Solar Sails the Earth - Sun System

A. Farrés[1] À. Jorba[2]

[1]Institut de Mathématiques de Bourgogne,
Université de Bourgogne

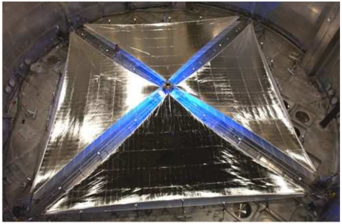
[2]Departament de Matemàtica Aplicada i Ànalisi,
Universitat de Barcelona

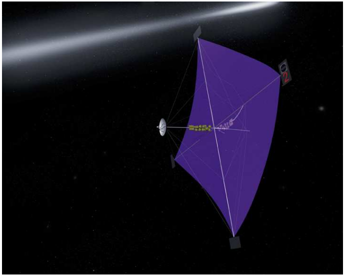
5th November 2012

Séminaire étudiants (DMD)

What is a Solar Sail ?

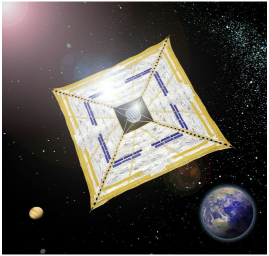
- Solar Sails are a new concept of spacecraft propulsion that takes advantage of the Solar radiation pressure to propel a satellite.
- The impact of the photons emitted by the Sun on the surface of the sail and its further reflection produce momentum on it.
- Solar Sails open a wide new range of possible missions that are not accessible by a traditional spacecraft.

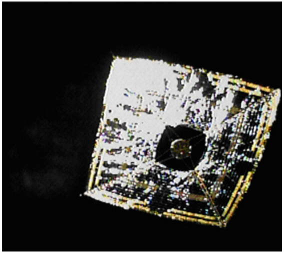




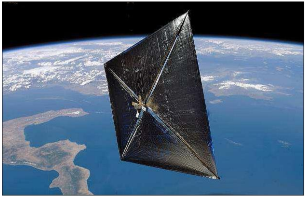
There have recently been two successful deployments of solar sails in space.

- IKAROS: in June 2010, JAXA managed to deploy the first solar sail in space.





- NanoSail-D2: in January 2011, NASA deployed the first solar sail that would orbit around the Earth.



The Solar Sail

We have considered a flat and perfectly reflecting Solar Sail. Hence, the force due to the solar radiation pressure is in the normal direction to the surface of the sail ($\vec{n}$).

The force due to the sail is defined by the *sail's orientation* and the *sail's lightness number*.

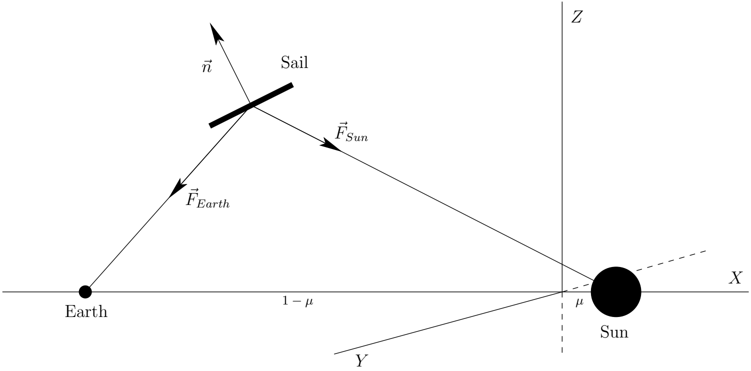
- The *sail's orientation* is given by the normal vector to the surface of the sail, $\vec{n}$. It is parametrised by two angles, α and δ.
- The *sail's lightness number* is given in terms of the dimensionless parameter β. It measures effectiveness of the sail.

Hence, the force is given by:

$$\vec{F}_{sail} = \beta \frac{m_s}{r_{ps}^2} \langle \vec{r}_s, \vec{n} \rangle^2 \vec{n}.$$

The Dynamical Model

We use the Restricted Three Body Problem (RTBP) taking the Sun and Earth as primaries and including the solar radiation pressure to model the motion of the sail.



