WEIGHTED WEAK-TYPE $(1,1)$ ESTIMATES FOR RADIAL FOURIER MULTIPLIERS VIA EXTRAPOLATION THEORY

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Abstract. In this paper, we prove a weighted estimate for the Bochner-Riesz operator at the critical index that is stronger than the weak-type (1,1) for A_1 weights, in the sense that the latter can be obtained via extrapolation arguments from the former. In addition, this estimate can be transferred to averages in order to deduce weighted weak-type (1,1) results for general radial Fourier multipliers.

1. Introduction

Let us start by giving the general definition of a Bochner-Riesz operator:

Definition 1.1. *Given $\lambda > 0$ and $r > 0$, we define the Bochner-Riesz operator B_λ^r on $\mathbb{R}^n$ by*

$$\widehat{B_\lambda^r f}(\xi) = (1 - |r\xi|^2)_+^\lambda \widehat{f}(\xi).$$

Notice that the term $(1 - |r\xi|^2)_+^\lambda$ restricts the support of $\widehat{f}$ to the ball $B(0, 1/r)$. However, the larger the value of λ, the smoother this truncation is, and thus, the better the operator B_λ^r will behave. More precisely, it is easy to see that if $\lambda > \frac{n-1}{2}$, then $B_\lambda^r f$ is essentially controlled by the Hardy-Littlewood maximal operator M (see, for instance, [14, Sec. 10.2]). However, for the so-called critical index $\lambda = \frac{n-1}{2}$, we do not have such a control. We will focus on this critical case with $r = 1$, so for the sake of simplicity, we will drop the indices λ or r whenever they are $\frac{n-1}{2}$ or 1 respectively.

Despite the fact that B is no longer controlled by the Hardy-Littlewood maximal operator, when it comes to its boundedness on weighted L^p-spaces, it satisfies the same estimates as M. Namely, in 1988, M. Christ [5] showed that B is of weak-type (1,1) with respect to the Lebesgue measure. Later on, in 1992, X. Shi and Q. Sun [24] proved that it was of strong-type (p,p) for every weight in A_p and every $1 < p < \infty$, and finally, in 1996, A. Vargas [28] extended the weak-type (1,1) estimate to A_1 weights. The main purpose of this paper is to show that B satisfies a certain restricted weak-type (p,p) estimate that, in particular, will imply its weak-type (1,1) for A_1 weights. The main advantage of this new estimate is that it will allow us to use extrapolation arguments on operators that can be written as an average of Bochner-Riesz operators $\{B^r\}_{r>0}$. The

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extrapolation that we will need follows the ideas in [3], but with a weaker assumption, especially aimed at obtaining estimates for the endpoint $p = 1$.

Let us recall that a locally integrable function $u > 0$ is said to be an A_1 weight if $Mu(x) \leq Cu(x)$ almost everywhere, and its A_1-constant $\|u\|_{A_1}$ is the infimum of all possible $C > 0$ in such an inequality. This class characterizes the weighted weak-type $(1,1)$ for the Hardy-Littlewood maximal operator M, as shown by B. Muckenhoupt in [20], and for every $f \in L^1(u)$, it holds that:

$$\|Mf\|_{L^{1,\infty}(u)} \lesssim \|u\|_{A_1} \|f\|_{L^1(u)}.$$

This paper is organized as follows. In Section 2, we prepare the framework and prove the key lemmas that will be needed in Section 3, where we present our main result for the Bochner-Riesz operator B. Namely, we shall prove that, for every $u \in A_1$, there exists $1 < p_0 < \infty$ such that, for each measurable set $E \subseteq \mathbb{R}^n$,

$$\|T\chi_E\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)} \lesssim \varphi_{p_0}(\|u\|_{A_1}) u(E)^{1/p_0}, \tag{1.1}$$

where $T = B$ (Theorem 3.2) and $T = B^r$ (Corollary 3.3). Moreover, in this second case the constant is uniform in $r > 0$. Then, in Section 4, we introduce the extrapolation technique and prove (Theorem 4.1) that if an operator T satisfies condition (1.1) then, for every $u \in A_1$ and every measurable set $E \subseteq \mathbb{R}^n$,

$$\|T\chi_E\|_{L^{1,\infty}(u)} \leq \|u\|_{A_1}^{1-\frac{1}{p_0}} \varphi_{p_0}(\|u\|_{A_1}) u(E).$$

Finally, in Section 5, we will transfer the weighted estimate (1.1) from $T = B^{1/s}$ to radial Fourier multipliers T_m by means of the following relation:

$$T_m f(x) = \int_0^\infty B^{1/s} f(x) \Phi_m(s) ds, \quad \Phi_m \in L^1(0,\infty), \tag{1.2}$$

and using the crucial fact that, contrary to what happens with $L^{1,\infty}$, the space $L^{p,\infty}$ is a Banach space for $p > 1$ and Minkowski's integral inequality holds. Then, using the extrapolation argument from Section 4 we obtain weighted results at the endpoint $p = 1$ for the operator T_m. The condition that we will require on m so that Φ_m is integrable and (1.2) holds will be

$$\int_0^\infty t^{\frac{n-1}{2}} |D^{\frac{n+1}{2}} m(t)| dt < \infty,$$

where $D^{\frac{n+1}{2}}$ is a suitable definition of fractional derivative. Before tackling this problem, we will illustrate the method by applying it to Fourier multipliers on $\mathbb{R}$, where the role of the Bochner-Riesz operator in (1.2) will be played by the Hilbert transform instead.

As usual, the symbol $f \lesssim g$ will indicate the existence of a constant $C > 0$ so that $f \leq Cg$. When both $f \lesssim g$ and $g \lesssim f$, we will write $f \approx g$. The implicit constants may depend on the dimension n and on values of p, but never on the weights, functions or sets involved. In all our results, we will make explicit the dependence on the weights of the constants, although we will not be concerned about their sharpness.

2. Preliminaries and lemmas

Let us consider the classical decomposition of B. Arguing as in [5], it is enough to study the operator (which we will call again B):

$$f \longmapsto \Big(\sum_{j=1}^{\infty} K_j\Big) * f,$$

where

$$K_j(x) = \eta\left(\frac{x}{|x|}\right)\psi(x)\varphi(2^{-j}x)|x|^{-n},$$

and:

- η is a fixed element from a finite $\mathcal{C}^\infty$ partition of the unity on the sphere $\mathbb{S}^{n-1}$, which we can assume to have very small support.
- $\psi(x) = \cos(2\pi|x| - \pi(n-1)/4)$.
- $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$, real-valued, radial, supported on $\{x \in \mathbb{R}^n : |x| \in [1/4, 1]\}$, and such that
$$\sum_{j\in\mathbb{Z}} \varphi(2^j x) \equiv 1, \quad \text{on } \mathbb{R}^n \setminus \{0\}.$$

The only properties of the kernels K_j that we will explicitly use have to do with their size and support, although at some point we will also need some estimates from [5] for which the author needs a deeper understanding of them. Namely, we will use that for every $j \geq 1$,

$$|K_j(x)| \lesssim 2^{-nj}\chi_{B(0,2^j)}(x). \tag{2.1}$$

This is a direct consequence of their definition, and trivially implies a uniform bound for the associated convolution operators by the Hardy-Littlewood maximal operator:

$$|K_j * f(x)| \lesssim Mf(x). \tag{2.2}$$

Once we have settled the decomposition of the kernel, we will need three lemmas to reach our goal. The first one will allow us to decompose a measurable set E by means of a simplified Calderón-Zygmund decomposition.

Lemma 2.1. *Let $0 < \alpha < 1$. Let $E \subseteq \mathbb{R}^n$ be a measurable set. Then there exists a family of pairwise disjoint dyadic cubes $\{Q_i\}_{i=0}^\infty$ such that*

$$\frac{|E \cap Q_i|}{|Q_i|} \approx \alpha,$$

and $E \subseteq \bigcup_{i=0}^\infty Q_i$.

