THE HILBERT TRANSFORM ALONG THE PARABOLA

Masters Final Project

Author: Carlos Domingo.
Supervisor: María Jesús Carro.
Tutor: F. Javier Soria.

Masters in Advanced and Professional Mathematics - Research Pathway.
Universitat de Barcelona - Year 2011/2012.

Contents

Preface

The classical Hilbert transform is given, at least for "nice" functions, by the following expression:

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|t|>\varepsilon} f(x-t) \frac{dt}{t}.$$

Its natural generalization[1] to functions defined on $\mathbb{R}^2$ is

$$Hf(x_1, x_2) = \lim_{\varepsilon \to 0} \int_{|t|>\varepsilon} f(x_1 - t, x_2 - t) \frac{dt}{t}, \tag{1}$$

but one can consider a whole family of operators $\{H_\Gamma\}_\Gamma$ by replacing the function

$$f((x_1, x_2) - (t, t)) \quad \text{by} \quad f((x_1, x_2) - \Gamma(t)),$$

where $\Gamma(t)$ is a flat curve in the plane. The resulting operator H_Γ is called the "Hilbert transform along Γ". The motivation for these generalizations comes from the field of partial differential equations. For instance, let L be the parabolic operator

$$Lu = \frac{\partial u}{\partial x_2} - \frac{\partial^2 u}{\partial x_1^2}.$$

It is easily checked that Lu can be written as

$$Lu = T_1(Lu) - T_2(Lu),$$

where

$$\widehat{T_1 f}(\xi_1, \xi_2) = \frac{2\pi i \xi_2}{2\pi i \xi_2 + 4\pi^2 \xi_1^2} \widehat{f}(\xi_1, \xi_2)$$
$$\widehat{T_2 f}(\xi_1, \xi_2) = \frac{4\pi^2 \xi_1^2}{2\pi i \xi_2 + 4\pi^2 \xi_1^2} \widehat{f}(\xi_1, \xi_2).$$

Notice that T_1 and T_2 are two operators with multipliers m_1 and m_2 satisfying

$$m_i(\lambda \xi_1, \lambda^2 \xi_2) = m_i(\xi_1, \xi_2), \quad \lambda > 0, \; i = 1, 2,$$

[1]The factor $\frac{1}{\pi}$ in the classical Hilbert transform is usually included so that H is an isometry on $L^2(\mathbb{R})$. However, it makes no difference in terms of boundedness and that's why we no longer include it when generalizing H.

and whose Fourier transforms $K_i = \widehat{m_i}$ have the following homogeneity property:

$$K_i(\lambda x_1, \lambda^2 x_2) = \lambda^{-3} K_i(x_1, x_2), \quad \lambda > 0,\ i = 1, 2. \tag{2}$$

In this setting, and after some computations involving the method of rotations, we observe that studying the solutions of boundary problems associated with parabolic operators such as L boils down to the study of operators like

$$Tf(x_1, x_2) = \int_0^\pi \Omega(\theta) H_\theta f(x_1, x_2)(1 + \sin^2(\theta)) d\theta,$$

where $\Omega(\theta) = K(\cos(\theta), \sin(\theta))$, K satisfies the condition in (2) and

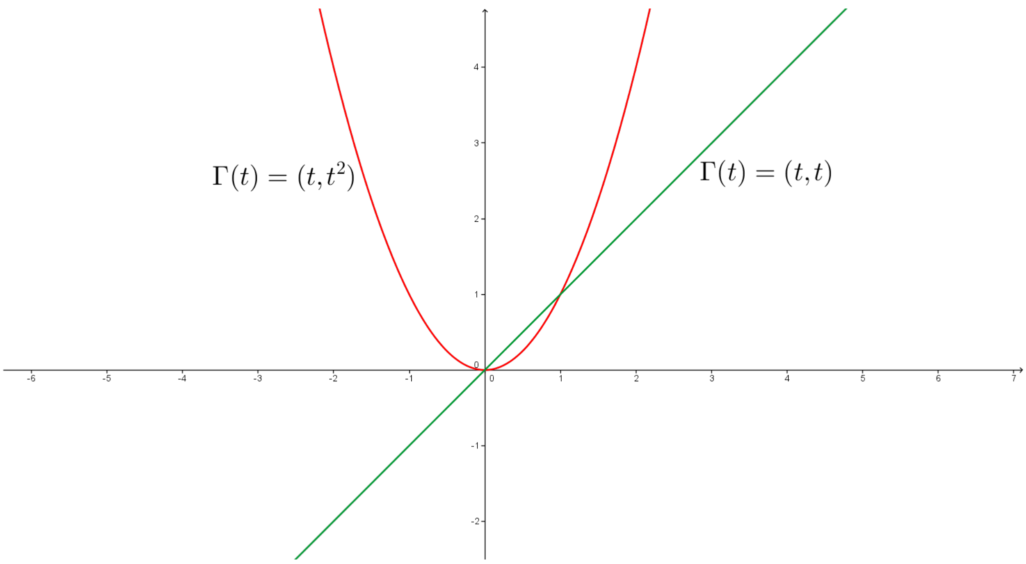
$$H_\theta f(x_1, x_2) = \lim_{\varepsilon \to 0} \int_{|r|>\varepsilon} f(x_1 - r\cos(\theta), x_2 - r^2 \operatorname{sgn}(r) \sin(\theta)) \frac{dr}{r}.$$

Notice that, for a fixed $\theta \in [0, \pi]$, H_θ is the Hilbert transform along the curve

$$\Gamma(t) = (t\cos(\theta), t^2 \operatorname{sgn}(t) \sin(\theta)).$$

Summing up, the Hilbert transform along flat curves is not just a pointless generalization of (1). In this case, when $\Gamma(t) = (t, t)$, pretty much everything that we know for the one-dimensional Hilbert transform can be extended to H_Γ. However, if we consider more general curves, things are not that easy. This is why the study of the Hilbert transform along flat curves has become a question of high interest in Harmonic Analysis. See, for instance, [6], [8], [11] and [12].

The aim of this project is to study the boundedness of the Hilbert transform along the most elementary curve: the parabola (t, t^2). However, it is not until the third chapter that this operator appears. The thorough study of several techniques in Harmonic Analysis is needed before one is ready to start examining the Hilbert transform along the parabola.



In the first chapter, we introduce notions and tools that will be subsequently used. For instance, we state the two main theorems in interpolation theory. If we have an operator that is bounded on two different spaces, this theory allows us to deduce its boundedness on an "intermediate" space, so it goes without saying how important this theory is when dealing with a boundedness problem like ours. Another tool that is discussed in this chapter is the Calderón-Zygmund decomposition. This decomposition is generally used to deduce that an operator is of weak-type $(1,1)$ from its L^2-boundedness. In order to so, one needs that the operator satisfies a "well-localization" property as well. This is the main difference between the classical Hilbert transform and the one along the parabola: the first one is well-localized, and hence it will be of weak-type $(1,1)$, whereas the latter fails to fulfill this condition, and in fact, the question of whether or not it is of weak-type $(1,1)$ remains open nowadays. Finally, we will explain the most basic result concerning the control of oscillatory integrals, Van der Corput's lemma, and we will make a short introduction to Littlewood-Paley theory, which in some sense, tries to find extensions of Plancherel's theorem for functions in L^p.

In the second chapter[2], we define the classical Hilbert transform and prove that it is of weak-type $(1,1)$ and strong-type (p,p) for $1 < p < \infty$. The case $p = 2$ is obvious from the definition. The shortest way to show the rest of the estimates is to start proving the weak-type $(1,1)$ by means of the Calderón-Zygmund decompostion and then use interpolation and duality to obtain the strong-type for $1 < p < \infty$. Nevertheless, we present a curious result which implies the strong-type $(2p, 2p)$ of the Hilbert transform assuming strong-type (p,p). Therefore, since we have the hypothesis for $p = 2$, we can use a recursive argument to obtain strong-type for all powers of 2. Then, using interpolation and duality we conclude that H is of strong-type (p,p) for $1 < p < \infty$. We prove the weak $(1,1)$ estimate as before, by the Calderón-Zygmund decomposition. Finally, since the Hilbert transform is initially defined on a dense set, we must make sure that its extensions to the L^p-spaces are pairwise consistent. In the last section of this chapter we give the proof of a result presented in Chapter 1 for which the classical Hilbert transform is needed.

In the last chapter, we introduce the Hilbert transform along the parabola. We start proving its L^2-boundedness, which is no longer direct. For this section, we follow some notes by A. Carbery [1], where he uses Van der Corput's lemma and Benedeck-Calderón-Panzone's theorem to obtain the result. A similar approach is given in J. Duoandikoetxea's book [9], where he tackles the problem of the L^p-boundedness of H_Γ. In this case, Littlewood-Paley theory is also required to obtain L^p-estimates for $p \neq 2$. Finally, we introduce Yano's extrapolation theorem and we see how it can be used in order to obtain boundedness results near $L^1(\mathbb{R}^2)$.

To conclude this preface, I would like to thank J. Soria, with whom I shared many enlightening meetings at the beginning of this journey of mine called "Analysis", and M. J. Carro, who gladly picked up the baton, for their great support and dedication.

Barcelona, June 2012
Carlos Domingo

[2]The definition of the classical Hilbert transform and its strong estimates for $1 < p < \infty$ are based on a final assignment carried out in the masters course "Functional Analysis", under the supervision of J. Cerdà.

Chapter 1

Preliminaries And Tools

1.1 Convolution With Tempered Distributions

First of all, let us recall some definitions and fix some notation:

Definition 1.1. *We define the Schwartz class by*

$$\mathcal{S}(\mathbb{R}) = \{\varphi \in \mathcal{C}^\infty(\mathbb{R}) : q_N(\varphi) < \infty \text{ for all } N \geq 0\},$$

where

$$q_N(\varphi) = \max_{\alpha \leq N} \|(1+x^2)^N \varphi^{(\alpha)}(x)\|_\infty .$$

The sequence $\{q_N\}_{N\geq 0}$ *is an increasing sequence of norms defining the topology of* $\mathcal{S}(\mathbb{R})$.

Definition 1.2. *We now define the space of tempered distributions by*

$$\mathcal{S}'(\mathbb{R}) = \{u : \mathcal{S}(\mathbb{R}) \to \mathbb{C} \text{ linear and continuous}\},$$

endowed with the weak-topology. Hence,*

$$u_n \to u \text{ in } \mathcal{S}'(\mathbb{R}) \iff u_n(\varphi) \to u(\varphi) \text{ in } \mathbb{C} \text{ for all } \varphi \in \mathcal{S}(\mathbb{R}).$$

If $u \in \mathcal{S}'(\mathbb{R})$ is a tempered distribution, we write its action on a Schwartz function $\varphi \in \mathcal{S}(\mathbb{R})$ by

$$u(\varphi) = \langle \varphi, u \rangle.$$

Definition 1.3. *If* $f \in L^1(\mathbb{R})$, *we define its Fourier transform by*

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx,$$

and if $u \in \mathcal{S}'(\mathbb{R})$,

$$\langle \varphi, \widehat{u} \rangle = \langle \widehat{\varphi}, u \rangle, \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}).$$

The two definitions are compatible, in the sense that if we think of an integrable function f as a tempered distribution in the following way,

$$\langle \varphi, f\rangle = \int_{\mathbb{R}} f(x)\varphi(x)dx, \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}),$$

then both definitions of $\widehat{f}$ coincide. It is also easy to show that

$$\mathcal{F} : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}) \ \text{ and } \ \mathcal{F} : \mathcal{S}'(\mathbb{R}) \longrightarrow \mathcal{S}'(\mathbb{R})$$

are linear bijections. Having settled this, let us now see how we can define a convolution in the setting of tempered distributions. Assume for a moment that $f \in L^1(\mathbb{R}) \subseteq \mathcal{S}'(\mathbb{R})$ and let $\varphi \in \mathcal{S}(\mathbb{R})$. Then,

$$(f * \varphi)(x) = \int_{\mathbb{R}} \varphi(x-y)f(y)dy = \langle \tau_x\tilde{\varphi}, f\rangle,$$

where we use the notation $(\tau_a\varphi)(x) = \varphi(x-a)$ and $\tilde{\varphi}(x) = \varphi(-x)$. Motivated by this, we give the following definition:

Definition 1.4. *Given $u \in \mathcal{S}'(\mathbb{R})$ and $\varphi \in \mathcal{S}(\mathbb{R})$, we define their convolution by*

$$(u * \varphi)(x) := \langle \tau_x\tilde{\varphi}, u\rangle.$$

Proposition 1.5. *Given $u \in \mathcal{S}'(\mathbb{R})$ and $\varphi \in \mathcal{S}(\mathbb{R})$, we have that:*

(i) The function $(u * \varphi) : \mathbb{R} \to \mathbb{C}$ is continuous on $\mathbb{R}$ and it defines a tempered distribution.

(ii) For every fixed $x \in \mathbb{R}$, $(u * \varphi)(x)$ is continuous with respect to u and φ.

(iii) For all $\psi \in \mathcal{S}(\mathbb{R})$, we have that $\langle \psi, u * \varphi\rangle = \langle \tilde{\varphi} * \psi, u\rangle$.

(iv) The correspondence

$$\begin{array}{ccc} \mathcal{S}'(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) & \longrightarrow & \mathcal{S}'(\mathbb{R}) \\ (u, \varphi) & \longmapsto & u * \varphi \end{array}$$

is continuous with respect to u and φ.

(v) It holds that $\widehat{u * \varphi} = \widehat{u} \cdot \widehat{\varphi}$.

Proof. (i) $u * \varphi \in \mathcal{C}(\mathbb{R})$ since it is the composition of two continuous maps:

$$u * \varphi : \mathbb{R} \xrightarrow{\tau.\tilde{\varphi}} \mathcal{S}(\mathbb{R}) \xrightarrow{u} \mathbb{C}.$$

Now, let us check that $u * \varphi \in \mathcal{S}'(\mathbb{R})$. Consider the family of norms in $\mathcal{S}(\mathbb{R})$

$$q_N(\varphi) := \max_{\alpha \le N} \|(1+x^2)^N \varphi^{(\alpha)}(x)\|_\infty, \quad \text{for all } N \ge 0, \ \varphi \in \mathcal{S}(\mathbb{R}),$$

which define the topology in $\mathcal{S}(\mathbb{R})$. If we fix $x\in\mathbb{R}$, we have that for all $N\geq 0$,

$$\begin{aligned} q_N(\tau_x\varphi) =& \max_{\alpha\leq N}\|(1+y^2)^N\tau_x\varphi^{(\alpha)}(y)\|_\infty \\ =& \max_{\alpha\leq N}\|(1+y^2)^N\varphi^{(\alpha)}(y-x)\|_\infty \\ \leq& 2^N(1+x^2)^N\max_{\alpha\leq N}\|(1+(y-x)^2)^N\varphi^{(\alpha)}(y-x)\|_\infty \\ =& 2^N(1+x^2)^N q_N(\varphi). \end{aligned}$$

In the last step, we use that $(1+y^2)\leq 2(1+x^2)(1+(y-x)^2)$ for all $x,y\in\mathbb{R}$. Indeed,

$$\begin{aligned} 2(1+x^2)(1+(y-x)^2) =& 2+2(y-x)^2+2x^2+2(xy-x^2)^2 \\ =& 1+y^2+[1+y^2-4yx+2x^2+2x^2+2(xy-x^2)^2] \\ =& 1+y^2+[1+(2x-y)^2+2(xy-x^2)^2] \\ \geq& 1+y^2. \end{aligned}$$

Now, since $u\in\mathcal{S}'(\mathbb{R})$, we have that there exist an integer $N\geq 0$ and a positive constant $C>0$ such that

$$|(u*\varphi)(x)| = |\langle\tau_x\tilde{\varphi},u\rangle| \leq Cq_N(\tau_x\tilde{\varphi}) \leq C(1+x^2)^N q_N(\tilde{\varphi}) = \bar{C}(1+x^2)^N,$$

using also the relationship between $q_N(\tau_x\varphi)$ and $q_N(\varphi)$ that we have found. Finally, given $\psi\in\mathcal{S}(\mathbb{R})$,

$$\begin{aligned} |\langle\psi,u*\varphi\rangle| \leq& \int_{\mathbb{R}}|\psi(x)||(u*\varphi)(x)|dx \\ \leq& \bar{C}\int_{\mathbb{R}}|\psi(x)|(1+x^2)^{N+1}(1+x^2)^{-1}dx \\ \leq& \bar{C}q_{N+1}(\psi)\int_{\mathbb{R}}(1+x^2)^{-1}dx = \tilde{C}q_{N+1}(\psi), \end{aligned}$$

and therefore, $u*\varphi\in\mathcal{S}'(\mathbb{R})$.

(ii) Fix $x\in\mathbb{R}$. Assume that $\varphi\in\mathcal{S}(\mathbb{R})$ and take a sequence $\{u_n\}_n\subseteq\mathcal{S}'(\mathbb{R})$ converging to $u\in\mathcal{S}'(\mathbb{R})$. Then, $\tau_x\tilde{\varphi}$ is another Schwartz function and by the weak*-topology of $S'(\mathbb{R})$ we have that

$$(u_n*\varphi)(x) = \langle\tau_x\tilde{\varphi},u_n\rangle \xrightarrow{n} \langle\tau_x\tilde{\varphi},u\rangle = (u*\varphi)(x).$$

Now, fix $u\in\mathcal{S}'(\mathbb{R})$ and take a sequence $\{\varphi_n\}_n\subseteq\mathcal{S}(\mathbb{R})$ converging to $\varphi\in\mathcal{S}(\mathbb{R})$. Therefore,

$$\tau_x\tilde{\varphi}_n \xrightarrow{n} \tau_x\tilde{\varphi} \quad \text{in } \mathcal{S}(\mathbb{R}),$$

and by the continuity of u,

$$(u*\varphi_n)(x) = \langle\tau_x\tilde{\varphi}_n,u\rangle \xrightarrow{n} \langle\tau_x\tilde{\varphi},u\rangle = (u*\varphi)(x).$$

(iii) Suppose for a moment that we have proved the identity for functions $\psi \in \mathcal{D}(\mathbb{R})$, that is, for $\mathcal{C}^\infty$-functions with compact support. We know that $\mathcal{D}(\mathbb{R})$ is a dense subspace of $\mathcal{S}(\mathbb{R})$. Now, take $\psi \in \mathcal{S}(\mathbb{R})$ and consider a sequence $\{\psi_n\}_n \subseteq \mathcal{D}(\mathbb{R})$ such that

$$\psi_n \xrightarrow{n} \psi \quad \text{in } \mathcal{S}(\mathbb{R}).$$

Then, by our assumption,

$$\langle \psi_n, u * \varphi \rangle = \langle \tilde{\varphi} * \psi_n, u \rangle, \quad \forall n \in \mathbb{N},$$

so taking limits when n tends to infinity and using that u and $u * \varphi$ are tempered distributions, we obtain the sought-after formula. Therefore, we can assume that $\psi \in \mathcal{D}(\mathbb{R})$. Now,

$$\begin{aligned}
\langle \psi, u * \varphi \rangle &= \int_{\mathbb{R}} \psi(x) u(\tau_x \tilde{\varphi}) dx \\
&= \int_{\mathbb{R}} u(\psi(x) \tau_x \tilde{\varphi}) dx \\
&= \lim_{h \to 0} \sum_{k \in \mathbb{Z}} h u(\psi(kh) \tau_{kh} \tilde{\varphi}) \\
&= u\left(\lim_{h \to 0} \sum_{k \in \mathbb{Z}} h \psi(kh) \tau_{kh} \tilde{\varphi} \right) \\
&= u\left(\int_{\mathbb{R}} \psi(x) \tau_x \tilde{\varphi} dx \right) = \langle \tilde{\varphi} * \psi, u \rangle,
\end{aligned}$$

using that the Riemann sums are finite (since ψ is compactly supported) and u is linear and continuous.

(iv) Fix $\varphi \in \mathcal{S}(\mathbb{R})$ and take a sequence $\{u_n\}_n \subseteq \mathcal{S}'(\mathbb{R})$ converging to $u \in \mathcal{S}'(\mathbb{R})$. We want to check that

$$u_n * \varphi \xrightarrow{n} u * \varphi \quad \text{in } \mathcal{S}'(\mathbb{R}).$$

Indeed, take $\psi \in \mathcal{S}(\mathbb{R})$ and by (iii),

$$\langle \psi, u_n * \varphi \rangle = \langle \tilde{\varphi} * \psi, u_n \rangle \xrightarrow{n} \langle \tilde{\varphi} * \psi, u \rangle = \langle \psi, u * \varphi \rangle.$$

Similarly, for a fixed $u \in \mathcal{S}'(\mathbb{R})$ and a sequence $\{\varphi_n\}_n \subseteq \mathcal{S}(\mathbb{R})$ converging to $\varphi \in \mathcal{S}(\mathbb{R})$, we have

$$\langle \psi, u * \varphi_n \rangle = \langle \tilde{\varphi}_n * \psi, u \rangle \xrightarrow{n} \langle \tilde{\varphi} * \psi, u \rangle = \langle \psi, u * \varphi \rangle,$$

so we are done.

(v) Let $\psi \in \mathcal{S}(\mathbb{R})$. Using (iii) and the fact that the Fourier transform is a linear bijection from $\mathcal{S}(\mathbb{R})$ to itself,

$$\begin{aligned}
\langle \psi, \widehat{u * \varphi} \rangle &= \langle \widehat{\psi}, u * \varphi \rangle = \langle \tilde{\varphi} * \widehat{\psi}, u \rangle = \langle \widehat{\check{\tilde{\varphi}} \psi}, u \rangle \\
&= \langle \check{\tilde{\varphi}} \psi, \widehat{u} \rangle = \langle \widehat{\varphi} \psi, \widehat{u} \rangle = \langle \psi, \widehat{\varphi} \widehat{u} \rangle.
\end{aligned}$$

Since this happens for every $\psi \in \mathcal{S}(\mathbb{R})$, we conclude the proof.

□

1.2 Weak L^p-Spaces

Definition 1.6. *Given a measurable function f on a measure space*[1] (X, μ), *we define the distribution function of f as the function d_f defined on $[0, \infty)$ by*

$$d_f(a) = \mu(\{x \in X : |f(x)| > a\}).$$

Definition 1.7. *For $0 < p < \infty$, the space weak-$L^p(X, \mu)$ is defined as the set of all measurable functions f such that*

$$\|f\|_{p,\infty} = \inf\left\{C > 0 : d_f(\alpha) \leq \frac{C^p}{\alpha^p} \text{ for all } \alpha > 0\right\} = \sup_{\gamma>0}\{\gamma d_f(\gamma)^{1/p}\}$$

is finite. The weak-$L^\infty(X, \mu)$ is by definition $L^\infty(X, \mu)$.

These spaces are denoted by $L^{p,\infty}(X, \mu)$ and the functions in $L^{p,\infty}(X, \mu)$ are considered equal if they coincide almost everywhere. It can be shown that

- $\|\lambda f\|_{p,\infty} = |\lambda|\|f\|_{p,\infty}$,
- $\|f + g\|_{p,\infty} \leq c_p(\|f\|_{p,\infty} + \|g\|_{p,\infty})$, with $c_p = \max\{2, 2^{1/p}\}$,
- $\|f\|_{p,\infty} = 0 \iff f = 0$

Therefore, $L^{p,\infty}$ is a quasinormed linear space for $0 < p < \infty$. These spaces are complete with respect to $\|\cdot\|_{p,\infty}$, so they are quasi-Banach spaces. Moreover, if $p > 1$, the $L^{p,\infty}$-spaces can be normed to become Banach spaces. Finally, we will prove an easy result that connects weak L^p-spaces to the classical ones:

Proposition 1.8. *For any $0 < p < \infty$ and any $f \in L^p(X, \mu)$, we have $\|f\|_{p,\infty} \leq \|f\|_p$. Hence, $L^p(X, \mu) \subseteq L^{p,\infty}(X, \mu)$.*

Proof. This is a trivial consequence of Chebyshev's inequality:

$$\alpha^p d_f(\alpha) \leq \int_{\{|f|>\alpha\}} |f(x)|^p d\mu(x) \leq \|f\|_p^p.$$

Taking the supremum over all $\alpha > 0$, we obtain $\|f\|_{p,\infty}^p \leq \|f\|_p^p$. □

The important fact about the $L^{p,\infty}$-spaces is that they are larger than the L^p-spaces. The inclusion, moreover, is strict. For instance, we have that $1/t \notin L^1(0, \infty)$ and

$$d_{1/t}(\gamma) = \left|\left\{t \in (0, \infty) : \frac{1}{t} > \gamma\right\}\right| = \frac{1}{\gamma} \quad \text{for all } \gamma > 0.$$

Therefore,

$$\left\|\frac{1}{t}\right\|_{1,\infty} = \sup_{\gamma>0}\{\gamma d_{1/t}(\gamma)\} = 1,$$

so the function $1/t$ belongs to $L^{1,\infty}(0, \infty)$. For more information concerning $L^{p,\infty}$-spaces, see [10].

[1] Throughout this project, all measure spaces will be assumed to be σ-finite.

