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Periodic orbits of renormalisation for the correlations of strange nonchaotic attractors

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Abstract

We calculate all piecewise-constant periodic orbits (with values ± 1) of the renormalisation recursion arising in the analysis of correlations of the orbit of a point on a strange nonchaotic attractor. Our results make rigorous and generalise previous numerical results.

1 Introduction

The occurrence and robustness of strange nonchaotic attractors was first noted by Grebogi *et al* in their seminal paper [4]. A strange nonchaotic attractor is an attractor whose geometry is "strange", and on which the dynamics is "nonchaotic" (i.e. for which there is no positive Lyapunov exponent). Grebogi *et al* [4] considered quasiperiodically forced systems of the type

$$x_{n+1} = f(x_n, \theta_n)\,, \tag{1.1}$$

$$\theta_{n+1} = \theta_n + \omega \quad (\mathrm{mod}\ 1)\,, \tag{1.2}$$

in which ω is irrational, the dynamical variable x (and f) may be scalar or higher dimensional, and f satisfies $f(x, \theta + 1) = f(x, \theta)$. (Such systems are examples of skew-product systems.) Strange nonchaotic attractors have since been reported in other theoretical and experimental situations. References to such occurrences may be found in [7].

In the scalar example studied in some detail in [4] the function f in equation (1.1) takes the form

$$f(x, \theta) = 2\lambda \tanh(x) \sin(2\pi\theta)\,. \tag{1.3}$$

For $|\lambda| < 1$ the invariant line $x = 0$ is the attractor. When $|\lambda| > 1$ this invariant line is no longer an attractor; however, since orbits are confined to a bounded region of phase space an attractor does exist. This is shown to be strange and nonchaotic in [4].

In [9] the autocorrelation of the orbit on the strange attractor is seen to be self-similar and possess a singular continuous spectrum. As in [2], however, we shall confine our attention to a coarser description of the dynamics. Namely we consider only the sign of the variable x, defining

$$y = -\mathrm{sign}(x)\,. \tag{1.4}$$

For the systems under consideration the dynamics are thereby reduced to the linear circle map (1.2) together with a recording (y) of whether θ is in $[0, 1/2)$ or $(1/2, 1)$.

In the case of golden mean forcing, the autocorrelation function of y is seen to be self-similar with structure determined by the renormalisation recursion relation

$$Q_n(x) = Q_{n-1}(-\omega x) Q_{n-2}(\omega^2 x + \omega)\,, \tag{1.5}$$

where $\omega = (\sqrt{5} - 1)/2$ is the golden mean. For completeness we shall include from [2] the derivation of this equation, in section 2.

In [2] Feudel *et al* numerically found a piecewise-constant period-6 orbit of this recursion. This periodic orbit is shown in figure 1.

In this paper we shall give an explicit construction of this periodic orbit, and moreover analyse all piecewise-constant periodic orbits. These periodic orbits correspond to taking a different coarse-grained description from merely noting in which half of the interval θ lies.

In a different but related work, Kuznetsov *et al* [7] have given an elegant analysis of the birth of a strange nonchaotic attractor. The same recursion is used to explain the occurrence of universal scaling factors. In this case however periodic orbits of (1.5) of a different nature are considered.

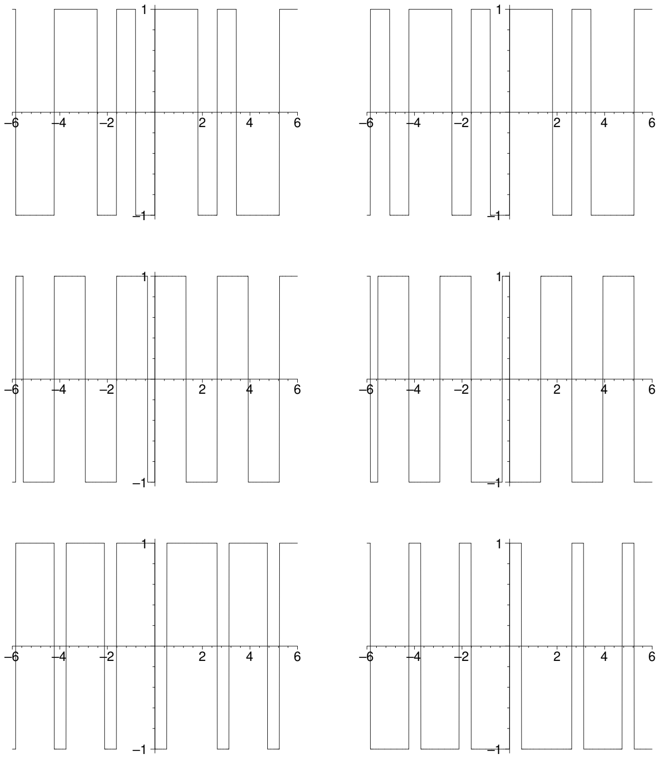

Figure 1: The period-6 orbit discovered by Feudel *et al* ([2])

Remarkably Ketoja and Satija [5] also derive this same equation in their analysis of the self-similar fluctuations of the localized eigenstates of the golden mean Harper equation (also known as the almost Mathieu equation)

$$\psi_{n+1} + \psi_{n-1} + 2\lambda\cos(2\pi(n\omega + \phi))\psi_n = E\psi_n \tag{1.6}$$

in the supercritical regime $\lambda > 1$. This finite difference eigenvalue equation is valuable in the study the localization transition in incommensurate systems. The recursion (1.5) helps explain the universality of the supercritical regime. Note that in [5] the iteration occurs in the form

$$\tilde{Q}_n(x) = -\tilde{Q}_{n-1}(-\omega x)\tilde{Q}_{n-2}(\omega^2 x + \omega)\,, \tag{1.7}$$

but the substitution $\tilde{Q}_n = -Q_n$ renders it equivalent to (1.5). A fixed point of this recursion characterises the universal fluctuations, and this is numerically found in [5]. The same recursion is also used in [5] to analyse a generalised Harper equation describing Bloch electrons on a square lattice with nearest neighbour anisotropy as in (1.6), and the addition of a next-nearest neighbour coupling term. Many periodic orbits are found, and Ketoja and Satija [5] conjecture the existence of a universal strange attractor under the action of the renormalization operator.

In a recent paper [8] we have proved that indeed there is a fixed point of the type numerically found in [5]. (See also [6].) We hope to be able to extend our results on smooth solutions to shed more light on the work of Kuznetsov *et al* [7] on the scenario of the birth of a strange nonchaotic attractor.

In [6] these two seemingly distinct scenarios are linked, and indeed an analogy with the critical dissipative standard map is also drawn.

In this paper we study periodic orbits of (1.5) for which Q_n is piecewise-constant with Q_n taking values ± 1 for all $x \in \mathbb{R}$. By piecewise-constant we mean that for each n the function $Q_n(x) = \pm 1$ has finitely many discontinuities in any bounded interval of $\mathbb{R}$, although Q_n may, and generally will, have infinitely many discontinuities on $\mathbb{R}$. Although this condition might appear somewhat restrictive, we shall see in section 2 that this is the appropriate condition for the renormalisation analysis given by Feudel *et al* in [2] of the correlation function of the sign of orbits in strange nonchaotic attractors. Moreover, as we shall see, the periodic orbit structure for (1.5) is already very rich in this case.

Let us define, for $x \in \mathbb{R}$, the discontinuity function

$$R_n(x) = \frac{Q_n(x+)}{Q_n(x-)}\,, \tag{1.8}$$

the ratio of the right-hand limit to the left-hand limit of Q_n at x. Then, since every discontinuity of Q_n is isolated, R_n is well defined.

Because we are not primarily interested in the value of Q_n at the discontinuity points, we shall identify any two functions having the same discontinuity points (i.e. those x with $R_n(x) = -1$) and agreeing at all continuity points (i.e. at those x with $R_n(x) = 1$).

We are now in a position to give a summary of the main results of the paper. In what follows we shall show that, if Q_n is a periodic orbit of (1.5) of period p, then R_n is also periodic with period m where $m \mid p$. (Here, and subsequently, the *period* is understood to refer to the minimal period.) Moreover we shall see that $p = m$, $2m$, or $3m$. Reducing the study of periodic orbits of (1.5) on $\mathbb{R}$ to a neighbourhood of the fundamental interval $[-\omega, 1]$, we shall identify the set of discontinuities on $[-\omega, 1]$ for the orbit and show that it is a finite union of periodic orbits of the map $F : [-\omega, 1] \to [-\omega, 1]$ given by

$$F(x) = \begin{cases} -\omega^{-1}x, & x \in [-\omega, \omega^2]\,; \\ \omega^{-2}x - \omega^{-1}, & x \in [\omega^2, 1]\,. \end{cases} \tag{1.9}$$

Such periodic orbits are classified by their codes (also called itineraries or kneading sequences) and we shall determine the possible values of m in terms of the codes of these orbits. We shall also identify in detail the cases in which $p = m$, $2m$ and $3m$ can occur. This latter analysis is somewhat complicated and involves some non-intuitive number-theoretic conditions on the codes. A consequence of this analysis is that we shall show *inter alia*

Theorem 1. *For every positive integer $p \geq 1$ there is a periodic orbit Q_n of (1.5) of period p.*

The paper is organised as follows. In the next section, closely following Feudel *et al* [2], we briefly review how the recursion (1.5) arises in the renormalisation analysis of the autocorrelation function for a strange nonchaotic attractor. In section 3 we establish some notation and indicate how an iterated function system and its 'inverse', the function F above (1.9), naturally arise in the recursion. The iterated function system has as invariant set the interval $[-\omega, 1]$, and we show in section 4 that it suffices to consider the recursion (1.5) restricted to this interval. Since we are solely concerned with piecewise-constant functions Q_n taking the values ± 1, much of the nature of the recursion can be understood from a study of the discontinuity function R_n defined above (1.8). This we consider in detail in section 5. However an analysis of the discontinuity function is not in itself sufficient, and in section 6 we relate the periodicity of the discontinuities to that of Q_n itself. This relationship is nontrivial and requires a careful consideration of the orbits of the map F. The results are summarised in section 7. In section 8 we give an analysis of the construction of periodic orbits of (1.5). The period-6 orbit of Feudel *et al* [2] shown in figure 1 is seen to be but one example.