Remark 2.2. *Based on this lemma, given $0 < \alpha < 1$ and $E \subseteq \mathbb{R}^n$, we can define for every $k \geq 0$,*

$$E_k := E \cap \left(\bigcup_{i=0}^\infty Q_i^k\right),$$

where $\{Q_i^k\}_{i=0}^\infty$ *is the subfamily of cubes of size* $|Q_i^k| = 2^{nk}$ *if* $k > 0$*, and* $|Q_i^0| \leq 1$*. Since the set* E *is contained in the union of all the cubes* $\{Q_i^k\}_{i,k=0}^\infty$*, we have that*

$$E = \bigcup_{k=0}^{\infty} E_k,$$

and for every $k, i \geq 0$*:*

$$\frac{|E_k \cap Q_i^k|}{|Q_i^k|} = \frac{|E \cap Q_i^k|}{|Q_i^k|} \approx \alpha.$$

We will also need the following property for weights of the form $(Mh)^\alpha$ when $\alpha < 0$ (see [4, 8]):

Lemma 2.3. *Given a locally integrable function* h *and* $\alpha < 0$*, we have that for every cube* $Q \subseteq \mathbb{R}^n$*,*

$$\sup_{x \in Q} (Mh)^\alpha(x) \lesssim \frac{1}{|Q|} \int_Q (Mh)^\alpha(y) dy.$$

In particular, if $Q \subseteq Q'$*,*

$$\frac{1}{|Q|} \int_Q (Mh)^\alpha(y) dy \lesssim \frac{1}{|Q'|} \int_{Q'} (Mh)^\alpha(y) dy.$$

The next lemma will be the cornerstone of our argument, and is inspired by the ideas in [28]. For technical reasons regarding interpolation, not only will we need estimates for E, but also for subsets $G \subseteq E$. Notice that if $G_k = G \cap E_k$, we still have the inequality $\frac{|G_k \cap Q_i^k|}{|Q_i^k|} \lesssim \alpha$, and this will suffice to get the right estimates.

Lemma 2.4. *Let* $0 < \alpha < 1$ *and let* $E = \bigcup_{k=0}^\infty E_k$ *be a measurable set decomposed as in Remark 2.2. Let* $G \subseteq E$ *be a measurable subset and define for every* $k \geq 0$*,* $G_k = G \cap E_k$*. Then for every* $1 \leq s < \infty$*:*

(a)

$$\left\| \sum_{j=s}^{\infty} K_j * \chi_{G_{j-s}} \right\|_2^2 \lesssim 2^{-s\frac{n-1}{2}} \alpha |G|.$$

(b) *For every weight* $u \in A_1$*,*

$$\left\| \sum_{j=s}^{\infty} K_j * \chi_{G_{j-s}} \right\|_{L^2(u)}^2 \lesssim \|u\|_{A_1}^2 \alpha u(G).$$

(c)

$$\left\| \sum_{j=s}^{\infty} K_j * \chi_{G_{j-s}} \right\|_{L^2((M\chi_E)^{-1})}^2 \lesssim |G|.$$

Proof. The proof of (a) is exactly the same as that of [5, Estimate (3.1)], where the author proves an estimate for the bad part of a Calderón-Zygmund decomposition without using its cancellation property (which allows us to adapt it to our case). In fact, this estimate is conveniently stated in [28, Section 2, Lemma 2] in the following way:

• Let $v = \sum_{Q\in\mathcal{F}} v_Q$, where $\mathcal{F}$ is a family of disjoint dyadic cubes, with $\operatorname{supp} v_Q \subseteq Q$ and $\int |v_Q| \lesssim \alpha|Q|$. Define $\mathcal{F}_k = \{Q \in \mathcal{F} : |Q| = 2^{nk}\}$ for $k \geq 1$, $\mathcal{F}_0 = \{Q \in \mathcal{F} : |Q| \leq 1\}$ and $V_k = \sum_{Q\in\mathcal{F}_k} v_Q$. Then

$$\left\| \sum_{j=s}^{\infty} K_j * V_{j-s} \right\|_2^2 \lesssim 2^{-s\frac{n-1}{2}} \alpha \|v\|_1.$$

For our purposes, take the function $v = \chi_G$, the family $\mathcal{F} = \{Q_i^k\}_{k,i=0}^{\infty}$, the subfamily $\mathcal{F}_k = \{Q_i^k\}_{i=0}^{\infty}$, and (a) follows.

Let us prove (b). Writing the left-hand side as an inner product in $L^2(u)$ and using its bilinearity and symmetry, we get that it can be essentially majorized by

$$\sum_{j=s}^{\infty} \sum_{i=s}^{j} \int |K_j * \chi_{G_{j-s}}(x)||K_i * \chi_{G_{i-s}}(x)|u(x)dx.$$

Since $\chi_{G_k} = \sum_{l=0}^{\infty} \chi_{G_k \cap Q_l^k}$ for every $k \geq 0$, we can write the previous expression as

$$\sum_{j=s}^{\infty} \sum_{l=0}^{\infty} \left(\sum_{i=s}^{j} \sum_{m=0}^{\infty} \int |K_j| * \chi_{G_{j-s}\cap Q_l^{j-s}}(x) |K_i| * \chi_{G_{i-s}\cap Q_m^{i-s}}(x) u(x) dx \right). \tag{2.3}$$

Now, let us look at the term in parentheses, where Q_l^{j-s} is fixed. Using (2.1), we know that the support of the first convolution is contained in

$$Q_l^{j-s} + B(0, 2^j) \subseteq \overline{Q}_l,$$

where $|\overline{Q}_l| = 2^{(j+2)n}$, and

$$|K_j| * \chi_{G_{j-s}\cap Q_l^{j-s}}(x) \leq 2^{-jn} |G_{j-s} \cap Q_l^{j-s}|.$$

Similarly, for every $s \leq i \leq j$ and every $m \geq 0$, the support of the second convolution is contained in $\overline{Q}_m$ with $|\overline{Q}_m| = 2^{(i+2)n}$ and $Q_m^{i-s} \subseteq \overline{Q}_m$. Moreover, since $x \in \overline{Q}_l$ (for the first convolution to be non-zero), we have that

$$\begin{aligned} |K_i| * \chi_{G_{i-s}\cap Q_m^{i-s}}(x) &= \int_{G_{i-s}\cap Q_m^{i-s}} |K_i(x-z)|dz = \int_{G_{i-s}\cap Q_m^{i-s}\cap 2\overline{Q}_l} |K_i(x-z)|dz \\ &\leq 2^{-in} |G_{i-s} \cap Q_m^{i-s} \cap 2\overline{Q}_l| \leq 2^{-in} |G_{i-s} \cap Q_m^{i-s}|, \end{aligned}$$

keeping in mind that we only need to consider the cubes $Q_m^{i-s} \subseteq 4\overline{Q}_l$. Here we used again (2.1) to see that $z \in \overline{Q}_l + B(0, 2^i) \subseteq 2\overline{Q}_l$ and $|K_i| \leq 2^{-in}$. Summing up, we have the following:

- $x \in \overline{Q}_l \cap \overline{Q}_m$,
- $|K_j| * \chi_{G_{j-s}\cap Q_l^{j-s}}(x) \leq 2^{-jn} |G_{j-s} \cap Q_l^{j-s}|$,
- $|K_i| * \chi_{G_{i-s}\cap Q_m^{i-s}}(x) \leq 2^{-in} |G_{i-s} \cap Q_m^{i-s}|$,
- $\bigcup_{i=s}^{j} \bigcup_{m=0}^{\infty} Q_m^{i-s} \subseteq 4\overline{Q}_l$.