1.3 Convergence In Measure

Definition 1.9. *Let f and f_n be measurable functions in a measure space (X, μ), for $n \geq 1$. It is said that $f_n \xrightarrow{n} f$ in measure if, for all $\varepsilon > 0$,*

$$\lim_{n\to\infty} \mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}) = 0.$$

Proposition 1.10. *Let $0 < p \leq \infty$ and f_n, f be in $L^{p,\infty}(X, \mu)$. If $f_n \xrightarrow{n} f$ in $L^{p,\infty}$, then f_n converges to f in measure.*

Proof. Take $\varepsilon > 0$, we have that

$$\mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\})^{1/p} = \frac{\varepsilon\mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\})^{1/p}}{\varepsilon} \leq \frac{\|f_n - f\|_{p,\infty}}{\varepsilon},$$

which tends to zero by hypothesis of $L^{p,\infty}$-convergence. □

Corollary 1.11. *If f_n, f are in L^p and $f_n \xrightarrow{n} f$ in L^p, we have convergence in measure.*

Proof. This is trivial since $f_n \xrightarrow{n} f$ in L^p implies $L^{p,\infty}$-convergence. □

Therefore, we conclude that convergence in measure is weaker than convergence in $L^{p,\infty}$. However, it still conserves the following good properties:

Proposition 1.12. *Let f_n and f be complex-valued measurable functions on a measure space (X, μ), $n \geq 1$, and assume that f_n converges to f and g in measure. Then, $f = g$ μ-almost everywhere.*

Proof. Fix $\varepsilon > 0$. We have, by hypothesis, that

$$\lim_{n\to\infty} \mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon/2\}) = 0 \quad \text{and} \quad \lim_{n\to\infty} \mu(\{x \in X : |f_n(x) - g(x)| > \varepsilon/2\}) = 0$$

Now,

$$\begin{aligned}
\mu(\{x : |f(x) - g(x)| > \varepsilon\}) &= \mu(\{x : |f(x) - f_n(x) + f_n(x) - g(x)| > \varepsilon\}) \\
&\leq \mu(\{x : |f(x) - f_n(x)| + |f_n(x) - g(x)| > \varepsilon\}) \\
&\leq \mu(\{x : |f(x) - f_n(x)| > \varepsilon/2\}) + \mu(\{x : |f_n(x) - g(x)| > \varepsilon/2\})
\end{aligned}$$

and taking limits when n tends to infinity, we deduce that

$$\mu(\{x \in X : |f(x) - g(x)| > \varepsilon\}) = 0.$$

Since this happens for every $\varepsilon > 0$, we conclude that

$$\mu(\{x \in X : f(x) \neq g(x)\}) = 0$$

and we complete the proof. □

Proposition 1.13. *If f_n converges to f in measure, then there exists a subsequence of $\{f_n\}_n$ which converges to f μ-almost everywhere.*

Proof. See [10, Th. 1.1.11] □

1.4 Interpolation Theorems

Theorem 1.14 (Riesz-Thorin). *Let (X,μ) and (Y,ν) be two measure spaces. Let T be a linear operator defined on the set of all simple functions on X and taking values in the set of all measurable functions on Y. Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and assume that*

$$\|Ts\|_{L^{q_0}(Y,\nu)} \leq M_0 \|s\|_{L^{p_0}(X,\mu)},$$
$$\|Ts\|_{L^{q_1}(Y,\nu)} \leq M_1 \|s\|_{L^{p_1}(X,\mu)},$$

for all simple functions s on X. Then, for every $0 < \theta < 1$, we have

$$\|Ts\|_{L^{q}(Y,\nu)} \leq M_0^{1-\theta} M_1^{\theta} \|s\|_{L^{p}(X,\mu)},$$

for all simple functions s on X, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \tag{1.1}$$

By density, T has a unique extension as a bounded operator from $L^p(X,\mu)$ to $L^q(Y,\nu)$ for all p and q as in (1.1).

Proof. The proof of this theorem can be found, for instance, in [10]. □

Definition 1.15. *Let T be an operator defined on the linear space of complex-valued, measurable functions on a measure space (X,μ) and taking values in the set of all complex-valued, finite almost everywhere measurable functions on a measure space (Y,ν). T is called sublinear if, for all f, g and all $\lambda \in \mathbb{C}$, we have*

$$|T(f+g)| \leq |Tf| + |Tg| \quad \text{and} \quad |T(\lambda f)| = |\lambda||Tf|.$$

Notice that if T is linear, in particular it is sublinear.

Theorem 1.16 (Marcinkiewicz). *Let (X,μ) and (Y,ν) be two measure spaces and let $0 < p_0 < p_1 \leq \infty$. Let T be a sublinear operator defined on the space $L^{p_0}(X) + L^{p_1}(X)$ and taking values in the space of measurable functions on Y. Assume that there exist two positive constants A_0 and A_1 such that*

$$\|Tf\|_{L^{p_0,\infty}(Y)} \leq A_0 \|f\|_{L^{p_0}(X)} \quad \text{for all } f \in L^{p_0}(X)$$
$$\|Tf\|_{L^{p_1,\infty}(Y)} \leq A_1 \|f\|_{L^{p_1}(X)} \quad \text{for all } f \in L^{p_1}(X).$$

Then, for all $p_0 < p < p_1$ and for all $f \in L^p(X)$, we have the estimate

$$\|Tf\|_{L^p(Y)} \leq A \|f\|_{L^p(X)},$$

where

$$A = 2\left(\frac{p}{p-p_0} + \frac{p}{p_1-p}\right)^{1/p} A_0^{\frac{(1/p)-(1/p_1)}{(1/p_0)-(1/p_1)}} A_1^{\frac{(1/p_0)-(1/p)}{(1/p_0)-(1/p_1)}}.$$

Proof. The proof of this theorem can also be found in [10]. □

1.5 A Basic Characterization Concerning Multipliers

Definition 1.17. *We denote by $\mathcal{M}^{2,2}(\mathbb{R}^n)$ the set of all bounded, linear operators from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ that commute with translations.*

It can be checked (see [10]) that every operator $T \in \mathcal{M}^{2,2}(\mathbb{R}^n)$ is given by convolution with a tempered distribution. The following theorem characterizes this space and allows us to check that a convolution operator from $L^2(\mathbb{R}^n)$ to itself is bounded just by studying the Fourier transform of the convolution kernel.

Theorem 1.18. *An operator T is in $\mathcal{M}^{2,2}(\mathbb{R}^n)$ if and only if it is given by convolution with some $u \in \mathcal{S}'(\mathbb{R}^n)$ whose Fourier transform $\widehat{u}$ is in $L^\infty(\mathbb{R}^n)$. In this case, the norm of T is equal to $\|\widehat{u}\|_\infty$.*

Proof. First assume that $\widehat{u} \in L^\infty(\mathbb{R}^n)$. Using Plancherel's theorem, for all $f \in L^2(\mathbb{R}^n)$,

$$\|Tf\|_2 = \|f * u\|_2 = \|\widehat{f} \cdot \widehat{u}\|_2 \leq \|\widehat{u}\|_\infty \|\widehat{f}\|_2 = \|\widehat{u}\|_\infty \|f\|_2,$$

so $\|T\| \leq \|\widehat{u}\|_\infty$ and $T \in \mathcal{M}^{2,2}(\mathbb{R}^n)$. Conversely, suppose that $T \in \mathcal{M}^{2,2}(\mathbb{R}^n)$ is given by convolution with $u \in \mathcal{S}'(\mathbb{R}^n)$. We will show that $\widehat{u}$ is a bounded function. Indeed, consider for every $R > 0$, a $\mathcal{C}^\infty$-function φ_R with compact support on the ball $B(0, 2R)$ and equal to 1 on $B(0, R)$. Since $\varphi_R \in \mathcal{S}(\mathbb{R}^n)$, we have that $\varphi_R^\vee \in \mathcal{S}(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n)$ and

$$\varphi_R \cdot \widehat{u} = \widehat{(\varphi_R^\vee * u)} = \widehat{T(\varphi_R^\vee)} \in L^2(\mathbb{R}^n).$$

Moreover, since the product of the function φ_R and the distribution $\widehat{u}$ coincides with $\widehat{u}$ on $B(0, R)$, we have that $\widehat{u} \in L^2(B(0, R))$ for all $R > 0$, and thus, $\widehat{u} \in L^2_{loc}(\mathbb{R}^n)$. Take now a function $f \in L^\infty(\mathbb{R}^n)$ with compact support. We have that the product function $f\widehat{u} \in L^2(\mathbb{R}^n)$, and Plancherel's theorem together with the boundedness[2] of T gives

$$\|f\widehat{u}\|_2^2 = \|f^\vee * u\|_2^2 = \|T(f^\vee)\|_2^2 \leq \|T\|^2 \|f\|_2^2.$$

Therefore, for every bounded function f with compact support,

$$\int_{\mathbb{R}^n} (\|T\|^2 - |\widehat{u}(x)|^2)|f(x)|^2 dx \geq 0.$$

For every $r > 0$ and $y \in \mathbb{R}^n$, define

$$f(x) = \frac{1}{|B(y, r)|^{1/2}} \chi_{B(y,r)}(x),$$

which is bounded and has compact support. Now, if we substitute this f in the previous estimate, we obtain

$$\frac{1}{|B(y, r)|} \int_{B(y,r)} (\|T\|^2 - |\widehat{u}(x)|^2) dx \geq 0,$$

[2]Notice that $f \in L^\infty_c(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n)$ and hence, $f^\vee \in L^2(\mathbb{R}^n)$.

and using the Lebesgue Differentiation Theorem together with the fact that $|\widehat{u}|^2$ is locally integrable, we deduce, taking limits when r tends to zero, that

$$\|T\|^2 - |\widehat{u}(y)|^2 \geq 0, \quad \text{a.e. } y \in \mathbb{R}^n.$$

Hence, $\widehat{u} \in L^\infty(\mathbb{R}^n)$ and

$$\|\widehat{u}\|_\infty \leq \|T\|.$$

Combining this fact with the estimate $\|T\| \leq \|\widehat{u}\|_\infty$, that holds whenever $\widehat{u} \in L^\infty(\mathbb{R}^n)$, we get the equality and finish the proof. □

1.6 Dyadic Cubes

Definition 1.19. *Define a dyadic cube in $\mathbb{R}^n$ as the set*

$$Q_{k,m_1,\dots,m_n} = [2^{-k}m_1, 2^{-k}(m_1+1)) \times \cdots \times [2^{-k}m_n, 2^{-k}(m_n+1)),$$

where $k, m_1, \dots, m_n \in \mathbb{Z}$.

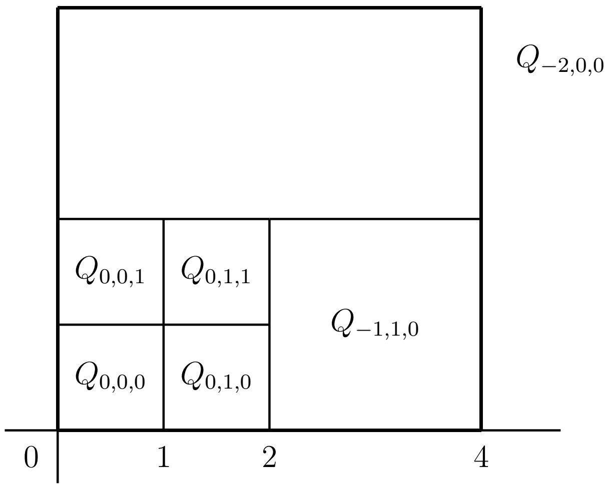
For a fixed $k \in \mathbb{Z}$, all the cubes $Q_{k,m_1,\dots,m_n}$ have side-length equal to 2^{-k} and their Lebesgue measure is 2^{-kn}. Moreover, their vertices are adjacent points of the lattice $(2^{-k}\mathbb{Z})^n$. Therefore, the set of all dyadic cubes of the same side-length form a covering of the whole space $\mathbb{R}^n$, and two different cubes in this set are always disjoint. We will write

$$\mathcal{D}_k = \{Q_{k,m_1,\dots,m_n} : m_1, \dots, m_n \in \mathbb{Z}\}, \text{ for every } k \in \mathbb{Z}.$$

We will also denote the set of all dyadic cubes by

$$\mathcal{D} = \bigcup_{k\in\mathbb{Z}} \mathcal{D}_k.$$

Given two dyadic cubes in $\mathcal{D}$, they are either disjoint or one is fully contained in the other. Moreover, a dyadic cube in $\mathcal{D}_k$ is contained in a unique cube of each family $\mathcal{D}_j$ for $j \leq k$, and contains exactly 2^n dyadic cubes of $\mathcal{D}_{k+1}$. The following picture represents some dyadic cubes in $\mathbb{R}^2$:



Notice that they are all either disjoint or related by inclusion, and that each dyadic cube in $\mathcal{D}_k$ contains exactly 4 dyadic cubes of $\mathcal{D}_{k+1}$.

1.7 The Calderón-Zygmund Decomposition

Theorem 1.20 (Calderón-Zygmund). *Let $f \in L^1(\mathbb{R}^n)$ and $\alpha > 0$. Then, there exist functions g and b on $\mathbb{R}^n$ such that*

(i) $f = g + b$.

(ii) $b = \sum_j b_j$, *where each* $b_j = \left(f - |Q_j|^{-1} \int_{Q_j} f\right)\chi_{Q_j}$ *is supported on a dyadic cube* Q_j. *Furthermore, the cubes* Q_k *and* Q_j *are disjoint when* $j \neq k$.

(iii) $\|g\|_1 \leq \|f\|_1$, $\|g\|_\infty \leq 2^n\alpha$ *and* $|g(x)| \leq \alpha$ *a.e. on* $\mathbb{R}^n \setminus \bigcup_j Q_j$.

(iv) $\int_{Q_j} b_j(x)dx = 0$.

(v) $\|b_j\|_1 \leq 2^{n+1}\alpha|Q_j|$.

(vi) $\sum_j |Q_j| \leq \alpha^{-1}\|f\|_1$.

Proof. Take the set of dyadic cubes $\mathcal{D}_k$ of side-length 2^{-k} such that

$$|Q| = 2^{-kn} \geq \frac{1}{\alpha}\|f\|_1, \quad \forall Q \in \mathcal{D}_k.$$

More precisely, take $k \in \mathbb{Z}$ with

$$k \leq \frac{\ln \alpha - \ln \|f\|_1}{n \ln 2}.$$

We will call the set $\mathcal{D}_k$, which decomposes the space $\mathbb{R}^n$ into a mesh of disjoint dyadic cubes of the same size, the *cubes of zero generation*. Now, if we subdivide each cube of zero generation into 2^n dyadic cubes, we obtain $\mathcal{D}_{k+1}$, which we call *of generation one*. We will select a cube Q of generation one if and only if

$$\frac{1}{|Q|}\int_Q |f(x)|dx > \alpha. \tag{1.2}$$

Let $S^{(1)} \subseteq \mathcal{D}_{k+1}$ be the set of all selected cubes of generation one. Now, subdivide each non-selected cube into 2^n subcubes, obtaining a subset of $\mathcal{D}_{k+2}$, which we will call *of generation two*. Select all cubes Q of generation two such that (1.2) holds, and call this set $S^{(2)}$. By repeating this procedure indefinitely, we obtain a countable collection of dyadic cubes

$$\{Q_j\}_{j\geq 0} := \bigcup_{m=1}^{\infty} S^{(m)}.$$

By construction, these cubes are disjoint. If we have not selected any cube, the sequence is empty and $b = 0$, $g = f$. Define for every $j \geq 0$,

$$b_j = \left(f - \frac{1}{|Q_j|}\int_{Q_j} f(y)dy\right)\chi_{Q_j},$$

and let

$$b = \sum_{j\geq 0} b_j, \quad g = f - b.$$

With this, we have (i) and (ii). Moreover,

$$\int_{Q_j} b_j(x)dx = \int_{Q_j} f(x)dx - \frac{1}{|Q_j|}\int_{Q_j} f(y)dy \cdot |Q_j| = 0,$$

so we also prove (iv). Now, for each cube Q_j, there exists a unique non-selected cube Q' of twice its side-length that contains Q_j. We will call this cube *the father of* Q_j. Since Q' was not selected, we know that

$$\frac{1}{|Q'|}\int_{Q'} |f(x)|dx \leq \alpha,$$

so, using that $Q_j \subseteq Q'$ with $|Q'| = 2^n|Q_j|$,

$$\frac{1}{|Q_j|}\int_{Q_j} |f(x)|dx \leq \frac{1}{|Q_j|}\int_{Q'} |f(x)|dx = \frac{2^n}{|Q'|}\int_{Q'} |f(x)|dx \leq 2^n\alpha. \tag{1.3}$$

Therefore, going back to the definition of b_j,

$$\int_{Q_j} |b_j(x)|dx = \int_{Q_j} \left| f(x) - \frac{1}{|Q_j|}\int_{Q_j} f(y)dy \right| dx \leq 2\int_{Q_j} |f(x)|dx \leq 2^{n+1}\alpha|Q_j|,$$

which proves (v). To prove (vi), recall that every cube Q_j satisfies (1.2) and they are disjoint, so

$$\sum_{j\geq 0} |Q_j| \leq \frac{1}{\alpha}\sum_{j\geq 0}\int_{Q_j} |f(x)|dx = \frac{1}{\alpha}\int_{\bigcup_j Q_j} |f(x)|dx \leq \frac{1}{\alpha}\|f\|_1.$$

Finally, we need to obtain the estimates in (iii) concerning the function g. We have that

$$\begin{aligned} g =& f - b = f - \sum_{j\geq 0} b_j = f - \sum_{j\geq 0} f\chi_{Q_j} + \sum_{j\geq 0} \frac{1}{|Q_j|}\int_{Q_j} f(y)dy\chi_{Q_j} \\ =& f \cdot \chi_{\mathbb{R}^n\setminus\bigcup_j Q_j} + \sum_{j\geq 0}\frac{1}{|Q_j|}\int_{Q_j} f(y)dy \cdot \chi_{Q_j}. \end{aligned}$$

Therefore, we deduce that

$$g(x) = \begin{cases} f(x) & \text{on } \mathbb{R}^n \setminus \bigcup_j Q_j, \\ \frac{1}{|Q_j|}\int_{Q_j} f(y)dy & \text{on } Q_j. \end{cases} \tag{1.4}$$

On Q_j, g is constant equal to $|Q_j|^{-1}\int_{Q_j} f$, and this is bounded by $2^n\alpha$ (see (1.3)). Now, take $x \in \mathbb{R}^n \setminus \bigcup_j Q_j$. For each $m \geq 0$, there exists a unique non-selected dyadic cube $Q_x^{(m)}$ of generation m that contains x. For each $m \geq 0$,

$$\left| \frac{1}{|Q_x^{(m)}|}\int_{Q_x^{(m)}} f(y)dy \right| \leq \frac{1}{|Q_x^{(m)}|}\int_{Q_x^{(m)}} |f(y)|dy \leq \alpha,$$

and the intersection of the closures of the cubes $Q_x^{(m)}$ is the singleton $\{x\}$,

$$\bigcap_{m=0}^{\infty} \overline{Q_x^{(m)}} = \{x\},$$

since their side-length is halved at each step and $Q_x^{(m)} \supseteq Q_x^{(m+1)}$ for every $m \geq 0$. Using Lebesgue's differentiation theorem, we deduce that, for almost every $x \in \mathbb{R}^n \setminus \bigcup_j Q_j$, we have

$$f(x) = \lim_{m\to\infty} \frac{1}{|Q_x^{(m)}|} \int_{Q_x^{(m)}} f(y)dy.$$

Since the modulus of these averages is at most α, we conclude that $|g(x)| = |f(x)| \leq \alpha$ a.e. on $\mathbb{R}^n \setminus \bigcup_j Q_j$. We had that $|g(x)| \leq 2^n\alpha$ on $\bigcup_j Q_j$, so we prove that $\|g\|_\infty \leq 2^n\alpha$. Going back to (1.4), we have that

$$\|g\|_1 \leq \int_{\mathbb{R}^n\setminus\bigcup_j Q_j} |f(x)|dx + \sum_{j\geq 0} \frac{1}{|Q_j|} \int_{Q_j} |f(y)|dy \cdot |Q_j| = \int_{\mathbb{R}^n} |f(x)|dx = \|f\|_1,$$

so we conclude the proof of (iii) and the theorem. □

This decomposition of f as the sum of g and b is called the *Calderón-Zygmund decomposition at height α*. The function g is called the *good function*, since it is both integrable and bounded. The function b stands for *bad function*. However, it follows from (i) and (ii) that the bad function is integrable with

$$\|b\|_1 \leq \|f\|_1 + \|g\|_1 \leq 2\|f\|_1,$$

and it is carefully chosen to have mean value zero. Indeed, since the support of b is the disjoint union of cubes $\bigcup_j Q_j$,

$$\int_{\mathbb{R}^n} b(x)dx = \int_{\bigcup_j Q_j} b(x)dx = \sum_j \int_{Q_j} b(x)dx = \sum_j \int_{Q_j} b_j(x)dx = 0.$$

Also, due to the fact that g is integrable and bounded, it lies in all the L^p-spaces for $1 \leq p \leq \infty$. More precisely, for $1 < p < \infty$,

$$\|g\|_p^p = \int_{\mathbb{R}^n} |g||g|^{p-1} \leq \|g\|_\infty^{p-1}\|g\|_1 \leq (2^n\alpha)^{p-1}\|f\|_1. \tag{1.5}$$

Definition 1.21. *We will say that an operator $T : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ is well-localized if, for every function $b \in L^2(\mathbb{R}^n)$ supported on a cube Q and such that $\int_{\mathbb{R}^n} b = 0$, we have*

$$\int_{\mathbb{R}^n\setminus 2Q} |Tb(x)|dx \leq C \int_Q |b(x)|dx.$$

Now we will present the first consequence of the Calderón-Zygmund decomposition:

Theorem 1.22. *If $T : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ is bounded and well-localized, then*

$$T : L^1(\mathbb{R}^n) \longrightarrow L^{1,\infty}(\mathbb{R}^n)$$

is bounded.

Proof. By density, it is enough to prove that for all $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and every $\lambda > 0$,

$$|\{x \in \mathbb{R}^n : |Tf(x)| > 2\lambda\}| \leq \frac{C}{\lambda}\|f\|_1.$$

If we consider the Calderón-Zygmund decomposition of f at height λ, we obtain a sequence of disjoint, dyadic cubes $\{Q_j\}_j$ and a couple of functions g and b such that $f = g + b$ and the conditions $(i) - (vi)$ of Theorem 1.20 are satisfied with constant $\alpha = \lambda$. Now, since[3] $Tf = Tg + Tb$, we have that

$$|\{x \in \mathbb{R}^n : |Tf(x)| > 2\lambda\}| \leq |\{x \in \mathbb{R}^n : |Tg(x)| > \lambda\}| + |\{x \in \mathbb{R}^n : |Tb(x)| > \lambda\}|.$$

Using Chebyshev's inequality, the L^2-boundedness of T and the estimate (1.5) for the L^p-norm of g, we get

$$|\{x \in \mathbb{R}^n : |Tg(x)| > \lambda\}| \leq \frac{1}{\lambda^2}\int_{\mathbb{R}^n} |Tg(x)|^2 dx \leq \frac{C}{\lambda^2}\|g\|_2^2 \leq \frac{C}{\lambda}\|f\|_1,$$

so we are done with the "good part" g. Let us see what happens with the bad one:

Let $2Q_j$ be the dilation of the cube Q_j, that is, the cube with the same center and twice its side-length. Write

$$\Omega := \bigcup_j Q_j \quad \text{and} \quad \Omega^* := \bigcup_j 2Q_j.$$

Clearly, $|\Omega^*| \leq C|\Omega|$ and

$$\begin{aligned} |\{x \in \mathbb{R}^n : |Tb(x)| > \lambda\}| &\leq |\Omega^*| + |\{x \notin \Omega^* : |Tb(x)| > \lambda\}| \\ &\leq \frac{C}{\lambda}\|f\|_1 + \frac{1}{\lambda}\int_{\mathbb{R}^n \setminus \Omega^*} |Tb(x)| dx, \end{aligned} \tag{1.6}$$

where we use Chebyshev's inequality and the fact that the measures $|Q_j|$ satisfy

$$|\Omega^*| \leq C|\Omega| \leq C\sum_j |Q_j| \leq \frac{C}{\lambda}\|f\|_1.$$

Now, by definition, we have pointwise convergence of the series $b(x) = \sum_j b_j(x)$. Moreover, since the cubes Q_j are disjoint,

$$\left|\sum_{j=1}^N b_j(x)\right| = |b(x)|\chi_{\bigcup_{j=1}^N Q_j}(x) \leq |b(x)| \in L^2(\mathbb{R}^n),$$

so we deduce that the series $\sum_j b_j$ converges to b in $L^2(\mathbb{R}^n)$. Using the continuity of T, we obtain that[4]

$$\sum_j Tb_j = Tb \quad \text{in } L^2(\mathbb{R}^n).$$

[3] We know that $g \in L^2(\mathbb{R}^n)$ and, since $f \in L^2(\mathbb{R}^n)$, we get $b = f - g \in L^2(\mathbb{R}^n)$. Therefore, Tg and Tb make perfect sense.