Equations of Motion

The equations of motion are:

$$\begin{aligned}
\ddot{x} &= 2\dot{y} + x - (1-\mu)\frac{x-\mu}{r_{ps}^3} - \mu\frac{x+1-\mu}{r_{pe}^3} + \beta\frac{1-\mu}{r_{ps}^2}\langle\vec{r}_s,\vec{n}\rangle^2 n_x,\\
\ddot{y} &= -2\dot{x} + y - \left(\frac{1-\mu}{r_{ps}^3} + \frac{\mu}{r_{pe}^3}\right) y + \beta\frac{1-\mu}{r_{ps}^2}\langle\vec{r}_s,\vec{n}\rangle^2 n_y,\\
\ddot{z} &= -\left(\frac{1-\mu}{r_{ps}^3} + \frac{\mu}{r_{pe}^3}\right) z + \beta\frac{1-\mu}{r_{ps}^2}\langle\vec{r}_s,\vec{n}\rangle^2 n_z,
\end{aligned}$$

where,

$$\begin{aligned}
n_x &= \cos(\phi(x,y)+\alpha)\cos(\psi(x,y,z)+\delta),\\
n_y &= \sin(\phi(x,y,z)+\alpha)\cos(\psi(x,y,z)+\delta),\\
n_z &= \sin(\psi(x,y,z)+\delta),
\end{aligned}$$

with $\phi(x,y)$ and $\psi(x,y,z)$ defining the Sun - Sail direction in spherical coordinates ($\vec{r}_s = \vec{r}_{ps}/r_{ps}$).

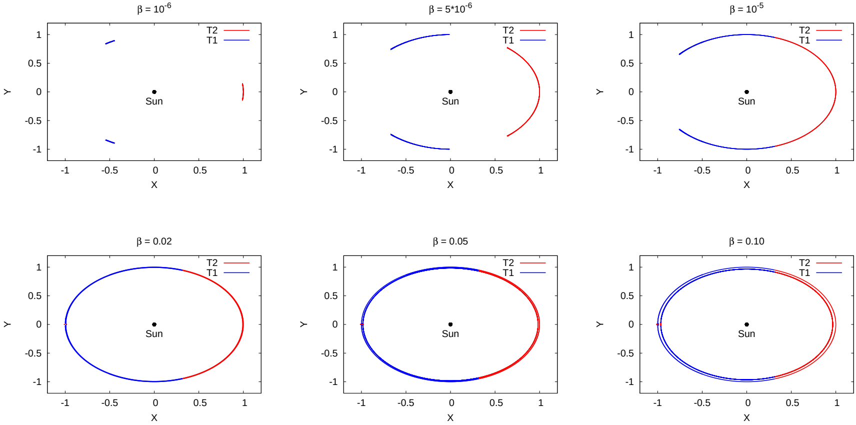
Dynamics of a Solar Sail in the RTBP

Equilibrium Points (I)

- The RTBP has 5 equilibrium points (L_i). For small β, these 5 points are replaced by 5 continuous families of equilibria, parametrised by α and δ.
- For a fixed small value of β, we have 5 disconnected family of equilibria around the classical L_i.
- For a fixed and larger β, these families merge into each other. We end up having two disconnected surfaces, S_1 and S_2. Where S_1 is like a sphere and S_2 is like a torus around the Sun.
- All these families can be computed numerically by means of a continuation method.

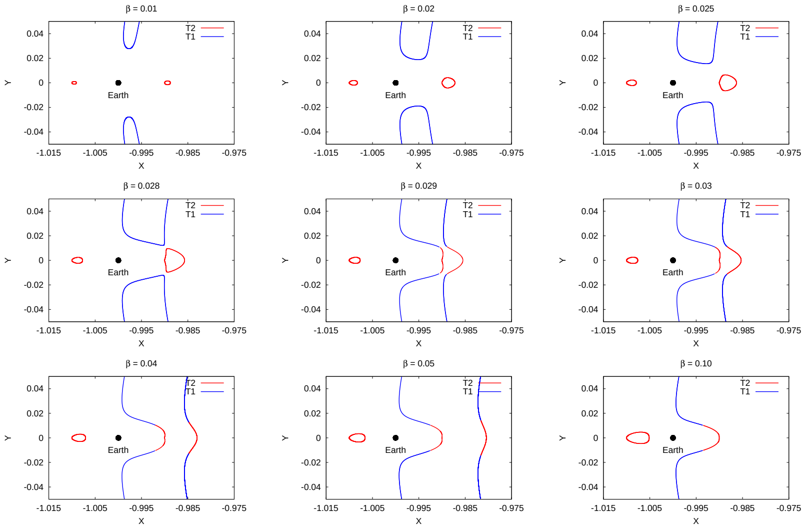
Equilibrium Points (II)

Family of Equilibria on the $\{X, Y\}$ plane.