With this, we can finish the proof of (b). We bound the expression in parentheses in (2.3) by

$$\begin{aligned}
&2^{-jn}|G_{j-s}\cap Q_l^{j-s}|\sum_{i=s}^{j}\sum_{m=0}^{\infty}|G_{i-s}\cap Q_m^{i-s}|\frac{u(\overline{Q}_l\cap\overline{Q}_m)}{2^{in}}\\
&\qquad\lesssim \alpha 2^{-jn}|G_{j-s}\cap Q_l^{j-s}|\sum_{i=s}^{j}\sum_{m=0}^{\infty}|Q_m^{i-s}|\frac{u(\overline{Q}_m)}{2^{in}}\\
&\qquad\le \alpha\|u\|_{A_1}2^{-jn}|G_{j-s}\cap Q_l^{j-s}|\sum_{i=s}^{j}\sum_{m=0}^{\infty}u(Q_m^{i-s})\le \alpha\|u\|_{A_1}2^{-jn}|G_{j-s}\cap Q_l^{j-s}|u\left(4\overline{Q}_l\right)\\
&\qquad\le \alpha\|u\|_{A_1}^2u(G_{j-s}\cap Q_l^{j-s}),
\end{aligned}$$

recalling that $|G_{i-s}\cap Q_m^{i-s}|\lesssim\alpha|Q_m^{i-s}|$ and that $|\overline{Q}_m|\approx 2^{in}$, $|4\overline{Q}_l|\approx 2^{jn}$. Finally, we can plug it into (2.3) to get the sought-after estimate:

$$\alpha\|u\|_{A_1}^2\sum_{j=s}^{\infty}\sum_{l=0}^{\infty}u(G_{j-s}\cap Q_l^{j-s})=\alpha\|u\|_{A_1}^2\sum_{j=s}^{\infty}u(G_{j-s})=\alpha\|u\|_{A_1}^2u(G).$$

Exactly as in (b), to show (c) it is enough to bound

$$\sum_{j=s}^{\infty}\sum_{l=0}^{\infty}\left(\sum_{i=s}^{j}\sum_{m=0}^{\infty}\int|K_j|*\chi_{G_{j-s}\cap Q_l^{j-s}}(x)|K_i|*\chi_{G_{i-s}\cap Q_m^{i-s}}(x)(M\chi_E)^{-1}(x)dx\right), \tag{2.4}$$

where the expression in parentheses is controlled by

$$2^{-jn}|G_{j-s}\cap Q_l^{j-s}|\sum_{i=s}^{j}\sum_{m=0}^{\infty}|G_{i-s}\cap Q_m^{i-s}|\frac{(M\chi_E)^{-1}(\overline{Q}_l\cap\overline{Q}_m)}{2^{in}}.$$

Now, since $Q_m^{i-s}\subseteq 4\overline{Q}_l$, $|\overline{Q}_l|=2^{(j+2)n}$ and $|\overline{Q}_m|=2^{(i+2)n}$, we deduce that $\overline{Q}_m\subseteq 5\overline{Q}_l$, and hence by Lemma 2.3,

$$\frac{(M\chi_E)^{-1}(\overline{Q}_m)}{2^{in}}\lesssim\frac{(M\chi_E)^{-1}(5\overline{Q}_l)}{2^{jn}}.$$

Using this, we obtain

$$(2^{-jn})^2(M\chi_E)^{-1}(5\overline{Q}_l)|G_{j-s}\cap Q_l^{j-s}|\sum_{i=s}^{j}\sum_{m=0}^{\infty}|G_{i-s}\cap Q_m^{i-s}|.$$

Assuming without loss of generality that $G\cap 4\overline{Q}_l$ has positive measure, we use the $A_2^{\mathcal{R}}$ condition of the weight $(M\chi_E)^{-1}$ with the subset $G\cap 4\overline{Q}_l\subseteq 5\overline{Q}_l$, and that $\bigcup_{i=s}^{j}\bigcup_{m=0}^{\infty}Q_m^{i-s}\subseteq 4\overline{Q}_l$ to get to

$$|G\cap 4\overline{Q}_l)|^{-1}|G_{j-s}\cap Q_l^{j-s}||G\cap 4\overline{Q}_l|.$$

Finally we can simplify and sum over $s\le j<\infty$ and $l\ge 0$ to obtain that the expression in (2.4) is majorized by $|G|$, as we claimed. □

The third and last lemma will be an interpolation argument on the estimates in Lemma 2.4 that will yield the right control of the L^2 norm with respect to the desired weights. Let us just remark that the first estimate will be used to prove the second one, so in this case we still need to consider subsets $G \subseteq E$.

Lemma 2.5. *Let $0 < \alpha < 1$ and let $E = \bigcup_{k=0}^{\infty} E_k$ be a measurable set decomposed as in Remark 2.2. Let $G \subseteq E$ be a measurable subset and define for every $k \geq 0$, $G_k = G \cap E_k$. Then for every $1 \leq s < \infty$ and every $u \in A_1$:*

(d)
$$\left\| \sum_{j=s}^{\infty} K_j * \chi_{G_{j-s}} \right\|_{L^2(u)}^2 \lesssim \|u\|_{A_1}^2 2^{-s\epsilon} \alpha u(G),$$
with $\epsilon = \frac{n-1}{2} \left(\frac{1}{1+2^{n+1}\|u\|_{A_1}} \right)$.

(e)
$$\left\| \sum_{j=s}^{\infty} K_j * \chi_{E_{j-s}} \right\|_{L^2((M\chi_E)^{-\theta} u)}^2 \lesssim \|u\|_{A_1}^2 2^{-s\beta} \alpha^{1-\theta} u(E).$$
with $\theta = \frac{1}{1+2^{n+1}\|u\|_{A_1}}$ *and* $\beta = \frac{n-1}{2} \left(\frac{2^{n+1}\|u\|_{A_1}}{(1+2^{n+1}\|u\|_{A_1})^2} \right)$

Proof. For $a, b > 0$, define $w_{a,b}(x) = \min\{au(x), b\}$. Fix $t > 0$ and write
$$B^1 = \{x \in \mathbb{R}^n : \|u\|_{A_1}^2 u(x) \leq 2^{-s\frac{n-1}{2}} t\},$$
and $B^2 = \mathbb{R}^n \setminus B^1$. For every $k \geq 0$, we set $G_k = G_k^1 \cup G_k^2$, where $G_k^i = G_k \cap B^i \subseteq E_k$ and $G^i = \bigcup_{k=0}^{\infty} G_k^i = G \cap B^i$, for $i = 1, 2$. Using (a) and (b) in Lemma 2.4 and the definitions we just introduced, we get
$$\begin{aligned} \left\| \sum_{j=s}^{\infty} K_j * \chi_{G_{j-s}} \right\|_{L^2(w_{1,t})}^2 &\lesssim \left\| \sum_{j=s}^{\infty} K_j * \chi_{G_{j-s}^1} \right\|_{L^2(u)}^2 + t \left\| \sum_{j=s}^{\infty} K_j * \chi_{G_{j-s}^2} \right\|_2^2 \\ &\lesssim \|u\|_{A_1}^2 \alpha u(G^1) + 2^{-s\frac{n-1}{2}} t\alpha |G^2| = \alpha w_{a,bt}(G), \end{aligned}$$
with $a = \|u\|_{A_1}^2$ and $b = 2^{-s\frac{n-1}{2}}$. Now, we integrate both sides with respect to $t \in (0, \infty)$ equipped with the measure $\frac{dt}{t^{\theta+1}}$, where $0 < \theta < 1$. Using Fubini and the definition of the weight, we obtain
$$\left\| \sum_{j=s}^{\infty} K_j * \chi_{G_{j-s}} \right\|_{L^2(u^{1-\theta})}^2 \lesssim \alpha a^{1-\theta} b^{\theta} u^{1-\theta}(G).$$
But we know (see [21]) that if $u \in A_1$ and $r = 1 + \frac{1}{2^{n+1}\|u\|_{A_1}}$, then $u^r \in A_1$ and $\|u^r\|_{A_1} \lesssim \|u\|_{A_1}$, so applying what we have shown to u^r and choosing $\theta = (r-1)/r$, we obtain
$$\left\| \sum_{j=s}^{\infty} K_j * \chi_{G_{j-s}} \right\|_{L^2(u)}^2 \lesssim \|u\|_{A_1}^{\frac{2^{n+2}\|u\|_{A_1}}{1+2^{n+1}\|u\|_{A_1}}} 2^{-s\frac{n-1}{2}\left(\frac{1}{1+2^{n+1}\|u\|_{A_1}}\right)} \alpha u(G).$$

Notice that the exponent in $\|u\|_{A_1}$ is always less than or equal to 2, so we conclude (d). To prove (e), define $v_{a,b}(x) = \min\{au(x), b(M\chi_E)^{-1}(x)\}$. Fix $t > 0$ and write

$$C^1 = \{x \in \mathbb{R}^n : \alpha\|u\|_{A_1}^2 2^{-s\epsilon}u(x) \leq (M\chi_E)^{-1}(x)t\},$$

