H(a)}(\omega). \tag{5.15}$$

It is surjective, and so its kernel is subgroup of $\mathbb{Z}^{\mathcal{A}}/\mathrm{Ker}(H)$ of index d.

Now set $r = \mathrm{rank}(H(\mathbb{Z}^{\mathcal{A}}))$. We have that $\mathbb{Z}^{\mathcal{A}}/\mathrm{Ker}(H) \simeq H(\mathbb{Z}^{\mathcal{A}})$, and so we can choose an isomorphism

$$\mathbb{Z}^{\mathcal{A}}/\mathrm{Ker}(H) \simeq \mathbb{Z}^r. \tag{5.16}$$

This identifies the kernel of the homomorphism in (5.15) with a subgroup $\Lambda \subset \mathbb{Z}^r$.

Denote by $\|\cdot\|$ the max-norm on $\mathbb{Z}^r$. Since Λ is a subgroup of index d, by Minkowski's first theorem there is $m_0 \in \Lambda \setminus \{0\}$ such that $\|m_0\| \le d^{1/r}$. Considering the linear map $\mathbb{Z}^{\mathcal{A}} \to \mathbb{Z}^r$ induced by the isomorphism in (5.16) and the Smith normal form of its associated matrix, we deduce that there is $a \in \mathbb{Z}^{\mathcal{A}}$ with $H(a) = m_0$ satisfying

$$\max_{m\in\mathcal{A}} |a_m| \le c\,\|m_0\| \le c\, d^{1/r}$$

for a constant $c > 0$ depending only on the coefficients of this matrix, as desired. □

5.C. Proof of Theorem 5.2. By Proposition 5.3 we can reduce to the case when the supports of f and g agree up to a translation because otherwise $I_v(\eta) = 0$ for all $v \in \mathfrak{M}\setminus\mathfrak{S}$ and $\eta \in O(\omega_\ell)_v$, and so the double sum in (5.1) vanishes for all ℓ. Assuming that this is the case, up to multiplying by a suitable monomial we can also suppose that $\mathrm{supp}(f) = \mathrm{supp}(g)$. Recall that this support contains the lattice point $0 \in M$.

Consider then the finite subset $\mathcal{A} = \mathrm{supp}(f) \setminus \{0\} = \mathrm{supp}(g) \setminus \{0\} \subset M$, which is nonempty because of the hypothesis that f and g are absolutely irreducible. Set also

$$\gamma_m = \frac{\alpha_m \beta_0}{\alpha_0 \beta_m} \in \mathbb{C}_v \quad \text{for } m \in \mathcal{A}.$$

Let $\ell \ge 1$. In our present situation, for each $v \in \mathfrak{M} \setminus \mathfrak{S}$ and $\eta \in O(\omega_\ell)_v$ we have

$$\max_{m,m'\in M} |\alpha_m\beta_{m'}\chi^{m'}(\eta) - \alpha_{m'}\beta_m\chi^m(\eta)|_v = \max_{m\in\mathcal{A}} |\chi^m(\eta) - \gamma_m|_v, \tag{5.17}$$

as shown in (5.6). Hence using the notation in (5.9) and Proposition 5.3 we can write

$$\sum_{v\in\mathfrak{M}\setminus\mathfrak{S}} \frac{n_v}{\#O(\omega_\ell)} \sum_{\eta\in O(\omega_\ell)_v} I_v(\eta) = \deg_{D_0,\dots,D_{n-2}}(Z) \sum_{v\in\mathfrak{M}\setminus\mathfrak{S}} n_v K_v(\omega_\ell). \tag{5.18}$$

Thus to prove the statement it is enough to show the asymptotic vanishing of the sum in the right-hand side of this equality.

Recall the linear map H in (5.10) and denote by r the rank of the subgroup $H(\mathbb{Z}^{\mathcal{A}})$ of M. When $r = 1$, the fact that f and g are absolutely irreducible implies that $f = \alpha_m\chi^m + \alpha_0$ and $g = \beta_m\chi^m + \beta_0$ with $m \in M$ primitive. Hence in this case $\mathcal{A} = \{m\}$ and so

$$\begin{aligned}\sum_{v\in\mathfrak{M}\setminus\mathfrak{S}} n_v K_v(\omega_\ell) &= \sum_{v\in\mathfrak{M}\setminus\mathfrak{S}} \frac{n_v}{\#O(\omega_\ell)} \sum_{\eta\in O(\omega_\ell)_v} \log|\chi^m(\eta) - \gamma_m|_v \\ &= \sum_{v\in\mathfrak{S}} \frac{-n_v}{\#O(\omega_\ell)} \sum_{\eta\in O(\omega_\ell)_v} \log|\chi^m(\eta) - \gamma_m|_v,\end{aligned}$$

where the second equality follows from the product formula. Since $\mathfrak{S}$ is finite, this sum converges to 0 as $\ell \to \infty$ as a consequence of the local vanishing (Theorem 4.4).

Hence from now on we suppose that $r \ge 2$. Write $d_\ell = \mathrm{ord}(\phi(\omega_\ell))$, and denote by $\mathcal{P}_\ell$ the finite set of prime divisors of this integer and by $\mathfrak{P}_\ell$ for the finite subset of $\mathfrak{M}$ of places extending the p-adic place of $\mathbb{Q}$ for some $p \in \mathcal{P}_\ell$. We then split the sum in the right hand side of (5.18) as

$$\sum_{v\in\mathfrak{M}\setminus\mathfrak{S}} n_v K_v(\omega_\ell) = S_\ell + T_\ell \tag{5.19}$$

where S_ℓ denotes the sum over the places in $\mathfrak{P}_\ell \setminus \mathfrak{S}$, and T_ℓ the complementary sum.

For the first sum, we deduce from Proposition 5.5 that

$$0 \geq S_\ell = \sum_{v \in \mathfrak{P}_\ell \setminus \mathfrak{S}} n_v K_v(\omega_\ell) \geq \sum_{v \in \mathfrak{P}_\ell \setminus \mathfrak{S}} \frac{-n_v \log(p_v)}{\#O(\phi(\omega_\ell))} \geq \frac{-1}{\#O(\phi(\omega_\ell))} \sum_{p \in \mathcal{P}_\ell} \log(p), \tag{5.20}$$

where the last inequality comes from the fact that $\sum_{v|v_0} n_v = 1$ for each place v_0 of $\mathbb{Q}$. Furthermore we have

$$\#O(\phi(\omega_\ell)) = [\mathbb{K}(\phi(\omega_\ell)) : \mathbb{K}] = \frac{[\mathbb{K}(\phi(\omega_\ell)) : \mathbb{Q}]}{[\mathbb{K} : \mathbb{Q}]} \geq \frac{[\mathbb{Q}(\phi(\omega_\ell)) : \mathbb{Q}]}{[\mathbb{K} : \mathbb{Q}]} = \frac{\varphi(d_\ell)}{[\mathbb{K} : \mathbb{Q}]}, \tag{5.21}$$

where φ denotes the Euler totient function. We conclude that

$$0 \geq S_\ell \geq -[\mathbb{K} : \mathbb{Q}] \, \frac{\log(d_\ell)}{\varphi(d_\ell)}. \tag{5.22}$$

For the second sum, suppose first that there is $a \in \mathbb{Z}^{\mathcal{A}}$ such that $H(a) \neq 0$ and $\gamma^a = 1$. Since the sequence $(\omega_\ell)_\ell$ is strict, there is $\ell_0 \geq 1$ such that $\chi^{H(a)}(\omega_\ell) \neq 1$ for $\ell \geq \ell_0$, and so by Proposition 5.6(2) we have that $T_\ell = 0$ for all such ℓ's.

