Borderline variants of the Muckenhoupt-Wheeden inequality

Carlos Domingo-Salazar – Universitat de Barcelona

joint work with M. Lacey and G. Rey

UNIVERSITAT DE BARCELONA
Facultat de Matemàtiques

3CJI
Murcia, Spain
September 10, 2015
We consider the Hardy-Littlewood maximal operator defined by

\[ Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_{Q} |f(y)|dy. \]
It is easy to see that

\[ M : L^1(\mathbb{R}^n) \not\rightarrow L^1(\mathbb{R}^n), \]
It is easy to see that

\[ M : L^1(\mathbb{R}^n) \not\leftrightarrow L^1(\mathbb{R}^n), \]

but if we make the range a little bigger, then

\[ M : L^1(\mathbb{R}^n) \rightarrow L^{1,\infty}(\mathbb{R}^n). \]

✓✓
It is easy to see that

\[ M : L^1(\mathbb{R}^n) \not\to L^1(\mathbb{R}^n), \]

but if we make the range a little bigger, then

\[ M : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n). \quad \checkmark \checkmark \]

Now we add a weight \((dx \sim w(x)dx)\), and

\[ M : L^1(w) \to L^{1,\infty}(w) \iff w \in A_1. \]
It is easy to see that

\[ M : L^1(\mathbb{R}^n) \not\rightarrow L^1(\mathbb{R}^n), \]

but if we make the range a little bigger, then

\[ M : L^1(\mathbb{R}^n) \rightarrow L^{1,\infty}(\mathbb{R}^n). \]

✓✓

Now we add a weight \( (dx \rightsquigarrow w(x)dx) \), and

\[ M : L^1(w) \rightarrow L^{1,\infty}(w) \iff w \in A_1. \]

But... if I want to have ANY weight \( w \) on the right-hand side, then what?

For every weight \( w \), \( M : L^1(??) \rightarrow L^{1,\infty}(w) \)
It is easy to see that

\[ M : L^1(\mathbb{R}^n) \not\rightarrow L^1(\mathbb{R}^n), \]

but if we make the range a little bigger, then

\[ M : L^1(\mathbb{R}^n) \rightarrow L^{1,\infty}(\mathbb{R}^n). \quad \checkmark \checkmark \]

Now we add a weight \((dx \sim w(x)dx)\), and

\[ M : L^1(w) \rightarrow L^{1,\infty}(w) \quad \iff \quad w \in A_1. \]

But... if I want to have ANY weight \(w\) on the right-hand side, then what?

For every weight \(w\), \( M : L^1(??) \rightarrow L^{1,\infty}(w) \)

?? must be "larger" than \(w\)!
In 1971, C. Fefferman and E. M. Stein, proved the following inequality:

**Theorem**

For every weight $w$, it holds that

$$M : L^1(Mw) \rightarrow L^{1,\infty}(w).$$

Or equivalently,

$$\lambda w(Mf > \lambda) \leq C \int_{\mathbb{R}^n} |f(x)| Mw(x) dx, \quad \forall \lambda > 0.$$
Muckenhoupt-Wheeden conjecture

The conjecture of B. Muckenhoupt and R. Wheeden was that "the same held for every Calderón-Zygmund operator $T$":

\[ M : L^1(Mw) \rightarrow L^{1,\infty}(w). \quad \text{(FS inequality)} \]

\[ T : L^1(Mw) \rightarrow L^{1,\infty}(w). \quad \text{(MW conjecture)} \]
Muckenhoupt-Wheeden conjecture

This was shown to be **FALSE** by M. C. Reguera and C. Thiele in 2012

\[ H : L^1(Mw) \not\rightarrow L^{1,\infty}(w). \]
Muckenhoupt-Wheeden conjecture

This was shown to be FALSE by M. C. Reguera and C. Thiele in 2012

\[ H : L^1(Mw) \not\longrightarrow L^{1,\infty}(w). \]

**Question**

*How can we modify the maximal operator \( M \leadsto M\varphi \) in the weight so that*

\[ T : L^1(M\varphi w) \longrightarrow L^{1,\infty}(w), \]