$C^2 = \mathbb{R}^n \setminus C^1$. Now we decompose for every $k \geq 0$, $E_k = E_k^1 \cup E_k^2$, with $E_k^i = E_k \cap C^i$ and $E^i = \bigcup_{k=0}^{\infty} E_k^i = E \cap C^i$, for $i = 1, 2$. To finish the proof, we argue as in (d), but this time interpolating estimates (c) in Lemma 2.4 and (d). □

3. Main result

Before stating our main result, let us recall some definitions and properties concerning weights that will be needed in what follows. For every $1 \leq p < \infty$, we define $A_p^{\mathcal{R}}$ weights w by

$$\|w\|_{A_p^{\mathcal{R}}} = \sup_{F \subseteq Q} \frac{|F|}{|Q|}\left(\frac{w(Q)}{w(F)}\right)^{1/p} < \infty,$$

where the supremum is taken over all cubes $Q \subseteq \mathbb{R}^n$ and all measurable sets $F \subseteq Q$. R. Kerman and A. Torchinsky [16] showed that this class characterizes the restricted weak-type (p, p) for the Hardy-Littlewood maximal operator and, for every measurable set $E \subseteq \mathbb{R}^n$:

$$\|M\chi_E\|_{L^{p,\infty}(w)} \lesssim \|w\|_{A_p^{\mathcal{R}}} w(E)^{1/p}. \tag{3.1}$$

When $p = 1$, this class coincides with $A_1 = A_1^{\mathcal{R}}$, entailing that the weighted weak-type and restricted weak-type (1,1) for M are equivalent. In [3], the authors prove the following result:

Proposition 3.1 ([3, Corollary 2.8])**.** *For every $u \in A_1$, every positive and locally integrable function f and every $1 \leq p < \infty$, the weight $(Mf)^{1-p}u \in A_p^{\mathcal{R}}$ and*

$$\|(Mf)^{1-p}u\|_{A_p^{\mathcal{R}}}^p \lesssim \|u\|_{A_1}.$$

We shall also need the weighted strong-type result in [24] for B but with the dependence on the weight. A simple argument to obtain this result is using [5, Lemma 3.1] to show that, for every $j > 0$,

$$\|K_j * f\|_2 \lesssim 2^{-\frac{n-1}{4}j}\|f\|_2,$$

and that (2.2) and the classical boundedness for M give

$$\|K_j * f\|_{L^2(w)} \lesssim \|Mf\|_{L^2(w)} \lesssim \|w\|_{A_2}\|f\|_{L^2(w)}, \qquad w \in A_2.$$

Interpolation with change of measure and the sharp Reverse Hölder property of A_2 weights in [21] allow us to sum in $j > 0$ and conclude that, for $w \in A_2$:

$$\|Bf\|_{L^2(w)} \lesssim \|w\|_{A_2}^2\|f\|_{L^2(w)}. \tag{3.2}$$

Now we are ready to present our main result for B:

Theorem 3.2. *Given $n > 1$, the Bochner-Riesz operator at the critical index B satisfies that, for every $u \in A_1$, there exists $1 < p_0 < \infty$ depending on u such that, for each measurable set $E \subseteq \mathbb{R}^n$,*

$$\|B\chi_E\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)} \lesssim \|u\|_{A_1}^{4/p_0} u(E)^{1/p_0}. \tag{3.3}$$

More precisely, the exact dependence is $p_0(\|u\|_{A_1}) = 1 + \frac{1}{1+2^{n+1}\|u\|_{A_1}}$.

Proof. Let $\theta \in (0,1)$ be as in (e) from Lemma 2.5. If $\alpha \geq 1$, then we use (3.2) for the A_2 weight $w_\theta := (M\chi_E)^{-\theta} u$:

$$\alpha^{1+\theta} w_\theta\left(\{x : |B\chi_E(x)| > \alpha\}\right) \leq \alpha^2 w_\theta\left(\{x : |B\chi_E(x)| > \alpha\}\right) \leq \|B\chi_E\|^2_{L^2(w_\theta)}$$
$$\lesssim \|w_\theta\|^4_{A_2} \|\chi_E\|^2_{L^2(w_\theta)} \lesssim \|u\|^4_{A_1} u(E).$$

In the last inequality we used the classical properties for A_p weights (see [14, Chap. 9]) and the fact that $0 < \theta = \frac{1}{1+2^{n+1}\|u\|_{A_1}} \leq \frac{1}{1+2^{n+1}} < 1$ is bounded away from 1:

$$\|w_\theta\|_{A_2} \leq \|(M\chi_E)^\theta\|_{A_1} \|u\|_{A_1} \approx \frac{\|u\|_{A_1}}{1-\theta} \approx \|u\|_{A_1}.$$

If $0 < \alpha < 1$, we decompose E as in Remark 2.2 and

$$\alpha^{1+\theta} w_\theta\left(\{x : |B\chi_E(x)| > \alpha\}\right) \lesssim \alpha^{1+\theta} w_\theta\Bigg(\bigcup_{i,k=0}^{\infty} 3Q_i^k \Bigg)$$
$$+ \alpha^{1+\theta} w_\theta\Bigg(\Bigg\{ x \notin \bigcup_{i,k=0}^{\infty} 3Q_i^k : |B\chi_E(x)| > \alpha \Bigg\}\Bigg).$$

For the first term, we use that $w_\theta \in A^{\mathcal{R}}_{1+\theta}$ and by Proposition 3.1, $\|w_\theta\|^{1+\theta}_{A^{\mathcal{R}}_{1+\theta}} \lesssim \|u\|_{A_1}$. Also, recall that $w_\theta = u$ on E:

$$\alpha^{1+\theta} w_\theta\Bigg(\bigcup_{i,k=0}^{\infty} 3Q_i^k \Bigg) \lesssim \alpha^{1+\theta} \|u\|_{A_1} \sum_{i,k=0}^{\infty} w_\theta(Q_i^k)$$
$$\approx \|u\|_{A_1} \sum_{i,k=0}^{\infty} \frac{w_\theta(Q_i^k)}{u(E_k \cap Q_i^k)} \left(\frac{|E_k \cap Q_i^k|}{|Q_i^k|} \right)^{1+\theta} u(E_k \cap Q_i^k)$$
$$\lesssim \|u\|^2_{A_1} u(E).$$

On the other hand, looking at the intersection of the supports of K_j and χ_{E_k}, it is easy to see that if $x \notin \bigcup_{i,k=0}^{\infty} 3Q_i^k$, then

$$B\chi_E = \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} K_j * \chi_{E_k} = \sum_{k=0}^{\infty} \sum_{j=k+1}^{\infty} K_j * \chi_{E_k} = \sum_{s=1}^{\infty} \sum_{j=s}^{\infty} K_j * \chi_{E_{j-s}},$$

so using Chebyshev and (e) in Lemma 2.5:

$$\alpha^{1+\theta} w_\theta\Bigg(\Bigg\{ x \notin \bigcup_{i,k=0}^{\infty} 3Q_i^k : |B\chi_E(x)| > \alpha \Bigg\}\Bigg) \leq \alpha^{1+\theta} w_\theta\Bigg(\Bigg\{ x : \Bigg| \sum_{s=1}^{\infty} \sum_{j=s}^{\infty} K_j * \chi_{E_{j-s}} \Bigg| > \alpha \Bigg\}\Bigg)$$
$$\leq \alpha^{\theta-1} \Bigg\| \sum_{s=1}^{\infty} \sum_{j=s}^{\infty} K_j * \chi_{E_{j-s}} \Bigg\|^2_{L^2(w_\theta)} \leq \alpha^{\theta-1} \Bigg(\sum_{s=1}^{\infty} \Bigg\| \sum_{j=s}^{\infty} K_j * \chi_{E_{j-s}} \Bigg\|_{L^2(w_\theta)} \Bigg)^2$$
$$\lesssim \alpha^{\theta-1} \Bigg(\sum_{s=1}^{\infty} \|u\|_{A_1} 2^{-s\frac{\beta}{2}} \alpha^{\frac{1-\theta}{2}} u(E)^{1/2} \Bigg)^2 \approx \|u\|^2_{A_1} (2^{\beta/2} - 1)^{-2} u(E) \lesssim \|u\|^4_{A_1} u(E),$$

since $(2^{\beta/2}-1)^{-2} \approx \|u\|_{A_1}^2$. So taking supremum over $\alpha > 0$, we have shown that (3.3) holds for $p_0 = 1 + \frac{1}{1+2^{n+1}\|u\|_{A_1}} > 1$. □

For later purposes, we will need the following fact stating that Theorem 3.2 holds for B^r uniformly in $r > 0$. This is an easy computation that we shall omit. The only detail that needs to be pointed out is that the dependence on u of the p_0 coming from Theorem 3.2 is in terms of $\|u\|_{A_1}$. Hence, if $u_r(x) = r^n u(rx)$, we have that $\|u_r\|_{A_1} = \|u\|_{A_1}$ and the same p_0 also works for u_r.