[4] Again, since $b_j = (f + C)\chi_{Q_j}$ and $f \in L^2(\mathbb{R}^n)$, we have that $b_j \in L^2(\mathbb{R}^n)$ and Tb_j is well defined.

As a consequence, there exists a subsequence which converges almost everywhere, say

$$Tb(x) = \lim_{k\to\infty} \sum_{j=1}^{N_k} Tb_j(x) \quad \text{a.e. } x,$$

and therefore,

$$|Tb(x)| = \left| \lim_{k\to\infty} \sum_{j=1}^{N_k} Tb_j(x) \right| \leq \lim_{k\to\infty} \sum_{j=1}^{N_k} |Tb_j(x)| = \sum_j |Tb_j(x)| \quad \text{a.e. } x.$$

Using this fact in (1.6), we deduce that

$$|\{x \in \mathbb{R}^n : |Tb(x)| > \lambda\}| \leq \frac{C}{\lambda}\|f\|_1 + \frac{1}{\lambda} \int_{\mathbb{R}^n \setminus \Omega^*} \sum_j |Tb_j(x)| dx.$$

We need to study this last integral, which by the monotone convergence theorem, satisfies

$$\begin{aligned}\int_{\mathbb{R}^n \setminus \Omega^*} \sum_j |Tb_j(x)| dx &= \sum_j \int_{\mathbb{R}^n \setminus \Omega^*} |Tb_j(x)| dx \leq \sum_j \int_{\mathbb{R}^n \setminus 2Q_j} |Tb_j(x)| dx \\ &\leq \sum_j C \int_{Q_j} |b_j(x)| dx \leq C \sum_j \|b_j\|_1 \leq C \sum_j 2^{n+1}\lambda |Q_j| \\ &\leq C\|f\|_1,\end{aligned}$$

where we use that T is well-localized together with some properties of the Calderón-Zygmund decomposition. Summing up, we have proved that

$$\begin{aligned}|\{x \in \mathbb{R}^n : |Tf(x)| > 2\lambda\}| &\leq |\{x \in \mathbb{R}^n : |Tg(x)| > \lambda\}| + |\{x \in \mathbb{R}^n : |Tb(x)| > \lambda\}| \\ &\leq \frac{C}{\lambda}\|f\|_1 + \frac{1}{\lambda} \int_{\mathbb{R}^n \setminus \Omega^*} \sum_j |Tb_j(x)| dx \leq \frac{C}{\lambda}\|f\|_1,\end{aligned}$$

so we complete the proof. □

Definition 1.23. *If K is a locally integrable function defined on $\mathbb{R}^n \setminus \{0\}$, we say that it satisfies the Hörmander condition if*

$$\int_{|x|>2|y|} |K(x-y) - K(x)| dx \leq C, \quad \textit{for all } y \in \mathbb{R}^n.$$

Proposition 1.24. *If $K \in \mathcal{C}^1(\mathbb{R}^n \setminus \{0\})$ and, for every $x \neq 0$,*

$$|\nabla K(x)| \leq \frac{C}{|x|^{n+1}},$$

then the Hörmander condition holds with constant $2^n C$. We call this stronger condition the gradient condition.

Proof. Let $y \in \mathbb{R}^n$. We can assume that $y \neq 0$, otherwise there is nothing to prove. Now, using the mean value theorem,

$$|K(x-y) - K(x)| \leq |\nabla K((x-y) + \lambda y)||y| \leq \frac{C|y|}{|x-y+\lambda y|^{n+1}},$$

for some $0 < \lambda < 1$ depending on x. Now, if we assume $|x| > 2|y|$, we have

$$|x-y+\lambda y| = |x-(1-\lambda)y| \geq |x| - (1-\lambda)|y| \geq |x| - \frac{1-\lambda}{2}|x| \geq \frac{|x|}{2},$$

and hence

$$\int_{|x|>2|y|} |K(x-y) - K(x)| dx \leq 2^{n+1} C|y| \int_{|x|>2|y|} \frac{1}{|x|^{n+1}} dx.$$

Using polar coordinates,

$$2^{n+1} C|y| \int_{|x|>2|y|} \frac{1}{|x|^{n+1}} dx = 2^{n+1} C|y| \int_{2|y|}^{\infty} \frac{r^{n-1}}{r^{n+1}} dr = 2^{n+1} C|y| \frac{1}{2|y|} = 2^n C,$$

so we conclude that

$$\int_{|x|>2|y|} |K(x-y) - K(x)| dx \leq 2^n C.$$

□

Finally, we will show yet another consequence of the Calderón-Zygmund decomposition:

Theorem 1.25. *Let K be a tempered distribution in $\mathbb{R}^n$ which coincides with a locally integrable function on $\mathbb{R}^n \setminus \{0\}$ and such that*

$$|\widehat{K}(\xi)| \leq A. \tag{1.7}$$

Assume that it also satisfies the Hörmander condition with constant $B > 0$. Then, for every $1 < p < \infty$,

$$\|K * f\|_p \leq C_p \|f\|_p, \quad f \in L^p(\mathbb{R}^n),$$

and

$$\|K * f\|_{1,\infty} \leq C\|f\|_1, \quad f \in L^1(\mathbb{R}^n).$$

Proof. We will show that these inequalities hold for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$, and by density we can extend them to an arbitrary $f \in L^p(\mathbb{R}^n)$. So let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and

$$T\varphi := K * \varphi.$$

Since by hypothesis $\widehat{K} \in L^\infty(\mathbb{R}^n)$, using Theorem 1.18, we deduce that

$$\|T\varphi\|_2 \leq A\|\varphi\|_2.$$

Assume for a moment that we have proved the weak $(1,1)$ inequality. Using Marcinkiewicz's interpolation theorem, we deduce the strong (p,p) inequality for $1 < p < 2$. Now, let us compute the adjoint operator T^*:

$$\begin{aligned}\int_{\mathbb{R}^n} T\varphi(x)\psi(x)dx &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(y)\varphi(x-y)dy\psi(x)dx \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(-y)\varphi(x+y)\psi(x)dydx \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(-y)\varphi(z)\psi(z-y)dydz \\ &= \int_{\mathbb{R}^n} \varphi(x)T^*\psi(x)dx,\end{aligned}$$

where

$$T^*\psi = K^* * \psi, \quad \text{with } K^*(x) = K(-x),$$

which satisfies the same conditions as T and thus, it is of strong type (p,p) for $1 < p < 2$. Therefore we can use duality to prove the strong (p,p) estimate for $p > 2$:

$$\begin{aligned}\|T\varphi\|_p &= \sup_{\psi\in\mathcal{S}(\mathbb{R}),\|\psi\|_{p'}\le 1} \left|\int_{\mathbb{R}^n} T\varphi(x)\psi(x)dx\right| \\ &= \sup_{\psi\in\mathcal{S}(\mathbb{R}),\|\psi\|_{p'}\le 1} \left|\int_{\mathbb{R}^n} \varphi(x)T^*\psi(x)dx\right| \\ &\le \sup_{\psi\in\mathcal{S}(\mathbb{R}),\|\psi\|_{p'}\le 1} \|\varphi\|_p\|T^*\psi\|_{p'} \le C_{p'}\|\varphi\|_p,\end{aligned}$$

using the fact that $1 < p' < 2$. Now, we have left to show that T is of weak-type $(1,1)$. By Theorem 1.22, since we already have the L^2-boundedness, we only have to check that T is well-localized. Take $b \in L^2(\mathbb{R})$ supported on a cube Q and with zero integral. We have to prove that

$$\int_{\mathbb{R}^n\setminus 2Q} |Tb(x)|dx \le C\int_Q |b(x)|dx.$$

Let $c \in \mathbb{R}^n$ be the center of Q and take $x \notin 2Q$. Using that b has zero integral,

$$Tb(x) = \int_Q K(x-y)b(y)dy = \int_Q (K(x-y)-K(x-c))b(y)dy,$$

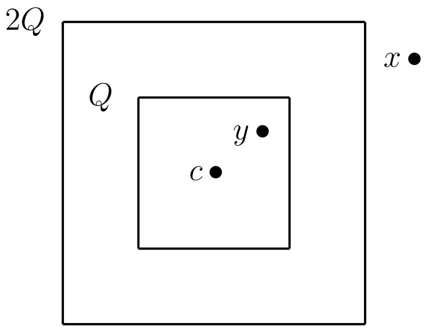
and hence, by Fubini,

$$\int_{\mathbb{R}^n\setminus 2Q} |Tb(x)|dx \le \int_Q |b(y)|\int_{\mathbb{R}^n\setminus 2Q} |K(x-y)-K(x-c)|dxdy \le B\int_Q |b(y)|dy,$$

since

$$\mathbb{R}^n \setminus 2Q \subseteq \{x\in\mathbb{R}^n : |x-c| > 2|y-c|\}$$

and we can use the Hörmander condition.



□

For later purposes, we will also include two vector-valued versions of this result. The proofs will be omitted, since they fall beyond the scope of this project.

Theorem 1.26. *Let*

$$\begin{array}{rcl}\vec{T}: L^2(\mathbb{R}) & \longrightarrow & L^2(\ell^2)\\ f & \longmapsto & \{T_j f\}_j = \{K_j * f\}_j\end{array}$$

be a bounded operator. Assume that the kernel $\{K_j\}_j$ *satisfies the gradient condition*

$$\|\{K_j'(x)\}\|_{\ell^2} \leq \frac{C}{|x|^2}, \quad x \in \mathbb{R}\setminus\{0\}.$$

Then, it holds that for $1 < p < \infty$*,*

$$\|\vec{T}f\|_{L^p(\ell^2)} = \left\| \left(\sum_{j\in\mathbb{Z}} |T_j f|^2 \right)^{1/2} \right\|_p \leq C_p \|f\|_p,$$

and for $p = 1$*,*

$$\left| \left\{ x \in \mathbb{R} : \left(\sum_{j\in\mathbb{Z}} |T_j f(x)|^2 \right)^{1/2} > \lambda \right\} \right| \leq \frac{C}{\lambda}\|f\|_1.$$

Theorem 1.27. *Let* T *be a convolution operator which is bounded from* $L^2(\mathbb{R}^n)$ *to itself and whose associated kernel* K *satisfies the Hörmander condition as in Theorem 1.25. Then, for* $1 < r < \infty$ *and* $1 < p < \infty$*,*

$$\left\| \left(\sum_{j\in\mathbb{Z}} |Tf_j|^r \right)^{1/r} \right\|_p \leq C_{p,r} \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^r \right)^{1/r} \right\|_p$$

and for $p = 1$*,*

$$\left| \left\{ x \in \mathbb{R}^n : \left(\sum_{j\in\mathbb{Z}} |Tf_j(x)|^r \right)^{1/r} > \lambda \right\} \right| \leq \frac{C_r}{\lambda} \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^r \right)^{1/r} \right\|_1.$$

Finally, we will also need the following corollary which can be deduced from this last theorem:

Corollary 1.28. *Let $\{I_j\}$ be a sequence of intervals on the real line and let $\{S_j\}_j$ be the sequence of operators defined by $\widehat{S_j f}(\xi) = \chi_{I_j}(\xi)\widehat{f}(\xi)$. Then, for $1 < r, p < \infty$,*

$$\left\| \left(\sum_{j\in\mathbb{Z}} |S_j f_j|^r \right)^{1/r} \right\|_p \leq C_{p,r} \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^r \right)^{1/r} \right\|_p .$$

Proof. In order to show this result, we need to combine Theorem 1.27 with the classical Hilbert transform. For this reason, this proof will be put off until the end of Chapter 2 (see Section 2.7). □

1.8 The Van der Corput Lemma

An *oscillatory integral* is an expression of the form

$$\int e^{ih(x)}\psi(x)dx.$$

Van der Corput's lemma is the most basic estimate for oscillatory integrals and deals with the case when $\psi \equiv 1$ and $x \in \mathbb{R}$. The result is the following:

Lemma 1.29 (Van der Corput)**.** *Let*

$$I(a,b) = \int_a^b e^{ih(t)}dt,$$

where h is a real-valued function.

(i) If $h \in \mathcal{C}^1[a,b]$, h' is monotonic and $|h'(t)| \geq \lambda > 0$, then

$$|I(a,b)| \leq \frac{C_1}{\lambda}.$$

(ii) If $h \in \mathcal{C}^k[a,b]$ with $k \geq 2$ and $|h^{(k)}(t)| \geq \lambda > 0$, then

$$|I(a,b)| \leq \frac{C_k}{\lambda^{1/k}}.$$

The constant $C_k = 3 \cdot 2^k - 2$ only depends on k.

Proof. (i) Integrating by parts,

$$I(a,b) = \int_a^b ih'(t)e^{ih(t)}\frac{dt}{ih'(t)} = \frac{e^{ih(b)}}{ih'(b)} - \frac{e^{ih(a)}}{ih'(a)} + i\int_a^b e^{ih(t)}d\left(\frac{1}{h'(t)}\right),$$

where the last (Riemann-Stieltjes) integral makes sense since the integrator $1/h'(t)$ is monotonic. Using precisely this fact, we know that $d(1/h'(t))$ has constant sign on $[a,b]$

and therefore,

$$\begin{aligned}|I(a,b)| \leq & \frac{1}{|h'(b)|} + \frac{1}{|h'(a)|} + \int_a^b \left| d\left(\frac{1}{h'(t)}\right)\right| \\ \leq & \frac{2}{\lambda} + \int_a^b \left| d\left(\frac{1}{h'(t)}\right)\right| \\ = & \frac{2}{\lambda} + \left| \int_a^b d\left(\frac{1}{h'(t)}\right)\right| \\ = & \frac{2}{\lambda} + \left| \frac{1}{h'(b)} - \frac{1}{h'(a)}\right| \leq \frac{4}{\lambda},\end{aligned}$$

so we get the desired estimate with $C_1 = 4$.

(ii) We will proceed by induction over $k \geq 1$. The case $k = 1$ is (i). Now, assume that our estimate holds for k, that is,

$$\left\{|h^{(k)}(t)| \geq \lambda > 0 \ \text{ (and } h^{(k)} \text{ monotonic if } k = 1)\right\} \Longrightarrow |I(a,b)| \leq \frac{C_k}{\lambda^{1/k}}.$$

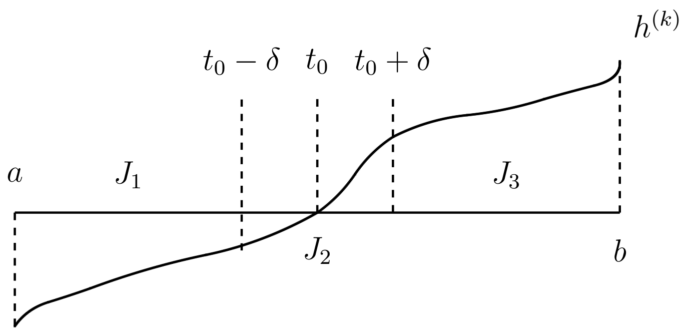
Our hypothesis is that $|h^{(k+1)}(t)| \geq \lambda > 0$. This means that $h^{(k+1)}$ is either strictly positive or negative on $[a,b]$, so we can assume, replacing h by $-h$ if necessary, that

$$h^{(k+1)}(t) \geq \lambda > 0, \quad t \in [a,b].$$

Hence, $h^{(k)}$ is increasing and it vanishes at, at most, one point $t_0 \in [a,b]$. If such t_0 exists, take $\delta > 0$ (to be defined later) and let

$$\begin{aligned} J_1 &= \{t \in (a,b) : t < t_0 - \delta\}, \\ J_2 &= (t_0 - \delta, t_0 + \delta) \cap (a,b), \\ J_3 &= \{t \in (a,b) : t > t_0 + \delta\}, \end{aligned}$$

which are allowed to be empty.



On J_1 and J_3, using the mean value theorem, we obtain that

$$|h^{(k)}(t)| = |h^{(k)}(t) - h^{(k)}(t_0)| = |h^{(k+1)}(\xi)| \cdot |t - t_0| \geq \lambda\delta,$$

where ξ is an intermediate point between t and t_0. Therefore, by our induction hypothesis, we deduce that

$$\left|\int_{J_1\cup J_3} e^{ih(t)}dt\right| \le \left|\int_{J_1} e^{ih(t)}dt\right| + \left|\int_{J_3} e^{ih(t)}dt\right| \le 2\frac{C_k}{(\lambda\delta)^{1/k}}.$$

Notice that the fact that $h^{(k)}$ is monotonic is essential when $k = 1$. On the other hand, for the set J_2 we have that δ controls the value of the integral:

$$\left|\int_{J_2} e^{ih(t)}dt\right| \le \int_{t_0-\delta}^{t_0+\delta} dt = 2\delta.$$

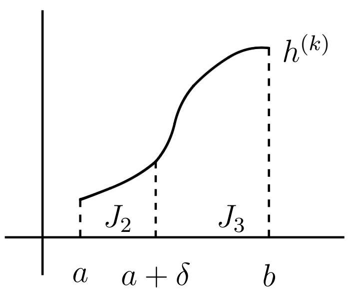
So fixing $\delta := \lambda^{-1/(k+1)}$, we get that

$$|I(a,b)| \le 2\frac{C_k}{(\lambda\delta)^{1/k}} + 2\delta = \frac{2C_k+2}{\lambda^{1/(k+1)}},$$

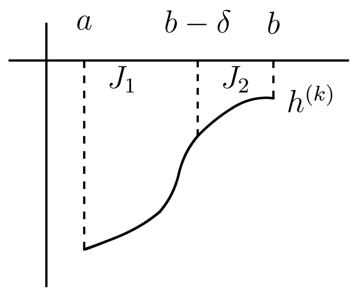
and $C_{k+1} = 2C_k + 2$. Since $C_1 = 4$, we can solve the recurrence equation and we obtain that

$$C_k = 3\cdot 2^k - 2.$$

Finally, if $h^{(k)}$ never vanishes (t_0 does not exist), then the same argument holds defining the sets J_i with $t_0 = a$ if $h^{(k)} > 0$ and $t_0 = b$ if $h^{(k)} < 0$.



$$J_1 = \emptyset$$
$$J_2 = (a, a+\delta)\cap(a,b)$$
$$J_3 = \{t\in(a,b) : t > a+\delta\}$$



$$J_1 = \{t\in(a,b) : t < b-\delta\}$$
$$J_2 = (b-\delta, b)\cap(a,b)$$
$$J_3 = \emptyset$$

□

1.9 Littlewood-Paley Theory

Littlewood-Paley theory is devoted to extending, in some sense, Plancherel's theorem to functions in L^p for values of $p \neq 2$. For example, we know that if we multiply by ± 1 the terms of the Fourier series of an L^2-function, we obtain another function in L^2, or if we multiply the Fourier transform of a function in L^2 by a function of modulus 1, the result is another L^2-function. Nevertheless, it can be shown that these properties do not hold for functions in L^p for $p \neq 2$, so we conclude that whether a function lies in L^p does not depend only on the size of its Fourier transform or its Fourier coefficients. Littlewood-Paley, however, states that the Fourier transform (or the Fourier series) of an L^p-function corresponds to that of another L^p-function if it is modified by ± 1 in dyadic blocks. For instance, if we multiply by the same factor $+1$ or -1 the coefficients whose indices range between 2^k and 2^{k+1}, then the resulting series corresponds to the Fourier series of another function in L^p. In this section, we will prove this result for the Fourier transform of functions defined on $\mathbb{R}$.

Define, for every $j \in \mathbb{Z}$,

$$I_j = (-2^{j+1}, -2^j] \cup [2^j, 2^{j+1}).$$

Define also the corresponding operator S_j as a Fourier multiplier with symbol χ_{I_j}, that is,

$$\widehat{S_j f}(\xi) = \chi_{I_j}(\xi)\widehat{f}(\xi).$$

If $f \in L^2(\mathbb{R})$, by Plancherel's theorem and the fact that the supports of $\widehat{S_j f}$ are pairwise disjoint, we have that

$$\begin{aligned} \|f\|_2^2 = \|\widehat{f}\|_2^2 &= \left\| \sum_{j\in\mathbb{Z}} \widehat{f}\chi_{I_j} \right\|_2^2 = \int_{\mathbb{R}} \left(\sum_{j\in\mathbb{Z}} \widehat{f}\chi_{I_j} \right)^2 \\ &= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} (\widehat{f}\chi_{I_j})^2 = \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} (S_j f)^2 = \int_{\mathbb{R}} \sum_{j\in\mathbb{Z}} (S_j f)^2, \end{aligned}$$

where in the last step we use the monotone convergence theorem. Hence, we have shown that

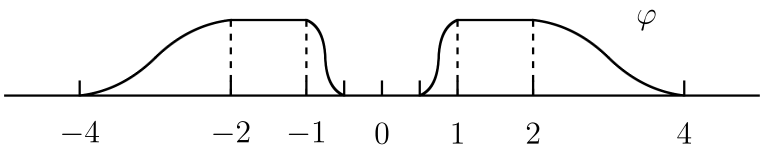
$$\|f\|_2 = \left\| \left(\sum_{j\in\mathbb{Z}} |S_j f|^2 \right)^{1/2} \right\|_2, \quad \forall f \in L^2(\mathbb{R}). \tag{1.8}$$

The next theorem states that, even though these quantities are no longer equal if we consider the p-norm for $p \neq 2$, they are still comparable:

Theorem 1.30 (Littlewood-Paley)**.** *Let $f \in L^p(\mathbb{R})$, $1 < p < \infty$. Then there exist positive constants c_p and C_p such that*

$$c_p\|f\|_p \leq \left\| \left(\sum_{j\in\mathbb{Z}} |S_j f|^2 \right)^{1/2} \right\|_p \leq C_p\|f\|_p.$$

We will prove this theorem as a consequence of the next result, which deals with smooth functions instead of characteristic functions. First, take a non-negative Schwartz function $\varphi \in \mathcal{S}(\mathbb{R})$, supported in the annulus $1/2 \leq |\xi| \leq 4$ and equal to 1 on $1 \leq |\xi| \leq 2$ (which corresponds to I_0).