Otherwise suppose that for all $a \in \mathbb{Z}^{\mathcal{A}}$ with $H(a) \neq 0$ we have that $\gamma^a \neq 1$. Applying Proposition 5.8 to the torsion point ω_ℓ we deduce that there is $a_\ell \in \mathbb{Z}^{\mathcal{A}}$ with $H(a_\ell) \neq 0$ such that

$$\chi^{H(a_\ell)}(\omega_\ell) = 1 \quad \text{and} \quad \|a_\ell\| \leq c \, d_\ell^{1/r} \tag{5.23}$$

for a constant $c > 0$ independent of ℓ. Then by Proposition 5.6(1)

$$\begin{aligned} 0 \geq T_\ell = \sum_{v \in \mathfrak{M} \setminus (\mathfrak{S} \cup \mathfrak{P}_\ell)} n_v K_v(\omega_\ell) &\geq \frac{1}{\#O(\phi(\omega_\ell))} \sum_{v \in \mathfrak{M} \setminus (\mathfrak{S} \cup \mathfrak{P}_\ell)} n_v \log|\gamma^{a_\ell} - 1|_v \\ &\geq \frac{-1}{\#O(\phi(\omega_\ell))} \sum_{v | \infty} n_v \log|\gamma^{a_\ell} - 1|_v, \end{aligned} \tag{5.24}$$

where the last inequality is a consequence of the product formula and the fact that $|\gamma^{a_\ell} - 1|_v \leq 1$ whenever v is non-Archimedean. It follows from (5.23) and (5.24) that

$$0 \geq T_\ell \geq \frac{-c' \, d_\ell^{1/r}}{\#O(\phi(\omega_\ell))} \geq -c' \, [\mathbb{K} : \mathbb{Q}] \, \frac{d_\ell^{1/r}}{\varphi(d_\ell)} \tag{5.25}$$

for a constant $c' > 0$ independent of ℓ, where the last inequality is ensured by (5.21).

To conclude, notice that the sequence of torsion points $(\phi(\omega_\ell))_\ell$ is strict in the torus $\phi(\mathbb{T})$, which has dimension at least 1, and therefore $\lim_{\ell \to \infty} d_\ell = \infty$. The result now follows from (5.22) and (5.25) by taking $\ell \to \infty$, using the hypothesis that $r \geq 2$ and the fact that for any $\varepsilon > 0$ we have

$$\lim_{\ell \to \infty} \frac{d_\ell^{1-\varepsilon}}{\varphi(d_\ell)} = 0.$$

6. Main results

We now put together the main results from the previous sections to prove Theorem B. In doing this, we need to treat separately the case in which the involved Laurent polynomials are binomials, due to the annoying hypothesis in the Archimedean logarithmic equidistribution of torsion points. As an application, we show how the statement of the main theorem affirmatively answers previous questions by the authors.

As usual, throughout the whole section $\mathbb{K}$ denotes a number field, and $\mathbb{T}$ a split algebraic torus over $\mathbb{K}$ of dimension $n \geq 2$ with character lattice M. Consider also a complete toric variety X which compactifies $\mathbb{T}$, and a family $\overline{D}_0, \dots, \overline{D}_{n-2}$ of semipositive toric metrized divisors on X, with corresponding collection $(\vartheta_{\overline{D}_i,v})_{v\in\mathfrak{M}}$ of v-adic roof functions, for $i = 0, \dots, n-2$.

6.A. Proof of Theorem B. We start by remarking that when the Laurent polynomials involved in Conjecture A have a large locus of bad intersection, its statement can be easily verified.

Lemma 6.1. *Let f, g be two absolutely irreducible Laurent polynomials in $\mathbb{K}[M]$ for which the closed subset $\Upsilon \subset \mathbb{T}$ from* (3.2) *has codimension at most* 1*. Then f and g are either monomials or binomials with the same support up to translation, and in this case Conjecture A holds true.*

Proof. It is clear that f and g must have the same support up to translation, as otherwise Υ is by definition the empty set. We can then restrict to the case in which $\mathrm{supp}(g) = \mathrm{supp}(f) + \{m_0\}$ for some $m_0 \in M$, and denote by Λ the linear span of the set $\mathrm{supp}(g) - \mathrm{supp}(g) = \{m - m' : m, m' \in \mathrm{supp}(g)\}$ in $M_{\mathbb{R}}$. We claim that

$$\mathrm{codim}(\Upsilon) \geq \dim(\Lambda), \tag{6.1}$$

which will be enough to prove the first statement because of the absolute irreducibility of f and g.

To show (6.1) notice that, denoting $r = \dim(\Lambda)$, there exist $m, m_1, \dots, m_r \in \mathrm{supp}(g)$ such that the lattice points $m_1 - m, \dots, m_r - m$ are linearly independent elements of M. We can then find a basis $e_1, \dots, e_n$ of M, and $\lambda_1, \dots, \lambda_r \in \mathbb{Z}$ for which

$$\lambda_i e_i = \sum_{j=1}^{n} a_{ij}(m_j - m)$$

for all $i = 1, \dots, r$, with each $a_{ij} \in \mathbb{Z}$. We then have that Υ is contained in the zero set

$$V_{\mathbb{T}}(\chi^{\lambda_i e_i} - \gamma_i : i = 1, \dots, r)$$

for an appropriate choice of $\gamma_1, \dots, \gamma_r \in \mathbb{K}$. Under the identification between $\mathbb{T}$ and $\mathbb{G}_{\mathrm{m}}^n$ given by the choice of the basis $\{e_1, \dots, e_n\}$, the $\overline{\mathbb{K}}$-points of this last set agree with

$$\big\{t \in (\overline{\mathbb{K}}^{\times})^n : t_1^{\lambda_1} = \gamma_1, \dots, t_r^{\lambda_r} = \gamma_r\big\},$$

and then it has codimension precisely r.