Corollary 3.3. *For every weight $u \in A_1$, there is some $1 < p_0 < \infty$ such that, for each measurable set $E \subseteq \mathbb{R}^n$,*

$$\|B^r \chi_E\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)} \lesssim \|u\|_{A_1}^{4/p_0} u(E)^{1/p_0},$$

uniformly in $r > 0$.

4. Extrapolation results

As mentioned in the introduction, our goal in this section is to prove that a condition of the type (3.3) can be extrapolated to obtain a weak-type $(1,1)$ estimate for every weight in A_1. We shall follow the ideas in [3, 4] where, motivated by Proposition 3.1, the authors introduced the subclass

$$\widehat{A}_p = \{w : w = (Mf)^{1-p}u, \text{ for some } f \in L^1_{\text{loc}} \text{ and } u \in A_1\} \subseteq A_p^{\mathcal{R}},$$

and proved that if we have a sublinear operator T that is of restricted weak-type (p_0, p_0) for some $1 < p_0 < \infty$ and every weight $w \in \widehat{A}_{p_0}$, then it is of restricted weak-type (p,p) for every weight $w \in \widehat{A}_p$ and every $1 \le p < \infty$. At this point we should emphasize that, unlike the classical Rubio de Francia extrapolation for A_p weights, this theory yields estimates at the endpoint $p = 1$. For more details on the classical Rubio de Francia theory, we refer to its modern presentation in [7, 11]. The following result states that if T satisfies a restricted weak-type estimate but only for a very particular subclass of $\widehat{A}_{p_0}$, then we obtain the analogous estimate for the whole range of $1 \le p < \infty$, and at $p = 1$, we still recover the whole A_1 class. We will also drop the sublinearity condition on T, since for the weight we are considering, we can avoid the interpolation step in the original result of [3].

Theorem 4.1. *Let $1 < p_0 < \infty$. If an operator T satisfies that, for every measurable set $E \subseteq \mathbb{R}^n$ and every weight $u \in A_1$,*

$$\|T\chi_E\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)} \le \varphi(\|u\|_{A_1}) u(E)^{1/p_0},$$

with φ an increasing function on $(0,\infty)$, then for every $1 \le p < \infty$,

$$\|T\chi_E\|_{L^{p,\infty}((M\chi_E)^{1-p}u)} \lesssim \varphi_p(\|u\|_{A_1}) u(E)^{1/p},$$

with

$$\varphi_p(t) = \begin{cases} t^{\frac{1}{p}-\frac{1}{p_0}}\varphi(t), & \text{if } 1 \le p \le p_0, \\ t^{\frac{p+1}{pp_0}\frac{p-p_0}{p-1}}\varphi(t), & \text{if } p_0 < p < \infty. \end{cases}$$

Proof. Let us start with the case $1 \leq p < p_0$. Following the ideas in [3, Theorem 3.10], we fix the weight $w = (M\chi_E)^{1-p}u$, a parameter $\gamma > 0$, and it holds that

$$\begin{aligned}\int_{\{|T\chi_E|>y\}} w(x)dx &\leq \int_{\{M\chi_E>\gamma y\}} w(x)dx + \gamma^{p_0-p}\frac{y^{p_0}}{y^p}\int_{\{|T\chi_E|>y\}} (M\chi_E)^{p-p_0}(x)w(x)dx \\ &= \int_{\{M\chi_E>\gamma y\}} w(x)dx + \gamma^{p_0-p}\frac{y^{p_0}}{y^p}\int_{\{|T\chi_E|>y\}} (M\chi_E)^{1-p_0}(x)u(x)dx.\end{aligned}$$

Now, we apply (3.1), Proposition 3.1 and our hypothesis to deduce that

$$y^p \int_{\{|T\chi_E>y|\}} w(x)dx \leq \frac{\|u\|_{A_1}u(E)}{\gamma^p} + \gamma^{p_0-p}\varphi(\|u\|_{A_1})^{p_0}u(E).$$

Finally, we take supremum over y and infimum over $\gamma > 0$, which is attained essentially at $\gamma = \|u\|_{A_1}^{\frac{1}{p_0}}\varphi(\|u\|_{A_1})^{-1}$, to conclude that

$$\|T\chi_E\|_{L^{p,\infty}((M\chi_E)^{1-p}u)} \lesssim \|u\|_{A_1}^{\frac{1}{p}-\frac{1}{p_0}}\varphi(\|u\|_{A_1})u(E)^{1/p}. \tag{4.1}$$

The case $p_0 < p < \infty$ is a little more involved. We shall follow [4, Theorem 3.1]. Choose β satisfying

$$1 < \beta < \frac{p_0'}{p'}, \quad \text{and} \quad \beta \leq 1 + \frac{1}{2^{n+1}\|u\|_{A_1}},$$

which by [21] ensures that $u^\beta \in A_1$ and $\|u^\beta\|_{A_1} \lesssim \|u\|_{A_1}$. Let $0 < \theta < 1$ such that

$$\beta\frac{p_0-1}{p-1} + \theta\frac{p-p_0}{p-1} = 1.$$

Now, as in [4], we get that for every $y > 0$,

$$\int_{\{|T\chi_E|>y\}} (M\chi_E)^{1-p}(x)u(x)dx \leq \int_{\{|T\chi_E|>y\}} (M\chi_E)^{1-p_0}(x)v(x)dx,$$

with

$$v(x) = u(x)^{\beta\frac{p_0-1}{p-1}}\left(M(u^\theta(M\chi_E)^{1-p}\chi_{\{|T\chi_E|>y\}})(x)\right)^{\frac{p-p_0}{p-1}} \in A_1,$$

and $\|v\|_{A_1} \lesssim \|u\|_{A_1}$ (using [4, Lemma 2.12] for this last fact). With this, our hypothesis yields

$$\int_{\{|T\chi_E|>y\}} (M\chi_E)^{1-p}(x)u(x)dx \lesssim \frac{1}{y^{p_0}}\varphi\left(\|u\|_{A_1}\right)^{p_0} v(E).$$

Finally, we need to estimate $v(E)$. This goes exactly as in the original proof, just recalling that $M\chi_E \equiv 1$ on E, so skipping the details, we get that

$$\|T\chi_E\|^p_{L^{p,\infty}((M\chi_E)^{1-p}u)} \lesssim C_{p,\theta}((M\chi_E)^{1-p}u)^{\frac{p(p-p_0)}{p_0(p-1)}}\varphi\left(\|u\|_{A_1}\right)^p u(E),$$

where the constant $C_{p,\theta}(\cdot)$ is the one appearing in [4, Lemma 2.6]. Using that in our case $\frac{1}{p'} < \theta < 1$, we can choose the best possible value for θ so that

$$C_{p,\theta}((M\chi_E)^{1-p}u) \lesssim \|u\|_{A_1}^{\frac{p+1}{p}}.$$

If we plug this into the previous estimate, we finish the proof:

$$\|T\chi_E\|_{L^{p,\infty}((M\chi_E)^{1-p}u)} \lesssim \|u\|_{A_1}^{\frac{p+1}{pp_0}\frac{p-p_0}{p-1}} \varphi\left(\|u\|_{A_1}\right) u(E)^{1/p}.$$