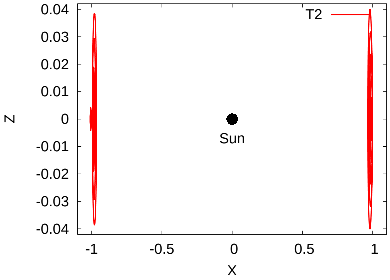
Equilibrium Points (III)

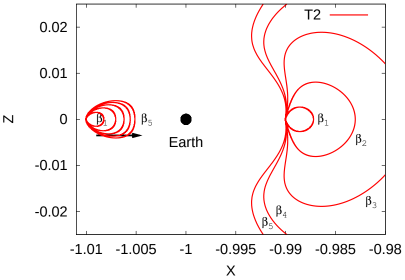
Family of Equilibria on the $\{X, Y\}$ plane (Zoom close to the Earth).



Equilibrium Points (IV)

Family of Equilibria on the $\{X, Z\}$ plane.





Fixed points for $\beta_1 = 0.02, \beta_2 = 0.04, \beta_4 = 0.06, \beta_5 = 0.1$. Left: Zoom close to the Earth.

Stability of the Equilibrium Points

We can classify the equilibrium points by their stability (i.e. the eigenvalues of the linearisation of the flow). Almost all of the equilibrium points belong to these two different classes:

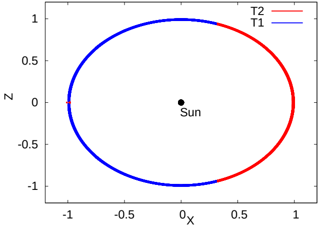
- $\mathcal{T}_1$: Three pairs of complex eigenvalues ($\nu_{1,2,3} \pm \mathrm{i}\omega_{1,2,3}$). Correspond to the blue points.
- $\mathcal{T}_2$: One pair of real eigenvalues ($\lambda_1 > 0, \lambda_2 < 0$) and two pairs of complex eigenvalues ($\nu_{1,2} \pm \mathrm{i}\omega_{1,2}$). Correspond to the red points.

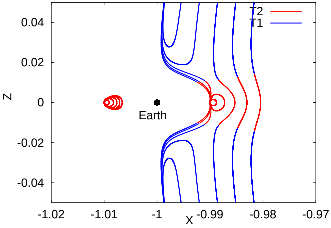
It can be seen that:

- close to $L_{1,2,3}$ the equilibria are of the type $\mathcal{T}_2$ with $|\nu_i| << |\lambda_i|$, hence the main instability is given by the saddle component.
- close to $L_{4,5}$ the equilibria are of the type $\mathcal{T}_1$ with $|\nu_i| << 1$, hence the instability due to the positive real part is very mild.

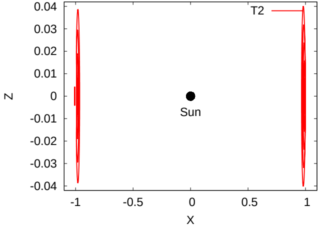
Equilibrium Points

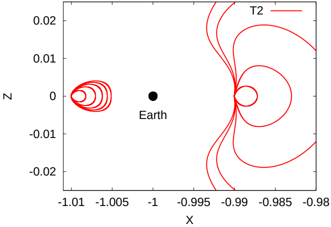
Equilibrium points in the XY plane



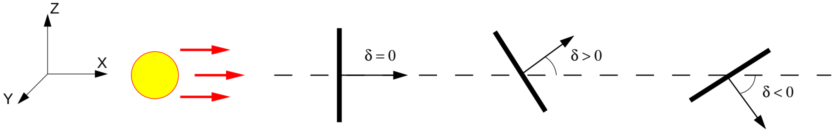


Equilibrium points in the XZ plane



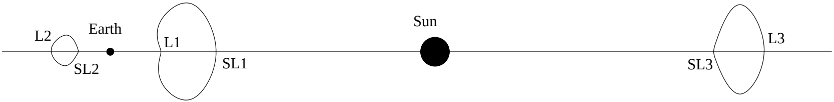


We must restrict ourselves to the case: $\alpha = 0$ and $\delta \in [-\pi/2, \pi/2]$ to find bounded motion around equilibria (i.e. only move the sail vertically w.r.t. the Sun - sail line):



- The system is time reversible $\forall\delta$ by $R : (x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \to (x, -y, z, -\dot{x}, \dot{y}, -\dot{z}, -t)$ and Hamiltonian only for $\delta = 0, \pm\pi/2$.
- There are 5 disconnected families of equilibrium points parametrised by δ, we call them $FL_{1,\dots,5}$ (each one related to one of the Lagrangian points $L_{1,\dots,5}$).
- Three of these families ($FL_{1,2,3}$) lie on the $Y = 0$ plane, and the linear behaviour around them is of the type *saddle* $\times$ *centre* $\times$ *centre*.
- The other two families ($FL_{4,5}$) are close to $L_{4,5}$ and are not fixed by R. The behaviour is of the type *sink* $\times$ *sink* $\times$ *source* or *sink* $\times$ *source* $\times$ *source*.