2 Renormalisation analysis of the autocorrelation function

In this section we review the work of Feudel *et al* [2] and show in particular how equation (1.5) arises in a renormalisation analysis of the autocorrelation function for a strange nonchaotic attractor.

In all that follows we shall take $\omega = (\sqrt{5}-1)/2$ and assume that $\lambda > 0$. Recall that the Fibonacci numbers are given by: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, for $n > 1$.

In terms of the discrete variable y defined above (1.4), our mapping (1.1)–(1.2), with the choice of f given by a function of the form (1.3), is now just

$$y_{n+1} = y_n \Phi(\theta_n)\,, \tag{2.1}$$

$$\theta_{n+1} = \theta_n + \omega \quad (\mathrm{mod}\ 1)\,, \tag{2.2}$$

where the "modulation function"

$$\Phi(\theta) = \begin{cases} -1\,, & 0 \le \theta < 1/2\,; \\ +1\,, & 1/2 \le \theta < 1\,. \end{cases} \tag{2.3}$$

Thus

$$y_n = \prod_{k=0}^{n-1} \Phi(\theta_k)\,, \tag{2.4}$$

$$\theta_n = \theta_0 + n\omega \quad (\mathrm{mod}\ 1)\,, \tag{2.5}$$

where we take $y_0 = 1$. The dynamics of y are nothing other than the recording of the location of iterates of the linear circle map, and depend only on the initial angle θ_0.

The autocorrelation function $C(t)$ of y (which has zero mean and unit variance) is the limit time average

$$C(t) = \lim_{T\to\infty} \frac{1}{T} \sum_{i=0}^{T-1} y_i y_{i+t}\,, \tag{2.6}$$

which in view of (2.4), and the fact that $\Phi = \pm 1$, is

$$C(t) = \lim_{T\to\infty} \frac{1}{T} \sum_{i=0}^{T-1} \prod_{k=0}^{i-1} \Phi(\theta_k) \prod_{k=0}^{i+t-1} \Phi(\theta_k) = \lim_{T\to\infty} \frac{1}{T} \sum_{i=0}^{T-1} \prod_{k=i}^{i+t-1} \Phi(\theta_k)\,. \tag{2.7}$$

Now the ergodicity of the linear circle map allows us to write

$$\lim_{T\to\infty} \frac{1}{T} \sum_{i=0}^{T-1} \prod_{k=i}^{i+t-1} \Phi(\theta_k) = \int_0^1 \prod_{k=i}^{i+t-1} \Phi(\theta_k) d\theta_0\,, \tag{2.8}$$

and, since Φ has unit period, we may also change the integration variable (initial condition θ_0) to $\theta_0 - i\omega$ resulting in

$$C(t) = \int_0^1 \prod_{k=0}^{t-1} \Phi(\theta_k) d\theta = \int_0^1 y_t(\theta) d\theta\,, \tag{2.9}$$

where we explicitly note the dependence of y_t on the (initial) angle θ.

The autocorrelation function is observed to have scaling about Fibonacci times, and so to analyse this we define $S_n(\theta) = y_{F_n}(\theta)$, and have, with $\theta_k = \theta + k\omega \pmod 1$,

$$S_n(\theta) = \prod_{k=0}^{F_n-1} \Phi(\theta_k) \tag{2.10}$$

$$= \prod_{k=0}^{F_{n-1}-1} \Phi(\theta_k) \prod_{k=F_{n-1}}^{F_n-1} \Phi(\theta_k) \tag{2.11}$$

$$= S_{n-1}(\theta) S_{n-2}(\theta + F_{n-1}\omega)\,, \tag{2.12}$$

which, using the fact that $F_{n-1}\omega = F_{n-2} - (-\omega)^{n-1}$, is

$$S_n(\theta) = S_{n-1}(\theta) S_{n-2}(\theta - (-\omega)^{n-1})\,. \tag{2.13}$$

To analyse the scaling, we define $Q_n(x) = S_n((-\omega)^n x)$ giving

$$Q_n(x) = Q_{n-1}(-\omega x) Q_{n-2}(\omega^2 x + \omega)\,, \tag{2.14}$$

which is equation (1.5). As noted in [2]

$$C(F_n) = \int_0^1 y_{F_n}(\theta) d\theta = \int_0^1 S_n(\theta) d\theta = \frac{1}{(-\omega)^{-n}} \int_0^{(-\omega)^{-n}} Q_n(x) dx\,. \tag{2.15}$$

Thus the autocorrelation function for Fibonacci times can be determined from the average of the function Q_n. For n not a multiple of three we have that F_n is odd which gives $C(F_n) = 0$. Indeed, as above, by changing the range of the product we may write

$$C(2m+1) = \int_0^1 \prod_{k=0}^{2m} \Phi(\theta_k) d\theta = \int_0^1 \prod_{k=-m}^{m} \Phi(\theta_k) d\theta = \int_0^1 \Phi(\theta) \prod_{k=1}^{m} \Phi(\theta_k)\Phi(\theta_{-k}) d\theta = 0\,, \tag{2.16}$$

since the integrand is odd about 1/2. When n is a multiple of three it is numerically observed in [2] that the average approaches approximately 0.55 for large n. This is the relative height of the secondary peaks in the autocorrelation function.

The results of this paper explain the periodic behaviour of the functions Q_n in the specific example studied by Feudel *et al*, and also determine the behaviour in the presence of more general modulation than equation (2.3).

3 Iterated function system and the inverse map F

We may write equation (1.5) in the form

$$Q_n(x) = Q_{n-1}(\phi_1(x)) Q_{n-2}(\phi_2(x))\,, \tag{3.1}$$

where

$$\phi_1(x) = -\omega x\;, \qquad \phi_2(x) = \omega^2 x + \omega\,, \tag{3.2}$$

and $\omega = (\sqrt{5}-1)/2$ is the golden mean satisfying $\omega^2 + \omega = 1$.

Associated with this equation is an iterated function system (IFS) on $\mathbb{R}$ given by the two contractions ϕ_1, ϕ_2 satisfying the following properties:

1. ϕ_1 and ϕ_2 are linear contractions with fixed points 0 and 1 respectively, and with $\phi_1'(x) = -\omega$ and $\phi_2'(x) = \omega^2$.

2. The interval $I = [-\omega, 1]$ is the fixed point set for the IFS. Indeed

$$\phi_1([-\omega, 1]) = [-\omega, \omega^2] , \qquad \phi_2([-\omega, 1]) = [\omega^2, 1] , \tag{3.3}$$

so that

$$\phi_1(I) \cup \phi_2(I) = I . \tag{3.4}$$

We shall henceforth refer to I as the *fundamental interval*.

3. The fundamental interval I is the attractor for the IFS. Indeed given any compact subset $K \subset \mathbb{R}$ and any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $k \geq N$ and any choice $i_1, \ldots, i_k \in \{1, 2\}$ we have

$$\phi_{i_1} \circ \cdots \circ \phi_{i_k}(x) \in [-\omega - \varepsilon, 1 + \varepsilon] \tag{3.5}$$

for any $x \in K$. This property will be important when we consider the behaviour of equation (1.5) outside the fundamental interval I.

We refer the reader to the book [1] for the theory of iterated function systems.

On the fundamental interval we may define a unique inverse map to the pair ϕ_1, ϕ_2. Let $F : [-\omega, 1] \to [-\omega, 1]$ be defined by

$$F(x) = \begin{cases} \phi_1^{-1}(x) = -\omega^{-1} x, & x \in [-\omega, \omega^2] ; \\ \phi_2^{-1}(x) = \omega^{-2} x - \omega^{-1}, & x \in [\omega^2, 1] , \end{cases} \tag{3.6}$$

as drawn in figure 2.

We shall see below that periodic points of F correspond to discontinuities of the periodic solutions of (1.5). It is therefore appropriate to study the periodic orbit structure of F, but, before so doing, it is worth noting that for any periodic point $y \in [-\omega, 1]$, precisely one of $\phi_1(y)$, $\phi_2(y)$ is also a periodic point of F. For suppose $F^\ell(y) = y$ for some $\ell \in \mathbb{N}$. Then $F(F^{\ell-1}(y)) = y$, so $\phi_i^{-1}(F^{\ell-1}(y)) = y$ for some $i \in \{1, 2\}$, which depends on whether the periodic point $F^{\ell-1}(y) \in [-\omega, \omega^2]$ (in which case $i = 1$), or $F^{\ell-1}(y) \in [\omega^2, 1]$ (in which case $i = 2$). We have that $F^{\ell-1}(y) \neq \omega^2$, since ω^2 is not periodic under F. Thus one of $\phi_1(y)$, $\phi_2(y)$ equals $F^{\ell-1}(y)$, which is periodic.