□

Notice that the most interesting feature of this result is that the conclusion at $p = 1$ holds for the whole A_1 class. In fact, if our goal is just to reach the endpoint, we can make yet another simplification. Namely, we can obtain the restricted weak-type (1,1) estimate for A_1 weights starting from a restricted weak-type (p_0, p_0) assumption in which p_0 may depend on the weight u. The key fact is that we always have $1 = p < p_0$. Therefore, regardless of the value of p_0, we must argue as for the first case in the proof of Theorem 4.1. Notice that in this case, to prove the estimate at level $p = 1$ for a fixed weight $u \in A_1$, we use the assumption at level p_0 with exactly the same weight u, so the dependence $p_0(u)$ does not affect the argument. The conclusion is (4.1) with $p = 1$, as we state in the following corollary. Here we make the dependence of φ on p_0 explicit, since it represents dependence on u and might need to be taken into account:

Corollary 4.2. *Let T be an operator. For every weight $u \in A_1$, if there is some $1 < p_0 < \infty$ such that*

$$\|T\chi_E\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)} \leq \varphi_{p_0}(\|u\|_{A_1})u(E)^{1/p_0},$$

then

$$\|T\chi_E\|_{L^{1,\infty}(u)} \leq \|u\|_{A_1}^{1-\frac{1}{p_0}} \varphi_{p_0}(\|u\|_{A_1})u(E).$$

Even though the results presented above only yield restricted weak-type estimates, it is known that for a large class of operators (as it happened for the Hardy-Littlewood maximal function M), this is equivalent to being of weak-type (1,1). We will need to define a notion introduced in [2] that gives a sufficient condition for operators to be of weak-type (1,1) just from a restricted weak-type estimate.

Definition 4.3. *Given $\delta > 0$, a function $a \in L^1(\mathbb{R}^n)$ is called a δ-atom if it satisfies the following properties:*

(i) *$\int_{\mathbb{R}^n} a = 0$, and*
(ii) *there exists a cube Q such that $|Q| \leq \delta$ and $\operatorname{supp} a \subseteq Q$.*

Moreover, a sublinear operator T is (ϵ, δ)-atomic if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|Ta\|_{L^1+L^\infty} \leq \epsilon\|a\|_1,$$

for every δ-atom a.

In [2], it was shown that this is not a strong property to assume on an operator. For instance, it is checked that if

$$Tf(x) = K * f(x), \tag{4.2}$$

with $K \in L^p(\mathbb{R}^n)$ for some $1 \leq p < \infty$, or K measurable and uniformly continuous on $\mathbb{R}^n$, then T is (ϵ, δ)-atomic. Many times we will not have (ϵ, δ)-atomic operators, but they will be approximable in some sense by them, which will be enough for our purposes. The result concerning the boundedness of this kind of operators is the following:

Theorem 4.4. *Let T be a sublinear operator (ϵ,δ)-atomic and let $u\in A_1$. Then, if there exists a constant $C_u>0$ such that, for every measurable set E,*

$$\|T\chi_E\|_{L^{1,\infty}(u)}\le C_u u(E),$$

we have that

$$T: L^1(u)\longrightarrow L^{1,\infty}(u)$$

with constant $2^nC_u\|u\|_{A_1}$.

This result was proved in [2] in the unweighted case, and extended to A_1 weights in [3]. Notice that by Plancherel's theorem, $K\in L^2(\mathbb{R}^n)$ and thus, B is (ϵ,δ)-atomic by (4.2). Therefore, Corollary 4.2 and Theorem 4.4 yield that the weighted estimate in Theorem 3.2 can be used to deduce A. Vargas [28] result:

Corollary 4.5. *Given $n>1$, the Bochner-Riesz operator at the critical index B is of weak-type $(1,1)$ for every weight in $u\in A_1$.*

Finally, the applications we will present are based on the transference of known estimates to operators that can be written as an average, in order to use extrapolation and get endpoint results. This idea will become clear after the following general proposition:

Proposition 4.6. *Let (Ω,μ) be a measure space and let $\{T_\omega\}_{\omega\in\Omega}$ be a collection of sublinear operators indexed by $\omega\in\Omega$ and such that, for every $u\in A_1$ there is some $1<p_0<\infty$ so that, for each $E\subseteq\mathbb{R}^n$ measurable set*

$$\|T_\omega\chi_E\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)}\lesssim\varphi_{p_0}(\|u\|_{A_1})u(E)^{1/p_0},$$

uniformly on $\omega\in\Omega$. Then, if $\Phi\in L^1(\Omega,|\mu|)$, the operator (that we assume to be well-defined)

$$Tf(x)=\int_\Omega T_\omega f(x)\Phi(\omega)d\mu(\omega)$$

is of restricted weak-type $(1,1)$ for every $u\in A_1$ with constant

$$\|u\|_{A_1}^{1-\frac{1}{p_0}}\varphi_{p_0}(\|u\|_{A_1})\|\Phi\|_{L^1(\Omega,|\mu|)}.$$

If T is in addition (ϵ,δ)-atomic, then it is of weak-type $(1,1)$ for every $u\in A_1$ with constant

$$\|u\|_{A_1}^{2-\frac{1}{p_0}}\varphi_{p_0}(\|u\|_{A_1})\|\Phi\|_{L^1(\Omega,|\mu|)}.$$

Proof. Given $u\in A_1$, take its associated $1<p_0=p_0(u)<\infty$ and by Minkowski's inequality

$$\begin{aligned}\|T\chi_E\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)}&\le\int_\Omega\|T_\omega\chi_E\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)}|\Phi(\omega)|d|\mu|(\omega)\\&\lesssim\varphi_{p_0}(\|u\|_{A_1})\|\Phi\|_{L^1(\Omega,|\mu|)}u(E)^{1/p_0}.\end{aligned}$$

Then, we apply Corollary 4.2 to obtain the restricted weak-type $(1,1)$ estimate with the right constant. If T is (ϵ,δ)-atomic, by Theorem 4.4 we complete the proof. □

Remark 4.7. *Notice that if we only had uniform restricted weak-type* $(1,1)$ *estimates for the family* $\{T_\omega\}_{\omega\in\Omega}$*, then the average operator* T *would not necessarily inherit that property, since* $L^{1,\infty}$ *is not a Banach space. The fact that we can transfer estimates from* T_ω *to* T *at level* $p_0 > 1$ *(where Minkowski's inequality is allowed) and then extrapolate down to* $p = 1$*, is the key ingredient in this result.*

5. Applications to Fourier multipliers

5.1. **Fourier multipliers on** $\mathbb{R}$**.** The first application will illustrate our technique with a very simple example. The weighted result that will play the role of Theorem 3.2 is the following:

Proposition 5.1. *Given* $1 < p < \infty$ *and a weight* $w \in A_p^{\mathcal{R}}$*, the Hilbert transform* H *satisfies the restricted weak-type estimate*

$$\|Hf\|_{L^{p,\infty}(w)} \lesssim \|w\|_{A_p^{\mathcal{R}}}^{p+1} \|f\|_{L^{p,1}(w)}.$$

This result has an easy proof based on the pointwise domination of Calderón-Zygmund operators by the so-called sparse operators, and is actually true for any operator with such a control, not just the Hilbert transform. The best known result in terms of domination by sparse operators is contained in [19], and includes all Calderón-Zygmund operators with a Dini-type condition on the modulus of continuity of the kernel. The first result concerning Fourier multipliers that we present is the following:

Theorem 5.2. *Let* m *be a function of bounded variation on* $\mathbb{R}$*. Then, the operator* T_m *defined by*

$$\widehat{T_m f}(\xi) = m(\xi)\widehat{f}(\xi)$$

is of weak-type $(1,1)$ *for every weight* $u \in A_1$ *and with constant controlled by* $\|dm\|\cdot\|u\|_{A_1}^3$*, where* $\|dm\|$ *denotes the total variation of the measure* dm*.*