We will focus on the family of equilibrium points from the FL_1 family for $\beta = 0.051689$ (i.e. *Lightness number for the Geostorm Mission*).



(schematic representation of the equilibrium points on $Y = 0$)

- We will describe the non-linear dynamics around different equilibrium points close to SL_1 (they correspond to a fixed sail orientation $\alpha = 0$ and $\delta \approx 0$).

- We will describe some of the numerical tools that have been developed to describe the non-linear dynamics.

[1] A. Farrés and À. Jorba, "Periodic and Quasi-Periodic motions of a Solar Sail close to SL_1 in the Earth - Sun system.", *Celestial Mechanics and Dynamical Astronomy* Vol. 107, pp. 233-253, 2010.

Periodic Motion (I)

As the system is reversible, the *Devaney - Lyapunov Centre Theorem* ensures us the existence of periodic motion around the equilibrium points on the families $FL_{1,2,3}$.

Theorem (Devaney - Lyapunov)

Let $\dot{x} = f(x)$, with $f \in \mathcal{C}^2$ and $x \in \mathbb{R}^{2n}$ be an autonomous R-reversible dynamical system, where $\dim(Fix(R)) = n$. Let p_0 be a fixed point such that $R(p_0) = p_0$, and with $\pm \mathrm{i}\omega,\ \pm\lambda_2, \dots,\ \pm\lambda_n$ as eigenvalues.

Then, if $\forall \lambda_i$ we have that $\mathrm{i}\omega/\lambda_i \notin \mathbb{Z}$, there exists a one-parametric family of periodic orbits emanating from p_0, where the period of these orbits tends to $2\pi/\omega$ when approaching p_0.

Around each equilibrium point on the $FL_{1,2,3}$ families there are two different families of periodic orbits. Each one related to one of the frequencies ω_1 and ω_2.

Periodic Motion (II)

If we linearise around a certain equilibrium point for a fixed δ:

$$\begin{aligned}\phi(t) &= A_0[\cos(\omega_1 t + \psi_1)\vec{v}_1 + \sin(\omega_1 t + \psi_1)\vec{u}_1] \\ &+ B_0[\cos(\omega_2 t + \psi_2)\vec{v}_2 + \sin(\omega_2 t + \psi_2)\vec{u}_2] \\ &+ C_0 e^{\lambda t}\vec{v}_\lambda + D_0 e^{-\lambda t}\vec{v}_{-\lambda}\end{aligned}$$

From the *Devaney - Lyapunov Centre Theorem*, if $\omega_1/\omega_2 \notin \mathbb{Z}$ then we have two families of periodic orbits.

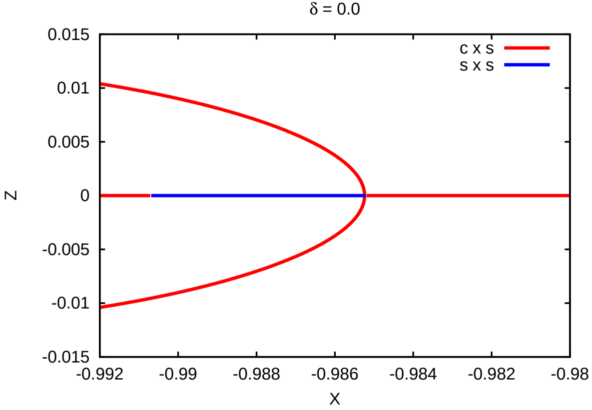
The periodic orbits emanating from ω_2 have a larger vertical oscillation than ω_1.

- We call $\mathcal{P}$ - Family, to the family emanating from ω_1.
- We call $\mathcal{V}$ - Family, to the family emanating from ω_2.

We have computed by means of a continuation method these two families, for different values of δ.

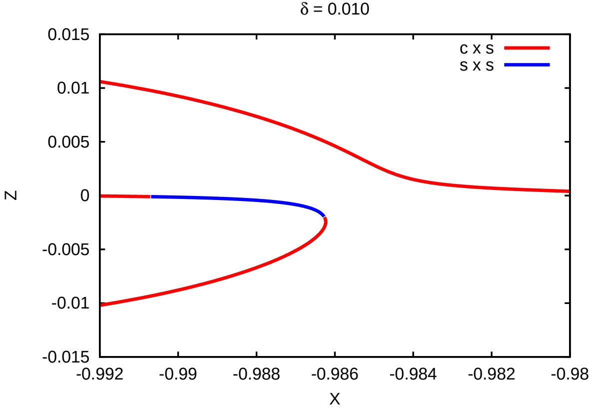
$\mathcal{P}$ - Family of Periodic Orbits (I)

We have computed the planar family for $\delta = 0$. (i.e. sail perpendicular to Sun - line direction).