Now suppose that both $\phi_1(y)$ and $\phi_2(y)$ are periodic. Then there exist ℓ_1, $\ell_2 \in \mathbb{N}$ such that $F^{\ell_1}(\phi_1(y)) = \phi_1(y)$, $F^{\ell_2}(\phi_2(y)) = \phi_2(y)$. Then $\phi_1(y) = F^{\ell_1 \ell_2}(\phi_1(y)) = F^{\ell_2 \ell_1}(\phi_2(y)) = \phi_2(y)$, where we have used the fact that $F(\phi_i(x)) = x$, for $i = 1$, 2, a simple consequence of the definition of F. Now the only solution of the equation $\phi_1(y) = \phi_2(y)$ is $y = -\omega$ so we must have $\phi_1(y) = \omega^2 = \phi_2(y)$, which is impossible since ω^2 is not a periodic point of F and the result is proved.

We now consider periodic orbits of the map F.

We may analyse the dynamics of F in terms of the code of a point $x \in I$. It is convenient for our purposes to define the code in terms of the symbols 1 and 2, rather than 0, 1, or $+1$, -1, as is usually done. Let

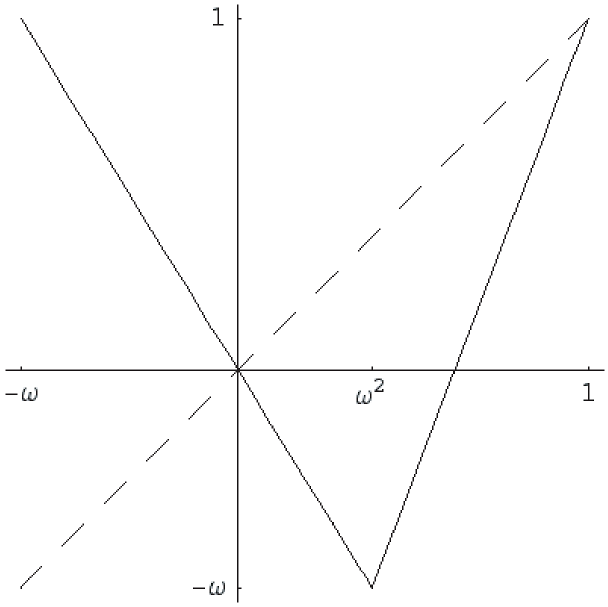

Figure 2: The function F.

the interval $[-\omega,\omega^2)$ be encoded with the symbol 1 and $(\omega^2,1]$ with the symbol 2. We define the code of $x\in I$ to be the sequence $(a_n)_{n\geq 0}$ in $\{1,2\}^{\mathbb{N}_0}$ given by

$$a_n = \begin{cases} 1, & F^n(x)\in[-\omega,\omega^2)\,; \\ 2, & F^n(x)\in(\omega^2,1]\,. \end{cases} \tag{3.7}$$

As is usual we ignore the (countable) set of points whose orbits under F include the point ω^2. (Such points are not periodic points of F.) Hence the codes are all infinite sequences. In terms of the code $a_0a_1a_2\ldots$ of a point $x\in[-\omega,1]$, we have

$$F(x) = (-\omega^{-1})^{a_0}x - (a_0-1)\omega^{-1}\,. \tag{3.8}$$

Since F is uniformly expanding ($|F'(x)|\geq\omega^{-1}$) every point $x\in I$ corresponds to a unique code and vice versa. In particular, periodic orbits of F correspond to periodic codes in $\{1,2\}^{\mathbb{N}_0}$ under the shift map σ:

$$\sigma(a_0a_1a_2\ldots) = a_1a_2\ldots \tag{3.9}$$

A periodic orbit $y_0,y_1,\ldots,y_{k-1}$ of period k of F is given uniquely by a periodic code

$$a_0a_1\ldots a_{k-1}a_0a_1\ldots a_{k-1}\ldots\,, \tag{3.10}$$

which we henceforth denote by $\overline{a_0a_1\ldots a_{k-1}}$.

It is straightforward to calculate the periodic orbit $y_0,y_1,\ldots,y_{k-1}$ of F corresponding to a given code $\overline{a_0a_1\ldots a_{k-1}}$. For we have $\phi^{-1}_{a_{k-1}}\circ\cdots\circ\phi^{-1}_{a_0}(y_0)=y_0$, or, equivalently, $\phi_{a_0}\circ\cdots\circ\phi_{a_{k-1}}(y_0)=y_0$. The (unique) solution of this equation is readily calculated to be

$$y_0 = \frac{-\sum_{j=0}^{k-1}(a_j-1)(-\omega)^{1+\sum_{i=0}^{j-1}a_i}}{1-(-\omega)^{\Sigma a_j}}\,, \tag{3.11}$$

where empty sums are defined to be zero. The other points of the orbit may be calculated by applying this formula with the code $a_0 a_1 \dots a_{k-1}$ cyclically permuted.

For example, $F(y)$ has two fixed points: $y = 0$ with code 1, and $y = 1$ with code 2. The period-2 orbit with code 21 is given by $y_0 = 1/2$ and $y_1 = -\omega/2$. It is the fixed point $y = 0$ and this period-2 orbit that are the discontinuity points in the fundamental interval of the period-6 orbit shown in figure 1. As an example of applying the formula (3.11), we calculate the period-4 orbit of F with code 1211: $y_0 = -\omega^2/(1+\omega^5)$, $y_1 = \omega/(1+\omega^5)$, $y_2 = -\omega^4/(1+\omega^5)$, $y_3 = \omega^3/(1+\omega^5)$.

In what follows it will be the code of the periodic orbit that is important, not the orbit itself. Therefore, from now on, we shall principally refer to periodic orbits of F just by their codes.

4 Reduction of Q_n on $\mathbb{R}$ to the fundamental interval

In this section we consider equation (3.1) outside the fundamental interval $[-\omega, 1]$ i.e., on the whole of $\mathbb{R}$. In what follows we restrict to Q_n taking values ± 1.

Because the fundamental interval I attracts points under the IFS (3.2), we have the following lemma:

Lemma 1. *Let Q_0, Q_1 be initial conditions on $\mathbb{R}$ and let $\varepsilon > 0$ be such that $Q_0(x) = Q_1(x) = 1$, for all $x \in [-\omega-\varepsilon, 1+\varepsilon]$, and let Q_n satisfy equation (3.1). Then for each $L > 1$, there exists $N > 0$ (depending only on L) such that $Q_n(x) = 1$ for all $x \in [-L, L]$ and all $n > N$.*

Proof. Let Q_0, Q_1 satisfy the hypotheses of the lemma, and let $L > 0$ be given. Since $\phi_1([-\omega-\varepsilon, 1+\varepsilon])$, $\phi_2([-\omega-\varepsilon, 1+\varepsilon]) \subseteq [-\omega-\varepsilon, 1+\varepsilon]$, we have that $Q_n(x) = 1$ for all $n \geq 0$ and all $x \in [-\omega-\varepsilon, 1+\varepsilon]$.

Now, from the properties of iterated function systems (section 3), it follows that there exists $N_1 \in \mathbb{N}$ such that for any $k \geq N_1$ and any choice $i_1, \dots, i_k \in \{1, 2\}$ we have

$$\phi_{i_1} \circ \cdots \circ \phi_{i_k}(x) \in [-\omega-\varepsilon, 1+\varepsilon] \tag{4.1}$$

for all $x \in [-L, L]$. Iterating (3.1), we see that $Q_n(x)$ may be written as a product

$$Q_n(x) = \prod_{i_1, i_2, \dots, i_k \in \{1,2\}} Q_{n-\sum_{j=1}^{k} i_j}(\phi_{i_1} \circ \cdots \circ \phi_{i_k}(x)). \tag{4.2}$$

Hence setting $N = 2N_1$, we have for $n > N$ and $x \in [-L, L]$,

$$Q_n(x) = \prod_{i_1, i_2, \dots, i_k \in \{1,2\}} Q_{n-\sum_{j=1}^{k} i_j}(\phi_{i_1} \circ \cdots \circ \phi_{i_k}(x)) = 1. \tag{4.3}$$

This completes the proof of the lemma. □

From the lemma we may prove the following proposition:

Proposition 1. *Let Q_n be a piecewise-constant periodic solution of (3.1) of period p on $\mathbb{R}$ with $Q_n(1+) = Q_n(1)$. Then Q_n is periodic with period p on the fundamental interval I. Conversely, suppose that Q_n is periodic with period p on I. Then there is a unique extension $\tilde{Q}_n$ of Q_n to $\mathbb{R}$ such that $\tilde{Q}_n$ is periodic on $\mathbb{R}$ with period p.*

Proof. First of all let Q_n be periodic on $\mathbb{R}$ with period p, and with $Q_n(1+) = Q_n(1)$ for all $n \geq 0$. Then, clearly, Q_n is periodic on I with period p' dividing p. Let $\varepsilon > 0$ be chosen so that for all $n \geq 0$ there are no discontinuities of Q_n in the intervals $[-\omega - \varepsilon, -\omega)$ and $(1, 1 + \varepsilon]$, and such that ϕ_1 and ϕ_2 map $[-\omega - \varepsilon, -\omega)$ into I. Such an ε exists since the discontinuities are isolated on $\mathbb{R}$, Q_n is periodic, and $\phi_1(-\omega) = \phi_2(-\omega) = \omega^2$, which is not a discontinuity of any Q_n. Furthermore, since Q_n has no discontinuities on $(1, 1+\varepsilon]$, we have that $Q_n(x) = Q_n(1)$ for $x \in [1, 1+\varepsilon]$ and so Q_n is periodic on $[1, 1+\varepsilon]$ with period dividing p'. Now for $x \in [-\omega - \varepsilon, -\omega)$, we have that $\phi_1(x)$, $\phi_2(x) \in I$, so from (3.1) we have that $Q_n(x)$ is periodic with period dividing p'. By the multiplicative property of equation (3.1), the functions $\tilde{Q}_n = Q_{n+p'}/Q_n$ satisfy equation (3.1) and $\tilde{Q}_0$, $\tilde{Q}_1$ are identically 1 on $[-\omega - \varepsilon, 1 + \varepsilon]$. Applying lemma 1, we have that $Q_{n+p'}(x) = Q_n(x)$ for all $x \in \mathbb{R}$, since Q_n is periodic on $\mathbb{R}$. Hence $p = p'$ and Q_n is periodic with period p on I.

