1) Prove that $\hat{u}(\xi) = -|\xi|^{-2}$ in $\mathbb{R}^3$ is a tempered distribution and satisfies $\Delta u = \delta$. Show that $u$ is in fact a function in $L^\infty + L^2$.

2) Let $n \geq 3$ and let $E(x) = |x|^{2-n}$. Prove that:
   a) $E$ is a tempered distribution.
   b) Compute $\Delta E$ whenever exists.
   c) Show that $\Delta E = \delta$ in the sense of distributions.

Recall Green’s identity: If $R$ is a bounded domain with smooth boundary $S$ and $f$ and $g$ are $C^1$ functions on $\mathbb{R}$, then:

$$\int_R (f\Delta g - g\Delta f) dx = \int_S (f\delta_v g - g\delta_v f) d\sigma,$$

where $\delta_v$ is the directional derivative with respect to the outward normal vector to $R$.

3) Let us consider the Heat equation

$$\frac{\partial u}{\partial t} = \Delta u, \quad t > 0, \ x \in \mathbb{R}^n. $$

a) Prove that if $P(x,t) = t^{-n/2} e^{-|x|^2/4t}$ then $P$ satisfies the Heat equation.

b) Let $P_\epsilon(x,t) = \chi_{(\epsilon,\infty)}(t) P(x,t)$. Prove that $\lim_{\epsilon \to 0} P_\epsilon$ is the fundamental solution of the Heat equation in $\mathbb{R}^{n+1}$.

4) Prove that if $P(x,t)$ is as in exercise 3 with $t > 0, \ x \in \mathbb{R}^n$, then:
   a) $S'(\mathbb{R}^n) - \lim_{\epsilon \to 0} P(x,t) = 0$.
   b) For every $\lambda > 0$,

$$F(x,\lambda) = S'(\mathbb{R}^n) - \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} e^{-\lambda t} P(x,t) dt$$

is the fundamental solution of $\Delta + \lambda I$.

5) a) For which $s \in \mathbb{R}$, we have that $1 \in H^s(\mathbb{R}^n)$.
   b) For which $s \in \mathbb{R}$, we have that $\delta \in H^s(\mathbb{R}^n)$.
   c) For which $s \in \mathbb{R}$, we have that $\chi_{[0,1]} \in H^s(\mathbb{R})$.
   d) For which $s \in \mathbb{R}$, we have that $\chi_{[0,1]} \times \chi_{[0,1]} \in H^s(\mathbb{R}^2)$.

6) Prove that if $\varphi \in S(\mathbb{R}^n)$ and $f \in H^s(\mathbb{R}^n)$, then $\varphi f \in H^s(\mathbb{R}^n)$.

7) Prove, justifying all the steps, that if $f, g \in L^2(\mathbb{R})$ are such that their derivatives in the sense of distributions $f', g' \in L^2(\mathbb{R})$, then the following integration by parts formula holds

$$\int_{\mathbb{R}} f'g' = -\int_{\mathbb{R}} f'g.$$

8) Given a function $f \in H^s(\mathbb{R}^n)$ prove that if $f_R$ is such that $\hat{f_R} = \chi_{B(0,R)} \hat{f}$, then $H^2 - \lim_{R \to \infty} f_R = f$ and deduce that $C^\infty \cap L^2$ is dense in $H^2$. 