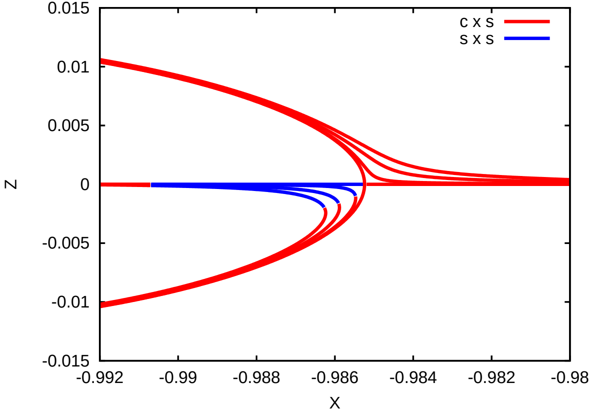
$\mathcal{P}$ - Family of Periodic Orbits (II)

We have computed the planar family for $\delta = 0.01$. (i.e. sail is no longer perpendicular to Sun - line direction).



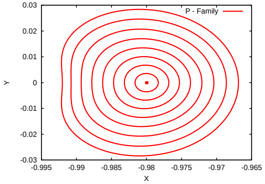
$\mathcal{P}$ - Family of Periodic Orbits (III)

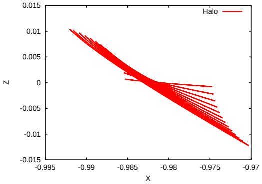
Continuation scheme for different values of δ (in this plot, from $\delta = 0$ to $\delta = 0.01$.

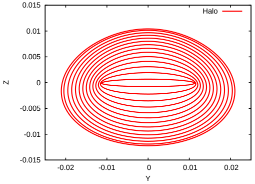


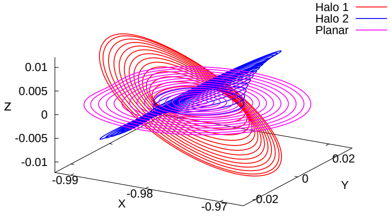
$\mathcal{P}$ - Family of Periodic Orbits (IV)

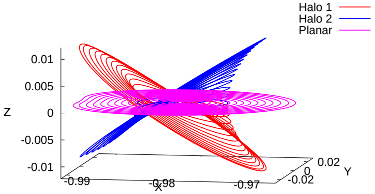
Periodic Orbits for $\delta = 0$.







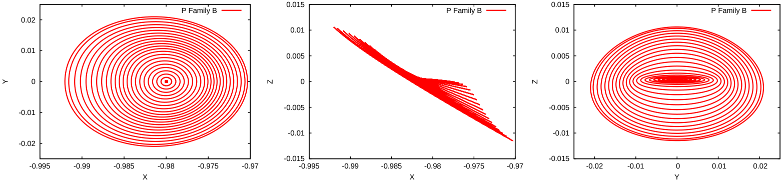




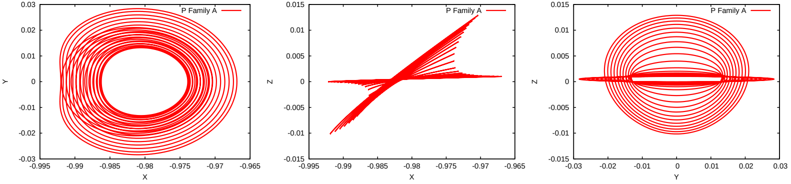
$\mathcal{P}$ - Family of Periodic Orbits (V)

Periodic Orbits for $\delta = 0.01$.