Conversely, suppose Q_n is periodic with period p on I. Let $\varepsilon > 0$ be given as above such that $\phi_1([-\omega - \varepsilon, -\omega])$ and $\phi_2([-\omega - \varepsilon, -\omega])$ do not contain discontinuities of Q_n for $n \geq 0$. (Such an ε exists since $\omega^2 = \phi_1(-\omega) = \phi_2(-\omega)$ is not a discontinuity of any Q_n, and there are only finitely many discontinuities of the Q_n on I since Q_n is periodic.) Then we may extend Q_0, Q_1 to $[-\omega - \varepsilon, 1+\varepsilon]$ by setting $Q_0(x) = Q_0(-\omega)$, $Q_1(x) = Q_1(-\omega)$ for $x \in [-\omega - \varepsilon, -\omega)$ and $Q_0(x) = Q_0(1)$, $Q_1(x) = Q_1(1)$ for $x \in (1, 1+\varepsilon]$. Moreover, since $\phi_1([-\omega - \varepsilon, 1+\varepsilon])$, $\phi_2([-\omega - \varepsilon, 1+\varepsilon]) \subseteq [-\omega - \varepsilon, 1+\varepsilon]$, for $n \geq 2$ we may define Q_n on $[-\omega - \varepsilon, -\omega)$ and $(1, 1+\varepsilon]$ by equation (3.1) and then Q_n is periodic with period p on $[-\omega - \varepsilon, 1+\varepsilon]$. Consider now, the initial conditions $\hat{Q}_0 = Q_p/Q_0$, $\hat{Q}_1 = Q_{p+1}/Q_1$. Then $\hat{Q}_0$, $\hat{Q}_1$ satisfy lemma 1. Let $x' \in \mathbb{R}$ and let $L > 1$ be such that $x' \in [-L, L]$. Applying lemma 1, and using the multiplicative property of equation (3.1), we have that there exists $N > 0$, depending only on L, such that $Q_{n+p}(x) = Q_n(x)$ for all $x \in [-L, L]$ and all $n > N$. We define $\tilde{Q}_0(x') = Q_{kp}(x')$ and $\tilde{Q}_1(x') = Q_{kp+1}(x')$, where k is an integer such that $kp > N$. (Note that any such k will give the same value for $\tilde{Q}_0(x')$ and $\tilde{Q}_1(x')$.) Now, for $n \geq 2$, we define $\tilde{Q}_n$ by equation (3.1) (with the Q's replaced by the $\tilde{Q}$'s). We observe that for $n \geq 2$, $\tilde{Q}_n(x') = Q_{kp+n}(x')$. This follows by induction: $\tilde{Q}_n(x') = \tilde{Q}_{n-1}(\phi_1(x'))\tilde{Q}_{n-2}(\phi_2(x')) = Q_{kp+n-1}(\phi_1(x'))Q_{kp+n-2}(\phi_2(x')) = Q_{kp+n}(x')$, by equation (3.1). (Here we have used the fact that $\phi_1(x')$, $\phi_2(x') \in [-L, L]$ so the same value of k is applicable for these points.) Thus it follows that $\tilde{Q}_{n+p}(x') = Q_{kp+n+p}(x') = Q_{kp+n}(x') = \tilde{Q}_n(x')$ for all $n \geq 0$. This defines $\tilde{Q}_n$ on $\mathbb{R}$ such that equation (3.1) holds, $\tilde{Q}_n = Q_n$ on I, $\tilde{Q}_0$, $\tilde{Q}_1$ are right continuous at 1, and $\tilde{Q}_n$ is periodic on $\mathbb{R}$ with period dividing p. Since Q_n is periodic on I with period p, it follows that $\tilde{Q}_n$ has period p.

This completes the proof of the proposition. □

In fact the fundamental interval I 'drives' the recurrence (3.1) as we see from the following proposition, which follows from lemma 1.

Proposition 2. *Let Q_0, Q_1 be piecewise-constant initial conditions on $\mathbb{R}$ and let $Q_0(1+) = Q_0(1)$, $Q_1(1+) = Q_1(1)$. Let Q_n satisfy (3.1), and be periodic of period p on the fundamental interval I. Then the sequence Q_n converges to the unique periodic extension $\tilde{Q}_n$ given by proposition 1, i.e., for all integers $r \geq 0$ we have $Q_{r+np}(x) \to \tilde{Q}_r(x)$ as $n \to \infty$.*

From the results of this section we see that, without loss of generality, we may restrict our analysis of the periodic orbits of (3.1) to the fundamental interval I, as we shall henceforth do.

5 Analysis of the discontinuities

In order to study the piecewise-constant periodic orbits of the recurrence (1.5) with $Q_n(x) = \pm 1$ and with initial conditions Q_0, Q_1, it is helpful to consider the dynamics of the discontinuities of Q_n. We may define, for each $x \in \mathbb{R}$ and $n \geq 0$,

$$R_n(x) = \frac{Q_n(x+)}{Q_n(x-)}\,, \tag{5.1}$$

the ratio of the right-hand limit at x to the left-hand limit at x. Since $Q_n(x) = \pm 1$, we have $R_n(x) = \pm 1$, and it is clear that $R_n(x) = -1$ if and only if Q_n has a discontinuity at x. Since Q_n has at most finitely many discontinuities in any compact interval we have that R_n is well defined. Because of the multiplicative nature of the recurrence (1.5), and because ϕ_1, ϕ_2 are orientation reversing and preserving respectively, we have

$$R_n(x) = \frac{Q_n(x+)}{Q_n(x-)} = \frac{Q_{n-1}(\phi_1(x)-)}{Q_{n-1}(\phi_1(x)+)}\frac{Q_{n-2}(\phi_2(x)+)}{Q_{n-2}(\phi_2(x)-)}\,, \tag{5.2}$$

so, using $R_n(x) = 1/R_n(x)$, we obtain

$$R_n(x) = R_{n-1}(\phi_1(x))R_{n-2}(\phi_2(x))\,, \tag{5.3}$$

and R_n satisfies the same recurrence relation as Q_n. However $R_n(x) = 1$ except at points of discontinuity of Q_n, where $R_n(x) = -1$.

We first of all discuss the dynamics of R_n and then relate the dynamics of Q_n to those of R_n. Indeed, it is clear that if Q_n is periodic with period $p \in \mathbb{N}$, then R_n is also periodic with period m dividing p. Our task is to determine the possible periods m of R_n and relate m to p, the period of Q_n.

From now on we assume that Q_n is periodic with period p and that R_n is periodic with period m, and, in view of proposition 1, we only consider the behaviour of Q_n and R_n on the fundamental interval $[-\omega, 1]$. We denote by

$$D = \{x \in [-\omega, 1] : R_n(x) = -1 \text{ for some } n \geq 0\}\,, \tag{5.4}$$

the *restricted discontinuity set.* Then D is the set of points in the fundamental interval $[-\omega, 1]$ for which Q_n has a discontinuity for at least one $n \geq 0$. One important observation is that since each Q_n is piecewise-constant (and so the set of discontinuities of Q_n on $[-\omega, 1]$ is finite), and since Q_n is periodic, it follows that D is a finite set.

5.1 The restricted discontinuity set and the map F

In this section we show that the restricted discontinuity set D consists of finitely many periodic orbits of the map F. Indeed we have the following result:

Proposition 3. *Let Q_n be a periodic orbit of (1.5) with $Q_n(x) = \pm 1$, and let D be the restricted discontinuity set. Then D consists of a finite collection of periodic orbits of the map F.*

For suppose $y \in D$. Then $R_n(y) = -1$ for some $n \geq 0$. From (5.3) we have that $R_{n-i_1}(\phi_{i_1}(y)) = -1$ for some $i_1 \in \{1, 2\}$. We therefore have $\phi_{i_1}(y) \in D$. Continuing in this way, we obtain a sequence

$i_1, i_2, \ldots \in \{1,2\}$ such that $\phi_{i_k} \circ \cdots \circ \phi_{i_1}(y) \in D$. Since D is finite there exist ℓ, $\ell' \in \mathbb{N}$ with $\ell > \ell'$ and $\phi_{i_{\ell'}} \circ \cdots \circ \phi_{i_1}(y) = \phi_{i_\ell} \circ \cdots \circ \phi_{i_1}(y)$. Applying F^ℓ to this equation gives $F^{\ell-\ell'}(y) = y$, so that y is a periodic point of F of period k dividing $\ell - \ell'$.