Let us now prove the second statement. When f and g are monomials, Proposition A.1 ensures that the conjecture holds. Suppose then that f and g are binomials with the same support up to translation. After multiplying each of them by a suitable monomial, Proposition A.1 and Remark A.2 allow to reduce the proof to the case in which $f = \chi^m - \alpha_0$ and $g = \chi^m - \beta_0$. For all $\omega \in \mathbb{T}(\overline{\mathbb{K}})$ the Laurent polynomials $\omega_1^* f, \omega_2^* g$ meet properly if and only if

$$\alpha_0 \chi^m(\omega_2) \neq \beta_0 \chi^m(\omega_1),$$

in which case $Z_{\mathbb{T}}(\omega_1^* f, \omega_2^* g) = 0$. Therefore, the left hand side of Conjecture A vanishes for f and g. The right hand one is also zero, as can be shown by noticing that the v-adic Ronkin functions of f and g are explicitly computable piecewise affine functions [Gua18b, Example 2.12], and by the properties of mixed integrals. □

We are finally ready to prove the main result of the article, that is the following special case of Conjecture A.

Theorem 6.2 (Theorem B). *Let f, g be two nonzero Laurent polynomials in $\mathbb{K}[M]$. Then, for all quasi-strict sequence $(\boldsymbol{\omega}_\ell)_\ell$ of torsion points in $\mathbb{T}(\overline{\mathbb{K}})^2$ we have that*

$$\lim_{\ell\to\infty} \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{n-2}}(Z_{\mathbb{T}}(\boldsymbol{\omega}_{\ell,1}^* f, \boldsymbol{\omega}_{\ell,2}^* g)) = \sum_{v\in\mathfrak{M}} n_v \, \mathrm{MI}_M(\vartheta_{\overline{D}_0,v}, \dots, \vartheta_{\overline{D}_{n-2},v}, \rho_{f,v}^\vee, \rho_{g,v}^\vee).$$

Proof. Thanks to Proposition 2.6 it is enough to consider the case in which f, g are absolutely irreducible Laurent polynomials with coefficients in $O_{\mathbb{K}}$, and whose Newton polytopes contain the lattice point $0 \in M$. By the same result, we can also assume that X is smooth and projective, with fan compatible with the Newton polytopes of f and g, that $\overline{D}_0, \dots, \overline{D}_{n-2}$ are very ample toric divisors on X equipped with smooth toric metrics at the Archimedean places and algebraic toric metrics at the non-Archimedean ones, and that $(\boldsymbol{\omega}_\ell)_\ell$ is of the form $(1, \omega_\ell)_\ell$ for a strict sequence $(\omega_\ell)_\ell$ of torsion points in $\mathbb{T}(\overline{\mathbb{K}})$.

Under these assumptions, the Laurent polynomials f and g define nef toric divisors D_f and D_g on X, and sections $s_f \in \Gamma(X, \mathcal{O}(D_f))$ and $s_g \in \Gamma(X, \mathcal{O}(D_g))$, respectively.

When the closed subset $\Upsilon \subset \mathbb{T}$ from (3.2) has codimension at most 1, the validity of the conjecture follows from Lemma 6.1. Therefore, we can assume for the remaining of the proof that Υ has codimension at least 2 in $\mathbb{T}$, and in particular the base change Υ_v is an essentially atoral subset of $\mathbb{T}_v$ for all Archimedean place v of $\mathbb{K}$.

We can now apply the strategy outlined in Section **2.B**. Since the sequence $(1, \omega_\ell)_\ell$ is quasi-strict, by Corollary 1.9 we have that for ℓ sufficiently large the global sections $s_f, \omega_\ell^* s_g$ meet properly and with no components outside of the principal open subset of X. Therefore, for such ℓ, and under our assumptions, Proposition 2.5 is written as

$$\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{n-2}}(Z_{\mathbb{T}}(f, \omega_\ell^* g)) = \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{n-2},\overline{D}_g^{\mathrm{Ron}}}(Z_{\mathbb{T}}(f)) + \sum_{v\in\mathfrak{M}} \frac{n_v}{\#O(\omega_\ell)} \sum_{\eta\in O(\omega_\ell)_v} I_v(\eta), \tag{6.2}$$

where for each place v the function $I_v \colon \mathbb{T}(\mathbb{C}_v) \to \mathbb{R} \cup \{-\infty\}$ is defined by

$$I_v(t) = \int_{X_v^{\mathrm{an}}} \log \|t^* s_g\|_{\mathrm{Ron},v} \, \mathrm{c}_1(\overline{D}_{0,v}) \wedge \dots \wedge \mathrm{c}_1(\overline{D}_{n-2,v}) \wedge \delta_{Z_X(s_f)_v^{\mathrm{an}}}.$$

Consider now the subset $\mathfrak{S}$ of $\mathfrak{M}$ as in beginning of Section 5. Then, combining Theorem 2.2 and (6.2) gives

$$\begin{aligned}\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{n-2}}(Z_{\mathbb{T}}(f, \omega_\ell^* g)) = & \sum_{v\in\mathfrak{M}} n_v \, \mathrm{MI}_M\left(\vartheta_{0,v}, \dots, \vartheta_{n-2,v}, \rho_{f,v}^\vee, \rho_{g,v}^\vee\right) \\ & + \sum_{v\in\mathfrak{S}} \frac{n_v}{\#O(\omega_\ell)} \sum_{\eta\in O(\omega_\ell)_v} I_v(\eta) \\ & + \sum_{v\in\mathfrak{M}\setminus\mathfrak{S}} \frac{n_v}{\#O(\omega_\ell)} \sum_{\eta\in O(\omega_\ell)_v} I_v(\eta).\end{aligned}$$

Since $\mathfrak{S}$ is finite by Lemma 5.1, the second sum in the right-hand-side converges to 0 as $\ell \to \infty$ because of Theorem 4.4. The third sum also converges to 0 as $\ell \to \infty$ because of the adelic equidistribution result in Theorem 5.2.

Therefore, the expression is convergent, and the limit is the desired one. □

As an application, we can affirmatively answer [GS23, Conjecture 11.8].