Main family of periodic orbits for $\delta = 0.01$

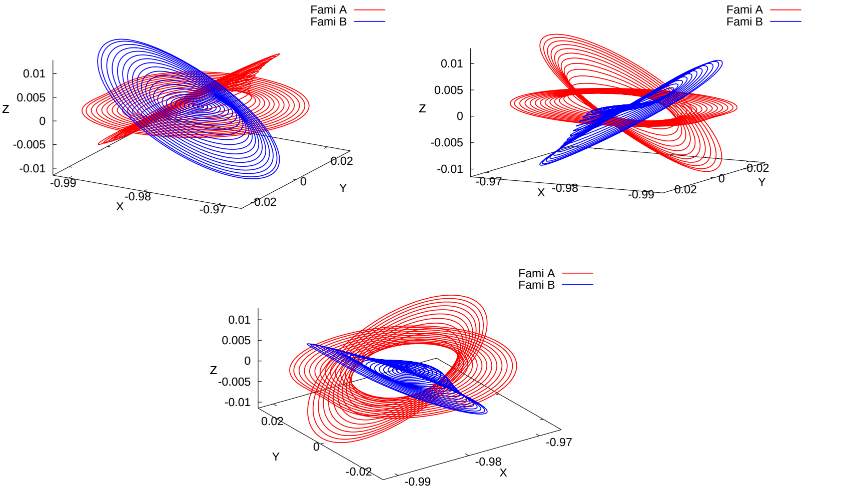


Secondary family of periodic orbits for $\delta = 0.01$

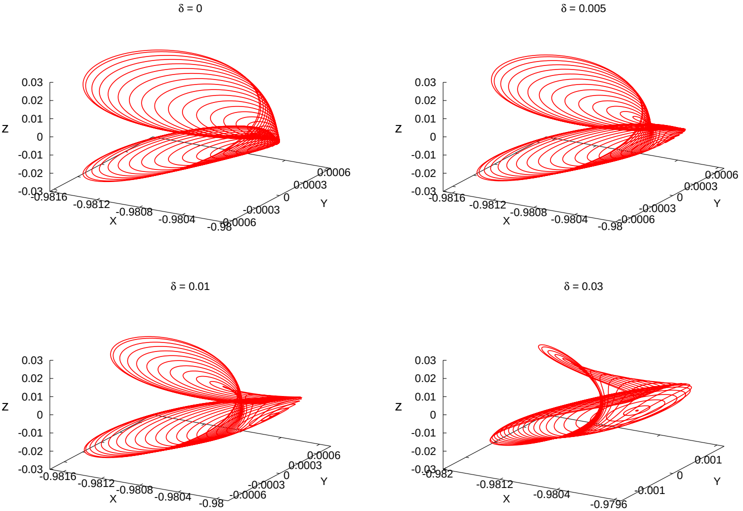


$\mathcal{P}$ - Family of Periodic Orbits (VI)

Periodic Orbits for $\delta = 0.01$.



$\mathcal{V}$ - Family of Periodic Orbits



Nonlinear Dynamics around Equilibria

- We want to understand the dynamics in an extended neighbourhood of an equilibrium point. We are interested in the trajectories that remain close to the equilibrium point.
- Due to the instability of the fixed point, we cannot take arbitrary initial conditions and integrate them numerically, as they will quickly escape from the vicinity of the fixed point.
- For this reason we will perform the reduction to the centre manifold, an invariant manifold tangent to the two centre directions. Although it is not unique, its Taylor expansion around the equilibrium point is well defined and unique.
- The main idea is to decouple up to high order the elliptic from the hyperbolic behaviour, and use this high order approximation of the centre manifold to understand the dynamics.

Reduction to the Centre Manifold

Using an appropriate linear transformation, the equations around the fixed point can be written as,

$$\begin{aligned} \dot{x} &= Ax + f(x,y), \quad x \in \mathbb{R}^4, \\ \dot{y} &= By + g(x,y), \quad y \in \mathbb{R}^2, \end{aligned}$$

where A is an elliptic matrix and B an hyperbolic one, and $f(0,0) = g(0,0) = 0$ and $Df(0,0) = Dg(0,0) = 0$.

- We want to obtain $y = v(x)$, with $v(0) = 0$, $Dv(0) = 0$, the local expression of the centre manifold.
- The flow restricted to the invariant manifold is

$$\dot{x} = Ax + f(x, v(x)).$$

Approximating the Centre Manifold

To find $y = v(x)$ we substitute this expression on the differential equations. Hence, $v(x)$ must satisfy,

$$Dv(x)Ax - Bv(x) = g(x, v(x)) - Dv(x)f(x, v(x)). \tag{1}$$

We take,

$$v(x) = \sum_{|k|\geq 2} v_k x^k, \qquad k \in (\mathbb{N} \cup \{0\})^4,$$

its expansion as power series. Then we solve equation (1) to find the coefficients v_k up to high degree ($|k| = N$).

- $\widehat{v}(x) = \sum_{|k|=2}^{N} v_k x^k$ is a high order approximation of the centre manifold.
- $\dot{x} = Ax + f(x, \widehat{v}(x))$ gives a high order approximation of the motion on the centre manifold.

Dynamics on the Centre Manifold

We have computed the centre manifold around different equilibrium points of the FL_1 family up to degree 16.