*for every \( w \) and Calderón-Zygmund operator \( T \)?*

We need to make \( M \) "larger"...
Orlicz versions of the maximal operator

The Hardy-Littlewood maximal operator can be written as

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy = \sup_{x \in Q} \int_{\mathbb{R}^n} |f(y)| \frac{\chi_Q(y)dy}{|Q|}.$$
Orlicz versions of the maximal operator

The Hardy-Littlewood maximal operator can be written as

\[ Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \, dy = \sup_{x \in Q} \|f\|_{L^1(\chi_Q/|Q|)}. \]
Orlicz versions of the maximal operator

The Hardy-Littlewood maximal operator can be written as

\[ Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \, dy = \sup_{x \in Q} \|f\|_{L^1(\chi_Q/|Q|)}. \]

We define

\[ M_\varphi f(x) = \sup_{x \in Q} \|f\|_{\varphi(L)(\chi_Q/|Q|)}, \]

where \( \varphi \) is a Young function such as:

- \( \varphi(t) = t \quad \leadsto \quad L^1 \) norm and \( M_\varphi = M \),
- \( \varphi(t) = t \log t \quad \leadsto \quad L \log L \) norm,
- ...

Larger \( \varphi \) \( \Rightarrow \) Larger operator \( M_\varphi \) \( \Rightarrow \) Smaller space \( L^1(M_\varphi \omega) \) \( \Rightarrow \) More likely \( T: L^1(M_\varphi \omega) \to L^1,\infty(\omega) \).
Orlicz versions of the maximal operator

The Hardy-Littlewood maximal operator can be written as

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy = \sup_{x \in Q} \|f\|_{L^1(\chi_Q/|Q|)}.$$ 

We define

$$M\varphi f(x) = \sup_{x \in Q} \|f\|_{\varphi(L)(\chi_Q/|Q|)},$$

where $\varphi$ is a Young function such as:

- $\varphi(t) = t \quad \leadsto \quad L^1$ norm and $M\varphi = M$,
- $\varphi(t) = t \log t \quad \leadsto \quad L \log L$ norm,
- ...

Larger $\varphi \Rightarrow$
Orlicz versions of the maximal operator

The Hardy-Littlewood maximal operator can be written as

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy = \sup_{x \in Q} \|f\|_{L^1(\chi_Q/|Q|)}.$$ 

We define

$$M_\varphi f(x) = \sup_{x \in Q} \|f\|_{\varphi(L)(\chi_Q/|Q|)},$$

where \(\varphi\) is a Young function such as:

- \(\varphi(t) = t\) \(\leadsto\) \(L^1\) norm and \(M_\varphi = M\),
- \(\varphi(t) = t \log t\) \(\leadsto\) \(L \log L\) norm,
- ...

Larger \(\varphi\) \(\Rightarrow\) Larger operator \(M_\varphi\) \(\Rightarrow\)
Orlicz versions of the maximal operator

The Hardy-Littlewood maximal operator can be written as

\[ Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \, dy = \sup_{x \in Q} \|f\|_{L^1(\chi_Q/|Q|)}. \]

We define

\[ M_\varphi f(x) = \sup_{x \in Q} \|f\|_{\varphi(L)(\chi_Q/|Q|)}, \]

where \( \varphi \) is a Young function such as:

- \( \varphi(t) = t \Rightarrow L^1 \) norm and \( M_\varphi = M \),
- \( \varphi(t) = t \log t \Rightarrow L \log L \) norm,
- ...

Larger \( \varphi \) \( \Rightarrow \) Larger operator \( M_\varphi \) \( \Rightarrow \) Smaller space \( L^1(M_\varphi w) \).
Orlicz versions of the maximal operator

The Hardy-Littlewood maximal operator can be written as

\[ Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \, dy = \sup_{x \in Q} \|f\|_{L^1(\chi_Q/|Q|)}. \]

We define

\[ M_\varphi f(x) = \sup_{x \in Q} \|f\|_{\varphi(L)(\chi_Q/|Q|)}, \]

where \( \varphi \) is a Young function such as:

- \( \varphi(t) = t \quad \leadsto \quad L^1 \) norm and \( M_\varphi = M \),
- \( \varphi(t) = t \log t \quad \leadsto \quad L \log L \) norm,
- ...