Now, define the symbols φ_j and the operators $\tilde{S}_j$:

$$\varphi_j(\xi) = \varphi(2^{-j}\xi) \quad \text{and} \quad \widehat{\tilde{S}_j f}(\xi) = \varphi_j(\xi)\widehat{f}(\xi).$$

It is clear that

$$S_j \circ \tilde{S}_j = S_j,$$

since $\chi_{I_j} \cdot \varphi_j = \chi_{I_j}$. Having settled this, we are ready to state our second result:

Theorem 1.31. *Let $f \in L^p(\mathbb{R})$, $1 < p < \infty$. Then there exists a constant C_p such that*

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\tilde{S}_j f|^2 \right)^{1/2} \right\|_p \leq C_p \|f\|_p$$

Proof. Write $\widehat{\Phi} = \varphi$ and $\Phi_j(x) = 2^j \Phi(2^j x)$. Then

$$\widehat{\Phi}_j = 2^j \widehat{\Phi(2^j\cdot)} = \widehat{\Phi}(2^{-j}\cdot) = \varphi(2^{-j}\cdot) = \varphi_j,$$

and thus,

$$\tilde{S}_j f = \varphi_j^\vee * f = \Phi_j * f.$$

Now, consider the vector-valued operator

$$\begin{array}{ccc} L^p & \longrightarrow & L^p(\ell^2) \\ f & \longmapsto & \{\tilde{S}_j f\}_j. \end{array}$$

We want to prove that this operator is bounded. If $p = 2$, using Plancherel's theorem and monotone convergence,

$$\begin{aligned} \left\| \left(\sum_{j\in\mathbb{Z}} |\tilde{S}_j f|^2 \right)^{1/2} \right\|_2^2 &= \int_{\mathbb{R}} \sum_{j\in\mathbb{Z}} |\tilde{S}_j f|^2 = \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} |\tilde{S}_j f|^2 \\ &= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} |\varphi_j(\xi)|^2 |\widehat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}} \sum_{j\in\mathbb{Z}} |\varphi_j(\xi)|^2 |\widehat{f}(\xi)|^2 d\xi. \end{aligned}$$

But by the definition of the φ_j's, at any value of $\xi \in \mathbb{R}$, at most three of them are non-zero, and always less than 1, so

$$\sum_{j\in\mathbb{Z}} |\varphi_j(\xi)|^2 \leq 3, \quad \xi \in \mathbb{R}.$$

Therefore, using again Plancherel's theorem,

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\tilde{S}_j f|^2 \right)^{1/2} \right\|_2^2 \leq 3\|f\|_2^2.$$

In order to extend this to every $1 < p < \infty$, we will make use of Theorem 1.26. Since we already have the boundedness for $p = 2$, we only have left to show that the kernel $\{\Phi_j\}_j$ satisfies the gradient condition

$$\|\Phi_j'(x)\|_{\ell^2} \leq \frac{C}{|x|^2}, \quad \text{for every } x \in \mathbb{R} \setminus \{0\}.$$

First notice that $\Phi_j = \varphi_j^\vee \in \mathcal{S}(\mathbb{R})$, since $\varphi_j \in \mathcal{S}(\mathbb{R})$. Now,

$$\|\Phi_j'(x)\|_{\ell^2} \leq \|\Phi_j'(x)\|_{\ell^1} = \sum_{j\in\mathbb{Z}} |\Phi_j'(x)| = \sum_{j\in\mathbb{Z}} 2^{2j}|\Phi'(2^j x)|.$$

Moreover, since we are dealing with Schwartz functions, we have that

$$|\Phi'(z)| \leq C\min\{1, |z|^{-3}\}. \tag{1.9}$$

Now, take $i \in \mathbb{Z}$ so that $2^{-i} \leq |x| < 2^{-i+1}$. Then, using (1.9),

$$\begin{aligned}\sum_{j\in\mathbb{Z}} 2^{2j}|\Phi'(2^j x)| &= \sum_{j\leq i} 2^{2j}|\Phi'(2^j x)| + \sum_{j>i} 2^{2j}|\Phi'(2^j x)| \\ &\leq C\left[\sum_{j\leq i} 2^{2j} + \sum_{j>i} 2^{2j}2^{-3j}|x|^{-3}\right] \\ &= C|x|^{-2}\left[\sum_{j\leq i} 2^{2j}|x|^2 + \sum_{j>i} 2^{-j}|x|^{-1}\right] \\ &\leq C|x|^{-2}\left[\sum_{j\leq i} 2^{2j}2^{-2i+2} + \sum_{j>i} 2^{-j}2^{i}\right],\end{aligned}$$

where in the last step we use that $2^{-i} \leq |x| < 2^{-i+1}$. In order to complete the proof, we only need to check that the expression in brackets is a constant:

$$\sum_{j\leq i} 4^{j-i+1} + \sum_{j>i} 2^{i-j} = 4 + \sum_{k=0}^{\infty} 4^{-k} + \sum_{k=1}^{\infty} 2^{-k} = \frac{19}{3}.$$

□

Now we are ready to tackle the proof of Littlewood-Paley's theorem:

Proof of Theorem 1.30. Using Corollary 1.28 with $r = 2$ and the identity $S_j \circ \tilde{S}_j = S_j$, we have that

$$\left\|\left(\sum_{j\in\mathbb{Z}} |S_j f|^2\right)^{1/2}\right\|_p = \left\|\left(\sum_{j\in\mathbb{Z}} |S_j\tilde{S}_j f|^2\right)^{1/2}\right\|_p \leq C_{p,2}\left\|\left(\sum_{j\in\mathbb{Z}} |\tilde{S}_j f|^2\right)^{1/2}\right\|_p,$$

and applying now Theorem 1.31, we deduce the first inequality

$$\left\|\left(\sum_{j\in\mathbb{Z}} |S_j f|^2\right)^{1/2}\right\|_p \leq C_p\|f\|_p. \tag{1.10}$$

Now, if we write

$$\begin{aligned}\vec{T} : L^2 &\longrightarrow L^2(\ell^2) \\ f &\longmapsto \{S_j f\}_j,\end{aligned}$$

then, the equation (1.8) can be interpreted as

$$\|\vec{T} f\|_{L^2(\ell^2)} = \|f\|_2, \quad \forall f \in L^2(\mathbb{R}),$$

and thus, from the polarization identity, we obtain that

$$(\vec{T}f, \vec{T}g)_{L^2(\ell^2)} = (f,g)_{L^2},$$

where $(\cdot,\cdot)_H$ denotes the inner product in the corresponding Hilbert space H. In particular, we have this identity for functions in $\mathcal{S}(\mathbb{R})$, and it can be written as

$$\int_{\mathbb{R}} \sum_{j\in\mathbb{Z}} S_j f \cdot \overline{S_j g} = \int_{\mathbb{R}} f\bar{g}.$$

With this expression, Hölder's inequality[5] and the estimate in (1.10),

$$\begin{aligned}
\|f\|_p &= \sup_{\|g\|_{p'}\le 1, g\in\mathcal{S}(\mathbb{R})} \left\{ \left| \int_{\mathbb{R}} f\bar{g} \right| \right\} \\
&= \sup_{\|g\|_{p'}\le 1, g\in\mathcal{S}(\mathbb{R})} \left\{ \left| \int_{\mathbb{R}} \sum_{j\in\mathbb{Z}} S_j f \cdot \overline{S_j g} \right| \right\} \\
&\le \sup_{\|g\|_{p'}\le 1, g\in\mathcal{S}(\mathbb{R})} \left\{ \left\| \left(\sum_{j\in\mathbb{Z}} |S_j f|^2 \right)^{1/2} \right\|_p \cdot \left\| \left(\sum_{j\in\mathbb{Z}} |S_j g|^2 \right)^{1/2} \right\|_{p'} \right\} \\
&\le C_p \left\| \left(\sum_{j\in\mathbb{Z}} |S_j f|^2 \right)^{1/2} \right\|_p .
\end{aligned}$$

Therefore, taking $c_p = C_p^{-1}$, we complete the proof. □

[5]We need to use Hölder's inequality twice:

$$\int_{\mathbb{R}} \sum_{j\in\mathbb{Z}} |S_j f \cdot \overline{S_j g}| \le \int_{\mathbb{R}} \left(\sum_{j\in\mathbb{Z}} |S_j f|^2 \right)^{1/2} \left(\sum_{j\in\mathbb{Z}} |S_j g|^2 \right)^{1/2} \le \left\| \left(\sum_{j\in\mathbb{Z}} |S_j f|^2 \right)^{1/2} \right\|_p \cdot \left\| \left(\sum_{j\in\mathbb{Z}} |S_j g|^2 \right)^{1/2} \right\|_{p'} .$$

Chapter 2

The Classical Hilbert Transform

2.1 The Poisson And The Conjugate Poisson Kernels

Given $t \in \mathbb{R}^+$, consider the function $g_t(\xi) = e^{-2\pi t|\xi|} \in L^1(\mathbb{R})$. Since it is integrable, we can compute its inverse[1] Fourier transform by means of an integral. Define

$$P_t(x) := (e^{-2\pi t|\xi|})^\vee(x) = \int_{\mathbb{R}} e^{-2\pi t|\xi|} e^{2\pi i x\xi} d\xi. \tag{2.1}$$

Let us compute it:

$$P_t(x) = I_- + I_+,$$

where,[2]

$$I_- = \int_{-\infty}^{0} e^{2\pi\xi(t+ix)} d\xi = \frac{1}{2\pi(t+ix)} e^{2\pi\xi(t+ix)}\Big|_{\xi=-\infty}^{0} = \frac{1}{2\pi(t+ix)},$$

and

$$I_+ = \int_{0}^{\infty} e^{2\pi\xi(ix-t)} d\xi = \frac{1}{2\pi(ix-t)} e^{2\pi\xi(ix-t)}\Big|_{\xi=0}^{\infty} = -\frac{1}{2\pi(ix-t)}.$$

Therefore,

$$P_t(x) = \frac{1}{2\pi(t+ix)} - \frac{1}{2\pi(ix-t)} = \frac{1}{\pi}\frac{t}{t^2+x^2}.$$

Definition 2.1. *We define the Poisson kernel on $\mathbb{R}$ by*

$$P_t(x) = \frac{1}{\pi}\frac{t}{t^2+x^2}, \qquad t > 0.$$

Taking images by the Fourier transform in (2.1) and using that it is a bijection from $\mathcal{S}'(\mathbb{R})$ to itself extending the definition on $L^1(\mathbb{R})$, we have that $\widehat{P_t}(\xi) = e^{-2\pi t|\xi|}$. Moreover,

- $P_t(x)$ is harmonic on the upper half-plane $\mathbb{R}^2_+$,
- $\{P_t\}_t$ defines a summability kernel as $t \to 0$,

[1]Notice that $g_t(\xi)$ is an even function, so its Fourier transform and its inverse Fourier transform are equal.

[2]We have that $e^{2\pi\xi t} e^{2\pi\xi i x} \overset{\xi\to-\infty}{\longrightarrow} 0$ since $e^{2\pi\xi t} \overset{\xi\to-\infty}{\longrightarrow} 0$ and $|e^{2\pi\xi i x}| = 1$.

which can be checked by easy computations.

Analogously, for $t \in \mathbb{R}^+$, consider the function $h_t(\xi) = -i\,\mathrm{sgn}(\xi)e^{-2\pi t|\xi|} \in L^1(\mathbb{R})$ and define

$$Q_t(x) := (-i\,\mathrm{sgn}(\xi)e^{-2\pi t|\xi|})^\vee(x) = \int_{\mathbb{R}} -i\,\mathrm{sgn}(\xi)e^{-2\pi t|\xi|}e^{2\pi i x\xi}d\xi.$$

Following the same notation as in the computation of P_t, we can write $Q_t(x) = i(I_- - I_+)$. Substituting the values of I_- and I_+, we obtain the following:

Definition 2.2. *We define the conjugate Poisson kernel on $\mathbb{R}$ by*

$$Q_t(x) = \frac{1}{\pi}\frac{x}{t^2 + x^2}, \qquad t > 0.$$

Again, we have that $\widehat{Q}_t(\xi) = -i\,\mathrm{sgn}(\xi)e^{-2\pi t|\xi|}$. Moreover, if we put $z = x + it$, then

$$P_t(x) + iQ_t(x) = \frac{1}{\pi}\frac{t + ix}{t^2 + x^2} = \frac{i}{\pi z},$$

which is analytic on the upper half-plane. Therefore, we deduce that $Q_t(x)$ is harmonic on $\mathbb{R}^2_+$ and the conjugate of the Poisson kernel $P_t(x)$. However, unlike $\{P_t\}_t$, the family $\{Q_t\}_t$ is not a summability kernel, since Q_t is not integrable for any $t > 0$. Indeed,

$$\int_{\mathbb{R}} |Q_t(x)|dx = \frac{2}{\pi}\int_0^\infty \frac{x}{t^2 + x^2}dx = \frac{1}{\pi}\ln|t^2 + x^2|\Big|_{x=0}^{\infty} = \infty.$$

Remark 2.3. *Notice that, for every fixed $t > 0$, Q_t belongs to $L^2(\mathbb{R})$ (since Q_t^2 behaves like $1/x^2$ at infinity and thus is integrable). Therefore, we know that Q_t is a tempered distribution for all $t > 0$.*

2.2 The Principal Value Of $1/x$

Definition 2.4. *We define a tempered distribution called the principal value of $1/x$ by*

$$p.v.\frac{1}{x}(\varphi) := \lim_{\varepsilon \to 0}\int_{|x|>\varepsilon} \frac{\varphi(x)}{x}dx, \qquad \varphi \in \mathcal{S}(\mathbb{R}).$$

First of all, we have to make sure that the previous expression is well defined, that is, that the limit exists and it is finite. We can rewrite it as

$$\lim_{\varepsilon \to 0}\int_{|x|>\varepsilon} \frac{\varphi(x)}{x}dx = \lim_{\varepsilon \to 0}\int_{\varepsilon<|x|<1} \frac{\varphi(x) - \varphi(0)}{x}dx + \int_{|x|>1}\frac{\varphi(x)}{x}dx, \tag{2.2}$$

since the integral of $1/x$ on $\varepsilon < |x| < 1$ is zero. Then, using the mean value theorem,

$$\left|\frac{\varphi(x) - \varphi(0)}{x} \cdot \chi_{\{\varepsilon<|x|<1\}}(x)\right| \le \|\varphi'\|_\infty \cdot \chi_{\{|x|<1\}}(x) \in L^1(\mathbb{R}),$$

so, by the dominated convergence theorem, the limit exists and the first term on the right-hand side of (2.2) is finite:

$$\lim_{\varepsilon\to 0}\int_{\varepsilon<|x|<1}\frac{\varphi(x)-\varphi(0)}{x}dx=\int_{|x|<1}\frac{\varphi(x)-\varphi(0)}{x}dx\in\mathbb{C}.$$

To prove that the second integral in (2.2) is also finite, we use that $\varphi\in\mathcal{S}(\mathbb{R})$, and thus

$$\left|\int_{|x|>1}\frac{\varphi(x)}{x}dx\right|=\left|\int_{|x|>1}x\varphi(x)\frac{1}{x^2}dx\right|\leq\|x\varphi\|_\infty\int_{|x|>1}\frac{1}{x^2}dx=2\|x\varphi\|_\infty<\infty.$$

Secondly, in order to check that the principal value defines a tempered distribution, we need to prove that

$$\text{p.v.}\frac{1}{x}:\mathcal{S}(\mathbb{R})\longrightarrow\mathbb{C}$$

is linear and continuous. Linearity is obvious. Now, suppose that $\varphi_k\to 0$ in $\mathcal{S}(\mathbb{R})$, that is, for all $\alpha,\beta\in\mathbb{N}$,

$$\lim_{k\to\infty}\|x^\alpha\varphi_k^{(\beta)}\|_\infty=0.$$

Using the previous bounds, we have that

$$\begin{aligned}\left|\text{p.v.}\frac{1}{x}(\varphi_k)\right|=&\left|\lim_{\varepsilon\to 0}\int_{|x|>\varepsilon}\frac{\varphi_k(x)}{x}dx\right|\leq\int_{|x|<1}\|\varphi_k'\|_\infty dx+2\|x\varphi_k\|_\infty\\ =&2\|\varphi_k'\|_\infty+2\|x\varphi_k\|_\infty\xrightarrow{k}0.\end{aligned}$$

The next step will be to establish the connection between the principal value and the conjugate Poisson kernel.

Proposition 2.5. *In the class of tempered distributions $\mathcal{S}'(\mathbb{R})$, we have that*

$$\lim_{t\to 0}Q_t=\frac{1}{\pi}p.v.\frac{1}{x}.$$

Proof. Consider, for every $t>0$, the function $\psi_t(x)=\frac{1}{x}\chi_{\{|x|>t\}}$, which is bounded and locally integrable. Therefore, it defines a tempered distribution

$$\langle\cdot,\psi_t\rangle:\mathcal{S}(\mathbb{R})\longrightarrow\mathbb{C}.$$

Moreover, it holds that for all $\varphi\in\mathcal{S}(\mathbb{R})$,

$$\lim_{t\to 0}\langle\varphi,\psi_t\rangle=\lim_{t\to 0}\int_{\mathbb{R}}\frac{\varphi(x)}{x}\chi_{\{|x|>t\}}dx=\text{p.v.}\frac{1}{x}(\varphi),$$

so it follows[3] that in $\mathcal{S}'(\mathbb{R})$,

$$\lim_{t\to 0}\psi_t=\text{p.v.}\frac{1}{x}.$$

[3] Remember that $\mathcal{S}'(\mathbb{R})$ is endowed with the weak*-topology.

Consequently, it is enough to prove that in $\mathcal{S}'(\mathbb{R})$,

$$\lim_{t\to 0}(\pi Q_t - \psi_t) = 0.$$

Again, we only need to check that it tends to zero when applied to any function $\varphi \in \mathcal{S}(\mathbb{R})$. Indeed,

$$\begin{aligned}\langle \varphi, \pi Q_t - \psi_t\rangle &= \int_{\mathbb{R}} \frac{x\varphi(x)}{x^2+t^2}dx - \int_{|x|>t} \frac{\varphi(x)}{x}dx \\ &= \int_{|x|<t} \frac{x\varphi(x)}{x^2+t^2}dx + \int_{|x|>t} \left(\frac{x\varphi(x)}{x^2+t^2} - \frac{\varphi(x)}{x}\right)dx \\ &= \int_{|y|<1} \frac{y\varphi(yt)}{y^2+1}dy - \int_{|y|>1} \frac{\varphi(yt)}{y(1+y^2)}dy,\end{aligned}$$

which tends to zero as $t \to 0$, using the dominated convergence theorem and the fact that odd functions integrate zero on symmetric domains[4]. With this, we finish the proof. □

Corollary 2.6. *We have that*

$$\mathcal{F}\left(\frac{1}{\pi}p.v.\frac{1}{x}\right) = -i\,\mathrm{sgn}(\cdot).$$

Proof. Using the previous proposition and the fact that the Fourier transform is continuous on $\mathcal{S}'(\mathbb{R})$,

$$\mathcal{F}\left(\frac{1}{\pi}\mathrm{p.v.}\frac{1}{x}\right) = \mathcal{F}(\lim_{t\to 0} Q_t) = \lim_{t\to 0}\widehat{Q}_t = \lim_{t\to 0} -i\,\mathrm{sgn}(\cdot)e^{-2\pi t|\cdot|} = -i\,\mathrm{sgn}(\cdot),$$

since $e^{-2\pi t|\cdot|}$ tends to 1 in $\mathcal{S}'(\mathbb{R})$ as $t \to 0$ (one can easily check it using the dominated convergence theorem). □

2.3 Definition Of The Hilbert Transform

Definition 2.7. *The Hilbert transform is the linear map*

$$H : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$$

defined by

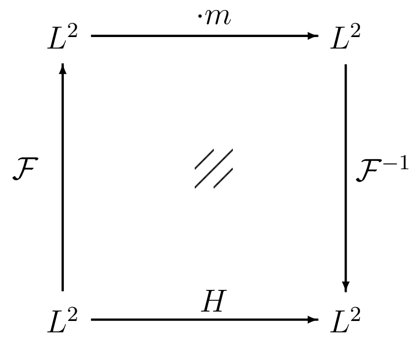
$$\widehat{Hf}(\xi) = -i\,\mathrm{sgn}(\xi)\widehat{f}(\xi), \quad \forall f \in L^2(\mathbb{R}).$$

Recall that Plancherel's theorem says that the Fourier transform is a bijective isometry from $L^2(\mathbb{R})$ onto itself. The Hilbert transform is the Fourier multiplier operator

$$H : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$$

with symbol $m(\xi) = -i\,\mathrm{sgn}(\xi)$.

[4] In the domination, we use that φ is bounded, and for the second integral, we also need to make sure that $\int_1^\infty \frac{1}{y(1+y^2)}dy < \infty$, which is immediate by comparing to $1/y^3 \in L^1(1,\infty)$.



Theorem 2.8. *The Hilbert transform is a bijective linear isometry such that*

$$H^2 = -Id \quad \textit{and} \quad H^* = -H.$$

Moreover, for every $\varphi \in \mathcal{S}(\mathbb{R})$,

$$H\varphi = \frac{1}{\pi} p.v. \frac{1}{x} * \varphi = \lim_{t\to 0} (Q_t * \varphi) \quad \textit{in } \mathcal{S}'(\mathbb{R}).$$

Proof. Put $m(\xi) = -i\,\mathrm{sgn}(\xi)$. It is clear that multiplying by m, $f \mapsto mf$, is a bijective, linear isometry from $L^2(\mathbb{R})$ onto itself. By Plancherel's theorem, the same holds for $\mathcal{F}$, so the composition

$$H = \mathcal{F}^{-1} \circ (\cdot m) \circ \mathcal{F}$$

is also a bijective linear isometry of $L^2(\mathbb{R})$. Moreover, since $m^2 = -1$, we deduce that $H^2 = -Id$. Now, let us compute the adjoint operator H^*. Since H is a linear isometry, we have that $(f, g)_2 = (Hf, Hg)_2$ for all $f, g \in L^2(\mathbb{R})$ (using the polarization identity). Therefore,

$$(Hf, g)_2 = (H^2 f, Hg)_2 = (-f, Hg)_2 = (f, -Hg)_2,$$

and we conclude that $H^* = -H$.

Finally, take $\varphi \in \mathcal{S}(\mathbb{R})$. On the one hand, using that the convolution is continuous with respect to the first variable[5] and recalling Proposition 2.5, we have that

$$\lim_{t\to 0} (Q_t * \varphi) = (\lim_{t\to 0} Q_t) * \varphi = \frac{1}{\pi} \text{p.v.} \frac{1}{x} * \varphi \quad \text{in } \mathcal{S}'(\mathbb{R}).$$

On the other hand, by Corollary 2.6,

$$\mathcal{F}(H\varphi) = -i\,\mathrm{sgn}(\xi)\widehat{\varphi}(\xi) = \mathcal{F}\left(\frac{1}{\pi}\text{p.v.}\frac{1}{x}\right)\mathcal{F}(\varphi) = \mathcal{F}\left(\frac{1}{\pi}\text{p.v.}\frac{1}{x} * \varphi\right),$$

so applying $\mathcal{F}^{-1}$, we obtain

$$H\varphi = \frac{1}{\pi}\text{p.v.}\frac{1}{x} * \varphi \quad \text{in } \mathcal{S}'(\mathbb{R})$$

and we finish the proof. The properties for the convolution that we have used are listed in Proposition 1.5. □

[5] Remember that, in Proposition 1.5, we showed that if $u_n \to u$ in $\mathcal{S}'$ then $u_n * \varphi \to u * \varphi$ in $\mathcal{S}'$.

With this result, we have that the Hilbert transform applied to a Schwartz function is a continuous function defined by

$$\begin{aligned} H\varphi(x) =& \left(\frac{1}{\pi}\text{p.v.}\frac{1}{y} * \varphi\right)(x) = \left(\frac{1}{\pi}\text{p.v.}\frac{1}{y}\right)(\tau_x\tilde{\varphi}) \\ =& \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|y|>\varepsilon}\frac{\varphi(x-y)}{y}dy = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}\frac{\varphi(y)}{x-y}dy. \end{aligned}$$

So far, the Hilbert transform is a bounded, linear operator defined on $L^2(\mathbb{R})$. Moreover, we have an explicit expression for $H\varphi$ whenever $\varphi \in \mathcal{S}(\mathbb{R})$. In the next section, we will prove that H is, in fact, of strong-type (p,p) for every $1 < p < \infty$ and of weak-type $(1,1)$.

2.4 L^p-Boundedness Of The Hilbert Transform

Lemma 2.9. *Let $\varphi \in \mathcal{S}(\mathbb{R})$ be a Schwartz function. Then,*

$$(H\varphi)^2 = \varphi^2 + 2H(\varphi H\varphi) \quad \text{in } L^2(\mathbb{R}).$$

Proof. First of all, notice that the hypothesis $\varphi \in \mathcal{S}(\mathbb{R})$ is important. If we take $\varphi \in L^2(\mathbb{R})$ instead, then $\varphi H\varphi \in L^1(\mathbb{R})$ and $H(\varphi H\varphi)$ would not be defined[6]. However, if $\varphi \in \mathcal{S}(\mathbb{R})$, then it is bounded and $\varphi H\varphi \in L^2(\mathbb{R})$.

Now, write $m(\xi) := -i\,\text{sgn}(\xi)$, which is the symbol of the Hilbert transform. Equivalently, we will prove the identity by taking Fourier transforms. That is,

$$\widehat{H\varphi} * \widehat{H\varphi} = \widehat{\varphi} * \widehat{\varphi} + 2m(\widehat{\varphi} * \widehat{H\varphi}).$$

Indeed,

$$\begin{aligned} (\widehat{\varphi} * \widehat{\varphi})(\xi) + 2m(\xi)(\widehat{\varphi} * \widehat{H\varphi})(\xi) &= \int_{\mathbb{R}} \widehat{\varphi}(\eta)\widehat{\varphi}(\xi-\eta)d\eta + 2m(\xi)\int_{\mathbb{R}} \widehat{\varphi}(\eta)\widehat{\varphi}(\xi-\eta)m(\xi-\eta)d\eta \\ &= \int_{\mathbb{R}} \widehat{\varphi}(\eta)\widehat{\varphi}(\xi-\eta)d\eta + 2m(\xi)\int_{\mathbb{R}} \widehat{\varphi}(\xi-\eta)\widehat{\varphi}(\eta)m(\eta)d\eta. \end{aligned}$$

Moreover, if we average the last two expressions, we obtain yet another equality

$$(\widehat{\varphi} * \widehat{\varphi})(\xi) + 2m(\xi)(\widehat{\varphi} * \widehat{H\varphi})(\xi) = \int_{\mathbb{R}} \widehat{\varphi}(\eta)\widehat{\varphi}(\xi-\eta)[1 + m(\xi)(m(\eta) + m(\xi-\eta))]d\eta.$$

Finally, using that $m(\xi) = -i\,\text{sgn}(\xi)$, we can check[7] that for every fixed $\xi \in \mathbb{R}$,

$$1 + m(\xi)(m(\eta) + m(\xi-\eta)) = m(\eta)m(\xi-\eta), \quad \text{for a.e. } \eta \in \mathbb{R},$$

and therefore, replacing this expression in the last integral, we conclude that

$$(\widehat{\varphi} * \widehat{\varphi})(\xi) + 2m(\xi)(\widehat{\varphi} * \widehat{H\varphi})(\xi) = \int_{\mathbb{R}} \widehat{\varphi}(\eta)m(\eta)\widehat{\varphi}(\xi-\eta)m(\xi-\eta)d\eta = (\widehat{H\varphi} * \widehat{H\varphi})(\xi),$$

as we wanted to show. □

[6] Actually, the Hilbert transform can be extended to $L^1(\mathbb{R})$ and it defines an operator of weak-type $(1,1)$, as we will see in the next section.

[7] If we consider all possible cases for the sign of ξ, η and $\xi-\eta$, (there are 13 of them), we conclude that the equality only fails when $\xi = \eta = 0$. Therefore, if $\xi \neq 0$ the identity holds for every $\eta \in \mathbb{R}$ and, if $\xi = 0$, it is true for all $\eta \in \mathbb{R}\setminus\{0\}$. At any rate, we have the identity for almost every $\eta \in \mathbb{R}$.