- After this reduction we are in a four dimensional phase space (x_1, x_2, x_3, x_4) that is difficult to visualise.
- We need to perform suitable Poincaré sections to reduce the phase space dimension and help us visualise the phase space.
- For $\delta = 0$ the system is Hamiltonian and we have a first integral. We will use it to reduce the phase space dimension.
- For $\delta \neq 0$ the system is no longer Hamiltonian. We will take a function that varies little along the trajectories as an "approximate first integral", and use it to visualise the phase space.

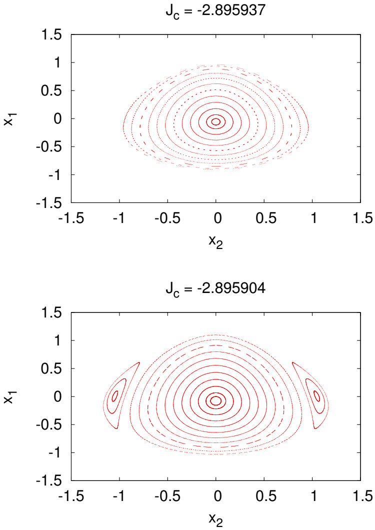
Dynamics for $\delta = 0$

Here the first integral is:

$$J_c = (\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) - 2\Omega(X, Y, Z).$$

- We fix a Poincaré section $x_3 = 0$ to reduce the system to a three dimensional phase space. *(Taking $x_3 = 0$ is like taking $Z = 0$).*
- We fix the energy level to determine x_4 and reduce the system to a two dimensional phase space that is easy to visualise. *(Taking $x_4(J_c, x)$ is like taking $\dot{Z}(J_c, x)$).*
- We have taken several initial conditions and computed their successive images on the Poincaré section.

Dynamics for $\delta = 0$ ($x_3 = 0$ section)



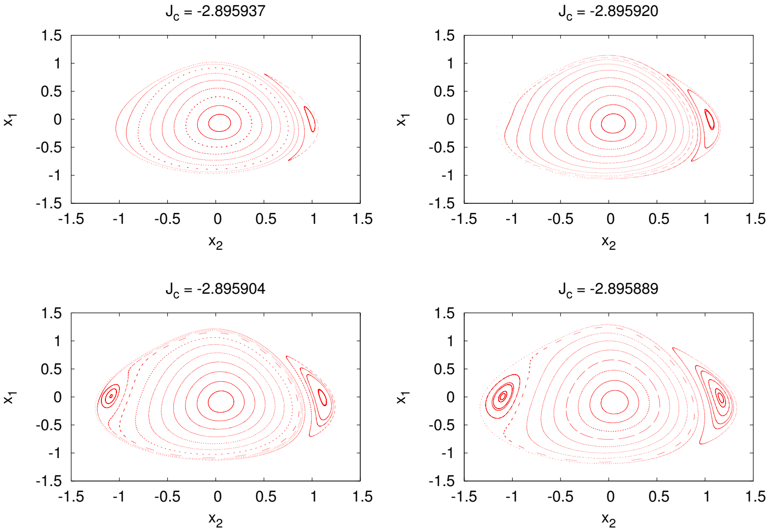
Dynamics for $\delta \neq 0$

Here we take an *"approximated first integral"* :

$$J_c = (\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) - 2\Omega(X, Y, Z) + \beta(1-\mu)\frac{Zr_2}{r_{PS}^3}\cos^2\delta\sin\delta$$

- We fix a Poincaré section $x_3 = 0$ to reduce the system to a three dimensional phase space. *(Taking $x_3 = 0$ is similar to taking $Z = Z^*$).*
- We fix J_c to determine x_4 and reduce the system to a two dimensional phase space that is easy to visualise. *(Taking $x_4(J_c, x)$ is like taking $\dot{Z}(J_c, x)$).*
- We have taken several initial conditions and computed their successive images on the Poincaré section.