Larger \( \varphi \Rightarrow \) Larger operator \( M_\varphi \Rightarrow \) Smaller space \( L^1(M_\varphi w) \Rightarrow \)
More likely \( T : L^1(M_\varphi w) \to L^{1,\infty}(w) \).
Current state of the problem

Question

What is the "least" Young function $\varphi$ such that

$$T : L^1(M\varphi w) \rightarrow L^{1,\infty}(w),$$

for every $w$ and Calderón-Zygmund operator $T$?
Current state of the problem

Question

*What is the "least" Young function $\varphi$ such that*

$$T : L^1(M \varphi w) \longrightarrow L^{1,\infty}(w),$$

*for every $w$ and Calderón-Zygmund operator $T$?*

$\varphi(t) = t$  \text{FALSE}

(Reguera 2011 / Reguera, Thiele 2012)
Question

What is the "least" Young function $\varphi$ such that

$$T : L^1(M_\varphi w) \longrightarrow L^{1,\infty}(w),$$

for every $w$ and Calderón-Zygmund operator $T$?

$$\varphi(t) = t(\log t)^\varepsilon \quad \text{TRUE, } \forall \varepsilon > 0$$

(Pérez 1994 / Hytönen, Pérez 2015, with $C = \frac{1}{\varepsilon}$)
Current state of the problem

Question

What is the "least" Young function $\varphi$ such that

$$T : L^1(M_\varphi w) \longrightarrow L^{1,\infty}(w),$$

for every $w$ and Calderón-Zygmund operator $T$?

$$\varphi(t) = t \log \log t \left(\log \log \log t\right)^{\alpha} \quad \text{TRUE, } \forall \alpha > 1$$

(D-S, Lacey, Rey 2015, and $C = \frac{1}{\alpha - 1}$)
Current state of the problem

**Question**

*What is the "least" Young function $\varphi$ such that* 

$$T : L^1(M\varphi w) \longrightarrow L^{1,\infty}(w),$$

*for every $w$ and Calderón-Zygmund operator $T$?*

$$\varphi(t) = o(t \log \log t) \quad \text{FALSE}$$

(Calderelli, Lerner, Ombrossi 2015)
Current state of the problem

\[ \varphi(t) = t \]

\[ t \log \log t \]

\[ t \log \log t (\log \log \log t)^\alpha \]

\[ t (\log t)^\epsilon \]
Current state of the problem

NEGATIVE RESULTS: With the Hilbert Transform.

POSITIVE RESULTS: With the reduction to sparse operators.
Suppose the Young function $\varphi$ satisfies
\[ c_\varphi = \sum_{k=1}^{\infty} \frac{1}{\psi^{-1}(2^{2^k})} < \infty. \]

Then, for all C-Z operator $T$, and any weight $w$, it holds that $T : L^1(M_\varphi w) \to L^{1,\infty}(w)$ with constant $c_\varphi$. That is,
\[ \sup_{\lambda > 0} \lambda w\{Tf > \lambda\} \lesssim c_\varphi \int_{\mathbb{R}^n} |f(x)| M_\varphi w(x) \, dx. \]
Theorem (D-S, Lacey, Rey 2015)

Suppose the Young function $\varphi$ satisfies

$$c_\varphi = \sum_{k=1}^{\infty} \frac{1}{\psi^{-1}(2^{2k})} < \infty.$$ 

Then, for all C-Z operator $T$, and any weight $w$, it holds that

$T : L^1(M_\varphi w) \longrightarrow L^{1,\infty}(w)$ with constant $c_\varphi$. That is,

$$\sup_{\lambda>0} \lambda w \{Tf > \lambda\} \lesssim c_\varphi \int_{\mathbb{R}^n} |f(x)| M_\varphi w(x) \, dx.$$ 