Example 6.3. Let $d_1, d_2 \geq 1$ and consider the Fermat polynomials

$$f = 1 + x_1^{d_1} + x_2^{d_1} \quad \text{and} \quad g = 1 + x_1^{d_2} + x_2^{d_2},$$

seen as bivariate Laurent polynomials in $\mathbb{Q}[x_1^{\pm 1}, x_2^{\pm 1}]$. Considering the canonically metrized hyperplane divisor on $\mathbb{P}^2_{\mathbb{Q}}$, and applying Theorem B we deduce that the limit naive height of the intersection between the twists of f and g is computed by

$$\lim_{\ell \to \infty} \mathrm{h}(Z_{\mathbb{T}}(\boldsymbol{\omega}_{\ell,1}^* f, \boldsymbol{\omega}_{\ell,2}^* g)) = \sum_{v \in \mathfrak{M}} \mathrm{MI}_{\mathbb{Z}^2}(0_\Delta, \rho_{f,v}^\vee, \rho_{g,v}^\vee),$$

where 0_Δ denotes the indicator function of the standard simplex Δ of $\mathbb{R}^2$. In the same line of what was done in [GS23] for $d_1 = d_2 = 1$, it would be interesting to explore whether this limit can be expressed it in terms of special values of some relevant function.

6.B. A consequences for strict sequences of sets. We conclude the article by showing how Theorem B can be adapted to compute the asymptotic behaviour of average heights over certain growing sets of torsion points.

To explain this, recall from [GS23, Definition 6.4] that a sequence $(E_\ell)_\ell$ of nonempty finite subsets of $\mathbb{T}(\overline{\mathbb{K}})^k$ is said to be *strict* if for every proper algebraic subgroup $G \subset \mathbb{T}^k$ we have that

$$\lim_{\ell \to \infty} \frac{\#(E_\ell \cap G(\overline{\mathbb{K}}))}{\# E_\ell} = 0.$$

This following can be seen as a generalization of Theorem 6.2, from which it actually follows.

Proposition 6.4. *Let f, g be two nonzero Laurent polynomials in $\mathbb{K}[M]$, and denote by H the set of $t \in \mathbb{T}(\overline{\mathbb{K}})^2$ for which $t_1^* f, t_2^* g$ do not meet properly. Let also $(E_\ell)_\ell$ be a strict sequence of nonempty finite subsets of torsion points in $\mathbb{T}(\overline{\mathbb{K}})^2$. Then*

$$\lim_{\ell \to \infty} \frac{1}{\# E_\ell} \sum_{\omega \in E_\ell \setminus H} \mathrm{h}_{\overline{D}_0, \dots, \overline{D}_{n-2}}(Z_{\mathbb{T}}(\omega_1^* f, \omega_2^* g)) = \sum_{v \in \mathfrak{M}} n_v \, \mathrm{MI}_M(\vartheta_{\overline{D}_0, v}, \dots, \vartheta_{\overline{D}_{n-2}, v}, \rho_{f,v}^\vee, \rho_{g,v}^\vee).$$

Moreover, if we denote by L the previous limit, for all $\varepsilon > 0$ we have that

$$\lim_{\ell \to \infty} \frac{\#\{\omega \in E_\ell \setminus H : |\, \mathrm{h}_{\overline{D}_0, \dots, \overline{D}_{n-2}}(Z_{\mathbb{T}}(\omega_1^* f, \omega_2^* g)) - L| < \varepsilon\}}{\# E_\ell} = 1.$$

Proof. Take the set of torsion points $\mathbb{T}(\overline{\mathbb{K}})^{\mathrm{tors}} \subset \mathbb{T}(\overline{\mathbb{K}})$ and consider the function $\varphi \colon \mathbb{T}(\overline{\mathbb{K}})^{\mathrm{tors}} \to \mathbb{R}$ defined by

$$\varphi(\omega) = \begin{cases} \mathrm{h}_{\overline{D}_0, \dots, \overline{D}_{n-2}}(Z_{\mathbb{T}}(\omega_1^* f, \omega_2^* g)) & \text{if } \omega \notin H, \\ 0 & \text{otherwise.} \end{cases}$$

This function is bounded above because of [Gua18a, Proposition 6.4.1], see also [MS19]. By considering a finite number of families $s_{0,i}, \dots, s_{n-2,i}$ of global sections of $\mathscr{O}(D_0), \dots, \mathscr{O}(D_{n-2})$ respectively, such that for each $t \in \mathbb{T}(\overline{\mathbb{K}})^2$ there is i for which $s_{0,i}, \dots, s_{n-2,i}, t_1^* f, t_2^* g$ meet properly, we can also show that φ is bounded below.

Then, the statement follows readily from [GS23, Lemma 6.7] and Theorem 6.2. □

As the sequence $(\tau(\ell))_\ell$ of the full sets of ℓ-torsion points in $\mathbb{T}(\overline{\mathbb{K}})^2$ is strict [GS23, Example 6.5], we obtain the following particular case of [Gua18a, Conjecture 6.4.4].

Corollary 6.5. *For two nonzero Laurent polynomials f, g in $\mathbb{K}[M]$ we have that*

$$\lim_{\ell\to\infty} \frac{1}{\ell^2} \sum_{\omega\in\tau(\ell)\setminus H} \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{n-2}}(Z_{\mathbb{T}}(\omega_1^* f, \omega_2^* g)) \\ = \sum_{v\in\mathfrak{M}} n_v \, \mathrm{MI}_M(\vartheta_{\overline{D}_0,v},\dots,\vartheta_{\overline{D}_{n-2},v},\rho^\vee_{f,v},\rho^\vee_{g,v}),$$

where H denotes the set of $t\in\mathbb{T}(\overline{\mathbb{K}})^2$ such that t_1^*f, t_2^*g do not meet properly.

Appendix A. Proof of the reductions

This appendix is dedicated to the proof of Proposition 2.6 about the reduction steps that one can make when proving the general form of Conjecture A. They concern assumptions on different objects playing a role in the statement: the family of Laurent polynomials, the ambient toric variety, the sequence of torsion points and the collection of metrized divisors. We show each of the corresponding reductions in a dedicated subsection.

1.A. Proof of Item 1 in Proposition 2.6. We start by proving that the conjecture is verified in a trivial situation.

Proposition A.1. *Conjecture A holds true when one of the Laurent polynomial of the family $(f_1,\dots,f_k)$ is a monomial.*

Proof. Assume, without loss of generality, that $f_1 = \alpha\chi^m$ for some $\alpha\in\mathbb{K}$ and $m\in M$. We can show that under this hypothesis both sides in Conjecture A are equal to zero.