Dynamics for $\delta = 0.01$ $(x_3 = 0$ *section)*

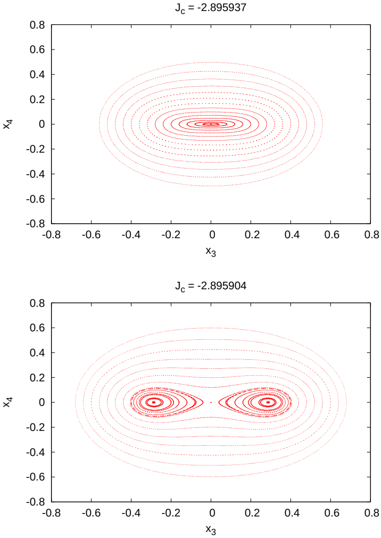


A different Poincaré section: $x_2 = 0$

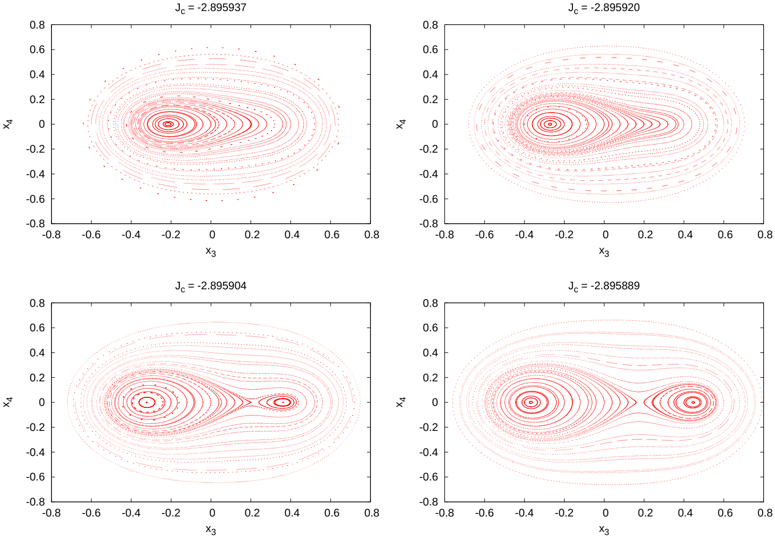
Notice that $x_3 = 0$ is not the only section one can take.

- Now we fix the Poincaré section $x_2 = 0$ to reduce the system to a three dimensional phase space. *(Taking* $x_2 = 0$ *is similar to taking* $Y = 0$*).*
- We fix J_c to determine x_1 and reduce the system to a two dimensional phase space that is easy to visualise. *(Taking* $x_1(J_c, x)$ *is similar to taking* $\dot{Y}(J_c, x)$*).*
- We have taken several initial conditions and computed their successive images on the Poincaré section.

Dynamics for $\delta = 0 \quad (x_2 = 0$ *section)*



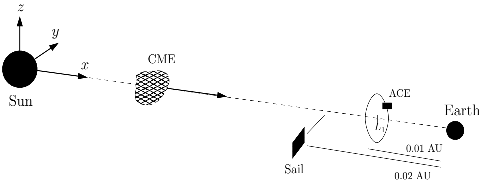
Dynamics for $\delta = 0.01$ $(x_2 = 0$ *section*$)$



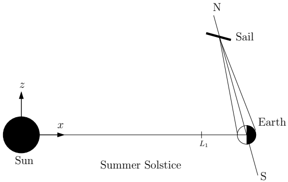
Cool Mission Applications for Solar Sails

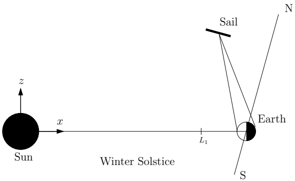
Interesting Missions Applications (I)

Observations of the Sun provide information of the geomagnetic storms, as in the Geostorm Warning Mission.



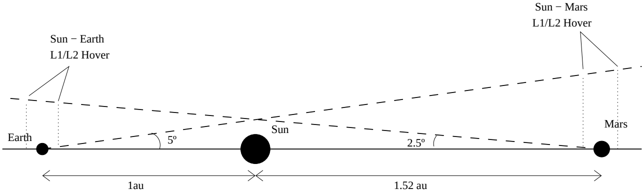
Observations of the Earth's poles, as in the Polar Observer.





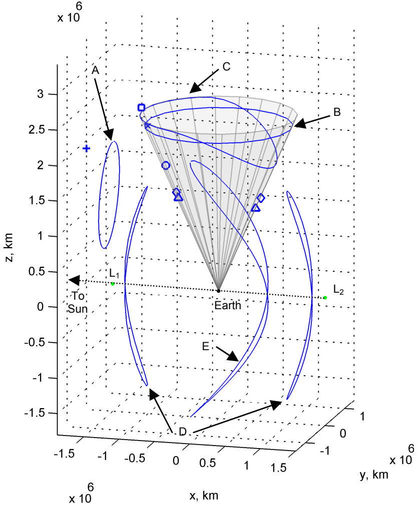
Interesting Missions Applications (II)

To ensure reliable radio communication between Mars and Earth even when the planets are lined up at opposite sides of the Sun.



Interesting Missions Applications (III)

Observations of the Earth's poles, as in the Polar Observer.



Thank you for your attention