A straightforward consequence of this result is the following lemma, which combined with the L^2-boundedness of the Hilbert transform, will allow us to prove that H is, in fact, of strong-type (p,p) for all $1<p<\infty$.

Lemma 2.10. *Given $1\le p<\infty$, if we assume that*

$$H : L^p(\mathbb{R}) \longrightarrow L^p(\mathbb{R})$$

is of strong-type (p,p) with $\|H\|_{L^p\to L^p}\le C_p$, then H is also of strong-type $(2p,2p)$ with

$$\|H\|_{L^{2p}\to L^{2p}} \le 2C_p+1.$$

Proof. Take $\varphi\in\mathcal{S}(\mathbb{R})$. If $\|H\varphi\|_{2p}\le\|\varphi\|_{2p}$, then trivially

$$\|H\varphi\|_{2p}\le (2C_p+1)\|\varphi\|_{2p}.$$

On the other hand, if $\|\varphi\|_{2p}\le\|H\varphi\|_{2p}$, using the previous lemma, our hypothesis of L^p-boundedness and Hölder's inequality, we get that

$$\begin{aligned}\|H\varphi\|_{2p}^2 =&\|(H\varphi)^2\|_p \le \|\varphi^2\|_p + 2\|H(\varphi H\varphi)\|_p \le \|\varphi\|_{2p}^2 + 2C_p\|\varphi H\varphi\|_p\\ \le&\|\varphi\|_{2p}^2 + 2C_p\|\varphi\|_{2p}\|H\varphi\|_{2p} \le \|\varphi\|_{2p}\|H\varphi\|_{2p} + 2C_p\|\varphi\|_{2p}\|H\varphi\|_{2p}\\ =&(2C_p+1)\|\varphi\|_{2p}\|H\varphi\|_{2p}.\end{aligned}$$

Simplifying $\|H\varphi\|_{2p}$ (which we can suppose to be non-zero), we obtain that

$$\|H\varphi\|_{2p}\le(2C_p+1)\|\varphi\|_{2p} \quad \text{for all } \varphi\in\mathcal{S}(\mathbb{R}).$$

Finally, using that $\mathcal{S}(\mathbb{R})\subseteq L^{2p}(\mathbb{R})$ is dense, we deduce that the Hilbert transform is of strong-type $(2p,2p)$ and

$$\|H\|_{L^{2p}\to L^{2p}} \le 2C_p+1.$$

□

Finally, let us state and prove the main theorem of this section:

Theorem 2.11 (M. Riesz)**.** *Given $1<p<\infty$, the Hilbert transform*

$$H : L^p(\mathbb{R}) \longrightarrow L^p(\mathbb{R})$$

is linear and bounded.

Proof. We know that the Hilbert transform is linear and bounded from $L^2(\mathbb{R})$ onto itself and $\|Hf\|_2=\|f\|_2$ for all $f\in L^2(\mathbb{R})$. Therefore, starting at $p=2$ with $C_2=1$, we can apply the previous lemma repeatedly and we obtain that

$$H : L^{2^n} \longrightarrow L^{2^n}$$

is linear and bounded with[8] $\|H\|_{2^n}\le 2^n-1$ for every $n\ge1$.

[8] From $C_2=1$ and $C_{2^n}\le 2C_{2^{n-1}}+1$ we prove, by a trivial induction, that $C_{2^n}=2^n-1$.

Now, given $p \geq 2$, we have that $p \in [2^n, 2^{n+1}]$ for some $n \geq 1$. Therefore, we can apply the Riesz-Thorin interpolation theorem with $p_0 = q_0 = 2^n$ and $p_1 = q_1 = 2^{n+1}$ to prove that H is also bounded from $L^p(\mathbb{R})$ to itself and

$$\|Tf\|_p \leq C_p \|f\|_p, \quad \forall f \in L^p(\mathbb{R}),$$

where $C_p = (2^n - 1)^{1-\theta}(2^{n+1} - 1)^{\theta}$ with $0 \leq \theta \leq 1$ satisfying

$$\frac{1}{p} = \frac{1-\theta}{2^n} + \frac{\theta}{2^{n+1}}.$$

So far, we have proved that the Hilbert transform, originally defined on $L^2(\mathbb{R})$, can be extended to functions in $L^p(\mathbb{R})$ with $2 \leq p < \infty$. Finally, we prove the boundedness for $1 < p < 2$. Since the conjugate exponent is $p' > 2$, we have that H is a bounded linear operator from $L^{p'}(\mathbb{R})$ to itself. We also know[9] that

$$\int_{\mathbb{R}} H\varphi(x)\overline{\psi(x)}dx = -\int_{\mathbb{R}} \varphi(x)\overline{H\psi(x)}dx, \quad \varphi, \psi \in \mathcal{S}(\mathbb{R}).$$

Now, using duality and the fact that $\mathcal{S}(\mathbb{R}) \subseteq L^{p'}(\mathbb{R})$ is dense, we can write

$$\begin{aligned} \|H\varphi\|_p &= \sup_{\psi \in \mathcal{S}(\mathbb{R}), \|\psi\|_{p'} \leq 1} \left| \int_{\mathbb{R}} H\varphi(x)\overline{\psi(x)}dx \right| \\ &= \sup_{\psi \in \mathcal{S}(\mathbb{R}), \|\psi\|_{p'} \leq 1} \left| \int_{\mathbb{R}} \varphi(x)\overline{H\psi(x)}dx \right| \\ &\leq \sup_{\psi \in \mathcal{S}(\mathbb{R}), \|\psi\|_{p'} \leq 1} \|\varphi\|_p \|H\psi\|_{p'} \leq C_{p'} \|\varphi\|_p, \end{aligned}$$

for all $\varphi \in \mathcal{S}(\mathbb{R})$. Finally, we use again that $\mathcal{S}(\mathbb{R}) \subseteq L^p(\mathbb{R})$ is dense to conclude that

$$\|Hf\|_p \leq C_{p'} \|f\|_p, \quad \forall f \in L^p(\mathbb{R}),$$

and thus, finish the proof. □

2.5 The Weak $(1,1)$ Inequality For The Hilbert Transform

Theorem 2.12 (Kolmogorov)**.** *The Hilbert transform can be extended to* $L^1(\mathbb{R})$ *and it is an operator of weak-type* $(1,1)$.

Proof. Since we know that $H : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$, by Theorem 1.22 it is enough to check that H is well-localized. Take $b \in L^2(\mathbb{R})$ supported on an interval I and with zero integral. We have to prove that

$$\int_{\mathbb{R}\setminus 2I} |Hb(x)|dx \leq C \int_I |b(x)|dx.$$

[9] Remember that for $H : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, we had $H^* = -H$.

Now, even though $b \notin \mathcal{S}(\mathbb{R})$, we state that, for almost every $x \notin 2I$, the following formula holds:

$$Hb(x) = \frac{1}{\pi}\int_I \frac{b(y)}{x-y}dy.$$

Indeed, by density, we can take a sequence $\{\varphi_n\}_n$ of $\mathcal{C}^\infty$-functions with compact support in I such that

$$\varphi_n \xrightarrow{n} b \quad \text{in } L^2.$$

Consequently, $H\varphi_n \xrightarrow{n} Hb$ in $L^2(\mathbb{R})$. We can assume, passing to a subsequence if necessary, that there is a.e. convergence. Now, take $x \notin 2I$ such that $H\varphi_n(x) \xrightarrow{n} Hb(x)$ (this happens for almost every $x \notin 2I$). Since $\varphi_n \in \mathcal{S}(\mathbb{R})$ are supported on I and $|x-y| \geq |I|/2 > 0$ whenever $y \in I$, we have that for every $n \geq 0$,

$$H\varphi_n(x) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} \frac{\varphi_n(y)}{x-y}dy = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon,\ y\in I} \frac{\varphi_n(y)}{x-y}dy = \frac{1}{\pi}\int_I \frac{\varphi_n(y)}{x-y}dy.$$

Therefore, using Hölder's inequality,

$$\begin{aligned}
\left|H\varphi_n(x) - \frac{1}{\pi}\int_I \frac{b(y)}{x-y}dy\right| &\leq \frac{1}{\pi}\int_I \frac{|\varphi_n(y)-b(y)|}{|x-y|}dy \\
&\leq \frac{2}{\pi|I|}\int_I |\varphi_n(y)-b(y)|dy \\
&\leq \frac{2}{\pi|I|}\|\varphi_n - b\|_2\|\chi_I\|_2 \xrightarrow{n} 0,
\end{aligned}$$

and we conclude that

$$Hb(x) = \lim_{n\to\infty} H\varphi_n(x) = \frac{1}{\pi}\int_I \frac{b(y)}{x-y}dy, \qquad \text{a.e. } x \notin 2I.$$

With this expression, denoting by c the center of I, using Fubini's theorem and the fact that b has zero integral, we get

$$\begin{aligned}
\int_{\mathbb{R}\setminus 2I} |Hb(x)|dx &= \frac{1}{\pi}\int_{\mathbb{R}\setminus 2I}\left|\int_I \frac{b(y)}{x-y}dy\right|dx \\
&= \frac{1}{\pi}\int_{\mathbb{R}\setminus 2I}\left|\int_I b(y)\left(\frac{1}{x-y} - \frac{1}{x-c}\right)dy\right|dx \\
&\leq \frac{1}{\pi}\int_I |b(y)|\left(\int_{\mathbb{R}\setminus 2I} \frac{|y-c|}{|x-y||x-c|}dx\right)dy \\
&\leq \frac{1}{\pi}\int_I |b(y)|\left(\int_{\mathbb{R}\setminus 2I} \frac{|I|}{|x-c|^2}dx\right)dy,
\end{aligned}$$

since $|y-c| \leq |I|/2$ and $|x-y| \geq |x-c|/2$. Moreover, the inner intregral can be computed,

$$\int_{\mathbb{R}\setminus 2I} \frac{|I|}{|x-c|^2}dx = 2|I|\int_{|I|}^\infty \frac{1}{z^2}dz = 2,$$

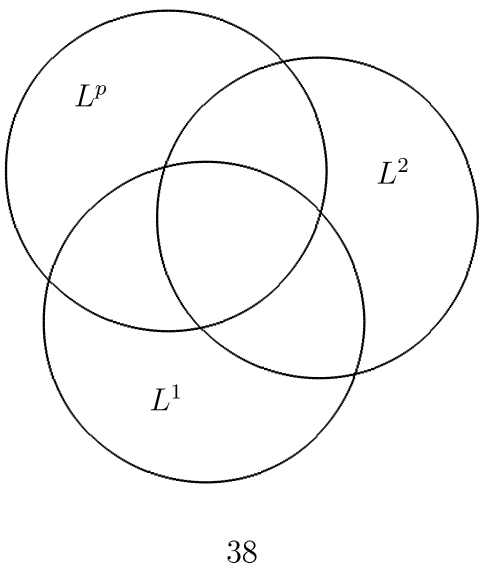
so we obtain the estimate
$$\int_{\mathbb{R}\setminus 2I} |Hb(x)|dx \le \frac{2}{\pi}\int_I |b(y)|dy$$
and we finish the proof. □

In this chapter, we have seen that the Hilbert transform, originally defined for functions in $L^2(\mathbb{R})$, can be extended to functions in $L^p(\mathbb{R})$ for $1 \le p < \infty$. In our case, we have checked that H is of strong-type (p,p) first and then, we have seen that it is also of weak-type $(1,1)$. However, had we started proving the weak $(1,1)$ inequality, by Marcinkiewicz interpolation theorem, we would have had the strong-type (p,p) for $1 < p < 2$, and with a duality argument as in the proof of Riesz's theorem, we would have been able to extend the boundedness to $2 < p < \infty$. Even though this is a shorter way to solve the problem, we decided to include the proof of Riesz's theorem independently of the weak $(1,1)$ inequality since it involves an original idea which is worth to mention.

2.6 Consistency Of Definitions

Finally, we want to check that the extensions of the Hilbert transform that we have made in the preceding sections are coherent.

- If $f \in L^2(\mathbb{R})$, we have the original definition of Hf. In particular, we have an explicit formula for functions in $\mathcal{S}(\mathbb{R}) \subseteq L^2(\mathbb{R})$.
- If $f \in L^p(\mathbb{R})$ for some $1 < p < \infty$, we have that
$$H^{(p)}f := \lim_{n\to\infty} H\varphi_n \quad \text{in } L^p,$$
where $\{\varphi_n\}_n \subseteq \mathcal{S}(\mathbb{R})$ and $\varphi_n \xrightarrow{n} f$ in L^p.
- If $f \in L^1(\mathbb{R})$
$$H^{(1)}f := \lim_{n\to\infty} H\varphi_n \quad \text{in } L^{1,\infty},$$
where $\{\varphi_n\}_n \subseteq \mathcal{S}(\mathbb{R})$ and $\varphi_n \xrightarrow{n} f$ in L^1.



First of all, we need to prove the following lemma:

Lemma 2.13. *Let* $1 \leq p, q < \infty$. *If* $f \in L^p(\mathbb{R}) \cap L^q(\mathbb{R})$, *then there is a sequence of functions* $\{\varphi_n\}_n \subseteq \mathcal{S}(\mathbb{R})$ *such that* $\varphi_n \xrightarrow{n} f$ *in* L^p *and in* L^q.

Proof. Let $\{K_n\}_n$ be a $\mathcal{C}^\infty$-class approximate identity. We know that

$$f * K_n \xrightarrow{n} f \quad \text{in } L^p \text{ and } L^q.$$

Define, for every $m \geq 1$, a $\mathcal{C}^\infty$-function ψ_m with compact support such that

- $\psi_m \equiv 1$ on $B(0, m)$,
- $\psi_m \equiv 0$ on $\mathbb{R} \setminus B(0, 2m)$,
- $0 \leq \psi_m(x) \leq 1$ for all $x \in \mathbb{R}$.

We have that

$$\varphi_{n,m} := (f * K_n)\psi_m \in \mathcal{C}_c^\infty(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R}), \quad \forall n, m \geq 1.$$

Now, if $g \in L^p(\mathbb{R}) \cap L^q(\mathbb{R})$, then $g\psi_m$ converges to g in L^p and L^q as m tends to infinity, since there is pointwise convergence and

$$|g\psi_m| \leq |g| \in L^p(\mathbb{R}) \cap L^q(\mathbb{R}), \quad \forall m \geq 1.$$

Therefore, for every $n \geq 1$, we can take an $m_n \geq 1$ such that

$$\|(f * K_n)\psi_{m_n} - f * K_n\|_p < \frac{1}{n} \quad \text{and} \quad \|(f * K_n)\psi_{m_n} - f * K_n\|_q < \frac{1}{n}.$$

Define $\varphi_n := \varphi_{n,m_n}$ and

$$\|\varphi_n - f\|_p \leq \|(f * K_n)\psi_{m_n} - f * K_n\|_p + \|f * K_n - f\|_p \leq \frac{1}{n} + \|f * K_n - f\|_p,$$

which tends to zero as n tends to infinity. Analogously, $\|\varphi_n - f\|_q$ tends to zero and we complete the proof. □

With this fact, it is easy to see that the different definitions of the Hilbert transform of a function f coincide when f is in the intersection of different L^p-spaces. Indeed, assume that $f \in L^p(\mathbb{R}) \cap L^q(\mathbb{R})$ for some $1 < p < \infty$, $1 \leq q < \infty$. By the last lemma, take a sequence $\{\varphi_n\}_n \subseteq \mathcal{S}(\mathbb{R})$ converging to f in both L^p and L^q. We have that

$$H^{(p)}f := \lim_{n\to\infty} H\varphi_n \quad \text{in } L^p$$

and

$$H^{(q)}f := \lim_{n\to\infty} H\varphi_n \quad \text{in } L^q \text{ (or } L^{1,\infty} \text{ if } q = 1.)$$

In particular, as we have seen in Section 1.3, we have that $H\varphi_n$ converges in measure to $H^{(p)}f$ and $H^{(q)}f$, so we conclude that

$$H^{(p)}f(x) = H^{(q)}f(x) \quad \text{a.e. } x \in \mathbb{R}.$$

2.7 A Proof Still To Be Dealt With

At the end of Section 1.7 there is a corollary with no proof. However, we promised to include it once the classical Hilbert transform had been introduced.

Proof of Corollary 1.28. Recall that we want to prove that, for $1 < r, p < \infty$,

$$\left\| \left(\sum_{j\in\mathbb{Z}} |S_j f_j|^r \right)^{1/r} \right\|_p \leq C_{p,r} \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^r \right)^{1/r} \right\|_p,$$

where $\{S_j\}_j$ is the sequence of operators defined by $\widehat{S_j f}(\xi) = \chi_{I_j}(\xi)\widehat{f}(\xi)$ and $\{I_j\}$ is a sequence of intervals on the real line. First, we will check that H is under the hypotheses of Theorem 1.27. Indeed, it is a convolution operator, bounded from $L^2(\mathbb{R})$ to itself, and whose associated kernel satisfies the gradient condition: $K = \frac{1}{\pi}\text{p.v.}\frac{1}{x} \in \mathcal{C}^1(\mathbb{R} \setminus \{0\})$ and

$$\frac{1}{\pi}\left|\frac{-1}{x^2}\right| \leq \frac{C}{|x|^2}, \quad x \neq 0$$

with $C = \pi^{-1}$. In particular, by Proposition 1.24, it satisfies the Hörmander condition. Therefore,

$$\left\| \left(\sum_{j\in\mathbb{Z}} |H f_j|^r \right)^{1/r} \right\|_p \leq C_{p,r} \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^r \right)^{1/r} \right\|_p.$$

Now, if $I = (a, b) \subseteq \mathbb{R}$ and $S_{a,b}$ is the operator defined by

$$\widehat{S_{a,b} f}(\xi) = \chi_{(a,b)}(\xi)\widehat{f}(\xi),$$

then it holds that

$$S_{a,b} = \frac{i}{2}(M_a H M_{-a} - M_b H M_{-b}), \tag{2.3}$$

where M_a is the modulation operator

$$M_a f(x) = e^{2\pi i a x} f(x).$$

Now, using that the multiplier of H is $-i\,\text{sgn}(\xi)$ and the properties of the Fourier transform with respect to the modulation, we have that

$$i\widehat{M_a H M_{-a}} = \text{sgn}(\xi - a)\widehat{f}(\xi).$$

Hence,

$$\frac{1}{2}(i\widehat{M_a H M_{-a}} - i\widehat{M_b H M_{-b}}) = [\text{sgn}(\xi - a) - \text{sgn}(\xi - b)]\frac{\widehat{f}(\xi)}{2} = \chi_{(a,b)}\widehat{f}(\xi),$$

and we deduce (2.3). Finally, write $I_j = (a_j, b_j)$. Using the triangle inequality and our estimate for H,

$$\begin{aligned}
\Big\| \Big(\sum_{j\in\mathbb{Z}} |S_j f_j|^r \Big)^{1/r} \Big\|_p &\leq \frac{1}{2} \left[\Big\| \Big(\sum_{j\in\mathbb{Z}} |HM_{-a_j} f_j|^r \Big)^{1/r} \Big\|_p + \Big\| \Big(\sum_{j\in\mathbb{Z}} |HM_{-b_j} f_j|^r \Big)^{1/r} \Big\|_p \right] \\
&\leq \frac{C_{p,r}}{2} \left[\Big\| \Big(\sum_{j\in\mathbb{Z}} |M_{-a_j} f_j|^r \Big)^{1/r} \Big\|_p + \Big\| \Big(\sum_{j\in\mathbb{Z}} |M_{-b_j} f_j|^r \Big)^{1/r} \Big\|_p \right] \\
&= C_{p,r} \Big\| \Big(\sum_{j\in\mathbb{Z}} |f_j|^r \Big)^{1/r} \Big\|_p,
\end{aligned}$$

and we complete the proof. □

Chapter 3

The Hilbert Transform Along The Parabola

3.1 A Not-So-New Kernel

Consider the following map:

$$\begin{array}{rccl} K: & \mathcal{S}(\mathbb{R}^2) & \longrightarrow & \mathbb{C} \\ & \varphi & \longmapsto & \lim_{\varepsilon\to 0}\int_{|t|>\varepsilon}\frac{\varphi(t,t^2)}{t}dt. \end{array}$$

We have that K is well-defined, linear and continuous, and therefore, K is a tempered distribution. Indeed, K can be written as the composition of the maps

$$\begin{array}{ccccc} \mathcal{S}(\mathbb{R}^2) & \longrightarrow & \mathcal{S}(\mathbb{R}) & \longrightarrow & \mathbb{C} \\ \varphi & \longmapsto & \Phi_\varphi & \longmapsto & \text{p.v.}\frac{1}{x}(\Phi_\varphi), \end{array}$$

where $\Phi_\varphi(t) = \varphi(t,t^2)$. Observe that

$$\begin{aligned} \Phi_\varphi'(t) &= D^{(1,0)}\varphi(t,t^2) + 2tD^{(0,1)}\varphi(t,t^2), \\ \Phi_\varphi''(t) &= D^{(2,0)}\varphi(t,t^2) + 2tD^{(1,1)}\varphi(t,t^2) + 2D^{(0,1)}\varphi(t,t^2) + 2tD^{(1,1)}\varphi(t,t^2) + 4t^2D^{(0,2)}\varphi(t,t^2), \\ \Phi_\varphi'''(t) &= \cdots, \end{aligned}$$

so all the derivatives of Φ_φ are linear combinations of the form

$$\Phi_\varphi^{(m)}(t) = \sum_{|\alpha|\le m} c_\alpha t^{n_\alpha} D^\alpha\varphi(t,t^2),$$

where $m \in \mathbb{N}, \alpha \in \mathbb{N}^2, c_\alpha \in \mathbb{N}$ and $n_\alpha \in \mathbb{N}$. But

$$\begin{aligned} \|t^n\Phi_\varphi^{(m)}\|_\infty &\le \sum_{|\alpha|\le m} c_\alpha \sup_{t\in\mathbb{R}} |t^{n+n_\alpha}D^\alpha\varphi(t,t^2)| \\ &\le \sum_{|\alpha|\le m} c_\alpha \sup_{(x,y)\in\mathbb{R}^2} |x^{n+n_\alpha}D^\alpha\varphi(x,y)| \\ &= \sum_{|\alpha|\le m} c_\alpha \|x^{n+n_\alpha}D^\alpha\varphi\|_\infty < \infty, \end{aligned}$$

for any $n, m \in \mathbb{N}$, so we conclude that $\Phi_\varphi \in \mathcal{S}(\mathbb{R})$ for all $\varphi \in \mathcal{S}(\mathbb{R}^2)$. Moreover, using this estimate, we prove that $\Phi_{\varphi_n} \overset{n}{\to} 0$ in $\mathcal{S}(\mathbb{R})$ if $\varphi_n \overset{n}{\to} 0$ in $\mathcal{S}(\mathbb{R}^2)$, so we deduce that $K \in \mathcal{S}'(\mathbb{R}^2)$.

Definition 3.1. *Given a function $\varphi \in \mathcal{S}(\mathbb{R}^2)$, we define its Hilbert transform along the parabola $\Gamma(t) = (t, t^2)$ by*

$$H_\Gamma \varphi(x,y) = (K * \varphi)(x,y) = \lim_{\varepsilon \to 0} \int_{|t|>\varepsilon} \frac{\varphi(x-t, y-t^2)}{t} dt.$$

As we discussed[1] in Section 1.1, this defines a continuous function which is also a tempered distribution.

3.2 The Benedeck-Calderón-Panzone Theorem

Our first goal is to prove that H_Γ is bounded on $L^2(\mathbb{R}^2)$. In order to do so, we will introduce the Benedeck-Calderón-Panzone theorem, which ensures the L^2-boundedness of a convolution operator provided that the kernel satisfies certain properties.

Definition 3.2. *We say that a collection of matrices $\{A(t)\}_{t>0}$ satisfies the Rivière Condition if, for each $t > 0$, $A(t) \in GL(n, \mathbb{R})$ and there exist some constants $C \geq 1$ and $\varepsilon > 0$ such that*

$$\|A(st)^{-1}A(t)\| \leq \frac{C}{s^\varepsilon}, \quad \text{for all } s \geq 1, t > 0.$$

The norm $\|\cdot\|$ can be assumed to be the ℓ^2_n−operator norm.

Theorem 3.3 (Benedeck-Calderón-Panzone)**.** *Let T be a convolution operator with kernel K. Assume that there exists a collection of matrices $\{A(t)\}_{t>0}$ satisfying the Rivière condition and so that*

(i) $K = \sum_{j\in\mathbb{Z}} K_j$ *in $\mathcal{S}'(\mathbb{R}^n)$ with K_j supported on the dilation $A(2^{j+1})B(0,1)$ of the unit ball.*

(ii) If $\tilde{K}_j(x) := \det A(2^j) K_j(A(2^j)x)$ for all $x \in \mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} |\tilde{K}_j(x)| dx \leq C,$$

and for every $\xi \in \mathbb{R}^n$,

$$\left|\widehat{\tilde{K}}_j(\xi)\right| \leq \frac{C}{(1+|\xi|)^\varepsilon},$$

for some constants $C, \varepsilon > 0$.