The vanishing of the left hand side is immediate from the fact that $Z_{\mathbb{T}}(\boldsymbol{t}^*\boldsymbol{f}) = 0$ for all admissible choice of $\boldsymbol{t}\in\mathbb{T}(\overline{\mathbb{K}})^k$. Concerning the right hand side, notice that in virtue of [Gua18b, Proposition 2.9 and Example 2.11] and [BPS14, Proposition 2.3.3] we have that

$$\rho^\vee_{f_1,v} = \iota_{\{m\}} + \log|\alpha|_v$$

for all $v\in\mathfrak{M}$, where $\iota_{\{m\}}$ is the function taking the value 0 on $\{m\}$, and $-\infty$ elsewhere. Applying [Gua18b, Corollary 1.10 and Proposition 1.3] and [Gua18a, Proposition 1.1.13], we obtain the equality

$$\mathrm{MI}_M(\vartheta_{0,v},\dots,\vartheta_{n-k,v},\rho^\vee_{f_1,v},\dots,\rho^\vee_{f_k,v}) \\ = \log|\alpha|_v\cdot \mathrm{MV}_M(\Delta_{D_0},\dots,\Delta_{D_{n-k}},\mathrm{NP}(f_2),\dots,\mathrm{NP}(f_k))$$

for all $v\in\mathfrak{M}$. Therefore, the right hand side of Conjecture A also vanishes, in virtue of the product formula on $\mathbb{K}$. □

Another useful observation is that the statement of the conjecture behaves well under multiplication of the involved Laurent polynomials.

Remark A.2. Let $f, g, f_2,\dots,f_k\in\mathbb{K}[M]$ be nonzero Laurent polynomials. Assume that Conjecture A holds for the family $(f, f_2,\dots,f_k)$ and for the family $(g, f_2,\dots,f_k)$. Then it also holds for $(f\cdot g, f_2,\dots,f_k)$.

Indeed, it is enough to show that for the family $\boldsymbol{h} = (f\cdot g, f_2,\dots,f_k)$ both sides of the equality in Conjecture A coincide with the sum of the corresponding side for

$\boldsymbol{f} = (f, f_2, \dots, f_k)$ and $\boldsymbol{g} = (g, f_2, \dots, f_k)$. First, the definition of the twist implies that whenever $\boldsymbol{t} = (t_1, \dots, t_k) \in \mathbb{T}(\overline{\mathbb{K}})^k$ is such that $t_1^*(fg), t_2^*f_2, \dots, t_k^*f_k$ meet properly, then also $t_1^*f, t_2^*f_2, \dots, t_k^*f_k$ meet properly and $t_1^*g, t_2^*f_2, \dots, t_k^*f_k$ meet properly, and moreover

$$Z_{\mathbb{T}}(\boldsymbol{t}^*\boldsymbol{h}) = Z_{\mathbb{T}}(\boldsymbol{t}^*\boldsymbol{f}) + Z_{\mathbb{T}}(\boldsymbol{t}^*\boldsymbol{g}).$$

The linearity of the height function yields then, by passage to the limit, the claimed additivity for the left hand side of Conjecture A. The one for the right hand side follows from [Gua18b, Proposition 2.9], [Roc70, Theorem 16.4] and the multilinearity of mixed integrals with respect to sup-convolution.

As a result of Proposition A.1 and Remark A.2, when proving Conjecture A (or special cases of it), one can safely replace any Laurent polynomial by its product with a nonzero monomial. In particular, up to multiplying by a monomial with suitable support and with coefficient of high enough valuations, we can assume that for each $i = 1, \dots, k$ the Laurent polynomial f_i lies in $O_{\mathbb{K}}[M]$ and that its support contains the lattice point $0 \in M$. This proves the first part of Item 1 in Proposition 2.6.

For the second part, we need the following observation about the choice of the base field over which the Laurent polynomials are defined.

Remark A.3. The validity of Conjecture A for the family $(f_1, \dots, f_k)$ over $\mathbb{K}$ is equivalent to its validity over any finite field extension $\mathbb{L}$ of $\mathbb{K}$, as both sides in the statement of the conjecture are not affected by the change of base field. Indeed, on the one hand, the height of a cycle does not depend on the choice of the number field over which it is defined. On the other hand, for each place w of $\mathbb{L}$ dividing a given place $v \in \mathfrak{M}$, the w-adic roof function of the extension of $\overline{D}$ to $\mathbb{L}$ agrees with its v-adic roof function, and $\rho_{f_i,w} = \rho_{f_i,v}$ for all $i \in \{1, \dots, k\}$. These facts, combined with the equality $\sum_{w|v} n_w = n_v$, yield the invariance of the right hand side.

Now, we can see each Laurent polynomial f_i as an element of the unique factorization domain $\overline{\mathbb{K}}[M]$, and take

$$f_i = \prod_{j=1}^{r_i} f_{ij}$$

to be its factorization into irreducible elements of $\overline{\mathbb{K}}[M]$. Denote by $\mathbb{L} \subset \overline{\mathbb{K}}$ the smallest field containing both $\mathbb{K}$ and the coefficients of the Laurent polynomials f_{ij} for all $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, r_i\}$. It is a finite field extension of $\mathbb{K}$. By using Remark A.3 and by repeatedly applying Remark A.2 over $\mathbb{L}$, we can reduce the proof of Conjecture A for the family $(f_1, \dots, f_k)$ to the proof of its validity for each of the family $(f_{1j_1}, \dots, f_{kj_k})$. Since each of these consists of absolutely irreducible Laurent polynomials, the second part of Item 1 in Proposition 2.6 follows.

1.B. Proof of Item 2 in Proposition 2.6. Let Σ denote the fan of the toric variety X. After subdividing its cones, we can refine Σ to a fan that is compatible with the Newton polytopes of the Laurent polynomials $f_1, \dots, f_k$. It is possible to further refine the obtained fan in $N_{\mathbb{R}}$ to the fan Σ' of a projective toric variety [CLS11, Theorem 6.1.18] whose cones are generated by part of a basis of the lattice N [Ful93, Section 2.6]. As a result, by construction and by [Ful93, Section 2.1] the toric variety X' associated to Σ' is a smooth projective compactification of $\mathbb{T}$ and its fan is compatible with the Newton polytopes of $f_1, \dots, f_k$.

We want to show that the validity of Conjecture A for X' implies the one for X.

To do so, consider the toric morphism $\phi\colon X' \to X$ constructed from the identity map on N as in [Ful93, Section 1.4]. It is a regular morphism whose restriction to the torus $\mathbb{T}$ agrees with the identity.

The pullbacks $\phi^*\overline{D}_0, \dots, \phi^*\overline{D}_{n-k}$ are semipositive toric metrized divisors on X', and thanks to [BPS14, Proposition 4.8.10] their v-adic roof functions are the same as the ones of $\overline{D}_0, \dots, \overline{D}_{n-k}$ at all place v of $\mathbb{K}$.