The function $\psi$ is called the complementary function of $\varphi$, and whenever

$$\varphi(t) = tL(t),$$

with $L$ a logarithmic part ($\log t$, $\log \log t$, $\log \log t(\log \log \log t)^\alpha...$), then essentially

$$\psi^{-1}(t) \approx L(t).$$
Hence, for instance, when

\[ \varphi(t) = tL(t) = t \log \log t (\log \log \log t)\alpha, \]

we have

\[
c_\varphi = \sum_{k=1}^{\infty} \frac{1}{\psi^{-1}(2^{2k})} \approx \sum_{k=1}^{\infty} \frac{1}{\log \log (2^{2k}) (\log \log \log (2^{2k}))^\alpha}
\]

\[
\approx \sum_{k=1}^{\infty} \frac{1}{k(\log k)\alpha} \lesssim \frac{1}{\alpha - 1}.
\]
Hence, for instance, when

\[ \varphi(t) = tL(t) = t\log \log t(\log \log \log t)^\alpha, \]

we have

\[
\begin{align*}
c_\varphi &= \sum_{k=1}^{\infty} \frac{1}{\psi^{-1}(2^{2^k})} \\ &\approx \sum_{k=1}^{\infty} \frac{1}{\log \log (2^{2^k})(\log \log \log (2^{2^k}))^\alpha} \\ &\approx \sum_{k=1}^{\infty} \frac{1}{k(\log k)^\alpha} \lesssim \frac{1}{\alpha - 1}.
\end{align*}
\]

Therefore, the theorem states that

\[
\lambda w\{Tf > \lambda\} \lesssim \frac{1}{\alpha - 1} \int_{\mathbb{R}^n} |f(x)| M_\varphi w(x) \, dx.
\]
We also recover the sharp constant of Hytönen-Pérez’s result, with

$$\varphi(t) = t(\log t)^\epsilon$$

we have

$$c_\varphi \approx \sum_{k=1}^{\infty} \frac{1}{(\log 2^k)^\epsilon} \approx \sum_{k=1}^{\infty} \frac{1}{2^{k\epsilon}} \approx \frac{1}{\epsilon},$$

and hence

$$\lambda w\{Tf > \lambda\} \lesssim \frac{1}{\epsilon} \int_{\mathbb{R}^n} |f(x)| M_\varphi w(x) \, dx.$$
How to prove it

It is enough to show that, for every sparse operator $S$,

$$\lambda w\{Sf > \lambda\} \lesssim c_\varphi \int_{\mathbb{R}^n} |f(x)| M_\varphi w(x) \, dx,$$
How to prove it

It is enough to show that, for every \textit{sparse operator} $S$,

$$\lambda w\{Sf > \lambda\} \lesssim c\varphi \int_{\mathbb{R}^n} |f(x)| M\varphi w(x) \, dx,$$

where

$$Sf(x) = \sum_{Q \in S} \left( \frac{1}{|Q|} \int_Q |f| \right) \chi_Q(x),$$

and $S$ is a family of dyadic cubes such that, for every $Q \in S$,

$$\left| \bigcup_{Q' \in S : Q' \subsetneq Q} Q' \right| \leq \frac{|Q|}{8}.$
How to prove it

It is enough to show that, for every sparse operator $S$,

$$\lambda w\{Sf > \lambda\} \lesssim c_\varphi \int_{\mathbb{R}^n} |f(x)| M_\varphi w(x) \, dx,$$

where

$$Sf(x) = \sum_{Q \in S} \left( \frac{1}{|Q|} \int_Q |f| \right) \chi_Q(x),$$

and $S$ is a family of dyadic cubes such that, for every $Q \in S$,

$$\left| \bigcup_{Q' \in S : Q' \subsetneq Q} Q' \right| \leq \frac{|Q|}{8}.$$

In fact, by linearity, we reduce to (GOAL)

$$w\{1 < Sf \leq 2\} \lesssim c_\varphi \int_{\mathbb{R}^n} |f(x)| M_\varphi w(x) \, dx.$$
Our contribution

