THE HILBERT TRANSFORM ALONG THE PARABOLA

Masters Final Project

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1. The classical Hilbert transform
   - Interpolation theory
   - The Calderón-Zygmund decomposition
   - The Kolmogorov-Riesz theorem
2. The Hilbert transform along the parabola
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   - Van der Corput's lemma and L^2-boundedness
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Contents

Definition

We define the Hilbert transform

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This definition is such that, for test functions $f \in \mathcal{S}(\mathbb{R})$ (which is a dense subspace), we have

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The Kolmogorov-Riesz theorem states that H can be extended to an operator such that

$$H : L^p(\mathbb{R}) \to L^p(\mathbb{R}), \quad 1 < p < \infty,$$

and

$$H : L^1(\mathbb{R}) \to L^{1,\infty}(\mathbb{R}).$$

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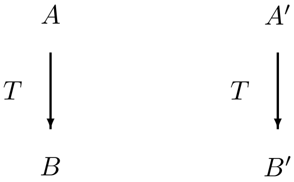
The idea behind interpolation theory



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$$\begin{array}{ccc} A & A'' & A' \\ T \Big\downarrow & T \Big\downarrow & T \Big\downarrow \\ B & B'' & B' \end{array}$$

Marcinkiewicz's interpolation theorem

The most important interpolation theorem is *Marcinkiewicz's interpolation theorem*, which essentially says that if T is a sublinear operator such that

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$$T : L^p \longrightarrow L^p$$

is also bounded for all $p_0 < p < p_1$.

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Good and bad parts

Given an integrable function $f \in L^1(\mathbb{R}^n)$, its Calderón-Zygmund decomposition at height $\alpha > 0$ is given by

$$f = g + b,$$

where g lies in all the L^p-spaces $(1 \leq p \leq \infty)$ and b can be written as

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In addition, the b_j's have integral zero and are supported on dyadic cubes Q_j which are pairwise disjoint and satisfy

$$\sum_j |Q_j| \leq \alpha^{-1} \|f\|_1.$$

Its payoff

Let us present the first consequence of the CZ decomposition:

We say that an operator $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is *well-localized* if

$$\int_{\mathbb{R}^n \setminus 2Q} |Tb(x)|dx \leq C \int_Q |b(x)|dx,$$

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For example, the classical Hilbert Transform is well-localized.

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– We use a duality argument to obtain boundedness for the rest of p's, $2 \leq p < \infty$.

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– We use interpolation between each couple of powers to conclude boundedness for $2 \leq p < \infty$.
– Again, by a duality argument we get boundedness for $1 < p \leq 2$.
– Finally, we prove that $H : L^1 \to L^{1,\infty}$ by showing that H is well-localized as before.

References about the classical Hilbert transform:

- J. Duoandikoetxea, *Fourier Analysis*, AMS (2000).
- L. Grafakos, *Classical Fourier Analysis*, Springer (2008).

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The Hilbert transform along curves

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If the function f is defined on $\mathbb{R}^2$, its natural generalization is

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However, we can consider a whole family of operators $\{H_\Gamma\}_\Gamma$ if we write

$$H_\Gamma f(x_1, x_2) = \lim_{\varepsilon \to 0} \int_{|t|>\varepsilon} f((x_1, x_2) - \Gamma(t)) \frac{dt}{t},$$

where $\Gamma(t)$ is a flat curve in the plane.

Motivation

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$$Lu = \frac{\partial u}{\partial x_2} - \frac{\partial^2 u}{\partial x_1^2}.$$

It is easily checked that Lu can be written as

$$Lu = T_1(Lu) - T_2(Lu),$$

where $\widehat{T_i f} = m_i \widehat{f}$ and the multipliers satisfy the *Homogeneity Condition*

$$\widehat{m_i}(\lambda x_1, \lambda^2 x_2) = \lambda^{-3} \widehat{m_i}(x_1, x_2), \quad \lambda > 0, \ i = 1, 2.$$

Motivation

After some computations, we observe that studying the solutions of boundary problems associated with parabolic operators such as L boils down to the study of operators like

$$Tf(x_1, x_2) = \int_0^{\pi} \Omega(\theta) H_\theta f(x_1, x_2)(1 + \sin^2(\theta)) d\theta,$$

where $\Omega(\theta) = K(\cos(\theta), \sin(\theta))$, K satisfies the previous *Homogeneity Condition* and

$$H_\theta f(x_1, x_2) = \lim_{\varepsilon \to 0} \int_{|r| > \varepsilon} f(x_1 - r\cos(\theta), x_2 - r^2 \operatorname{sgn}(r) \sin(\theta)) \frac{dr}{r}.$$

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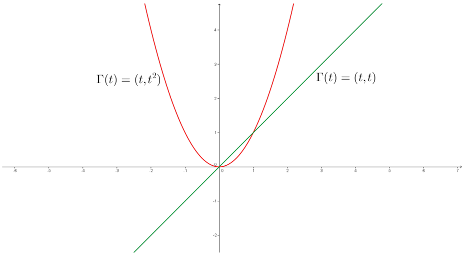
Notice that, for a fixed $\theta \in [0, \pi]$, H_θ is the Hilbert transform along the curve

$$\Gamma(t) = (t\cos(\theta), t^2 \operatorname{sgn}(t) \sin(\theta)).$$

Goal

Our goal is to study the boundedness of the Hilbert tranform along the parabola $\Gamma(t) = (t, t^2)$.