Then, if K satisfies the cancellation property

$$\left| \int_{\mathbb{R}^n} \sum_{\alpha \leq j \leq \beta} K_j(x) dx \right| \leq C, \quad \forall \alpha, \beta \in \mathbb{Z}, \tag{3.1}$$

the operator $T : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ is bounded.

[1] In Section 1.1, we only dealt with Schwartz functions on the line, but the same proofs hold for functions on $\mathbb{R}^n$.

Remark 3.4. *The $\{\tilde{K}_j\}_j$ are called the normalized pieces of K. Notice that by the change of variable formula on $\mathbb{R}^n$, we have that*

$$\int_{\mathbb{R}^n} |\tilde{K}_j(x)|dx = \int_{\mathbb{R}^n} |K_j(A(2^j)x)||\det A(2^j)|dx = \int_{\mathbb{R}^n} |K_j(x)|dx.$$

Proof. In this proof, we will use the characterization in Theorem 1.18. Suppose that K satisfies the cancellation property (3.1). We want to show that $\widehat{K}$ is in $L^\infty(\mathbb{R}^n)$. Equivalently, using that the Fourier transform is continuous on $\mathcal{S}'(\mathbb{R}^n)$ and $K = \sum_j K_j$ in $\mathcal{S}'(\mathbb{R}^n)$, we will prove that

$$|\widehat{K}(\xi)| = \left|\sum_{j\in\mathbb{Z}} \widehat{K}_j(\xi)\right| \leq C \quad \text{a.e. } \xi \in \mathbb{R}^n.$$

Fix $\xi \in \mathbb{R}^n$ and assume that there exists a $j_0 \in \mathbb{Z}$ being the largest integer such that $|A(2^{j_0})^*\xi| \leq 1$, where $\cdot^*$ denotes the transpose matrix. Consider the identity

$$\sum_{\alpha\leq j\leq j_0-1} \widehat{K}_j(\xi) = \sum_{\alpha\leq j\leq j_0-1} (\widehat{K}_j(\xi) - \widehat{K}_j(0)) + \int_{\mathbb{R}^n} \sum_{\alpha\leq j\leq j_0-1} K_j(x)dx,$$

where we use that by hypothesis (ii), the pieces $\{K_j\}_j$ are integrable functions and thus its Fourier transform is computed as an integral. Using (3.1) with $\beta = j_0 - 1$ and the fact that K_j is supported on $A(2^{j+1})B(0,1)$, we get

$$\left|\sum_{\alpha\leq j\leq j_0-1} \widehat{K}_j(\xi)\right| \leq \sum_{\alpha\leq j\leq j_0-1} \int_{A(2^{j+1})B(0,1)} |K_j(x)(e^{-2\pi i x\cdot\xi} - 1)|dx + C. \tag{3.2}$$

Now, for all $x \in A(2^{j+1})B(0,1)$ there exists a $u_x \in B(0,1)$ such that $x = A(2^{j+1})u_x$, so

$$x\cdot\xi = (A(2^{j+1})u_x)\cdot\xi = u_x\cdot(A(2^{j+1})^*\xi).$$

With this, and using that on the unit circle the lenght of a chord between two points is less than the arc-lenght distance between them, we conclude by Cauchy-Schwarz that

$$|e^{-2\pi i x\cdot\xi} - 1| < 2\pi|x\cdot\xi| = 2\pi|u_x\cdot(A(2^{j+1})^*\xi)| \leq 2\pi|u_x||A(2^{j+1})^*\xi| \leq 2\pi|A(2^{j+1})^*\xi|.$$

Going back with this estimate to (3.2), we obtain

$$\left|\sum_{\alpha\leq j\leq j_0-1} \widehat{K}_j(\xi)\right| \leq 2\pi \sum_{j\leq j_0-1} \int_{A(2^{j+1})B(0,1)} |K_j(x)|dx|A(2^{j+1})^*\xi| + C.$$

By hypothesis (ii), we have that[2]

$$\int_{\mathbb{R}^n} |\tilde{K}_j(x)|dx = \int_{\mathbb{R}^n} |K_j(x)|dx \leq C,$$

[2] We will assume, without loss of generality, that all the constants C appearing in this proof are the same. Otherwise, take the maximum.

so we can write

$$\left|\sum_{\alpha\le j\le j_0-1}\widehat{K}_j(\xi)\right|\le C\left(\sum_{j\le j_0-1}|A(2^{j+1})^*\xi|+1\right).$$

Finally, recalling the properties of the transpose matrix, making use of the Rivière condition and the fact that $|A(2^{j_0})^*\xi|\le 1$, we obtain for every $j\le j_0-1$

$$\begin{aligned}|A(2^{j+1})^*\xi| &= |A(2^{j+1})^*(A(2^{j_0})^*)^{-1}A(2^{j_0})^*\xi|\\ &\le \|A(2^{j+1})^*(A(2^{j_0})^*)^{-1}\||A(2^{j_0})^*\xi|\\ &\le \|A(2^{j_0})^{-1}A(2^{j+1})\|\\ &= \|A(2^{j_0-(j+1)}\cdot 2^{j+1})^{-1}A(2^{j+1})\|\le \frac{C}{2^{(j_0-(j+1))\varepsilon}},\end{aligned}$$

where $t=2^{j+1}>0$ and $s=2^{j_0-(j+1)}\ge 1$ are the parameters in the Rivière condition and $\varepsilon>0$ is a constant. With this estimate,

$$\sum_{j\le j_0-1}|A(2^{j+1})^*\xi|\le C\sum_{j\le j_0-1}\frac{1}{2^{(j_0-(j+1))\varepsilon}}=C\sum_{k=0}^{\infty}\left(\frac{1}{2^\varepsilon}\right)^k<\infty,$$

and therefore,

$$\left|\sum_{\alpha\le j\le j_0-1}\widehat{K}_j(\xi)\right|\le C.$$

Since this is independent of α, we actually have

$$\left|\sum_{j\le j_0-1}\widehat{K}_j(\xi)\right|\le C.$$

If $j=j_0$, by (ii)

$$|\widehat{K}_{j_0}(\xi)|\le\int_{\mathbb{R}^n}|K_{j_0}(x)|dx=\int_{\mathbb{R}^n}|\tilde{K}_{j_0}(x)|dx\le C.$$

Let us now study what happens for $j>j_0$. Recall the following property of the Fourier transform: if $A\in GL(n,\mathbb{R})$, then

$$\mathcal{F}(\det(A^{-1})f(A^{-1}\cdot))(\xi)=(\mathcal{F}f)(A^*\xi).$$

Therefore, with this formula

$$\widehat{\tilde{K}}_j(\eta)=\mathcal{F}(\det(A(2^j))K_j(A(2^j)\cdot))(\eta)=\widehat{K}_j((A(2^j)^{-1})^*\eta),$$

so if we write $\xi=(A(2^j)^{-1})^*\eta$ and use hypothesis (ii), we obtain

$$\sum_{j>j_0}|\widehat{K}_j(\xi)|=\sum_{j>j_0}\left|\widehat{\tilde{K}}_j(A(2^j)^*\xi)\right|\le C\sum_{j>j_0}\frac{1}{(1+|A(2^j)^*\xi|)^\varepsilon}\le C\sum_{j>j_0}\frac{1}{|A(2^j)^*\xi|^\varepsilon}$$

Now, for every $j > j_0$, we can use the properties of the transpose matrix and the Rivière condition as before to conclude that

$$\begin{aligned}|A(2^{j_0+1})^*\xi| &\leq \|A(2^{j_0+1})^*(A(2^j)^*)^{-1}\||A(2^j)^*\xi| = \|A(2^j)^{-1}A(2^{j_0+1})\||A(2^j)^*\xi| \\ &= \|A(2^{j-(j_0+1)}2^{j_0+1})^{-1}A(2^{j_0+1})\||A(2^j)^*\xi| \leq \frac{C|A(2^j)^*\xi|}{2^{(j-(j_0+1))\varepsilon}}. \end{aligned} \tag{3.3}$$

Therefore, since $|A(2^{j_0+1})^*\xi| > 1$ by the definition of j_0,

$$|A(2^j)^*\xi| \geq \frac{2^{(j-(j_0+1))\varepsilon}|A(2^{j_0+1})^*\xi|}{C} > \frac{2^{(j-(j_0+1))\varepsilon}}{C},$$

so, going back to the series we were studying, we obtain the following estimate:

$$\sum_{j>j_0} |\widehat{K}_j(\xi)| \leq C \sum_{j>j_0} \frac{1}{|A(2^j)^*\xi|^\varepsilon} \leq C^{\varepsilon+1} \sum_{j>j_0} \frac{1}{2^{(j-(j_0+1))\varepsilon^2}},$$

which is a convergent geometric series. Summing up, we have that

$$\left|\sum_{j\in\mathbb{Z}} \widehat{K}_j(\xi)\right| \leq \left|\sum_{j\leq j_0-1} \widehat{K}_j(\xi)\right| + |\widehat{K}_{j_0}(\xi)| + \sum_{j>j_0} |\widehat{K}_j(\xi)| \leq C,$$

which is the sought-after inequality. In order to finish the proof, we have left to study what happens if j_0 does not exist. Fix $j \in \mathbb{Z}$ and assume that there exists a sequence of integers $\{j_k\}_k \subseteq \mathbb{Z}$ such that $j_k \xrightarrow{k} \infty$, $j_k > j$ and $|A(2^{j_k})^*\xi| \leq 1$ for all $k \geq 1$. As in (3.3), we have that

$$|A(2^{j+1})^*\xi| \leq \frac{C|A(2^{j_k})^*\xi|}{2^{(j_k-(j+1))\varepsilon}} \leq \frac{C}{2^{(j_k-(j+1))\varepsilon}} \xrightarrow{k} 0.$$

Therefore, we conclude that $A(2^{j+1})^*\xi = 0$. Since $A(2^{j+1})^*$ is invertible, this can only happen when $\xi = 0$ and we obtain that for every $\xi \in \mathbb{R}^n \setminus \{0\}$, we are in the case where $j_0 \in \mathbb{Z}$ exists and thus

$$|\widehat{K}(\xi)| \leq C \quad \text{a.e. } \xi \in \mathbb{R}^n.$$

This implies that $\|\widehat{K}\|_\infty \leq C$ and completes the proof. □

Remark 3.5. *Notice that if a collection of matrices $\{A(t)\}_{t>0}$ satisfy the Rivière condition, then, for every $j \in \mathbb{Z}$, there exists $k_0 \in \mathbb{Z}$, $k_0 > j$, such that*

$$A(2^j)B(0,1) \subseteq A(2^k)B(0,1), \quad \forall k \geq k_0.$$

Moreover, if the constant C in the Rivière condition is 1, then $k_0 = j + 1$.

Indeed, take $x \in A(2^j)B(0,1)$, that is, $x = A(2^j)u$ for some $u \in B(0,1)$. If $k > j$, we have that $x = A(2^k)A(2^k)^{-1}A(2^j)u$ and using the Rivière condition,

$$|A(2^k)^{-1}A(2^j)u| \leq \|A(2^k)^{-1}A(2^j)\| \leq \frac{C}{2^{(k-j)\varepsilon}} < 1,$$

provided that $k \geq k_0$ for some $k_0 \in \mathbb{Z}$, or $k \geq j + 1$ if $C = 1$.

Remark 3.6. *Notice that the Benedeck-Calderón-Panzone theorem is still true if the pieces $K_j = \mu_j$ are finite measures. Indeed, recall that by definition, the Fourier transform of a finite measure μ is computed as*

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} d\mu(x), \quad \xi \in \mathbb{R}^n.$$

The fact that this expression is very similar to the Fourier transform of an integrable function allows us to extend the theorem to finite measures with no need to change the proof. The hypotheses in this case would be:

(i) $K = \sum_{j\in\mathbb{Z}} \mu_j$ *in* $\mathcal{S}'(\mathbb{R}^n)$ *as before, with* μ_j *supported on the dilation* $A(2^{j+1})B(0,1)$*, that is,*

$$\int_E d\mu_j(x) = 0 \quad \textit{for every measurable set } E \subseteq \mathbb{R}^n \setminus A(2^{j+1})B(0,1).$$

(ii) The normalized pieces would be $\tilde{\mu}_j(\cdot) = \det(A(2^j))(\mu_j \circ A(2^j))(\cdot)$*. Now recall that the composition of a distribution* u *with an invertible matrix* A *is the distribution*

$$\langle u \circ A, \varphi\rangle = |\det A|^{-1}\langle u, \varphi \circ A^{-1}\rangle.$$

Therefore, for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$*,*

$$\tilde{\mu}_j(\varphi) = \operatorname{sgn}(\det(A(2^j)))\mu_j(\varphi(A(2^j)^{-1}\cdot)).$$

Moreover, they would have to satisfy that

$$\int_{\mathbb{R}^n} d|\tilde{\mu}_j|(x) \leq C$$

and, for every $\xi \in \mathbb{R}^n$*,*

$$\left|\widehat{\tilde{\mu}}_j(\xi)\right| = \left|\int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} d\tilde{\mu}_j(x)\right| \leq \frac{C}{(1+|\xi|)^{\varepsilon}}$$

for some constants $C, \varepsilon > 0$*.*

The cancellation property can be written as

$$\left|\int_{\mathbb{R}^n} \sum_{\alpha\leq j\leq\beta} d\mu_j(x)\right| \leq C, \quad \forall \alpha, \beta \in \mathbb{Z}.$$

In the next section, we will see that the Hilbert transform H_Γ along the parabola $\Gamma(t) = (t, t^2)$ has a convolution kernel that can be written as a series of finite measures. Hence, in order to apply Benedeck-Calderón-Panzone's theorem and prove the L^2-boundedness of H_Γ, we will have to show that these finite measures satisfy all the conditions gathered in this last remark.

3.3 L^2-Boundedness Of The Hilbert Transform Along The Parabola

At the beginning of this chapter, we defined the following kernel $K \in \mathcal{S}'(\mathbb{R}^2)$:

$$\begin{array}{rccl} K: & \mathcal{S}(\mathbb{R}^2) & \longrightarrow & \mathbb{C} \\ & \varphi & \longmapsto & \lim_{\varepsilon\to 0} \int_{|t|>\varepsilon} \frac{\varphi(t,t^2)}{t} dt. \end{array}$$

We also defined the Hilbert transform along the parabola $\Gamma(t) = (t, t^2)$ applied to a Schwartz function $\varphi \in \mathcal{S}(\mathbb{R}^2)$ as its convolution with K,

$$H_\Gamma \varphi(x, y) = (K * \varphi)(x, y) = \lim_{\varepsilon\to 0} \int_{|t|>\varepsilon} \frac{\varphi(x - t, y - t^2)}{t} dt.$$

In this section, we will use Benedeck-Calderón-Panzone's theorem to prove that H_Γ is actually a convolution operator of strong-type $(2, 2)$. Consider the following parabolic dilations:

$$A(t) = \begin{pmatrix} t & 0 \\ 0 & t^2 \end{pmatrix}, \quad t > 0.$$

For every $j \in \mathbb{Z}$, let K_j be the distribution defined by

$$\begin{array}{rccl} K_j: & \mathcal{S}(\mathbb{R}^2) & \longrightarrow & \mathbb{C} \\ & \varphi & \longmapsto & \int_{2^j\leq|t|\leq 2^{j+1}} \frac{\varphi(t,t^2)}{t} dt. \end{array}$$

These $\{K_j\}_j$ will be the pieces into which we will decompose K. Let us now check that, with these ingredients, we are under the hypotheses of Benedeck-Calderón-Panzone's theorem.

- The parabolic dilations $\{A(t)\}_{t>0}$ satisfy the Rivière condition with constants $C = \varepsilon = 1$, since $A(t) \in GL(2, \mathbb{R})$ for all $t > 0$ and, if we take $s \geq 1$ and consider the Euclidean norm,

$$\|A(st)^{-1}A(t)\| = \left\| \begin{pmatrix} (st)^{-1} & 0 \\ 0 & (st)^{-2} \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t^2 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 1/s & 0 \\ 0 & 1/s^2 \end{pmatrix} \right\| = \frac{1}{s}.$$

- The pieces $K_j = \mu_j$ are finite measures, which is enough to apply the Benedeck-Calderón-Panzone theorem (see Remark 3.6). If $E \subseteq \mathbb{R}^2$ is a Lebesgue-measurable set,

$$\mu_j(E) = \int_{I_j} \chi_E(t, t^2) \frac{dt}{t},$$

where $I_j := [-2^{j+1}, -2^j] \cup [2^j, 2^{j+1}]$. Notice that

$$\mu_j(\mathbb{R}^2) = \int_{I_j} \frac{dt}{t} = 0,$$

since I_j is symmetric. Now, if $\varphi \in \mathcal{S}(\mathbb{R}^2)$, these measures act on φ as follows:

$$\mu_j(\varphi) := \int_{\mathbb{R}^2} \varphi(x) d\mu_j(x) = \int_{I_j} \varphi(t, t^2) \frac{dt}{t}. \tag{3.4}$$

To prove the second equality, recall that if $s = \sum_{i=1}^k a_i \chi_{E_i}$ is a simple function, then

$$\int_{\mathbb{R}^2} s(x)d\mu_j(x) = \sum_{i=1}^k a_i \mu_j(E_i) = \sum_{i=1}^k a_i \int_{I_j} \chi_{E_i}(t,t^2)\frac{dt}{t} = \int_{I_j} s(t,t^2)\frac{dt}{t},$$

so by the monotone convergence theorem, we get (3.4). Moreover, it is easy to see that the measures μ_j are supported on the dilation $A(2^{j+1})B(0,1)$, since

$$\{(t,t^2) \in \mathbb{R}^2 : t \in I_j\} \subseteq A(2^{j+1})B(0,1).$$

- We also have that

$$K = \sum_{j\in\mathbb{Z}} \mu_j \quad \text{in } \mathcal{S}'(\mathbb{R}^2).$$

Indeed, take $\varphi \in \mathcal{S}(\mathbb{R}^2)$,

$$\begin{aligned}
\lim_{n\to\infty} \sum_{j=-n}^{n} \mu_j(\varphi) &= \lim_{n\to\infty} \int_{2^{-n}\le|t|\le 2^{n+1}} \varphi(t,t^2)\frac{dt}{t} \\
&= \lim_{n\to\infty} \left[\int_{2^{-n}\le|t|<1} \varphi(t,t^2)\frac{dt}{t} + \int_{1<|t|\le 2^{n+1}} \varphi(t,t^2)\frac{dt}{t} \right] \\
&= \lim_{\varepsilon\to 0} \int_{\varepsilon\le|t|<1} \varphi(t,t^2)\frac{dt}{t} + \int_{1<|t|} \varphi(t,t^2)\frac{dt}{t} \\
&= \lim_{\varepsilon\to 0} \int_{|t|>\varepsilon} \varphi(t,t^2)\frac{dt}{t} \\
&= K(\varphi),
\end{aligned}$$

where, in the third equality, we use the dominated converge theorem together with

$$\left| \frac{\varphi(t,t^2)}{t} \chi_{\{1<|t|\le 2^{n+1}\}}(t) \right| \le \frac{\|x\varphi\|_\infty}{t^2} \chi_{\{|t|>1\}} \in L^1(\mathbb{R}).$$

- The normalized pieces $\tilde{\mu}_j$ act on a function $\varphi \in \mathcal{S}(\mathbb{R}^2)$ as we described in Remark 3.6:

$$\begin{aligned}
\tilde{\mu}_j(\varphi) &= \operatorname{sgn}(\det(A(2^j)))\mu_j(\varphi(A(2^j)^{-1}\cdot)) = \int_{2^j\le|t|\le 2^{j+1}} \varphi(2^{-j}t, 2^{-2j}t^2)\frac{dt}{t} \\
&= \int_{1\le|t|\le 2} \varphi(t,t^2)\frac{dt}{t}.
\end{aligned}$$

Moreover, if E denotes a measurable set,

$$\tilde{\mu}_j^+(E) = \int_1^2 \chi_E(t,t^2)\frac{dt}{t} \quad \text{and} \quad \tilde{\mu}_j^-(E) = -\int_{-2}^{-1} \chi_E(t,t^2)\frac{dt}{t},$$

and therefore,

$$|\tilde{\mu}_j|(E) = \int_1^2 (\chi_E(t,t^2) + \chi_E(-t,t^2))\frac{dt}{t}. \tag{3.5}$$

With this expression, we can check that

$$\int_{\mathbb{R}^2} d|\tilde{\mu}_j|(x) = |\tilde{\mu}_j|(\mathbb{R}^2) = 2\ln 2,$$

and thus, the hypothesis

$$\int_{\mathbb{R}^2} d|\tilde{\mu}_j|(x) \leq C$$

is satisfied.

- The cancellation hypothesis is trivially fulfilled:

$$\left| \int_{\mathbb{R}^2} \sum_{\alpha \leq j \leq \beta} d\mu_j(x) \right| = \left| \sum_{\alpha \leq j \leq \beta} \mu_j(\mathbb{R}^2) \right| = 0,$$

since we checked that $\mu_j(\mathbb{R}^2) = 0$ for all $j \in \mathbb{Z}$.

- The last condition that we need in order to apply Benedeck-Calderón-Panzone's theorem is the following:

$$\left|\widehat{\tilde{\mu}}_j(\xi)\right| = \left| \int_{\mathbb{R}^2} e^{-2\pi i \xi \cdot x} d\tilde{\mu}_j(x) \right| = \left| \int_{1 \leq |t| \leq 2} e^{-2\pi i \xi \cdot (t,t^2)} \frac{dt}{t} \right| \leq \frac{C}{(1+|\xi|)^\varepsilon}.$$

We claim that it is enough to prove that

$$\left|\widehat{\tilde{\mu}}_j(\xi)\right| \leq \frac{C}{|\xi|^\varepsilon}.$$

Indeed, assume that we have this last bound. Since $\tilde{\mu}_j$ is a finite measure, we know that $\widehat{\tilde{\mu}}_j$ is a bounded function, say by C. Then, if we define $C' := C2^\varepsilon$, let us check that we have

$$\left|\widehat{\tilde{\mu}}_j(\xi)\right| \leq \frac{C'}{(1+|\xi|)^\varepsilon}.$$

If $0 \leq |\xi| \leq 1$,

$$\left|\widehat{\tilde{\mu}}_j(\xi)\right| \leq C = \frac{C2^\varepsilon}{2^\varepsilon} \leq \frac{C'}{(1+|\xi|)^\varepsilon},$$

and if $|\xi| > 1$, then

$$\left|\widehat{\tilde{\mu}}_j(\xi)\right| \leq \frac{C}{|\xi|^\varepsilon} = \frac{C2^\varepsilon}{(2|\xi|)^\varepsilon} \leq \frac{C'}{(1+|\xi|)^\varepsilon}.$$

So our goal will be to show that

$$\left| \int_{1 \leq |t| \leq 2} e^{-2\pi i \xi \cdot (t,t^2)} \frac{dt}{t} \right| \leq \frac{C}{|\xi|^{1/2}}, \tag{3.6}$$

where ε is taken to be $1/2$. First, we will have to go back to Van der Corput's lemma (Lemma 1.29) to prove the following result:

Lemma 3.7. *Let*

$$I(t) = \int_1^t e^{-2\pi i(s\xi_1 + s^2\xi_2)} ds.$$

Then, for $1 < t < 2$ *and* $|\xi_1| > 1$*, we have that*

$$|I(t)| \leq \frac{8}{|\xi_1|^{1/2}}.$$

Proof. Define $h(s) := -2\pi(s\xi_1 + s^2\xi_2)$ for $1 \leq s \leq t$. We have that

$$|h'(s)| = |-2\pi\xi_1 - 4\pi s\xi_2| = 2\pi|\xi_1 + 2s\xi_2| \geq 2\pi||\xi_1| - 2s|\xi_2||.$$

(i) If $|\xi_1| \geq 8|\xi_2|$, then

$$|h'(s)| \geq 2\pi(|\xi_1| - 4|\xi_2|) \geq \pi|\xi_1|,$$

since $1 \leq s \leq t < 2$. Using that h' is monotonic, we can apply Van der Corput's lemma and we get that

$$|I(t)| \leq \frac{C_1}{\pi|\xi_1|} \leq \frac{C_1}{\pi|\xi_1|^{1/2}},$$

recalling our condition $|\xi_1| > 1$.