Proof. Since m is of bounded variation on $\mathbb{R}$, the limit of $m(t)$ as $t \to -\infty$ exists, so by adding a constant to m if necessary, we can assume this limit to be zero. Let $\{\varphi_j\}_j$ be a non-negative approximation to the identity as $j \to \infty$. It holds that $\|\widehat{\varphi}_j\|_\infty \le \|\varphi_j\|_1 = 1$, and we can furthermore assume that the total variation $\|d\widehat{\varphi}_j\| \le 2$ (take for instance the approximation associated with the Poisson kernel, which essentially satisfies $\widehat{\varphi}_j(t) = e^{-|t|/j}$ and has this property). For every $j > 0$, define

$$m_j(t) = m(t)\widehat{\varphi_j}(t).$$

This function is of bounded variation with $\|dm_j\| \le 3\|dm\|$, since

$$\|dm_j\| \le \|m\|_\infty \|d\widehat{\varphi}_j\| + \|\widehat{\varphi}_j\|_\infty \|\|dm\| \le 3\|dm\|.$$

We still have that m_j vanishes at $-\infty$, so we can write the Lebesgue-Stieltjes integral

$$m_j(\xi) = \int_{\mathbb{R}} \chi_{(t,\infty)}(\xi) dm_j(t).$$

The multiplier associated with $\chi_{(t,\infty)}$ is

$$f(x) \longmapsto \frac{1}{2}(e^{2\pi i t x} H(e^{-2\pi i t\cdot} f)(x) - f(x)),$$

which is essentially a modulated Hilbert transform that we will denote by H_t (see [10, Estimate (3.9)]). Then,

$$T_{m_j} f(x) = \int_{\mathbb{R}} H_t f(x) dm_j(t). \tag{5.1}$$

Now we use Proposition 5.1 with the weight $w = (M\chi_E)^{1-p}u$, for some $u \in A_1$ and $1 < p < \infty$, and Proposition 3.1, to conclude

$$\begin{aligned}\|H_t\chi_E\|_{L^{p,\infty}((M\chi_E)^{1-p}u)} &\approx \|H(e^{2\pi it\cdot}\chi_E)\|_{L^{p,\infty}((M\chi_E)^{1-p}u)} \\ &\lesssim \|u\|_{A_1}^{1+\frac{1}{p}} \|\chi_E\|_{L^{p,1}((M\chi_E)^{1-p}u)} = \|u\|_{A_1}^{1+\frac{1}{p}} u(E)^{1/p},\end{aligned}$$

uniformly in $t \in \mathbb{R}$. Therefore, the family $\{H_t\}_t$ is under the hypotheses of Proposition 4.6. Also, for every $j > 0$, the operator T_{m_j} is (ϵ, δ)-atomic (since m_j is integrable and hence, its associated convolution kernel is uniformly continuous as in (4.2)). With this, we conclude that T_{m_j} is of weak-type $(1,1)$ for every weight in A_1 with constant

$$\|u\|_{A_1}^{2-\frac{1}{p}} \|u\|_{A_1}^{1+\frac{1}{p}} \|dm_j\| \lesssim \|dm\| \|u\|_{A_1}^3.$$

Finally, since $\{\varphi_j\}_j$ is an approximation to the identity, at least for Schwartz functions f, there is a subsequence such that

$$T_{m_{j(i)}} f(x) = \varphi_{j(i)} * T_m f(x) \xrightarrow{i} T_m f(x) \quad \text{a.e. } x.$$

With this, we use the estimate for T_{m_j} and Fatou's lemma to finish the proof:

$$\|T_m f\|_{L^{1,\infty}(u)} \leq \liminf_{i\to\infty} \|T_{m_{j(i)}} f\|_{L^{1,\infty}(u)} \lesssim \|dm\| \|u\|_{A_1}^3 \|f\|_{L^1(u)}.$$

□

The idea of transferring estimates on Banach spaces from H to T_m based on (5.1) is not new. In [10, Corollary 3.8], this method is used to show that T_m is bounded on $L^p(\mathbb{R})$ for all $1 < p < \infty$. The only difference here is that the Banach estimate that we transfer from H to T_m is a weighted one, and this allows us to extrapolate and deduce a weak-type $(1,1)$ result for T_m that could not be obtained by means of (5.1) and Minkowski's inequality. These multipliers are closely related to the ones appearing in the Marcinkiewicz multiplier theorem (see [10, Theorem 8.13]). In that case, the result claims that if m has uniformly bounded variation on each dyadic interval in $\mathbb{R}$, then T_m maps $L^p(\mathbb{R})$ into itself for every $1 < p < \infty$. This is obtained by means of Littlewood-Paley theory, and can be extended to the weighted setting to prove the same result for A_p weights [17]. However, it is known that there are operators under the hypotheses of Marcinkiewicz's theorem that fail to be of weak-type (1,1), even in the unweighted case (see [25] for sharp results near L^1). Therefore, we know that our assumption for m to be of bounded variation on $\mathbb{R}$ cannot be relaxed to uniform bounded variation on dyadic intervals.

The next subsection will follow the same argument but using the estimate for the Bochner-Riesz operator in Theorem 3.2 to draw conclusions about radial Fourier multipliers on $\mathbb{R}^n$.

5.2. **Radial Fourier multipliers on $\mathbb{R}^n$.** For this part, we will need to recall fractional integration and derivation, in the sense of Weyl. The idea of using fractional calculus to obtain results for radial Fourier multipliers was already introduced in [26] and subsequently used in [9, 13], among others.

Definition 5.3. *Given $0 \leq \delta < 1$ and $w > 0$, we define the truncated fractional integral of order $1-\delta$ of a locally integrable function f on $\mathbb{R}$ by*

$$I_w^{1-\delta} f(t) := \frac{1}{\Gamma(1-\delta)} \int_{-w}^{w} (s-t)_+^{-\delta} f(s) ds, \quad t < w,$$

and 0 if $t \geq w$. Moreover, if $\alpha = [\alpha] + \delta > 0$, with $[\alpha]$ being its integer part and δ its fractional part, we define the fractional derivative of f of order α by

$$D^\alpha f(t) := -\left(\frac{d}{dt}\right)^{[\alpha]} \lim_{w\to\infty} \frac{d}{dt} I_w^{1-\delta} f(t),$$

whenever the right-hand side exists. In particular, if f has compact support, then

$$D^\alpha f(t) := -\left(\frac{d}{dt}\right)^{[\alpha]+1} I_\infty^{1-\delta} f(t).$$

Now we are ready to state the main result of this section. Before making its statement precise, let us briefly summarize how it is related to other results in the literature. The integrability condition that we will require on m will be

$$\int_0^\infty t^{\frac{n-1}{2}} |D^{\frac{n+1}{2}} m(t)| dt < \infty, \tag{5.2}$$

and we will obtain a weak-type $(1,1)$ estimate with respect to every weight in A_1 for the Fourier multiplier with symbol $m(|\xi|^2)$. This type of condition (5.2) on m is not new. For instance, in the unweighted setting, [9, 22] use Weyl's fractional calculus to obtain strong-type (p,p) and weak-type $(1,1)$ results for maximal operators associated with quasiradial Fourier multipliers. The condition that they require on m is also an integrability condition for $t^{\alpha-1} D^\alpha m$, but with $\alpha > \frac{n+1}{2}$ (see [22, Corollary 1]).

Another similar result to the one we present can be found in [18]. Here the authors deal with weights, but they consider general Fourier multipliers on $\mathbb{R}^n$, not necessarily radial ones. In terms of differentiability requirements, the condition that they need on m to get the weak-type $(1,1)$ for every weight in A_1 is up to order n. In our case, we only work with radial multipliers and require order $\frac{n+1}{2}$ instead. In the classical Hörmander theorem [15] without weights, it is enough to have differentiability up to order strictly larger than $\frac{n}{2}$, which is essentially optimal even in the radial case (see [6]). Therefore, the differentiability assumption in our result is not that far from the optimal order for the unweighted case. Another important reference is [1], where one can find sufficient conditions for radial Fourier multipliers to be bounded on $L^p(\mathbb{R}^2)$ for $4/3 \leq p \leq 4$. This limitation in the range of p (which totally excludes the endpoint $p = 1$) allows the authors to lower the order of differentiability of m to $\alpha > 1/2$, which corresponds to $\frac{n-1}{2}$ in $\mathbb{R}^2$.

The precise statement of our result is the following. AC_{loc} will denote the space of functions which are absolutely continuous on every compact subset of $(0, \infty)$.