Now let $y_0 = y, y_1, \ldots, y_{k-1}$ be the points on the orbit of y_0 under F with $y_{i+1 \pmod k} = F(y_i)$ for $i = 0, 1, \ldots, k-1$, and let $a_0 a_1 \ldots a_{k-1}$ be the code of the orbit. Then for $0 \le i \le k-1$ we have

$$\phi_{a_i}^{-1}(y_i) = y_{i+1} \quad \text{or, equivalently,} \tag{5.5}$$

$$\phi_{a_{i-1}}(y_i) = y_{i-1} \,, \tag{5.6}$$

where here, and in what follows, we assume that expressions relating to the periodic orbit $y_0, y_1, \ldots, y_{k-1}$ are reduced modulo k.

Moreover, by the results of section 3, we have that precisely one of $\phi_1(y_i)$, $\phi_2(y_i)$ is periodic, so that $\phi_2(y_i) \notin D$ if $a_{i-1} = 1$ and $\phi_1(y_i) \notin D$ if $a_{i-1} = 2$. It follows that the recurrence (5.3) becomes

$$R_n(y_i) = \begin{cases} R_{n-1}(y_{i-1}), & a_{i-1} = 1 \\ R_{n-2}(y_{i-1}), & a_{i-1} = 2 \,, \end{cases} \tag{5.7}$$

where we have used the facts that $R_{n-2}(\phi_2(y_i)) = 1$ if $a_{i-1} = 1$ and $R_{n-1}(\phi_1(y_i)) = 1$ if $a_{i-1} = 2$. This can be written as

$$R_n(y_i) = R_{n-a_{i-1}}(y_{i-1}) \,. \tag{5.8}$$

From this we see that $R_{n+a_0+\cdots+a_{i-1}}(y_i) = R_n(y_0)$, so that if $y_0 \in D$ and $R_n(y_0) = -1$ we have $y_i \in D$, since $R_{n+a_0+\cdots+a_{i-1}}(y_i) = R_n(y_0) = -1$.

We conclude that not only must every point y in D be a periodic point of F, but that every point on the periodic orbit of y also lies in D, so that D consists of complete orbits of F. Since D is finite, proposition 3 now follows.

From (5.8) we see that only one of the factors in the right-hand side of (5.3) is different from $+1$, although which one depends on the code $a_0 a_1 \ldots a_{k-1}$. We also observe that in (5.8) n decreases by a_{i-1}. Now, over the whole of the orbit $y_0, y_1, \ldots, y_{k-1}$ we have that n decreases by

$$\ell = \sum_{i=0}^{k-1} a_i \,, \tag{5.9}$$

i.e.,

$$R_n(y_i) = R_{n-\ell}(y_i) \,, \tag{5.10}$$

for $0 \le i \le k-1$. It follows that we must have $m \mid \ell$. We therefore conclude the following:

Proposition 4. *The period m of the discontinuity function R_n restricted to a periodic orbit $y_0, \ldots, y_{k-1}$ of F divides ℓ, the sum of the code over the orbit of F.*

We now introduce three examples of periodic orbits of F which we shall use to illustrate the theory as it develops.

Example 1. Period-4 orbit of F with code 1122. Then $\ell = 6$.

Example 2. Period-4 orbit of F with code 1211. Then $\ell = 5$.

Example 3. Period-6 orbit of F with code 111222. Then $\ell = 9$.

5.2 The discontinuity matrix

Let $y_0 y_1 \dots y_{k-1}$ be a periodic orbit of F with code $a_0 a_1 \dots a_{k-1}$, and with ℓ given by equation (5.9). We shall first consider the dynamics of R_n on this orbit.

It is helpful at this point to introduce an $\ell \times k$ matrix M, the *discontinuity matrix*, with entries ± 1 defined by

$$M_{n,i} = R_n(y_i)\,, \tag{5.11}$$

for $0 \le n \le \ell - 1$, $0 \le i \le k-1$. Then the entry in row n and column i is the value of R_n at the point y_i on the orbit $y_0, y_1, \dots, y_{k-1}$.

The relation (5.8) above gives a special structure to the matrix M. Indeed (5.8) translates to

$$M_{n,i} = M_{n-a_{i-1},i-1}\,, \tag{5.12}$$

where here, and in what follows, indices referring to the periodicity of R_n are reduced modulo ℓ.

The structure (5.12) can be more easily understood as follows. Column i of the matrix M is simply column $(i-1)$ cyclically permuted downwards by a_{i-1} single cyclic permutations. This observation also holds when $i = 0$, for then (5.12) becomes

$$M_{n,0} = M_{n-a_{k-1},k-1}\,. \tag{5.13}$$

Let us denote the column 0 by $(X_0, X_1, \dots, X_{\ell-1})$, i.e., $M_{n,0} = X_n$ for $0 \le n \le \ell - 1$. Then the relation (5.12) tells us that

$$M_{n,1} = M_{n-a_0,0} = X_{n-a_0}\,, \tag{5.14}$$

and, in general,

$$M_{n,i} = M_{n-\sum_{j=0}^{i-1} a_j,0} = X_{n-\sum_{j=0}^{i-1} a_j}\,, \tag{5.15}$$

so that the columns of M are simply cyclic permutations of the column 0 of M.

As an illustration consider example 2. Recall that the code is 1211, the period k is 4, and $\ell = 5$. The matrix M is

$$M = \begin{pmatrix} X_0 & X_4 & X_2 & X_1 \\ X_1 & X_0 & X_3 & X_2 \\ X_2 & X_1 & X_4 & X_3 \\ X_3 & X_2 & X_0 & X_4 \\ X_4 & X_3 & X_1 & X_0 \end{pmatrix}. \tag{5.16}$$

Note that the column i is obtained from the column $(i-1)$ by cyclically permuting the column $(i-1)$ downwards by a_{i-1}.

5.3 Periodicity of the discontinuities

Not only does any periodic orbit R_n with discontinuities at a periodic orbit $y_0, y_1, \dots, y_{k-1}$ of F have a discontinuity matrix M with the structure (5.15), but also, conversely, any matrix M satisfying (5.15) corresponds to a periodic orbit of R_n, by defining $R_n(x) = 1$ except on the points $y_0, \ldots, y_{k-1}$ where we define $R_n(y_i) = M_{n \bmod \ell, i}$ for $n \geq 0$. The period of R_n certainly divides ℓ, but may not actually be equal to ℓ. Indeed, trivially, setting $M_{n,i} = -1$ for all i, n gives a periodic orbit of period 1 for R_n. In fact the period of R_n depends only on the period of column 0 of M viewed as a sequence of ± 1. This is because this column is periodic with period m if and only if it is invariant under m single cyclic permutations and m is the least positive integer for which this is true, i.e., $X_{n+m} = X_n$ and $X_{n+j} \neq X_n$ for all n if $1 \leq j < m$. Now the other columns of M are obtained from column 0 by cyclically permuting and thus they will also have period m. In fact, since any column of M can be obtained from any other one by cyclic permutations it follows that all columns of M have the same period m. Indeed, for $r \in \mathbb{N}$

$$M_{n+r,i} = X_{n+r-\sum_{j=0}^{\ell-1} a_j} = X_{n-\sum_{j=0}^{\ell-1} a_j + r} = M_{n,i} \tag{5.17}$$

if, and only if, $m \mid r$. This is because the X_n have period m. Thus each column of M has period m. We conclude that the period m of R_n is the period of the first column $(X_0, X_1, \ldots, X_{\ell-1})$ of M. It is now clear that $m \mid \ell$ and that for every m dividing ℓ we can find a column $(X_0, X_1, \ldots, X_{\ell-1})$ with period m.

It is worth remarking that the first two rows $n = 0$ and $n = 1$ of M are not independent, so that, although the recursion (1.5) is second order, we cannot choose R_0 and R_1 arbitrarily on $y_0, \ldots, y_{k-1}$ and obtain a periodic orbit.

We have therefore solved the question of the periodic behaviour of the discontinuities R_n for a single periodic orbit $y_0, y_1, \ldots, y_{k-1}$ of F with code $a_0 a_1 \ldots a_{k-1}$. In summary, the period m of R_n corresponds precisely to the period of a single column of M, i.e., $R_n(y_i)$ for any $0 \leq i \leq k-1$. We have m divides $\ell = \sum_{j=0}^{k-1} a_j$, and conversely, for every m dividing ℓ, we can, for suitable choice of the column 0 of M, viz., $(X_0, X_1, \ldots, X_{\ell-1})$, arrange for $(X_0, X_1, \ldots, X_{\ell-1})$, and thus M and R_n, to have period m.

Consider example 2. Here the first column is $(X_0, X_1, X_2, X_3, X_4)$. The only positive integers dividing $\ell = 5$ are 1 and 5, so the only possible periods in this case are $m = 1$ and $m = 5$. Setting $X_0 = X_1 = X_2 = X_3 = X_4 = -1$ gives period 1, whilst any other choice (with at least one -1) gives period 5. Setting $X_0 = X_1 = X_2 = X_3 = X_4 = 1$ gives period 1, but then the orbit of F will not lie in D.

5.4 Multiple periodic orbits in D

Having considered the dynamics of the discontinuity function R_n on a single periodic orbit of F, we now consider the case in which the restricted discontinuity set D consists of more than one periodic orbit of F. To do this, we must establish some notation.