(ii) If $|\xi_1| < 8|\xi_2|$, we use that $|h''(s)| = |-4\pi\xi_2| = 4\pi|\xi_2|$ and again by Van der Corput's lemma,

$$|I(t)| \leq \frac{C_2}{2\sqrt{\pi}|\xi_2|^{1/2}} \leq \frac{\sqrt{8}C_2}{2\sqrt{\pi}|\xi_1|^{1/2}}.$$

Therefore, since[3]

$$\max\left\{\frac{C_1}{\pi}, \frac{\sqrt{8}C_2}{2\sqrt{\pi}}\right\} = \frac{\sqrt{8}C_2}{2\sqrt{\pi}} = \frac{10\sqrt{2}}{\sqrt{\pi}} \approx 7.98,$$

we obtain the desired estimate. □

Finally, we will show that (3.6) holds. Consider the $\mathbb{R}^2$-norm $|\xi| = \max\{|\xi_1|, |\xi_2|\}$. If we integrate by parts, we get that

$$\begin{aligned}
\left|\int_{1\leq|t|\leq 2} e^{-2\pi i\xi\cdot(t,t^2)}\frac{dt}{t}\right| &\leq \left|\int_{-2}^{-1} e^{-2\pi i(t\xi_1+t^2\xi_2)}\frac{dt}{t}\right| + \left|\int_1^2 e^{-2\pi i(t\xi_1+t^2\xi_2)}\frac{dt}{t}\right| \\
&= \left|\int_1^2 e^{-2\pi i(-t\xi_1+t^2\xi_2)}\frac{dt}{t}\right| + \left|\int_1^2 e^{-2\pi i(t\xi_1+t^2\xi_2)}\frac{dt}{t}\right| \\
&= \left|\frac{I_-(2)}{2} + \int_1^2 I_-(t)\frac{dt}{t^2}\right| + \left|\frac{I_+(2)}{2} + \int_1^2 I_+(t)\frac{dt}{t^2}\right| \\
&\leq \frac{|I_-(2)|}{2} + \sup_{t\in(1,2)}\frac{|I_-(t)|}{2} + \frac{|I_+(2)|}{2} + \sup_{t\in(1,2)}\frac{|I_+(t)|}{2},
\end{aligned} \tag{3.7}$$

where

$$I_\pm(t) = \int_1^t e^{-2\pi i(\pm s\xi_1 + s^2\xi_2)} ds$$

as in the previous lemma with ξ_1 and $-\xi_1$ respectively.

[3] Recall that the constant in Van der Corput's lemma is $C_k = 3\cdot 2^k - 2$.

(i) If $|\xi| = |\xi_2|$, defining $h_\pm(s) := -2\pi(\pm s\xi_1 + s^2\xi_2)$, we have that $|h''_\pm(s)| = 4\pi|\xi_2|$ and by Van der Corput's lemma, we obtain that

$$|I_\pm(t)| \leq \frac{C_2}{2\sqrt{\pi}|\xi_2|^{1/2}}, \quad 1 \leq t \leq 2.$$

Inserting this in (3.7),

$$\left| \int_{1\leq|t|\leq 2} e^{-2\pi i\xi\cdot(t,t^2)} \frac{dt}{t} \right| \leq \frac{C_2}{\sqrt{\pi}|\xi_2|^{1/2}}.$$

(ii) If $|\xi| = |\xi_1|$ and $|\xi_1| > 1$, by Lemma 3.7 we also have that

$$\left| \int_{1\leq|t|\leq 2} e^{-2\pi i\xi\cdot(t,t^2)} \frac{dt}{t} \right| \leq \frac{16}{|\xi_1|^{1/2}}.$$

(iii) If $|\xi| = |\xi_1|$ and $|\xi_1| \leq 1$, we can estimate roughly

$$\left| \int_{1\leq|t|\leq 2} e^{-2\pi i\xi\cdot(t,t^2)} \frac{dt}{t} \right| \leq \int_{1\leq|t|\leq 2} \frac{1}{|t|} dt = 2\ln 2 \leq \frac{2\ln 2}{|\xi_1|^{1/2}}.$$

Hence, since $C_2 = 10$, if we take

$$C = \max\left\{ \frac{C_2}{\sqrt{\pi}}, 16, 2\ln 2 \right\} = 16,$$

we obtain our last condition

$$\left| \int_{1\leq|t|\leq 2} e^{-2\pi i\xi\cdot(t,t^2)} \frac{dt}{t} \right| \leq \frac{C}{|\xi|^{1/2}}.$$

Summing up, in this section we have checked that the operator H_Γ satisfies all the hypotheses of Benedeck-Calderón-Panzone's theorem. Therefore, we have proved that

$$H_\Gamma : L^2(\mathbb{R}^2) \longrightarrow L^2(\mathbb{R}^2)$$

is bounded.

3.4 L^p-Boundedness Of The Hilbert Transform Along The Parabola

In the next pages, we will prove that the Hilbert transform along $\Gamma(t) = (t, t^2)$,

$$H_\Gamma f(x,y) = \lim_{\varepsilon\to 0} \int_{|t|>\varepsilon} f(x-t, y-t^2) \frac{dt}{t},$$

and the maximal function

$$M_\Gamma f(x,y) = \sup_{h>0} \frac{1}{2h} \left| \int_{-h}^{h} f(x-t, y-t^2) dt \right|,$$

are bounded operators from $L^p(\mathbb{R}^2)$ to itself for $1 < p < \infty$ and $1 < p \leq \infty$ respectively. In order to do so, we will follow a similar approach as when proving that H_Γ is of strong-type $(2,2)$. For every $j \in \mathbb{Z}$, recall the finite measures μ_j which acted on functions by

$$\mu_j(\varphi) = \int_{I_j} \varphi(t,t^2)\frac{dt}{t}, \tag{3.8}$$

where, following our notation, $I_j = [-2^{j+1}, -2^j] \cup [2^j, 2^{j+1}]$. We had that, for every $f \in \mathcal{S}(\mathbb{R}^2)$,

$$H_\Gamma f = \sum_{j\in\mathbb{Z}} \mu_j * f, \quad \text{in } \mathcal{S}'(\mathbb{R}^2).$$

Define also finite measures σ_j such that, for every $\varphi \in \mathcal{S}(\mathbb{R}^2)$,

$$\sigma_j(\varphi) = \frac{1}{2^{j+1}} \int_{I_j} \varphi(t,t^2)dt. \tag{3.9}$$

Let us check that

$$M_\Gamma f \leq 2 \sup_j \sigma_j * |f|. \tag{3.10}$$

Indeed,

$$\sigma_j * |f| = \frac{1}{2^{j+1}} \int_{I_j} |f(x-t, y-t^2)|dt.$$

Fix $2^j \leq h \leq 2^{j+1}$. Using the monotone convergence theorem, it holds that

$$\begin{aligned}
\frac{1}{2h}\left| \int_{-h}^{h} f(x-t,y-t^2)dt \right| &\leq \frac{1}{2^{j+1}} \int_{-2^{j+1}}^{2^{j+1}} |f(x-t,y-t^2)|dt \\
&= \frac{1}{2^{j+1}} \sum_{i\leq j} \int_{2^i \leq |t| \leq 2^{i+1}} |f(x-t,y-t^2)|dt \\
&= \frac{1}{2^{j+1}} \sum_{i\leq j} 2^{i+1}\sigma_i * |f| \leq \sup_{i\leq j}(\sigma_i * |f|) \sum_{i \leq j} \frac{2^{i+1}}{2^{j+1}},
\end{aligned}$$

therefore, since the last series equals 2, we deduce that

$$\begin{aligned}
M_\Gamma f &= \sup_{h>0} \frac{1}{2h}\left| \int_{-h}^{h} f(x-t,y-t^2)dt \right| \\
&= \sup_{j\in\mathbb{Z}} \sup_{2^j \leq h \leq 2^{j+1}} \frac{1}{2h}\left| \int_{-h}^{h} f(x-t,y-t^2)dt \right| \\
&\leq \sup_{j\in\mathbb{Z}} \sup_{i\leq j} 2\sigma_i * |f| = 2 \sup_{j\in\mathbb{Z}} \sigma_j * |f|,
\end{aligned}$$

as we claimed. The next step will be to prove a couple of general results from which we will deduce the boundedness of H_Γ and M_Γ. Given a measure μ, we will denote by $|\mu|$ its total variation measure and by $\|\mu\|$, its total variation.

Theorem 3.8. *Let $\{\mu_j\}_j$ be a sequence of finite Borel measures on $\mathbb{R}^2$ with $\|\mu_j\| \leq C$ and such that for some $a > 0$,*

$$|\widehat{\mu}_j(\xi)| \leq C \min\{|2^j \xi_1|^a, |2^j \xi_1|^{-a}\}.$$

If the operator

$$\mu^*(f) = \sup_j ||\mu_j| * f|$$

is a bounded operator on $L^q(\mathbb{R}^2)$ for some $q > 1$, then the operators

$$Tf = \sum_{j \in \mathbb{Z}} \mu_j * f \quad \text{and} \quad g(f) = \left(\sum_{j \in \mathbb{Z}} |\mu_j * f|^2 \right)^{1/2}$$

are bounded on $L^p(\mathbb{R}^2)$ for values of p satisfying

$$\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{1}{2q}.$$

Proof. Define the operator S_j by

$$\widehat{S_j f}(\xi) = \chi_{I_j}(\xi_1) \widehat{f}(\xi), \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2,$$

where $I_j = (-2^{j+1}, -2^j] \cup [2^j, 2^{j+1})$. With this notation, we can write

$$Tf = \sum_{k \in \mathbb{Z}} T_k f,$$

where

$$T_k f = \sum_{j \in \mathbb{Z}} \mu_j * S_{k-j} f.$$

Indeed, using that the Fourier transform is continuous on $\mathcal{S}'(\mathbb{R}^2)$,

$$\sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \widehat{\mu_j * S_{k-j} f} = \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \widehat{\mu}_j \chi_{I_{k-j}} \widehat{f} = \sum_{j \in \mathbb{Z}} \widehat{\mu}_j \widehat{f} = \sum_{j \in \mathbb{Z}} \widehat{\mu_j * f} = \widehat{Tf}.$$

Now, let us prove that

$$\|T_k f\|_2 \leq C 2^{-a|k|} \|f\|_2, \tag{3.11}$$

or equivalently, that the multiplier satisfies

$$\left\| \sum_{j \in \mathbb{Z}} \widehat{\mu}_j(\xi) \chi_{I_{k-j}}(\xi_1) \right\|_\infty \leq C 2^{-a|k|}.$$

Indeed, take $\xi \in \mathbb{R}^2$ and let $j_0 \in \mathbb{Z}$ such that $\xi_1 \in I_{k-j_0}$ (that is $|\xi_1| \approx 2^{k-j_0}$). Then, using that $I_i \cap I_j = \emptyset$ if $i \neq j$ and our hypothesis on $|\widehat{\mu}_j(\xi)|$, we get that

$$\begin{aligned} \left| \sum_{j \in \mathbb{Z}} \widehat{\mu}_j(\xi) \chi_{I_{k-j}}(\xi_1) \right| &= |\widehat{\mu}_{j_0}(\xi)| \leq C \min\{|2^{j_0} \xi_1|^a, |2^{j_0} \xi_1|^{-a}\} \\ &\approx C \min\{(2^{j_0} 2^{k-j_0})^a, (2^{j_0} 2^{k-j_0})^{-a}\} = C 2^{-a|k|}. \end{aligned}$$

Since this holds for every $\xi \in \mathbb{R}^2$, we deduce our claim. Now, define p_0 such that

$$\frac{1}{2} - \frac{1}{p_0} = \frac{1}{2q},$$

or in other words, such that $q = (p_0/2)'$. Given a function g, we have that

$$\begin{aligned}|\mu_j * g|^2 &= \left|\int_{\mathbb{R}^2} g(x-y)d\mu_j(y)\right|^2 \leq \left(\int_{\mathbb{R}^2} |g(x-y)|d|\mu_j|(y)\right)^2 \\ &= \|\mu_j\|^2 \left(\int_{\mathbb{R}^2} |g(x-y)|d\frac{|\mu_j|}{\|\mu_j\|}(y)\right)^2 \\ &\leq \|\mu_j\|^2 \int_{\mathbb{R}^2} |g(x-y)|^2 d\frac{|\mu_j|}{\|\mu_j\|}(y) \\ &= \|\mu_j\|(|\mu_j| * |g|^2) \leq C(|\mu_j| * |g|^2),\end{aligned}$$

making use of Jensen's inequality and our hypothesis $\|\mu_j\| \leq C$. Using duality and the previous estimate, we obtain

$$\begin{aligned}\left\|\left(\sum_{j\in\mathbb{Z}} |\mu_j * g_j|^2\right)^{1/2}\right\|_{p_0}^2 &= \left\|\sum_{j\in\mathbb{Z}} |\mu_j * g_j|^2\right\|_{p_0/2} = \int_{\mathbb{R}^2} \sum_{j\in\mathbb{Z}} |\mu_j * g_j|^2 u \\ &\leq C \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^2} (|\mu_j| * |g_j|^2)u,\end{aligned}$$

for some function $u \in L^q(\mathbb{R}^2)$ with $\|u\|_q = 1$. Now, an easy computation and Fubini's theorem yield

$$\int_{\mathbb{R}^2} (|\mu_j| * |g_j(x)|^2)u(x)dx = \int_{\mathbb{R}^2} (|\mu_j| * |\tilde{u}(-x)|^2)|g_j(x)|^2 dx \leq \int_{\mathbb{R}^2} |g_j|^2 \mu^*(\tilde{u}), \tag{3.12}$$

where $\tilde{u}(x) = u(-x)$ and $\mu^*(f) = \sup_j ||\mu_j| * f|$. Finally, if we go back to our previous computations with this estimate, Hölder's inequality and the fact that μ^* is bounded from $L^q(\mathbb{R}^2)$ to itself, we conclude that

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^2} (|\mu_j| * |g_j|^2)u &\leq \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^2} |g_j|^2 \mu^*(\tilde{u}) = \int_{\mathbb{R}^2} \sum_{j\in\mathbb{Z}} |g_j|^2 \mu^*(\tilde{u}) \\ &\leq \left\|\sum_{j\in\mathbb{Z}} |g_j|^2\right\|_{p_0/2} \|\mu^*(\tilde{u})\|_q \leq C \left\|\left(\sum_{j\in\mathbb{Z}} |g_j|^2\right)^{1/2}\right\|_{p_0}^2.\end{aligned}$$

Hence, we have proved that

$$\left\|\left(\sum_{j\in\mathbb{Z}} |\mu_j * g_j|^2\right)^{1/2}\right\|_{p_0} \leq C \left\|\left(\sum_{j\in\mathbb{Z}} |g_j|^2\right)^{1/2}\right\|_{p_0}. \tag{3.13}$$

Using the lower bound of Theorem 1.30, we have that

$$\|T_k f\|_{p_0} \leq C \left\| \left(\sum_{i\in\mathbb{Z}} |S_i T_k f|^2 \right)^{1/2} \right\|_{p_0},$$

but since $S_i S_j f = 0$ if $i \neq j$,

$$S_i T_k f = \sum_{j\in\mathbb{Z}} \mu_j * S_i(S_{k-j} f) = \mu_{k-i} * S_i f.$$

Therefore, using now (3.13) and the upper bound of Theorem 1.30,

$$\begin{aligned}
\|T_k f\|_{p_0} &\leq C \left\| \left(\sum_{i\in\mathbb{Z}} |\mu_{k-i} * S_i f|^2 \right)^{1/2} \right\|_{p_0} = C \left\| \left(\sum_{i\in\mathbb{Z}} |\mu_i * S_{k-i} f|^2 \right)^{1/2} \right\|_{p_0} \\
&\leq C \left\| \left(\sum_{i\in\mathbb{Z}} |S_{k-i} f|^2 \right)^{1/2} \right\|_{p_0} \leq C\|f\|_{p_0}.
\end{aligned}$$

Finally, interpolating between this estimate and (3.11), we get that for $2 \leq p \leq p_0$,

$$\|T_k f\|_p \leq C_p 2^{-a|k|(1-\theta)} C_{p_0}^{\theta} \|f\|_p,$$

with $\theta = 0$ if $p = 2$ and $\theta = 1$ if $p = p_0$. Therefore, if we take $2 \leq p < p_0$,

$$\|Tf\|_p \leq \sum_{k\in\mathbb{Z}} \|T_k f\|_p \leq C_p C_{p_0}^{\theta} \|f\|_p \sum_{k\in\mathbb{Z}} 2^{-a|k|(1-\theta)} = C_p \|f\|_p,$$

since $p < p_0$ implies $1 - \theta > 0$. Recall that we wanted to prove that T is bounded for every p satisfying

$$\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{1}{2q}.$$

If we compute these values, we observe that we want to have boundedness for every $p_0' < p < p_0$. If $2 \leq p < p_0$, we are done. If $p_0' < p \leq 2$, we know that T is bounded on $L^{p'}(\mathbb{R}^2)$, so we will proceed by duality and the same argument as in (3.12):

$$\begin{aligned}
\|Tf\|_p &= \sup_{\|g\|_{p'}=1} \left| \int_{\mathbb{R}^2} Tf \cdot g \right| = \sup_{\|g\|_{p'}=1} \left| \int_{\mathbb{R}^2} \sum_{j\in\mathbb{Z}} [\mu_j * f(x)] g(x) dx \right| \\
&= \sup_{\|g\|_{p'}=1} \left| \int_{\mathbb{R}^2} \sum_{j\in\mathbb{Z}} [\mu_j * \tilde{g}(-x)] f(x) dx \right| = \sup_{\|g\|_{p'}=1} \left| \int_{\mathbb{R}^2} T\tilde{g} \cdot \tilde{f} \right| \\
&\leq \sup_{\|g\|_{p'}=1} \|T\tilde{g}\|_{p'} \|\tilde{f}\|_p \leq C\|\tilde{f}\|_p = C\|f\|_p
\end{aligned}$$

In order to finish the proof, we have left to show that for $p_0' < p < p_0$, the operator

$$g(f) = \left(\sum_{j\in\mathbb{Z}} |\mu_j * f|^2 \right)^{1/2}$$

is also bounded. First note that given a sequence $\varepsilon = \{\varepsilon_j\}_j$ such that $\varepsilon_j = \pm 1$ for each $j \in \mathbb{Z}$, if we define the operator

$$T_\varepsilon f = \sum_{j\in\mathbb{Z}} \varepsilon_j \mu_j * f,$$

then T_ε is bounded on $L^p(\mathbb{R}^2)$ for $p_0' < p < p_0$, since the measures $\{\varepsilon_j\mu_j\}_j$ satisfy the same conditions as $\{\mu_j\}_j$ and the first part of the theorem applies. The bound for T_ε will be the same as for T, and thus independent of the sequence ε:

$$\|T_\varepsilon f\|p \le C_p \|f\|_p, \quad p_0' < p < p_0. \tag{3.14}$$

Now, recall that the Rademacher functions are defined by

$$r_0(t) = \begin{cases} -1 & \text{if } 0 \le t < 1/2, \\ 1 & \text{if } 1/2 \le t < 1, \end{cases}$$

and for $j \ge 1$, $r_j(t) = r_0(2^j t)$, where r_0 is extended to the whole real line periodically. We also know that these functions $\{r_j\}_{j\ge 0}$ form an orthonormal system in $L^2(0,1)$ and that given

$$F(t) = \sum_{j=0}^{\infty} a_j r_j(t) \in L^2(0,1),$$

we have that $F \in L^p(0,1)$ for $1 < p < \infty$ and there exist positive constants A_p and B_p such that

$$A_p \|F\|_p \le \left(\sum_{j=0}^{\infty} |a_j|^2 \right)^{1/2} \le B_p \|F\|_p.$$

This inequality is the so-called Khintchine's inequality and can be found in [13, p. 104]. Now, if we index the integers from 0 to infinity ($\mathbb{Z} = \{k_j\}_{j\ge 0}$), then for every $p_0' < p < p_0$,

$$g(f)(x)^p = \left(\sum_{j=0}^{\infty} |\mu_{k_j} * f(x)|^2 \right)^{p/2} \le B_p^p \int_0^1 \left| \sum_{j=0}^{\infty} \mu_{k_j} * f(x) r_j(t) \right|^p dt.$$

Integrating with respect to x and using Fubini's theorem, we conclude

$$\|g(f)\|_p^p \le B_p^p \int_{\mathbb{R}^2} \int_0^1 \left| \sum_{j=0}^{\infty} r_j(t) \mu_{k_j} * f(x) \right|^p dt dx = B_p^p \int_0^1 \int_{\mathbb{R}^2} \left| \sum_{j=0}^{\infty} r_j(t) \mu_{k_j} * f(x) \right|^p dx dt$$
$$= B_p^p \int_0^1 \|T_{\varepsilon_t} f\|_p^p dt \le B_p^p C_p^p \|f\|_p^p,$$

where the operator T_{ε_t} satisfies (3.14) with

$$\varepsilon_t = \{(\varepsilon_t)_{k_j}\}_{j\ge 0} = \{r_j(t)\}_{j\ge 0}.$$

With this, we prove that g is of strong-type (p,p) for $p_0' < p < p_0$ and we finish the proof of the theorem. □

Remark 3.9. *If we go over the proof of this last theorem, we notice that just from the hypotheses that $\{\mu_j\}_j$ is a sequence of finite Borel measures on $\mathbb{R}^2$ with $\|\mu_j\| \le C$ and*

$$|\widehat{\mu}_j(\xi)| \le C \min\{|2^j\xi_1|^a, |2^j\xi_1|^{-a}\}, \quad \text{for some } a > 0,$$

we deduce that both T and g are bounded on $L^2(\mathbb{R}^2)$.

Before turning to the second theorem, we will need to state the following majorization lemma:

Lemma 3.10. *Let φ be a function defined on $\mathbb{R}^n$ which is positive, radial, decreasing as a function of $|x|$ and integrable. Then,*

$$\sup_{t>0} |\varphi_t * f(x)| \le \|\varphi\|_1 Mf(x),$$

where $\varphi_t(x) = t^{-n}\varphi(t^{-1}x)$ and M is the Hardy-Littlewood maximal operator.

Proof. The proof of this lemma is straightforward: we first assume that φ is also a simple function and then we approximate an arbitrary φ by a sequence of simple functions which increase to it monotonically. The details can be found in [9, Prop 2.7]. □

In our case, we will need this lemma with $n = 1$ and $t = 2^j$ for $j \in \mathbb{Z}$.

Theorem 3.11. *Let $\{\sigma_j\}_j$ be a sequence of positive Borel measures on $\mathbb{R}^2$ with $\|\sigma_j\| \le C$ and such that for some $0 < a < 1$,*

$$|\widehat{\sigma}_j(\xi)| \le C|2^j\xi_1|^{-a},$$

and

$$|\widehat{\sigma}_j(\xi) - \widehat{\sigma}_j(0,\xi_2)| \le C|2^j\xi_1|^a.$$

Let

$$\mathcal{M}_2 f = \sup_{j\in\mathbb{Z}} |\sigma_j^2 * f|$$

be the maximal operator in the variable x_2, where the measures σ_j^2 have Fourier transforms $\widehat{\sigma}_j(0,\xi_2)$. If $\mathcal{M}_2$ is bounded on $L^p(\mathbb{R})$, $1 < p < \infty$, then the maximal operator

$$\mathcal{M}f(x) = \sup_{j\in\mathbb{Z}} |\sigma_j * f(x)|$$

is also bounded on $L^p(\mathbb{R}^2)$, for $1 < p \le \infty$.

Proof. Take a Schwartz function $\varphi \in \mathcal{S}(\mathbb{R})$ being positive, radial, decreasing on $(0,\infty)$ and such that $\|\varphi\|_1 = 1$ (notice that this implies $\widehat{\varphi}(0) = 1$). For each $j \in \mathbb{Z}$ define the measure $\tilde{\mu}_j$ by

$$\widehat{\tilde{\mu}}_j(\xi) = \widehat{\sigma}_j(\xi) - \widehat{\sigma}_j(0,\xi_2)\widehat{\varphi}(2^j\xi_1).$$

Let us check that the sequence $\{\tilde{\mu}_j\}_j$ satisfies the assumptions in Theorem 3.8.

- $\{\tilde{\mu}_j\}_j$ is a sequence of finite Borel measures with $\|\tilde{\mu}_j\| \leq C$.
  First of all, let us compute $\|\sigma_j^2\|$. The measure σ_j^2 is defined so that

$$\widehat{\sigma_j^2}(\xi_2) = \int_{\mathbb{R}} e^{-2\pi i y \xi_2} d\sigma_j^2(y) = \widehat{\sigma}_j(0,\xi_2) = \int_{\mathbb{R}^2} e^{-2\pi i y \xi_2} d\sigma_j(x,y),$$

  and therefore, $\sigma_j^2(E) = \sigma_j(\mathbb{R} \times E)$. Since σ_j is a positive measure, so is σ_j^2 and

$$\|\sigma_j^2\| = \sigma_j^2(\mathbb{R}) = \sigma_j(\mathbb{R}^2) = \|\sigma_j\|. \tag{3.15}$$

  Now, inverting the Fourier transform in the definition of $\tilde{\mu}_j$, we have that

$$\tilde{\mu}_j = \sigma_j - \sigma_j^2 2^{-j}\varphi(2^{-j}\cdot),$$

  and hence, using (3.15) and the assumption $\|\sigma_j\| \leq C$,

$$\|\tilde{\mu}_j\| \leq \|\sigma_j\| + \|\sigma_j^2\|\|2^{-j}\varphi(2^{-j}\cdot)\|_1 = \|\sigma_j\| + \|\sigma_j\|\|\varphi\|_1 \leq C.$$

- $|\widehat{\tilde{\mu}}_j(\xi)| \leq C \min\{|2^j\xi_1|^a, |2^j\xi_1|^{-a}\}$.