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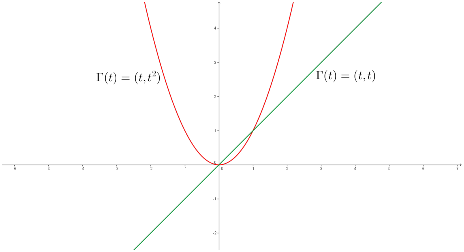

The problem is that H_Γ is not well-localized and it does not satisfy the property of

$$L^p - \text{boundedness} \Longrightarrow L^{2p} - \text{boundedness},$$

so the techniques that we used for the classical case are no longer useful.

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Van der Corput's lemma

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$$I(a,b) = \int_a^b e^{ih(t)}\,dt,$$

h is of class $\mathcal{C}^k$ and $|h^{(k)}(t)| \geq \lambda > 0$, then

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If $k = 1$, h is also required to be monotonic.

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The constants can be computed by $C_k = 3 \cdot 2^k - 2$.

L^2-boundedness

In order to show that H_Γ (which is initially defined on $\mathcal{S}(\mathbb{R}^2)$) can be extended to an operator

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One of these conditions is that

$$|\widehat{\tilde{K}_j}(\xi)| = \left| \int_{1\le|t|\le 2} e^{-2\pi i \xi\cdot(t,t^2)} \frac{dt}{t} \right| \le \frac{C}{|\xi|^\varepsilon},$$

so we can see why Van der Corput's lemma plays an essential role in the L^2-boundedness of H_Γ.

References about the L^2-boundedness of H_Γ:

A. Carbery, *An Introduction to the Oscillatory Integrals of Harmonic Analysis*, Personal communication.

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Difficulties

The main difference between the classical case and the one along the parabola is that now, the question of whether

$$H_\Gamma : L^1(\mathbb{R}^2) \to L^{1,\infty}(\mathbb{R}^2)$$

is bounded or not is an open problem. Therefore, we cannot use interpolation theory between L^1 and L^2 and we are forced to try a different approach. The main ingredient: *Littlewood-Paley theory*.

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Then, Plancherel's theorem yields

$$\|f\|_2 = \left\| \left(\sum_{j \in \mathbb{Z}} |S_j f|^2 \right)^{1/2} \right\|_2,$$

and Littlewood-Paley's theory says that, for all $1 < p < \infty$, these quantities are comparable:

$$c_p \|f\|_p \leq \left\| \left(\sum_{j \in \mathbb{Z}} |S_j f|^2 \right)^{1/2} \right\|_p \leq C_p \|f\|_p.$$

L^p-boundedness

We need to consider the maximal operator along the parabola as well:

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Now, we take sequences of measures $\{\mu_j\}_j$ and $\{\sigma_j\}_j$ in such a way that

$$H_\Gamma f = \sum_j \mu_j * f, \quad \text{and} \quad M_\Gamma f \leq 2 \sup_j \sigma_j * |f|.$$

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Finally, we prove a couple of results concerning sequences of measures and yielding boundedness for convolution operators as the ones above. With these theorems, we are able to obtain the sought-after boundedness estimate.

It is in the proofs of these results where we need to apply Littlewood-Paley theory.

References about the L^p-boundedness of H_Γ:

- M. Christ, *Hilbert transforms along curves I. Nilpotent groups*, Ann. of Math. (1985).
- J. Duoandikoetxea, *Fourier Analysis*, AMS (2000).
- J. Duoandikoetxea, J. L. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates*, Invent. Math. (1986).
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Extrapolation theory

Normally, when one has an operator which is bounded on L^p for $p > 1$ but the case $p = 1$ remains open, one tries to "get closer" to L^1 by means of extrapolation theory. The main result is Yano's theorem:

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We tried to use this approach, but our constant for $p > 1$ was not sharp enough near $p = 1$.

Results near $p = 1$

In 1987, M. Christ and E. M. Stein proved that

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This is used, together with Yano's extrapolation theorem, for the "bad part" of the decomposition. For the "good part", they only need the properties derived from the decomposition result.

Results near $p = 1$

In 2004, A. Seeger, T. Tao and J. Wright showed the best result near L^1 that is known so far, mainly that

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Here, they also use a new variant of the Calderón-Zygmund decomposition.

References about extrapolation and boundedness near L^1:

- M. J. Carro, *New extrapolation estimates*, J. Funct. Anal. (2000).
- M. J. Carro, *On the range space of Yano's extrapolation theorem and new extrapolation estimates at infinity*, Proceedings of the 6th International Conference on Harmonic Analysis and PDE's (2002).
- M. Christ, E. M. Stein, *A remark on singular Calderón-Zygmund theory*, Proc. Amer. Math. Soc. (1987).
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- S. Yano, *Notes on Fourier analysis XXIX: An extrapolation theorem*, J. Math. Soc. Japan (1951).

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A glimpse at the future

– It seems natural to think that if we improve Yano's theorem, we might achieve $L(\log\log\log L)$-estimates. With this motivation, one can try to work on this theory and later apply it to operators for which the case $p = 1$ is still open.

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– The study of the different variations of the Calderón-Zygmund decomposition seems also advisable, since the last two main results in this direction use this approach.

– Finally, the question of whether $H_\Gamma : L^1(\mathbb{R}^2) \to L^{1,\infty}(\mathbb{R}^2)$ or not would be another ambitious goal. An extrapolation argument would not work and one would have to find an original, new strategy.

=)

Thanks for your attention!