Firstly, let t be the number of periodic orbits of F in D. For $0 \leq s \leq t-1$, we consider the periodic orbit s of F in D. We make the general convention that superscript s refers to the orbit s. Let k^s denote its period and let the points $y_0^s, \ldots, y_{k^s-1}^s$ be the members of the orbit. We denote the code by $a_0^s \ldots a_{k^s-1}^s$. Let

$$\ell^s = \sum_{j=0}^{k^s-1} a_j^s \,. \tag{5.18}$$

Now, from the multiplicative structure of (5.3), we have that a product of solutions is again a solution of the equation. Moreover, because the periodic orbits in D are distinct, and are never mapped to each other under the two maps ϕ_1, ϕ_2, we have that the dynamics of R_n on each of the periodic orbits in D

are independent. Indeed, we may write

$$R_n(x) = \prod_{s=0}^{t-1} R_n^s(x)\,, \tag{5.19}$$

where R_n^s is the restriction of R_n to the periodic orbit s, i.e.,

$$R_n^s(x) = \begin{cases} R_n(x)\,, & x \in \{y_0^s, \dots, y_{k^s-1}^s\}\,; \\ 1\,, & \text{otherwise}\,. \end{cases} \tag{5.20}$$

We may apply the analysis of the previous subsections to each of the functions R_n^s. This is because $R_n^s(x) = 1$, except when x is one of the points on the periodic orbit $y_0^s \dots y_{k^s-1}^s$ of F. In particular, for each orbit in D we can formulate the $\ell^s \times k^s$ discontinuity matrix M^s, where, for $0 \le n \le \ell^s - 1$ and $0 \le i \le k^s - 1$,

$$M_{n,i}^s = R_n^s(y_i^s)\,. \tag{5.21}$$

We observe that these matrices are independent of each other since the dynamics of R_n on each periodic orbit in D are independent.

The theory for R_n that we discussed above carries over in a straightforward manner to the function R_n^s. To simplify notation, we adopt the convention that, when dealing with periodic orbit s and its matrix, expressions relating to the periodic orbit $y_0^s, \dots, y_{k^s-1}^s$ are reduced modulo k^s whilst those relating to the periodicities of R_n are reduced modulo ℓ^s. Thus, as in (5.12), we have

$$M_{n,i}^s = M_{n-a_{i-1}^s, i-1}^s\,, \tag{5.22}$$

for $0 \le n \le \ell^s - 1$ and $0 \le i \le k^s - 1$, and the matrix M^s is determined by its column 0: $(X_0^s, X_1^s, \dots, X_{\ell^s-1}^s)$. Indeed, as in (5.15),

$$M_{n,i}^s = X_{n-\sum_{j=0}^{i-1} a_j^s}^s\,, \tag{5.23}$$

and the period m^s of the column 0 is precisely the row period of M^s. We also have $m^s \mid \ell^s$. Conversely, let $\ell = \mathrm{lcm}(\ell^0, \dots, \ell^{t-1})$. Then for any $m \mid \ell$ we define $m^s = \gcd(m, \ell^s)$. Then $m^s \mid \ell^s$ and by appropriate choices of $(X_0^s, X_1^s, \dots, X_{\ell^s-1}^s)$ we may construct a matrix M^s with row period any m^s dividing ℓ^s, and, extending periodically to all $n \ge 0$, we have that R_n has period m^s restricted to the orbit $y_0^s, \dots, y_{k^s-1}^s$.

We therefore have the following proposition for piecewise-constant, right-continuous functions taking the values ± 1:

Proposition 5. *Let Q_n be a periodic orbit of (1.5). Then the period m of the discontinuity function R_n is given by*

$$m = lcm(m^0, \dots, m^{t-1})\,, \tag{5.24}$$

where m^s is the period of the function R_n^s and is given by the period of $(X_0^s, X_1^s, \dots, X_{\ell^s-1}^s)$, i.e., column 0 of the discontinuity matrix M^s. Furthermore, m divides

$$\ell = lcm(\ell^0, \dots, \ell^{t-1})\,. \tag{5.25}$$

Moreover, by appropriate choices of $(X_0^s, X_1^s, \dots, X_{\ell^s-1}^s)$, for any m dividing ℓ we may construct a periodic orbit of R_n with period m.

Let us illustrate this result when D is the union of examples 1–3 in subsection 5.1. Then $\ell = \mathrm{lcm}(6,5,9) = 90$. Hence R_n has period dividing 90 and, conversely, for any m dividing 90, we may ensure that R_n has period m.

6 The relationship between Q_n and R_n

Let D be the restricted discontinuity set. We now consider the period of the functions Q_n and relate it to that of the discontinuity functions R_n. We first of all note that R_n does not completely determine Q_n. However, Q_n *is* determined by R_n together with the value $Q_n(x)$ at a single point x. Although any choice of x would be sufficient, for our purposes it is convenient to take $x = 1+$, the right-hand limit at $x = 1$. We write

$$Q_n^{1+} = Q_n(1+)\,. \tag{6.1}$$

Indeed, since Q_n is right-continuous, this is just $Q_n(1)$, but we write Q_n^{1+} to emphasise the fact that it is the right-hand limit. Now, on the fundamental interval, we have

$$Q_n(x) = Q_n(x+) = Q_n^{1+} \prod_{\substack{x<y\le 1\\ y\in D}} R_n(y) \tag{6.2}$$

$$Q_n(x-) = Q_n^{1+} \prod_{\substack{x\le y\le 1\\ y\in D}} R_n(y)\,, \tag{6.3}$$

for $x \in [-\omega, 1]$. It follows that Q_n is periodic with period p if and only if R_n is periodic with period m dividing p and Q_n^{1+} is periodic with period p, or R_n is periodic with period $p = m$ and Q_n^{1+} is periodic with period dividing p. We can therefore reduce the problem of the periodicity of Q_n to that of Q_n^{1+} and of R_n on $[-\omega, 1]$.

To simplify the notation in what follows we introduce the quantities

$$D_n = \prod_{y\in D} R_n(y)\,, \qquad D_n^s = \prod_{i=0}^{k^s-1} R_n^s(y_i^s)\,. \tag{6.4}$$

We now evaluate (1.5) at $x = 1+$ to obtain

$$\begin{aligned} Q_n^{1+} = Q_n(1+) &= Q_{n-1}(-\omega-)Q_{n-2}(1+) && (6.5)\\ &= Q_{n-1}^{1+}Q_{n-2}^{1+}D_{n-1} && (6.6)\\ &= (Q_{n-2}^{1+})^2 Q_{n-3}^{1+} D_{n-1}D_{n-2} && (6.7)\\ &= Q_{n-3}^{1+} D_{n-1} D_{n-2}\,, && (6.8) \end{aligned}$$

where we have used the fact that $(Q_{n-2}^{1+})^2 = 1$ since $Q_{n-2}^{1+} = \pm 1$.

Now each of the products D_{n-1}, D_{n-2} in (6.8) is a product of entries in the matrices M^s for $0 \le s \le t-1$. Indeed for each n in the range $0 \le n \le m-1$ we have

$$D_n = \prod_{s=0}^{t-1} D_n^s = \prod_{s=0}^{t-1}\prod_{i=0}^{k^s-1} M_{n,i}^s\,, \tag{6.9}$$

so in (6.8) we have expressed Q_n^{1+} in terms of Q_{n-3}^{1+} and a product of entries in the matrices M^s, $0 \le s \le t-1$, and hence of the X_n^s.

Now an orbit of the second order recurrence (1.5) is periodic with period p if and only if $Q_0 = Q_p$, and $Q_1 = Q_{p+1}$, where p is the least such positive integer. We know that p is a multiple of m, the period of R_n. To obtain the relationship between p and m, we investigate Q_m^{1+}/Q_0^{1+} and Q_{m+1}^{1+}/Q_1^{1+}. We note that $p = m$ if and only if both of these ratios have value 1, i.e.,

$$\frac{Q_m^{1+}}{Q_0^{1+}} = \frac{Q_{m+1}^{1+}}{Q_1^{1+}} = 1\,. \tag{6.10}$$

In what follows we shall need to evaluate products of the form

$$\prod_{\substack{n \not\equiv r \bmod 3 \\ r \le n < m+r}} D_n \tag{6.11}$$

where $m \ge 1$ and $r \in \{0, 1, 2\}$. We therefore prove the following lemma, which we shall use in our subsequent work.

Lemma 2. *Let a' be a positive integer and let $D_n = \pm 1$ have period dividing a', i.e., $D_{n+a'} = D_n$ for all n. Let b' be a positive integer with $b' \equiv 0 \bmod 3$ and $a' \mid b'$, and let $r \in \{0, 1, 2\}$. Then:*

1. *if $a' \equiv 0 \bmod 3$ then*

$$\prod_{\substack{n \not\equiv r \bmod 3 \\ r \le n < b'+r}} D_n = \Big(\prod_{\substack{n \not\equiv r \bmod 3 \\ r \le n < a'+r}} D_n \Big)^{(b'/a')}; \tag{6.12}$$

2. *if $a' \equiv 0 \bmod 3$ then*

$$\prod_{\substack{n \not\equiv r \bmod 3 \\ r \le n < 2a'+r}} D_n = 1\,; \tag{6.13}$$

3. *if $a' \not\equiv 0 \bmod 3$ then*

$$\prod_{\substack{n \not\equiv r \bmod 3 \\ r \le n < 3a'+r}} D_n = 1\,; \tag{6.14}$$

4. *if $a' \not\equiv 0 \bmod 3$ then*

$$\prod_{\substack{n \not\equiv r \bmod 3 \\ r \le n < b'+r}} D_n = 1\,. \tag{6.15}$$

Proof. Suppose that $a' \equiv 0 \bmod 3$. Then

$$\prod_{\substack{n \not\equiv r \bmod 3 \\ r \le n < b'+r}} D_n = \prod_{j=0}^{b'/a'-1} \prod_{\substack{n \not\equiv r \bmod 3 \\ ja'+r \le n < (j+1)a'+r}} D_n \tag{6.16}$$

$$= \prod_{j=0}^{b'/a'-1} \prod_{\substack{n-ja' \not\equiv r \bmod 3 \\ r \le n-ja' < a'+r}} D_{n-ja'} \tag{6.17}$$

$$= \Big(\prod_{\substack{n \not\equiv r \bmod 3 \\ r \le n < a'+r}} D_n \Big)^{(b'/a')}. \tag{6.18}$$

In the above calculation we have used the fact that $3 \mid a' \mid b'$ and $D_{n+a'} = D_n$. This proves assertion 1. Assertion 2 follows immediately from 1, with $b'/a' = 2$.