  (i) Case $|2^j\xi_1| \geq 1$. Then, we want to show that $|\widehat{\tilde{\mu}}_j(\xi)| \leq C|2^j\xi_1|^{-a}$. Indeed, using our hypotheses on $|\widehat{\sigma}_j(\xi)|$ and $\|\sigma_j\|$,

$$\begin{aligned}|\widehat{\tilde{\mu}}_j(\xi)| &\leq |\widehat{\sigma}_j(\xi)| + |\widehat{\sigma}_j(0,\xi_2)\widehat{\varphi}(2^j\xi_1)| \leq C|2^j\xi_1|^{-a} + \|\sigma_j\||\widehat{\varphi}(2^j\xi_1)| \\ &\leq C|2^j\xi_1|^{-a},\end{aligned}$$

  since $\widehat{\varphi} \in \mathcal{S}(\mathbb{R})$ and thus $|\widehat{\varphi}(t)| \leq Ct^{-a}$.

  (ii) Case $|2^j\xi_1| < 1$. Then, we want to show that $|\widehat{\tilde{\mu}}_j(\xi)| \leq C|2^j\xi_1|^a$. We will use our assumptions on $|\widehat{\sigma}_j(\xi) - \widehat{\sigma}_j(0,\xi_2)|$ and $\|\sigma_j\|$ together with the mean value theorem:

$$\begin{aligned}|\widehat{\tilde{\mu}}_j(\xi)| &\leq |\widehat{\sigma}_j(\xi) - \widehat{\sigma}_j(0,\xi_2)| + |\widehat{\sigma}_j(0,\xi_2) - \widehat{\sigma}_j(0,\xi_2)\widehat{\varphi}(2^j\xi_1)| \\ &\leq C|2^j\xi_1|^a + \|\sigma_j\||1 - \widehat{\varphi}(2^j\xi_1)| \leq C|2^j\xi_1|^a + C\|\widehat{\varphi}'\|_\infty|2^j\xi_1| \\ &\leq C|2^j\xi_1|^a.\end{aligned}$$

  Recall that $\widehat{\varphi}(0) = 1$ and, for the last inequality, that $|2^j\xi_1| < 1$ and $0 < a < 1$.

Therefore, our sequence $\{\tilde{\mu}_j\}_j$ is under the hypotheses of Theorem 3.8. Let $\tilde{g}$ and $\tilde{\mu}^*$ be the operators associated with the sequence $\{\tilde{\mu}_j\}_j$ as in Theorem 3.8. Now, recall that $\sigma_j = \tilde{\mu}_j + \sigma_j^2\varphi_j$, where $\varphi_j(x) = 2^{-j}\varphi(2^{-j}x)$ follows the notation of Lemma 3.10. With this,

$$\begin{aligned}\mathcal{M}f(x) = \sup_{j\in\mathbb{Z}} |\sigma_j * f(x)| &\leq \sup_{j\in\mathbb{Z}} |\tilde{\mu}_j * f(x)| + \sup_{j\in\mathbb{Z}} |\sigma_j^2\varphi_j * f(x)| \\ &\leq \left(\sum_{j\in\mathbb{Z}} |\tilde{\mu}_j * f(x)|^2\right)^{1/2} + \sup_{j\in\mathbb{Z}} |\sigma_j^2\varphi_j * f(x)| \\ &= \tilde{g}(f)(x) + \sup_{j\in\mathbb{Z}} |\sigma_j^2\varphi_j * f(x)|.\end{aligned}$$

Let us see what happens with the second summand:

$$\begin{aligned}\sup_{j\in\mathbb{Z}}|\sigma_j^2\varphi_j * f(x)| &= \sup_{j\in\mathbb{Z}}\left|\int_{\mathbb{R}^2} f(x_1-y_1,x_2-y_2)\varphi_j(y_1)dy_1d\sigma_j^2(y_2)\right| \\ &\le \sup_{j\in\mathbb{Z}}\left|\int_{\mathbb{R}} |\varphi_j * f(x_1,x_2-y_2)|d\sigma_j^2(y_2)\right| \\ &\le \sup_{j\in\mathbb{Z}}\left|\int_{\mathbb{R}} M_1 f(x_1,x_2-y_2)d\sigma_j^2(y_2)\right| \\ &= \sup_{j\in\mathbb{Z}}|\sigma_j^2 * M_1 f(x_1,x_2)| = \mathcal{M}_2 M_1 f(x),\end{aligned}$$

where in the second inequality we use Lemma 3.10 with $\|\varphi\|_1 = 1$ and M_1 denotes the Hardy-Littlewood maximal operator in the first variable. Therefore, we have proved that

$$\mathcal{M}f(x) \le \tilde{g}(f)(x) + \mathcal{M}_2 M_1 f(x). \tag{3.16}$$

Using now that σ_j, σ_j^2 and φ_j are positive, we have that

$$|\tilde{\mu}_j| \le \sigma_j + \sigma_j^2\varphi_j.$$

By exactly the same argument as before, we show that

$$\begin{aligned}\tilde{\mu}^*(f)(x) = \sup_{j\in\mathbb{Z}}||\tilde{\mu}_j| * f(x)| &\le \sup_{j\in\mathbb{Z}}|\sigma_j * f(x)| + \sup_{j\in\mathbb{Z}}|\sigma_j^2\varphi_j * f(x)| \\ &\le \mathcal{M}f(x) + \mathcal{M}_2 M_1 f(x). \end{aligned}\tag{3.17}$$

Finally, from the fact that the measures $\{\tilde{\mu}_j\}_j$ satisfy the hypotheses of Theorem 3.8, it follows that $\tilde{g}$ is bounded on $L^2(\mathbb{R}^2)$ (see Remark 3.9). Therefore, using majorization in (3.16) together with our hypothesis on $\mathcal{M}_2$ and the fact that M_1 is bounded on $L^p(\mathbb{R})$ for $1 < p \le \infty$, we deduce that $\mathcal{M}$ is also bounded on $L^2(\mathbb{R}^2)$. By (3.17), we have then that $\tilde{\mu}^*$ is bounded on $L^2(\mathbb{R}^2)$ and by Theorem 3.8, $\tilde{g}$ will be bounded on $L^p(\mathbb{R}^2)$ for values of p satisfying

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{4},$$

that is, for $4/3 < p < 4$. Starting from this, we now have that $\mathcal{M}$ and $\tilde{\mu}^*$ are bounded on this range as well, and again, Theorem 3.8 states that $\tilde{g}$ is bounded on $L^p(\mathbb{R}^2)$ for $8/7 < p < 8$. We can apply this process repeatedly in order to conclude that $\tilde{\mu}^*$, $\tilde{g}$ and $\mathcal{M}$ are bounded on $L^p(\mathbb{R}^2)$ for every $1 < p < \infty$. A direct computation shows the boundedness of $\mathcal{M}$ on $L^\infty(\mathbb{R}^2)$:

$$\mathcal{M}f(x) = \sup_{j\in\mathbb{Z}}\left|\int_{\mathbb{R}^2} f(x-y)d\sigma_j(y)\right| \le \|f\|_\infty \sup_{j\in\mathbb{Z}}\|\sigma_j\| \le C\|f\|_\infty.$$

□

With these two theorems, we are ready to prove the following one, which is the goal of this section:

Theorem 3.12. *The operators H_Γ and M_Γ are bounded on $L^p(\mathbb{R}^2)$ for $1 < p < \infty$ and $1 < p \leq \infty$ respectively.*

Proof. Let μ_j and σ_j defined as in (3.8) and (3.9). We will want to apply Theorem 3.8 to the sequence $\{\mu_j\}_j$ and Theorem 3.11 to $\{\sigma_j\}_j$. Let us check the hypotheses: In both cases, they are finite Borel measures on $\mathbb{R}^2$. It is easily seen that

$$|\mu_j|(E) = \int_{2^j}^{2^{j+1}} (\chi_E(t,t^2) + \chi_E(-t,t^2))\frac{dt}{t}, \tag{3.18}$$

exactly as we did in (3.5), and therefore, for all $j \in \mathbb{Z}$

$$\|\mu_j\| = |\mu_j|(\mathbb{R}^2) = 2\ln 2 \leq C.$$

Analogously,

$$\sigma_j(E) = \frac{1}{2^{j+1}} \int_{2^j}^{2^{j+1}} \chi_E(t,t^2)dt,$$

and thus, $\{\sigma_j\}_j$ are positive measures satisfying

$$\|\sigma_j\| = \sigma_j(\mathbb{R}^2) = \frac{1}{2} \leq C.$$

Now, we want to check the estimates concerning the Fourier transforms of these measures. First, a direct application of Lemma 3.7 gives us that, for $|\xi_1| > 1$,

$$|\widehat{\sigma}_0(\xi)| = \frac{1}{2}\left|\int_{1\leq|t|\leq 2} e^{-2\pi i(\xi_1 t+\xi_2 t^2)}dt\right| \leq \frac{C}{|\xi_1|^{1/2}}.$$

Also, if we apply Lemma 3.7 to the identity in (3.7), we have that for $|\xi_1| > 1$,

$$|\widehat{\mu}_0(\xi)| = \left|\int_{1\leq|t|\leq 2} e^{-2\pi i(\xi_1 t+\xi_2 t^2)}\frac{dt}{t}\right| \leq \frac{C}{|\xi_1|^{1/2}}.$$

Then, from the relations

$$\widehat{\mu}_j(\xi) = \widehat{\mu}_0(2^j\xi_1, 2^{2j}\xi_2), \quad \widehat{\sigma}_j(\xi) = \widehat{\sigma}_0(2^j\xi_1, 2^{2j}\xi_2),$$

we get that, for $|2^j\xi_1| > 1$,

$$|\widehat{\mu}_j(\xi)|, |\widehat{\sigma}_j(\xi)| \leq C|2^j\xi_1|^{-1/2}. \tag{3.19}$$

Now, the estimates that we need are:

(i) $|\widehat{\mu}_j(\xi)| \leq C\min\{|2^j\xi_1|^{1/2}, |2^j\xi_1|^{-1/2}\}$.

Indeed, if $|2^j\xi_1| > 1$, by (3.19) we have $|\widehat{\mu}_j(\xi)| \leq C|2^j\xi_1|^{-1/2}$, which is the minimum of the two quantities. If $|2^j\xi_1| \leq 1$, using cancellation and the mean value theorem,

$$\begin{aligned}|\widehat{\sigma}_0(\xi)| &= \left|\int_{1\leq|t|\leq 2} e^{-2\pi i(\xi_1 t+\xi_2 t^2)}\frac{dt}{t}\right| = \left|\int_{1\leq|t|\leq 2} (e^{-2\pi i(\xi_1 t+\xi_2 t^2)} - e^{-2\pi i\xi_2 t^2})\frac{dt}{t}\right| \\ &\leq \int_{1\leq|t|\leq 2} |e^{-2\pi i\xi_2 t^2}|\cdot|e^{-2\pi i\xi_1 t} - 1|\frac{dt}{|t|} \leq \int_{1\leq|t|\leq 2} 2\pi|\xi_1|dt \leq C|\xi_1|.\end{aligned}$$

Therefore,

$$|\widehat{\mu}_j(\xi)| \leq C|2^j\xi_1| \leq C|2^j\xi_1|^{1/2},$$

recalling that we are in the case where $|2^j\xi_1| \leq 1$.

(ii) $|\widehat{\sigma}_j(\xi)| \leq C|2^j\xi_1|^{-1/2}$.
By (3.19), we have the estimate when $|2^j\xi_1| > 1$. If $|2^j\xi_1| \leq 1$,

$$|\widehat{\sigma}_j(\xi)| \leq \|\sigma_j\| \leq C \leq C|2^j\xi_1|^{-1/2}.$$

(iii) $|\widehat{\sigma}_j(\xi) - \widehat{\sigma}_j(0,\xi_2)| \leq C|2^j\xi_1|^{1/2}$.
We proceed as in (i), making use of the mean value theorem:

$$\begin{aligned}|\widehat{\sigma}_j(\xi) - \widehat{\sigma}_j(0,\xi_2)| &= \frac{1}{2^{j+1}}\left|\int_{2^j\leq|t|\leq 2^{j+1}} e^{-2\pi i(\xi_1 t+\xi_2 t^2)} - e^{-2\pi i\xi_2 t^2}dt\right| \\ &\leq \frac{1}{2^{j+1}}\int_{2^j\leq|t|\leq 2^{j+1}} 2\pi|\xi_1||t|dt \\ &\leq 4\pi|\xi_1|(2^{j+1}-2^j) = C|2^j\xi_1|.\end{aligned}$$

Now, if $|2^j\xi_1| \leq 1$,

$$|\widehat{\sigma}_j(\xi) - \widehat{\sigma}_j(0,\xi_2)| \leq C|2^j\xi_1| \leq C|2^j\xi_1|^{1/2},$$

whereas if $|2^j\xi_1| > 1$,

$$|\widehat{\sigma}_j(\xi) - \widehat{\sigma}_j(0,\xi_2)| \leq 2\|\sigma_j\| \leq C \leq C|2^j\xi_1|^{1/2}.$$

Next, notice that the operator $\mathcal{M}_2$ in Theorem 3.11 is given by

$$\mathcal{M}_2 g(x_2) = \sup_{j\in\mathbb{Z}} \frac{1}{2^{j+1}}\left|\int_{2^j\leq|t|\leq 2^{j+1}} g(x_2-t^2)dt\right|.$$

Using that the expression inside the integral is even and making the change of variables $s = t^2$ when $2^j \leq t \leq 2^{j+1}$, we get

$$\begin{aligned}\mathcal{M}_2 g(x_2) &= \sup_{j\in\mathbb{Z}} \frac{1}{2^j}\left|\int_{2^j}^{2^{j+1}} g(x_2-t^2)dt\right| = \sup_{j\in\mathbb{Z}} \frac{1}{2^{j+1}}\left|\int_{2^{2j}}^{2^{2(j+1)}} g(x_2-s)\frac{ds}{\sqrt{s}}\right| \\ &= \sup_{j\in\mathbb{Z}} \frac{1}{2}\left|\int_{\mathbb{R}} g(x_2-s)\tilde{\varphi}_j(s)ds\right|,\end{aligned}$$

where

$$\begin{aligned}\tilde{\varphi}_j(s) &= \frac{1}{2^j\sqrt{s}}\cdot\chi_{[2^{2j},2^{2(j+1)}]}(s) = \frac{1}{2^{2j}}\cdot\frac{1}{\sqrt{2^{-2j}s}}\cdot\chi_{[1,4]}(2^{-2j}s) \\ &\leq \frac{1}{2^{2j}}\left(\frac{1}{\sqrt{-2^{-2j}s}}\cdot\chi_{[-4,-1]}(2^{-2j}s) + \frac{1}{\sqrt{2^{-2j}s}}\cdot\chi_{[1,4]}(2^{-2j}s)\right) \\ &= 2^{-2j}\varphi(2^{-2j}s) =: \varphi_j(s),\end{aligned}$$

with

$$\varphi(s) = \frac{1}{\sqrt{-s}}\chi_{[-4,-1]}(s) + \frac{1}{\sqrt{s}}\chi_{[1,4]}(s)$$

being a positive, radial, decreasing on $(0,\infty)$, integrable function defined on $\mathbb{R}$. Therefore, by Lemma 3.10 with $\|\varphi\|_1 = 4$,

$$\mathcal{M}_2 g(x_2) \leq \frac{1}{2} \sup_{j\in\mathbb{Z}} |\varphi_j * g(x_2)| \leq 2Mg(x_2),$$

where M is the Hardy-Littlewood maximal operator. From this, we deduce that $\mathcal{M}_2$ is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$, so we can apply Theorem 3.11 to conclude that

$$\mathcal{M}f = \sup_{j\in\mathbb{Z}} |\sigma_j * f|$$

is bounded for all $1 < p \leq \infty$. Using now (3.10), we obtain that

$$M_\Gamma f \leq 2 \sup_{j\in\mathbb{Z}} \sigma_j * |f| = 2\mathcal{M}(|f|)$$

and consequently, $M_\Gamma : L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2)$ is also bounded for all $1 < p \leq \infty$.

Finally, using the identity in (3.18), we have that maximal operator μ^* in Theorem 3.8 can be written as

$$\mu^*(f) = \sup_{j\in\mathbb{Z}} ||\mu_j| * f(x,y)| = \sup_{j\in\mathbb{Z}} \left| \int_{2^j}^{2^{j+1}} [f(x-t, y-t^2) + f(x+t, y-t^2)] \frac{dt}{t} \right|.$$

Now, assume f to be a positive function and let $j \in \mathbb{Z}$, it holds that

$$\begin{aligned}
\int_{2^j}^{2^{j+1}} [f(x-t, y-t^2) + f(x+t, y-t^2)] \frac{dt}{t} &\leq \frac{1}{2^j} \int_0^{2^{j+1}} [f(x-t, y-t^2) + f(x+t, y-t^2)] dt \\
&= \frac{4}{2^{j+2}} \int_0^{2^{j+1}} [f(x-t, y-t^2) + f(x+t, y-t^2)] dt \\
&\leq \sup_{h>0} \frac{4}{2h} \int_0^h [f(x-t, y-t^2) + f(x+t, y-t^2)] dt \\
&= \sup_{h>0} \frac{4}{2h} \int_{-h}^h f(x-t, y-t^2) dt = 4M_\Gamma(f).
\end{aligned}$$

Therefore, for every positive function

$$\mu^*(f) \leq 4M_\Gamma(f)$$

and from the boundedness of M_Γ, we conclude that μ^* is bounded on $L^q(\mathbb{R}^2)$ for all $1 < q \leq \infty$. With this, we can apply Theorem 3.8 to get that

$$H_\Gamma : L^p(\mathbb{R}^2) \longrightarrow L^p(\mathbb{R}^2)$$

is bounded for all $1 < p < \infty$ and thus finish the proof. □

3.5 Extrapolation And Further Results

In 1951, S. Yano [15] proved the following result:

Theorem 3.13 (Yano)**.** *Let (X, μ), (Y, ν) be two finite measure spaces. Assume that*

$$T : L^p(\mu) \longrightarrow L^p(\nu)$$

is a bounded operator with constant

$$\frac{C}{(p-1)^k} \tag{3.20}$$

for some $k > 0$ and every $1 < p \leq p_0$, where $1 < p_0 \leq \infty$. Then,

$$T : L(\log L)^k(\mu) \longrightarrow L^1(\nu)$$

is also bounded.

Even though Yano's theorem assumes the measures to be finite, if they are σ-finite, the proof can be modified so that

$$T : L(\log L)^k(\mu) \longrightarrow L^1_{\text{loc}}(\nu).$$

In fact, in 2000, M. J. Carro [2] [3] proved that

$$T : L(\log L)^k(\mu) \longrightarrow B(\nu),$$

where $B(\nu)$ is a rearrangement invariant space such that

$$B(\nu) \subsetneq L^1(\nu) + L^\infty(\nu).$$

In this last section, we will see that the constant that we have obtained in this project acts up near L^1 and does not allow us to apply Yano's theorem. However, we will present some recent progress made in this direction yielding boundedness results for H_Γ near L^1.

After carefully analyzing the steps in the proof of Theorem 3.12, one realizes that the problem with the constant originates in the bootstrapping argument of Theorem 3.11. Without going into details, the idea is the following: We start from an L^2-estimate. After the k-th step, we have boundedness for values of p satisfying

$$\frac{2^{k+1}}{2^{k+1}-1} < p < 2^{k+1}.$$

However, every step in the process adds an "asymptotic" $\frac{1}{p-1}$ factor[4] to the constant. This factor comes from the interpolation theory behind Theorem 3.8. Finally, since we need an infinite number of steps to cover the whole range $1 < p < \infty$, we cannot obtain a constant of the type (3.20) as $p \to 1^+$ and thus we cannot make use of Yano's theorem. Therefore, even though we have managed to show that H_Γ is of strong-type (p, p) for $1 < p < \infty$, the bounds that we have obtained are not sharp enough to extrapolate.

In 1987, M. Christ and E. M. Stein [7] proved the following result:

[4] By asymptotic $\frac{1}{p-1}$ factor we mean that it behaves like it as $p \to 1^+$.

Theorem 3.14. *Let $a_1 < \cdots < a_n$ with $a_i \in \mathbb{Z}$ for all $i = 1, ..., n$. Let $B \subseteq \mathbb{R}^n$ be a bounded set. Then, if we define*

$$H_\Gamma f(x) = \lim_{\varepsilon\to 0} \int_{|t|>\varepsilon} f(x_1 - t^{a_1}, x_2 - t^{a_2}, \cdots, x_n - t^{a_n}) \frac{dt}{t},$$

we have that

$$H_\Gamma : L(\log L)(B) \to L^{1,\infty}(B).$$

The usual technique for this kind of operators is applying Calderón-Zygmund theory after proving an L^2-estimate. In order to do so, however, we need the operator to satisfy the Hörmander condition (see Theorem 1.25), which fails to be true in this case. In [7], the authors introduce a variant of the Calderón-Zygmund decomposition which allows them to prove that H_Γ satisfies a generalization of the Hörmander condition that yields the sought-after estimate. The statement is the following:

Lemma 3.15 (Variant of CZ decomposition, Christ and Stein)**.** *For every $f \in L^p(\mathbb{R}^n)$ and every $\alpha > 0$, there exist functions g and $\{b_i\}_i$ such that*

(i) $f = g + \sum_{i=1}^n b_i$,

(ii) $\|g\|_\infty < \alpha$,

(iii) The supports of b_i are contained in balls $B_i = B(x_i, 2^{j_i})$, which are pairwise disjoint.

(iv) $\sum_{i=1}^n |B_i| \le C\alpha^{-1}\|f\|_p^p$, $\quad \frac{1}{|B_i|}\int_{B_i} |b_i|^p \le C\alpha^p$,

(v) Every point of $\mathbb{R}^n$ is contained, at most, in C double balls $2B_i$.

This decomposition of a function into a "good part" and a "bad part", together with some technical lemmas, enables the authors to show that H_Γ maps $L(\log L)(B)$ into $L^{1,\infty}(B)$. For the "good part", they use the estimates in the decomposition to prove that the image of this part lies in $L^{1,\infty}(B)$. For the "bad part", they prove an L^p-estimate with a sharp enough bound so that it behaves like $\frac{1}{p-1}$ as $p \to 1^+$. Thanks to this fact, they can apply Yano's extrapolation theorem to deduce that the image of this other part is in $L^1(B)$. However, since for the "good part" they only manage to get weak-type estimates, their conclusion is that

$$H_\Gamma : L(\log L)(B) \to L^{1,\infty}(B).$$

In fact, they pose the question of whether H_Γ maps $L(\log L)(B)$ to $L^1(B)$, rather than merely to $L^{1,\infty}(B)$, as an open conjecture.

More recently, in 2004, A. Seeger, T. Tao and J. Wright [12] proved that

$$H_\Gamma : L(\log\log L)(\mathbb{R}^2) \to L^{1,\infty}(\mathbb{R}^2)$$

is bounded, where

$$H_\Gamma f(x) = \lim_{\varepsilon\to 0} \int_{|t|>\varepsilon} f(x_1 - t, x_2 - t^2) \frac{dt}{t}.$$

Again, they show this fact by means of a new modification of the Calderón-Zygmund decomposition. As far as we know, this is the best result concerning the boundedness of the Hilbert transform along the parabola near L^1 that has been published so far. Nevertheless, it seems natural to think that if we improve Yano's theorem, we might achieve $L(\log\log\log L)$-boundedness. The study of the different variations of the Calderón-Zygmund decomposition seems also advisable, since the last two main results in this direction use this approach. The question of whether H_Γ is of weak-type $(1,1)$ remains open.

Bibliography

[1] A. Carbery, *An Introduction to the Oscillatory Integrals of Harmonic Analysis*, Personal communication.

[2] M. J. Carro, *New extrapolation estimates*, J. Funct. Anal. **174** (2000), 155–166.

[3] M. J. Carro, *On the range space of Yano's extrapolation theorem and new extrapolation estimates at infinity*, Proceedings of the 6th International Conference on Harmonic Analysis and Partial Differential Equations **46** (2002), 27–37.

[4] J. Cerdà, *Linear Functional Analysis*, American Mathematical Society, 2010.

[5] J. Cerdà, *Análisis Real*, Edicions Universitat de Barcelona, 2000.

[6] M. Christ, *Hilbert transforms along curves I. Nilpotent groups*, Ann. of Math. (2) **122** (1985), 575–596.

[7] M. Christ, E. M. Stein, *A remark on singular Calderón-Zygmund theory*, Proc. Amer. Math. Soc. **99** (1987), 71–75.

[8] J. Duoandikoetxea, J. L. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates*, Invent. Math. **84** (1986), 541–561.

[9] J. Duoandikoetxea, *Fourier Analysis*, American Mathematical Society, 2000.

[10] L. Grafakos, *Classical Fourier Analysis*, Springer, 2008.

[11] A. Nagel, N. M. Rivière, S. Wainger, *On Hilbert transforms along curves II*, Amer. J. Math. **98** (1976), 395–403.

[12] A. Seeger, T. Tao, J. Wright, *Singular maximal functions and Radon transforms near* L^1, Am. J. of Math. **126** (2004), 607–647.

[13] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[14] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton University Press, 1993.

[15] S. Yano, *Notes on Fourier analysis XXIX: An extrapolation theorem*, J. Math. Soc. Japan **3** (1951), 296–305.