On the other hand, the arithmetic projection formula ensures that for all $\boldsymbol{t} = (t_1, \dots, t_k) \in \mathbb{T}(\overline{\mathbb{K}})^k$ for which $t_1^* f_1, \dots, t_k^* f_k$ meet properly

$$\mathrm{h}_{\phi^*\overline{D}_0,\dots,\phi^*\overline{D}_{n-k}}(Z_{\mathbb{T}}(\boldsymbol{t}^*\boldsymbol{f})) = \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{n-k}}(Z_{\mathbb{T}}(\boldsymbol{t}^*\boldsymbol{f})),$$

since $\phi_* \overline{Z_{\mathbb{T}}(\boldsymbol{t}^*\boldsymbol{f})}^{X'} = \overline{Z_{\mathbb{T}}(\boldsymbol{t}^*\boldsymbol{f})}^{X}$ because the cycle has no components outside of $\mathbb{T}$, and the restriction of ϕ to this torus is the identity map.

Item 2 in Proposition 2.6 follows then from the above considerations.

1.C. Proof of Item 3 in Proposition 2.6. The following observation on the well behaviour of Conjecture A under linear combination of semipositive toric metrized divisors comes in handy.

Remark A.4. Let $\overline{D}_0, \dots, \overline{D}_{n-k}, \overline{E}$ be semipositive toric metrized divisors, and let $i \in \{0, \dots, n-k\}$. If Conjecture A holds for the families $\overline{D}_0, \dots, \overline{D}_{n-k}$ and $\overline{D}_0, \dots, \overline{D}_{i-1}, \overline{E}, \overline{D}_{i+1}, \dots, \overline{D}_{n-k}$, then it also holds for the family

$$\overline{D}_0, \dots, \overline{D}_{i-1}, a\overline{D}_i + b\overline{E}, \overline{D}_{i+1}, \dots, \overline{D}_{n-k}$$

for all choice of $a, b \in \mathbb{Z}$ for which $a\overline{D}_i + b\overline{E}$ is semipositive. In fact, both sides of the equality in Conjecture A are multilinear in the choice of the metrized divisors.

To show this, thanks to the symmetry of the height and of the mixed integral operator, it is enough to show that linearity holds in the first argument; we can then assume $i = 0$. Whenever $\boldsymbol{t} = (t_1, \dots, t_k) \in \mathbb{T}(\overline{\mathbb{K}})^k$ is such that $t_1^* f_1, \dots, t_k^* f_k$ meet properly, we have

$$\mathrm{h}_{a\overline{D}_0+b\overline{E},\overline{D}_1,\dots,\overline{D}_{n-k}}(Z_{\mathbb{T}}(\boldsymbol{t}^*\boldsymbol{f})) = a\,\mathrm{h}_{\overline{D}_0,\overline{D}_1,\dots,\overline{D}_{n-k}}(Z_{\mathbb{T}}(\boldsymbol{t}^*\boldsymbol{f})) + b\,\mathrm{h}_{\overline{E},\overline{D}_1,\dots,\overline{D}_{n-k}}(Z_{\mathbb{T}}(\boldsymbol{t}^*\boldsymbol{f})),$$

which implies the linearity of the left hand side in Conjecture A. On the other hand, and assuming that $a, b \geq 0$, [BPS14, Propositions 4.3.14, 2.3.1 and 2.3.3] ensure that

$$\vartheta_{a\overline{D}_0+b\overline{E},v} = (\vartheta_{\overline{D}_0,v} \cdot a) \boxplus (\vartheta_{\overline{E},v} \cdot b)$$

for all place v of $\mathbb{K}$, with $\boxplus$ denoting the sup-convolution binary operation between concave functions, and $\cdot$ the right scalar multiplication. This equality, plugged in the right hand side of Conjecture A, yields its linearity because of the analogous properties of the mixed integral operator with respect to the sup-convolution of its arguments.

In the case in which $a \geq 0$ and $b < 0$, the same conclusion is similarly reached after noticing that

$$\vartheta_{\overline{D}_0,v} \cdot a = \vartheta_{a\overline{D}_0+b\overline{E}+|b|\overline{E},v} = \vartheta_{a\overline{D}_0+b\overline{E},v} \boxplus (\vartheta_{\overline{E},v} \cdot |b|).$$

Now, assume that Conjecture A is valid for families of semipositive toric metrized divisors with very ample underlying divisors, and let us show that we can prove it in general.

Consider an arbitrary family $\overline{D}_0, \dots, \overline{D}_{n-k}$ of semipositive toric metrized divisors. Because of Item 2 in Proposition 2.6, we can assume that X is projective, and therefore consider a very ample toric divisor A on X. Since every nef toric divisor is globally generated, and because of [Har77, Exercise 7.5(d) at page 169], we have that $D_i + A$ is also very ample for every $0 \leq i \leq n-k$. Moreover, by ampleness, we can equip A with a semipositive toric metric, and call $\overline{A}$ the corresponding semipositive toric metrized divisor.

By applying Remark A.4, the validity of Conjecture A for the family $\overline{D}_0, \dots, \overline{D}_{n-k}$ is implied by the one of the analogous statement for the two families

$$\overline{D}_0 + \overline{A}, \overline{D}_1, \dots, \overline{D}_{n-k} \quad \text{and} \quad \overline{A}, \overline{D}_1, \dots, \overline{D}_{n-k}.$$

Reasoning in the same manner on the subsequent entries of these new families, we can inductively reduce the validity of Conjecture A for $\overline{D}_0, \dots, \overline{D}_{n-k}$ to the one for families whose members are either $\overline{A}$ or of the form $\overline{D}_i + \overline{A}$, for $0 \leq i \leq n-k$. As these all have very ample underlying divisors, Item 3 in Proposition 2.6 is proved.

1.D. Proof of Item 4 in Proposition 2.6. We aim to show that the validity of Conjecture A is preserved under uniform limits of the metrics. We start by proving a lemma on the behaviour of mixed integrals with respect to approximations of concave functions, which can be of independent interest.

Lemma A.5. *Let μ be a Haar measure on a real vector space V of dimension n. For all $i \in \{0, \dots, n\}$, let $(g_{i,j})_j$ be a sequence of concave functions having a closed convex body Q_i of V as common domain, and which converges uniformly to a concave function g_i. Then*

$$\lim_{j \to \infty} \mathrm{MI}_\mu(g_{0,j}, \dots, g_{n,j}) = \mathrm{MI}_\mu(g_0, \dots, g_n).$$

Proof. By definition, the mixed integral $\mathrm{MI}_\mu(g_{0,j}, \dots, g_{n,j})$ is an alternating sum of integrals of the form

$$\int_{Q_{i_1} + \dots + Q_{i_k}} (g_{i_1,j} \boxplus \dots \boxplus g_{i_k,j}) \, d\mu,$$

with $\{i_1, \dots, i_k\} \subseteq \{0, \dots, n\}$ and $\boxplus$ denoting the sup-convolution between concave functions. One can show that this last operation preserves uniform limits, and therefore the sup-convolution of any subset of the $g_{i,j}$ converges uniformly to the sup-convolution of their respective limits. As a result, since the domain $Q_{i_1} + \dots + Q_{i_k}$ has finite μ-measure, each of the above integrals converges to

$$\int_{Q_{i_1} + \dots + Q_{i_k}} (g_{i_1} \boxplus \dots \boxplus g_{i_k}) \, d\mu,$$

which is enough to conclude. □

Assume now that Conjecture A holds for every choice of semipositive adelic metrized toric divisors whose metrics are smooth at Archimedean places and algebraic at non-Archimedean ones. We want to show that it holds true for an arbitrary family $\overline{D}_0, \dots, \overline{D}_{n-k}$ of semipositive toric metrized divisors. As usual, we refer to [BPS14] for the terminology employed in the following.