Now suppose $a' \not\equiv 0 \bmod 3$ and $r \leq n < a' + r$. Then precisely one of n, $n + a'$, $n + 2a'$ is congruent to $r \bmod 3$. Furthermore, because $D_n = D_{n+a'} = D_{n+2a'}$ each factor occurs precisely twice in the product

$$\prod_{\substack{n \not\equiv r \bmod 3 \\ r \leq n < 3a'+r}} D_n = \prod_{\substack{n \not\equiv r \bmod 3 \\ r \leq n < a'+r}} D_n \prod_{\substack{n \not\equiv r \bmod 3 \\ a'+r \leq n < 2a'+r}} D_n \prod_{\substack{n \not\equiv r \bmod 3 \\ 2a'+r \leq n < 3a'+r}} D_n \tag{6.19}$$

and since $D_n = \pm 1$ each factor cancels itself out. Thus assertion 3 follows, and assertion 4 follows easily from 1 and 3. □

We now return to evaluating the ratios Q^{1+}_m / Q^{1+}_0 and Q^{1+}_{m+1}/Q^{1+}_1. They can be obtained by iterating equation (6.8); their values depend on the residue of m modulo 3. Accordingly, we divide into three cases.

6.1 $m \equiv 1 \bmod 3$

In this case we have, iterating (6.8), and using (6.6),

$$\frac{Q^{1+}_m}{Q^{1+}_0} = \frac{Q^{1+}_1}{Q^{1+}_0} \prod_{\substack{n \not\equiv 1 \bmod 3 \\ 1 \leq n < m}} D_n \tag{6.20}$$

$$\frac{Q^{1+}_{m+1}}{Q^{1+}_1} = \frac{Q^{1+}_2}{Q^{1+}_1} \prod_{\substack{n \not\equiv 2 \bmod 3 \\ 2 \leq n < m+1}} D_n = Q^{1+}_0 \prod_{\substack{n \not\equiv 2 \bmod 3 \\ 1 \leq n < m+1}} D_n \,. \tag{6.21}$$

We see that whether (6.10) holds, or not, depends both on the product of the D_n (which themselves are products of entries from the matrices M^s) and on Q^{1+}_0, Q^{1+}_1. In the case (6.10) we have that Q_n has the same periodicity as R_n, i.e., $p = m$. Otherwise we have $p > m$. From (6.8) we have

$$\frac{Q^{1+}_{3m}}{Q^{1+}_0} = \prod_{\substack{n \not\equiv 0 \bmod 3 \\ 0 \leq n < 3m}} D_n \,, \qquad \frac{Q^{1+}_{3m+1}}{Q^{1+}_1} = \prod_{\substack{n \not\equiv 1 \bmod 3 \\ 1 \leq n < 3m+1}} D_n \,. \tag{6.22}$$

Now from lemma 2 (3), with $a' = m$ and $r = 0, 1$, we have $Q^{1+}_{3m}/Q^{1+}_0 = Q^{1+}_{3m+1}/Q^{1+}_1 = 1$, and hence we have $p = 3m$. (We cannot have $p = 2m$ for this would imply $p = m$, since 2 and 3 have greatest common divisor 1.)

6.2 $m \equiv 2 \bmod 3$

The analysis in the case $m \equiv 2 \bmod 3$ is similar, and again leads to the conclusion that either $p = m$ or $p = 3m$. Indeed, we have

$$\frac{Q^{1+}_m}{Q^{1+}_0} = Q^{+}_1 \prod_{\substack{n \not\equiv 2 \bmod 3 \\ 1 \leq n < m}} D_n \tag{6.23}$$

$$\frac{Q^{1+}_{m+1}}{Q^{1+}_1} = \frac{Q^{1+}_0}{Q^{1+}_1} \prod_{\substack{n \not\equiv 0 \bmod 3 \\ 1 \leq n < m+1}} D_n \,, \tag{6.24}$$

and a similar calculation to the above gives $Q^{1+}_{3m}/Q^{1+}_0 = Q^{1+}_{3m+1}/Q^{1+}_1 = 1$, even when (6.10) does not hold.

6.3 $m \equiv 0 \bmod 3$

We now consider the case $m \equiv 0 \bmod 3$. By iterating equation (6.8), we have

$$\frac{Q_m^{1+}}{Q_0^{1+}} = \prod_{\substack{n \not\equiv 0 \bmod 3 \\ 0 \leq n < m}} D_n \,, \qquad \frac{Q_{m+1}^{1+}}{Q_1^{1+}} = \prod_{\substack{n \not\equiv 1 \bmod 3 \\ 1 \leq n < m+1}} D_n \,. \tag{6.25}$$

We observe that, in this case, whether equation (6.10) holds or not is independent of Q_0^{1+} and Q_1^{1+}. Now, if (6.10) does not hold, then

$$\frac{Q_{2m}^{1+}}{Q_0^{1+}} = \prod_{\substack{n \not\equiv 0 \bmod 3 \\ 0 \leq n < 2m}} D_n = 1 \,, \qquad \frac{Q_{2m+1}^{1+}}{Q_1^{1+}} = \prod_{\substack{n \not\equiv 1 \bmod 3 \\ 1 \leq n < 2m+1}} D_n = 1 \,, \tag{6.26}$$

which follows from lemma 2 (2). We therefore conclude that, in this case, either $p = m$ or $p = 2m$.

We may therefore sum up these results as follows.

Proposition 6. *Let Q_n be periodic with period p and let R_n have period m. Then if $m \not\equiv 0 \bmod 3$, then either $p = m$ or $p = 3m$. Otherwise, if $m \equiv 0 \bmod 3$, then either $p = m$ or $p = 2m$.*

7 Theorem 2

We now compile the results of the previous sections into the following theorem.

Theorem 2. *Let Q_n, $n \geq 0$, be a periodic orbit of period p of (1.5) with $Q_n(x) = \pm 1$ for all x, Q_n right-continuous, and such that the restricted discontinuity set D is finite. Let m be the period of the discontinuity function R_n given by (5.1). Then*

1. *D is a finite set of t periodic orbits (y_i^s), $0 \leq s \leq t-1$, $0 \leq i \leq k^s - 1$ of F with codes $a_0^s \dots a_{k^s-1}^s$;*
2. *the period m of R_n divides $\ell = lcm(\ell^0, \dots, \ell^{t-1})$ where $\ell^s = \sum_{j=0}^{k^s-1} a_j^s$;*
3. *the period p of Q_n is either m, $2m$ or $3m$. If $m \not\equiv 0 \bmod 3$ then $p = m$ or $p = 3m$ depending on the values of R_n and Q_0^{1+}, Q_1^{1+}. However if $m \equiv 0 \bmod 3$ then either $p = m$ or $p = 2m$ and this depends only on the values of R_n.*

Theorem 2 gives only a partial classification of the periodic orbit structure of (1.5). It remains to determine what periods p for Q_n can actually be achieved for a given choice of restricted discontinuity set D. It is this question that we study in the rest of the paper.

8 The construction of periodic orbits

In this section we consider how, by an appropriate choice of the X_n^s, and Q_0^{1+} and Q_1^{1+}, we may construct periodic orbits with a given restricted discontinuity set D.

Let D be a finite collection of t periodic orbits of F. We adopt the notation of subsection 5.4 for the orbits y_i^s, viz., ℓ^s, ℓ etc. We know (by theorem 2) that the period m of R_n must divide ℓ. Now suppose that m is any positive integer dividing ℓ. Let $0 \le s \le t-1$ and let $m^s = \gcd(m, \ell^s)$. Then by choosing column 0 of M^s, i.e., $(X_0^s, X_1^s, \dots, X_{\ell^s-1}^s)$, to have period m^s, we may ensure that R_n^s has period m^s. Thereby we may ensure that R_n will have period $\operatorname{lcm}(m^0, \dots, m^{t-1})$ which is equal to m, and that the restricted discontinuity set is D.

However, in order to determine the possible values of p, the period of Q_n, we must be more careful in our choices, at least in the case $m \equiv 0 \bmod 3$. We now consider the two cases $m \not\equiv 0 \bmod 3$ and $m \equiv 0 \bmod 3$ separately.

Suppose first that $m \not\equiv 0 \bmod 3$. From subsections 6.1 and 6.2, and in particular from equations (6.20) and (6.23), we see that, if R_n is chosen to have period m, then we may choose Q_0^{1+} and Q_1^{1+} so that either equation (6.10) holds (in which case $p = m$) or else it does not hold (in which case $p = 3m$). We are therefore able to conclude the following result.