Theorem 5.4. *Fix $n \geq 2$ and $\alpha = \frac{n+1}{2}$. Let m be a bounded, continuous function on $(0,\infty)$ which vanishes at infinity and satisfies that*

$$D^{\alpha-j}m \in AC_{\text{loc}} \quad \forall j = 1, ..., [\alpha].$$

Then, if $D^{\frac{n+1}{2}}m$ exists and

$$\Phi(t) = t^{\alpha-1}D^\alpha m(t) \in L^1(0,\infty),$$

the operator T_m defined by

$$\widehat{T_m f}(\xi) = m(|\xi|^2)\widehat{f}(\xi), \quad \xi \in \mathbb{R}^n,$$

is of weak-type $(1,1)$ for every weight $u \in A_1$ with constant controlled by $C\|\Phi\|_{L^1(0,\infty)}\|u\|_{A_1}^5$.

Proof. First, we will use [27, Lemma 3.14] to write

$$m(t) = \frac{(-1)^{[\alpha]}}{\Gamma(\alpha)}\int_{\mathbb{R}}(s-t)_+^{\alpha-1}D^\alpha m(s)ds = C_\alpha\int_t^\infty (s-t)^{\alpha-1}D^\alpha m(s)ds,$$

which is valid under our hypotheses for m. With this identity, we prove that

$$T_m f(x) = \int_0^\infty B^{1/s}f(x)\overline{\Phi}(s)ds, \quad x \in \mathbb{R}^n, \tag{5.3}$$

with $\overline{\Phi} \in L^1(0,\infty)$. It is enough to check that for every $\xi \in \mathbb{R}^n$,

$$m(|\xi|^2) = \int_0^\infty \left(1 - \frac{|\xi|^2}{s^2}\right)_+^{\alpha-1}\overline{\Phi}(s)ds, \tag{5.4}$$

but this follows by the change of variables $s = r^2$ and taking $t = |\xi|^2$,

$$\begin{aligned} m(|\xi|^2) =& 2C_\alpha\int_{|\xi|}^\infty (r^2 - |\xi|^2)^{\alpha-1}D^\alpha m(r^2)rdr \\ =& 2C_\alpha\int_0^\infty r^{2\alpha-1}\left(1 - \frac{|\xi|^2}{r^2}\right)_+^{\alpha-1}D^\alpha m(r^2)dr. \end{aligned}$$

which is (5.4) with $\overline{\Phi}(r) = C_\alpha r^{2\alpha-1}D^\alpha m(r^2)$ and $\|\overline{\Phi}\|_{L^1(0,\infty)} \approx \|\Phi\|_{L^1(0,\infty)}$. The second ingredient in the proof is the uniform bound given in Corollary 3.3. More precisely, that

$$\|B^{1/s}\chi_E\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)} \lesssim \|u\|_{A_1}^{4/p_0}u(E)^{1/p_0}, \tag{5.5}$$

uniformly in $s \in (0,\infty)$. To conclude the argument, if $K_{1/s}$ is the kernel associated with $B^{1/s}$, define, for every $j > 0$,

$$K^j(x) = \int_0^j K_{1/s}(x)\overline{\Phi}(s)ds = \int_0^\infty K_{1/s}(x)\overline{\Phi}_j(s)ds,$$

with $\overline{\Phi}_j(s) = \overline{\Phi}(s)\chi_{(0,j)}(s) \in L^1(0,\infty)$ and $\|\overline{\Phi}_j\|_{L^1(0,\infty)} \leq \|\overline{\Phi}\|_{L^1(0,\infty)}$. Clearly, $K^j \in L^2(\mathbb{R}^n)$ and by (4.2),

$$T^j f(x) = K^j * f(x) = \int_0^\infty B^{1/s}f(x)\overline{\Phi}_j(s)ds$$

is an (ϵ,δ)-atomic operator. Now, we use Proposition 4.6 and (5.5) to deduce that T^j is of weak-type $(1,1)$ for every $u \in A_1$ and with constant

$$\|u\|_{A_1}^{2-\frac{1}{p_0}} \|u\|_{A_1}^{\frac{4}{p_0}} \|\overline{\Phi}_j\|_{L^1(0,\infty)} \leq \|u\|_{A_1}^5 \|\overline{\Phi}\|_{L^1(0,\infty)} \approx \|u\|_{A_1}^5 \|\Phi\|_{L^1(0,\infty)},$$

independently of $j>0$. Using Fatou's Lemma and (5.3), we conclude that for every $f \in L^1(u)$,

$$\|T_m f\|_{L^{1,\infty}(u)} \leq \liminf_{j\to\infty} \|T^j f\|_{L^{1,\infty}(u)} \lesssim \|\Phi\|_{L^1(0,\infty)} \|u\|_{A_1}^5 \|f\|_{L^1(u)}.$$

□

Let us finish this section by giving a particular example of application of Theorem 5.4. It will be related to the following conjecture stated in [23]:

Conjecture 5.5. *Assume that φ is a $\mathcal{C}^\infty$ function with compact support contained in $(-1/2,1/2)$ and, for every $0<\delta<1$, set*

$$h_\delta(s) := \varphi\left(\frac{1-s}{\delta}\right).$$

Then, for every $1<p<\frac{2n}{n+1}$, the operator T_{h_δ} defined by

$$\widehat{T_{h_\delta} f}(\xi) = h_\delta(|\xi|^2)\widehat{f}(\xi)$$

satisfies:

$$\|T_{h_\delta}\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim \delta^{-\lambda(p)}, \qquad \text{with } \lambda(p) = n\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{2}. \tag{5.6}$$

The result we present is the following:

Corollary 5.6. *Given $n\geq 2$, the operator T_{h_δ} is of weak-type (1,1) for every weight $u\in A_1$ and*

$$\|T_{h_\delta}\|_{L^1(u)\to L^{1,\infty}(u)} \lesssim \delta^{-\left(\frac{n-1}{2}\right)} \|u\|_{A_1}^5.$$

To prove this, it is enough to apply Theorem 5.4 (h_δ is under its hypotheses) together with the following computation at $\alpha = \frac{n+1}{2}$:

Lemma 5.7. *Given $\alpha>0$, it holds that for $\Phi(t) = t^{\alpha-1}D^\alpha h_\delta(t)$,*

$$\|\Phi\|_{L^1(0,\infty)} \leq C_{\varphi,\alpha}\delta^{-\alpha+1}.$$

Proof. First, we compute $D^\alpha h_\delta$. It can be easily checked from the definition that

$$D^\alpha h_\delta(t) = \frac{1}{\delta^\alpha} D^\alpha \widetilde{\varphi}\left(\frac{t-1}{\delta}\right),$$

with $\widetilde{\varphi}(s) = \varphi(-s)$ being the reflection of φ on $\mathbb{R}$. Now,

$$\int_0^\infty |\Phi(t)|dt = \delta^{-\alpha}\int_0^\infty t^{\alpha-1}\left|D^\alpha\widetilde{\varphi}\left(\frac{t-1}{\delta}\right)\right|dt = \delta^{-\alpha+1}\int_{-1/\delta}^\infty (r\delta+1)^{\alpha-1}|D^\alpha\widetilde{\varphi}(r)|dr.$$

Since $\widetilde{\varphi}$ has compact support in $(-1/2,1/2)$, it is easy to see that $|D^\alpha\widetilde{\varphi}(r)|$ is zero for $r>1/2$ and decays as $C_{\varphi,\alpha}/|r|^{\alpha+1}$ when $r\to-\infty$. Using this decay when $r\in(-1/\delta,-1)$

and that $|D^\alpha \widetilde{\varphi}(r)|$ is bounded when $r \in (-1, 1/2)$, we conclude the proof by showing that

$$\int_{-1/\delta}^{\infty} (r\delta + 1)^{\alpha - 1} |D^\alpha \widetilde{\varphi}(r)| dr \lesssim C_{\varphi,\alpha}.$$

□

Notice that $\lambda(1) = \frac{n-1}{2}$, and hence, Corollary 5.6 is the endpoint weighted weak-type version of estimate (5.6). The same result but with an ϵ loss in the exponent of δ can be derived from [12, Lemma 5.2], where the authors prove that for every $\epsilon > 0$,

$$|T_{h_\delta} f(x)| \le C_\epsilon \delta^{-(\frac{n-1}{2} + \epsilon)} Mf(x).$$

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