By definition, the v-adic metric of each of the $\overline{D}_0, \dots, \overline{D}_{n-k}$ is the canonical one for all v outside of a finite set $\mathfrak{P} \subset \mathfrak{M}$ containing the Archimedean places. Moreover, for each $i \in \{0, \dots, n-k\}$ there is a sequence of semipositive toric metrized divisors

$$\overline{D}_{i,j} = \big(D_i, (\| \cdot \|_{i,j,v})_{v \in \mathfrak{M}}\big)$$

such that the metric $\|\cdot\|_{i,j,v}$ is semipositive smooth if v Archimedean, it is associated to a semipositive algebraic model if v is non-Archimedean, it is the canonical one if $v \notin \mathfrak{P}$, and

$$\lim_{j\to\infty} \mathrm{dist}(\|\cdot\|_{i,j,v}, \|\cdot\|_{i,v}) = 0$$

for all v. Because of [BPS14, Proposition 4.3.14(3)] and the fact that uniform convergence of concave functions is preserved under Legendre–Fenchel duality, the sequence $(\vartheta_{\overline{D}_{i,j},v})_j$ of continuous concave functions on Δ_{D_i} converges uniformly to $\vartheta_{\overline{D}_i,v}$, for all $i \in \{0, \dots, n-k\}$.

Let now $(\boldsymbol{\omega}_\ell)_\ell$ be a quasi-strict sequence of torsion points in $\mathbb{T}(\overline{\mathbb{K}})^k$. In proving Conjecture A for this sequence, after approximating each $\overline{D}_i$ by $\overline{D}_{i,j}$ we are faced with a double limit, one over ℓ and the other over j. It will be crucial to be able to switch them, for which reason we need to prove that one of the convergence is uniform.

To do so, notice that for ℓ suitably large, the $(n-k)$-cycle $Z_{\mathbb{T}}(\boldsymbol{\omega}_\ell^* \boldsymbol{f})$ is well-defined thanks to the first part of Corollary 1.9, and we can choose rational sections s_i of $\mathcal{O}(D_i)$, possibly depending on ℓ, such that $s_0, \dots, s_{n-k}$ meet its closure $Z_X(\boldsymbol{\omega}_\ell^* s_{\boldsymbol{f}})$ in X properly. By the definition of the global height as sum of local contributions,

$$\begin{aligned}\Big|\mathrm{h}_{\overline{D}_{0,j},\dots,\overline{D}_{n-k,j}}(Z_{\mathbb{T}}(\boldsymbol{\omega}_\ell^* \boldsymbol{f})) - \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{n-k}}(Z_{\mathbb{T}}(\boldsymbol{\omega}_\ell^* \boldsymbol{f}))\Big| \\ \le \sum_{v\in\mathfrak{P}} n_v \Big| \mathrm{h}_{\overline{D}_{0,j,v},\dots,\overline{D}_{n-k,j,v}}(Z_X(\boldsymbol{\omega}_\ell^* s_{\boldsymbol{f}}); s_0, \dots, s_{n-k}) \\ - \mathrm{h}_{\overline{D}_{0,v},\dots,\overline{D}_{n-k,v}}(Z_X(\boldsymbol{\omega}_\ell^* s_{\boldsymbol{f}}); s_0, \dots, s_{n-k})\Big|.\end{aligned}$$

Each v-adic summand on the right hand side of the previous inequality can be upper bounded by the product between n_v and the sum

$$\begin{aligned}\sum_{r=0}^{n-k} \Big| \mathrm{h}_{\overline{D}_{0,v},\dots,\overline{D}_{r-1,v},\overline{D}_{r,j,v},\dots,\overline{D}_{n-k,j,v}}(Z_X(\boldsymbol{\omega}_\ell^* s_{\boldsymbol{f}}); s_0, \dots, s_{n-k}) \\ - \mathrm{h}_{\overline{D}_{0,v},\dots,\overline{D}_{r,v},\overline{D}_{r+1,j,v},\dots,\overline{D}_{n-k,j,v}}(Z_X(\boldsymbol{\omega}_\ell^* s_{\boldsymbol{f}}); s_0, \dots, s_{n-k})\Big|.\end{aligned}$$

Applying [BPS14, Theorem 1.4.17(4)], each of this new summands agrees with

$$\begin{aligned}\Big| \int_{X_v^{\mathrm{an}}} \log \frac{\|s_r\|_{r,j,v}}{\|s_r\|_{r,v}}\ \mathrm{c}_1(\overline{D}_{0,v}) \wedge \dots \wedge \mathrm{c}_1(\overline{D}_{r-1,v}) \\ \wedge \mathrm{c}_1(\overline{D}_{r+1,j,v}) \wedge \dots \wedge \mathrm{c}_1(\overline{D}_{n-k,j,v}) \wedge \delta_{Z_X(\boldsymbol{\omega}_\ell^* s_{\boldsymbol{f}})}\Big|.\end{aligned}$$

The notion of distance between metrics and the knowledge of the total volume of the involved Monge–Ampère measure, together with the second part of Corollary 1.9, allow to upper bound the previous expression by

$$\mathrm{dist}(\|\cdot\|_{r,j,v}, \|\cdot\|_{r,v}) \cdot \mathrm{MV}_M(\Delta_{D_0}, \dots, \Delta_{D_{r-1}}, \Delta_{D_{r+1}}, \dots, \Delta_{D_{n-k}}, \mathrm{NP}(f_1), \dots, \mathrm{NP}(f_k)).$$

Putting all these bounds together we finally obtain a constant $c \ge 0$ only depending on the polytopes $\Delta_{D_0}, \dots, \Delta_{D_{n-k}}$ and $\mathrm{NP}(f_1), \dots, \mathrm{NP}(f_k)$ for which

$$\Big|\mathrm{h}_{\overline{D}_{0,j},\dots,\overline{D}_{n-k,j}}(Z_{\mathbb{T}}(\boldsymbol{\omega}_\ell^* \boldsymbol{f})) - \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{n-k}}(Z_{\mathbb{T}}(\boldsymbol{\omega}_\ell^* \boldsymbol{f}))\Big| \le c \cdot \sum_{v\in\mathfrak{P}} n_v \sum_{r=0}^{n-k} \mathrm{dist}(\|\cdot\|_{r,j,v}, \|\cdot\|_{r,v}).$$