How to prove it

- We split \( S \) into

\[
S_k = \left\{ Q \in S : 2^{-k-1} < \frac{1}{|Q|} \int_Q |f| \leq 2^{-k} \right\}.
\]
How to prove it

- We split $S$ into

$$S_k = \left\{ Q \in S : 2^{-k-1} < \frac{1}{|Q|} \int_Q |f| \leq 2^{-k} \right\}.$$ 

- By Fefferman-Stein, we can assume that $S_k = \emptyset$ for $k < 2$, and, for every $k \geq 2$, there is a finite number of layers $S_k = S_{k,0} \cup \cdots \cup S_{k,2^k}$:

  $S_{k,0}$
  ____________  ______  ______  ______

  $S_{k,1}$
  ______  ___  ___  ___  ___  ___  ___
  :     :     :     :     :     :     :

  $S_{k,2^k}$
  ___  ___  ___  ___  ___  ___  ___  ___  ___  ___

  Figure: Layer decomposition of $S_k$. 
This gives a simpler description of the operator:

\[ S f(x) = \sum_{Q \in S} \left( \frac{1}{|Q|} \int_Q |f| \right) \chi_Q(x) \]
This gives a simpler description of the operator:

\[
Sf(x) = \sum_{Q \in S} \left( \frac{1}{|Q|} \int_{Q} |f| \right) \chi_{Q}(x)
\]

\[
= \sum_{k=2}^{\infty} \sum_{\nu=0}^{2^k} \sum_{Q \in S_{k,\nu}} \left( \frac{1}{|Q|} \int_{Q} |f| \right) \chi_{Q}(x)
\]
This gives a simpler description of the operator:

\[
Sf(x) = \sum_{Q \in S} \left( \frac{1}{|Q|} \int_Q |f| \right) \chi_Q(x)
\]

\[
= \sum_{k=2}^{\infty} \sum_{\nu=0}^{2^k} \sum_{Q \in S_{k,\nu}} \left( \frac{1}{|Q|} \int_Q |f| \right) \chi_Q(x)
\]

\[
\approx \sum_{k=2}^{\infty} \sum_{\nu=0}^{2^k} \sum_{Q \in S_{k,\nu}} 2^{-k} \chi_Q(x)
\]
This gives a simpler description of the operator:

\[
Sf(x) = \sum_{Q \in S} \left( \frac{1}{|Q|} \int_{Q} |f| \right) \chi_{Q}(x) \\
= \sum_{k=2}^{\infty} \sum_{\nu=0}^{2^{k}} \sum_{Q \in S_{k,\nu}} \left( \frac{1}{|Q|} \int_{Q} |f| \right) \chi_{Q}(x) \\
\approx \sum_{k=2}^{\infty} \sum_{\nu=0}^{2^{k}} \sum_{Q \in S_{k,\nu}} 2^{-k} \chi_{Q}(x) \\
= \sum_{k=2}^{\infty} 2^{-k} \sum_{\nu=0}^{2^{k}} \sum_{Q \in S_{k,\nu}} \chi_{Q}(x) \sim \text{overlapping}.
\]
The main lemma is the following:

If $S = \sum_{k \geq 2} S_k$, with

$$S_k f(x) = \sum_{Q \in S_k} \left( \frac{1}{|Q|} \int_Q |f| \right) \chi_Q(x).$$
The main lemma is the following:

If $S = \sum_{k \geq 2} S_k$, with

$$S_k f(x) = \sum_{Q \in S_k} \left( \frac{1}{|Q|} \int_Q |f| \right) \chi_Q(x).$$

**Lemma**

For each $k \geq 2$, if we denote $\mathcal{E} = \{1 < Sf \leq 2\}$,

$$\int_{\mathcal{E}} S_k f(x) w(x) dx \leq 2^{-k} w(\mathcal{E}) + \frac{C}{\psi^{-1}(2^{2k})} \int_{\mathbb{R}^n} |f(x)| M\varphi w(x) dx.$$

Recall our goal was

$$w\{1 < Sf \leq 2\} \lesssim c\varphi \int_{\mathbb{R}^n} |f(x)| M\varphi w(x) dx.$$
Muchas Gracias!