Therefore, the convergence

$$\lim_{j\to\infty} \mathrm{h}_{\overline{D}_{0,j},\dots,\overline{D}_{n-k,j}}(Z_{\mathbb{T}}(\boldsymbol{\omega}_\ell^* \boldsymbol{f})) = \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{n-k}}(Z_{\mathbb{T}}(\boldsymbol{\omega}_\ell^* \boldsymbol{f}))$$

is uniform in the choice of ℓ. Moreover, thanks to the assumptions, for all fixed j the limit

$$\lim_{\ell\to\infty} \mathrm{h}_{\overline{D}_{0,j},\dots,\overline{D}_{n-k,j}}(Z_{\mathbb{T}}(\boldsymbol{\omega}_\ell^* \boldsymbol{f}))$$

exists and agrees with the corresponding sum of mixed integrals. Therefore, the classical Moore–Osgood theorem allows to exchange limits, and we obtain

$$\begin{aligned}
\lim_{\ell\to\infty} \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{n-k}}(Z_{\mathbb{T}}(\boldsymbol{\omega}_\ell^* \boldsymbol{f})) &= \lim_{\ell\to\infty}\lim_{j\to\infty} \mathrm{h}_{\overline{D}_{0,j},\dots,\overline{D}_{n-k,j}}(Z_{\mathbb{T}}(\boldsymbol{\omega}_\ell^* \boldsymbol{f}))\\
&= \lim_{j\to\infty}\lim_{\ell\to\infty} \mathrm{h}_{\overline{D}_{0,j},\dots,\overline{D}_{n-k,j}}(Z_{\mathbb{T}}(\boldsymbol{\omega}_\ell^* \boldsymbol{f}))\\
&= \lim_{j\to\infty}\sum_{v\in\mathfrak{M}} n_v \,\mathrm{MI}_M(\vartheta_{\overline{D}_{0,j},v},\dots,\vartheta_{\overline{D}_{n-k,j},v},\rho_{f_1,v}^{\vee},\dots,\rho_{f_k,v}^{\vee})\\
&= \sum_{v\in\mathfrak{M}} n_v \,\mathrm{MI}_M(\vartheta_{\overline{D}_0,v},\dots,\vartheta_{\overline{D}_{n-k},v},\rho_{f_1,v}^{\vee},\dots,\rho_{f_k,v}^{\vee}),
\end{aligned}$$

where we have used Lemma A.5 in the last equality. This completes the proof of Item 4 in Proposition 2.6.

1.E. Proof of Item 5 in Proposition 2.6. First of all, notice that in the setting of Conjecture A the height of $Z_{\mathbb{T}}(\boldsymbol{\omega}_\ell^* \boldsymbol{f})$, whenever defined, only depends on the class of $\boldsymbol{\omega}_\ell \in \mathbb{T}(\overline{\mathbb{K}})^k$ in the quotient of $\mathbb{T}^k$ by the image of the torsion points of $\mathbb{T}$ under the diagonal embedding.

Indeed, let ω be a torsion point in $\mathbb{T}(\overline{\mathbb{K}})$, and consider $\boldsymbol{t} = (t_1,\dots,t_k) \in \mathbb{T}(\overline{\mathbb{K}})^k$ for which $t_1^* f_1,\dots,t_k^* f_k$ meet properly. The translation-by-ω map $\tau_\omega \colon X \to X$ restricts to an isomorphism of the torus $\mathbb{T}$. Hence, since by definition $(\tau_\omega\,|_{\mathbb{T}})^*(t_i^* f_i) = (\omega t_i)^* f_i$ for all $i \in \{1,\dots,k\}$ as regular functions on $\mathbb{T}$, we deduce that $(\omega t_1)^* f_1,\dots,(\omega t_k)^* f_k$ also meet properly in $\mathbb{T}$. Moreover, with the notation $\omega\boldsymbol{t} = (\omega t_1,\dots,\omega t_k)$, the projection formula implies that

$$(\tau_\omega)_*\overline{Z_{\mathbb{T}}((\omega\boldsymbol{t})^*\boldsymbol{f})} = \overline{(\tau_\omega\,|_{\mathbb{T}})_* Z_{\mathbb{T}}((\omega\boldsymbol{t})^*\boldsymbol{f})} = \overline{Z_{\mathbb{T}}(\boldsymbol{t}^*\boldsymbol{f})}.$$

The invariance of toric metrics under translation by torsion points and the projection formula for heights, see for instance [BPS14, Theorem 1.5.11(2)], yield the equality

$$\begin{aligned}
\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{n-k}}(Z_{\mathbb{T}}((\omega\boldsymbol{t})^*\boldsymbol{f})) &= \mathrm{h}_{\tau_\omega^*\overline{D}_0,\dots,\tau_\omega^*\overline{D}_{n-k}}(Z_{\mathbb{T}}((\omega\boldsymbol{t})^*\boldsymbol{f}))\\
&= \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{n-k}}((\tau_\omega)_*\overline{Z_{\mathbb{T}}((\omega\boldsymbol{t})^*\boldsymbol{f})}) = \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{n-k}}(Z_{\mathbb{T}}(\boldsymbol{t}^*\boldsymbol{f})),
\end{aligned}$$

proving the claimed invariance.

Now, consider a quasi-strict sequence $(\boldsymbol{\omega}_\ell)_\ell$. As seen, we can replace each $\boldsymbol{\omega}_\ell$ by

$$\omega_{\ell,1}^{-1}.\boldsymbol{\omega}_\ell = (1, \omega_{\ell,1}^{-1}\omega_{\ell,2},\dots,\omega_{\ell,1}^{-1}\omega_{\ell,k})$$

without affecting the value of the height of the corresponding intersection cycle. Because of the isomorphism in (1.6) and the quasi-strictness of $(\boldsymbol{\omega}_\ell)_\ell$, the sequence $((\omega_{\ell,1}^{-1}\omega_{\ell,2},\dots,\omega_{\ell,1}^{-1}\omega_{\ell,k}))_\ell$ is strict in $\mathbb{T}(\overline{\mathbb{K}})^{k-1}$. The validity of Conjecture A for $(\omega_{\ell,1}^{-1}\boldsymbol{\omega}_\ell)_\ell$ implies the one for $(\boldsymbol{\omega}_\ell)_\ell$, thus concluding the proof of Item 5 in Proposition 2.6.

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