Proposition 7. *Let D be a finite collection of t periodic orbits of F with the notation of subsection 5.4 and let m divide ℓ with $m \not\equiv 0 \bmod 3$. Then for both $p = m$ and for $p = 3m$, there exists a periodic orbit Q_n of (1.5) of period p and with restricted discontinuity set D.*

8.1 $m \equiv 0 \bmod 3$

The case $m \equiv 0 \bmod 3$ is more delicate, since in that case we see from equation (6.25) that p is unaffected by the choice of Q_0^{1+} and Q_1^{1+}; it depends only on R_n, and, in particular, the X_n^s. From subsection 6.3, we see that $p = m$ if and only if the products

$$P_0 = \prod_{\substack{n \not\equiv 0 \bmod 3 \\ 0 \le n < m}} D_n, \qquad P_1 = \prod_{\substack{n \not\equiv 1 \bmod 3 \\ 1 \le n < m+1}} D_n \tag{8.1}$$

are both equal to 1; otherwise $p = 2m$. We shall have to be more careful in the way the X_n^s are chosen. In particular, we must examine in more detail the products P_0 and P_1.

Writing

$$P_0^s = \prod_{\substack{n \not\equiv 0 \bmod 3 \\ 0 \le n < m}} D_n^s, \qquad P_1^s = \prod_{\substack{n \not\equiv 1 \bmod 3 \\ 1 \le n < m+1}} D_n^s, \tag{8.2}$$

we have that

$$P_0 = \prod_{s=0}^{t-1} P_0^s, \qquad P_1 = \prod_{s=0}^{t-1} P_1^s. \tag{8.3}$$

We now study P_0^s and P_1^s. Recall that $m^s = \gcd(m, \ell^s)$. Then we have $m^s \mid \ell^s$ and $m^s \mid m$. If $m^s \not\equiv 0 \bmod 3$ then $(m/m^s) \equiv 0 \bmod 3$. Therefore, from lemma 2 (using (1) with $b' = m$ and $a' = 3m^s$, and then (3) with $a' = m^s$), we have that

$$P_0^s = \Big(\prod_{\substack{n \not\equiv 0 \bmod 3 \\ 0 \le n < 3m^s}} D_n^s\Big)^{m/(3m^s)} = 1, \qquad P_1^s = \Big(\prod_{\substack{n \not\equiv 1 \bmod 3 \\ 1 \le n < 3m^s+1}} D_n^s\Big)^{m/(3m^s)} = 1. \tag{8.4}$$

We therefore conclude that, unless $m^s \equiv 0 \bmod 3$, we have $P_0^s = P_1^s = 1$.

Now suppose $m^s \equiv 0 \bmod 3$. Then from lemma 2 (1) (with $b' = m$ and $a' = m^s$), we have

$$P_0^s = (\prod_{\substack{n \not\equiv 0 \bmod 3 \\ 0 \le n < m^s}} D_n^s)^{(m/m^s)}, \qquad P_1^s = (\prod_{\substack{n \not\equiv 1 \bmod 3 \\ 1 \le n < m^s + 1}} D_n^s)^{(m/m^s)}. \tag{8.5}$$

We observe that we shall again have $P_0^s = P_1^s = 1$ unless m/m^s is odd. Hence we have:

Proposition 8. *Let $m \equiv 0 \bmod 3$. Then*

$$P_0 = \prod_{\substack{s=0 \\ 3 | m^s,\, 2 \nmid (m/m^s)}}^{t-1} P_0^s, \qquad P_1 = \prod_{\substack{s=0 \\ 3 | m^s,\, 2 \nmid (m/m^s)}}^{t-1} P_1^s. \tag{8.6}$$

(Here empty products are defined to equal 1*.)*

We now introduce a condition on the orbit s of F which is necessary for $P_0^s = P_1^s = 1$ not to hold. We call orbits satisfying this condition *active*.

8.2 Active periodic orbits of F

In view of the previous proposition, we make the following definition.

Definition. Let m be a positive integer dividing ℓ and divisible by 3. The periodic orbit $y_0^s, y_1^s, \ldots, y_{k^s-1}^s$ of F with code $a_0^s a_1^s \ldots a_{k^s-1}^s$ is said to be *active with respect to m* if

1. $3 \mid \ell^s = \sum_{j=0}^{k^s-1} a_j^s$ (so that $3 \mid m^s = \gcd(m, \ell^s)$);
2. m/m^s is odd; and
3. in the sequence $a_0^s, a_0^s + a_1^s, \ldots, a_0^s + a_1^s + \cdots + a_{k^s-1}^s$ the three residue classes modulo 3 do not all occur with the same parity, i.e., there is at least one residue class that occurs an even number of times and another which occurs an odd number of times.

Note that whether a given code is active or not with respect to m does not depend on the restricted discontinuity set D (and hence ℓ) directly, but only on the choice of integer m dividing ℓ.

This definition is not at all intuitive, and we shall illustrate it with reference to examples $1-3$ and with $m = 30$.

Example 1: The code is 1122 and the orbit is active with respect to m, since $3 \mid \ell^s = 6$, $m^s = \gcd(30, 6) = 6$ so $m/m^s = 5$ is odd. Furthermore $a_0^s, a_0^s + a_1^s, \ldots, a_0^s + a_1^s + \cdots + a_{k^s-1}^s = 1, 2, 4, 6 \equiv 1, 2, 1, 0 \bmod 3$, so the residue class 1 occurs an even number of times and the residue classes 0, 2 occur an odd number of times.

Example 2: The code is 1211 and the orbit is not active with respect to m, since $3 \nmid \ell^s = 5$.

Example 3: The code is 111222 and the orbit is not active with respect to m. Indeed it fails on two counts, since $3 \mid \ell^s = 9$, and $m^s = \gcd(30, 9) = 3$, so $m/m^s = 10$ is even. Moreover we have $a_0^s, a_0^s + a_1^s, \ldots, a_0^s + a_1^s + \cdots + a_{k^s-1}^s = 1, 2, 3, 5, 7, 9 \equiv 1, 2, 0, 2, 1, 0 \bmod 3$, so each residue class occurs an even number of times.

Now let $m^s \equiv 0 \bmod 3$, let m/m^s be odd, and suppose that $(X_0^s, X_1^s, \ldots, X_{m^s-1}^s)$ is periodic with period m^s. Then the rows of the matrix M^s are also periodic with period m^s, and all entries are one of $X_0^s, X_1^s, \dots, X_{m^s-1}^s$ which we regard as unknowns taking values ± 1. We are interested in evaluating P_0^s and P_1^s as functions of the unknowns $X_0^s, X_1^s, \dots, X_{m^s-1}^s$. Now each entry of M^s is one of $X_0^s, X_1^s, \dots, X_{m^s-1}^s$ and so P_0^s and P_1^s are each products of the unknowns $X_0^s, X_1^s, \dots, X_{m^s-1}^s$. Since each of these is ± 1, the important ones are those that occur to an odd power in the product.

In what follows, by the *parity* of a set, we mean the number reduced modulo 2 of elements in the set.

We may identify these using the following lemma:

Lemma 3. *Let the orbit s in D satisfy $m^s \equiv 0 \bmod 3$ and let m/m^s be odd. Then for $r = 1, 2$, we have*

$$P_r^s = \prod_{0 \le n' < m^s} (X_{n'}^s)^{\Gamma(n', r)} \tag{8.7}$$

where $\Gamma(n', r)$ is the parity of the set

$$\{i \mid 0 \le i < k^s,\, n' + \sum_{j=0}^{i-1} a_j^s \not\equiv r \bmod 3\}\,. \tag{8.8}$$

Proof. Let $r = 0, 1$. From (8.5) we have

$$P_r^s = \prod_{\substack{n \not\equiv r \bmod 3 \\ r \le n < r + m^s}} D_n^s = \prod_{i=0}^{k^s-1} \prod_{\substack{n \not\equiv r \bmod 3 \\ r \le n < r + m^s}} M_{n,i}^s = \prod_{\substack{n \not\equiv r \bmod 3 \\ r \le n < r + m^s}} \prod_{i=0}^{k^s-1} X^s_{n - \sum_{j=0}^{i-1} a_j^s}\,, \tag{8.9}$$

where the subscript is reduced modulo m^s. Writing $n' = n - \sum_{j=0}^{i-1} a_j^s \bmod m^s$, we obtain

$$P_r^s = \prod_{0 \le n' < m^s} \prod_{\substack{i=0 \\ n' + \sum_{j=0}^{i-1} a_j^s \not\equiv r \bmod 3}}^{k^s-1} X_{n'}^s = \prod_{0 \le n' < m^s} (X_{n'}^s)^{\Gamma(n', r)}\,, \tag{8.10}$$

since $X_{n'}^s = \pm 1$. This completes the proof of the lemma. $\square$

From this lemma it follows immediately that those of the X_n^s occurring to an odd power in P_0^s come in residue classes modulo 3, and similarly for P_1^s.

We let $\mathbb{Z}_3$ denote the additive group of residue classes modulo 3, and $\mathbb{Z}_2$ denote the residue classes modulo 2. Let $\tilde{\gamma} : \mathbb{Z}_3 \to \mathbb{Z}_2$ be the number modulo 2 of residue classes modulo 3 occurring in $a_0^s, a_0^s + a_1^s, \dots, a_0^s + a_1^s + \dots + a_{k-1}^s$. Then, from the above definition, the orbit is active if and only if $\tilde{\gamma}$ is surjective, i.e., there exists at least one residue class u with $\tilde{\gamma}(u) = 0$, and another, u', with $\tilde{\gamma}(u') = 1$.

Let $\gamma(n', r)$ be the parity of the set $\{i \mid 0 \le i < k^s,\, n' + \sum_{j=0}^{i-1} a_j^s \equiv r \bmod 3\}$. Then, since $\ell^s \equiv 0 \bmod 3$, $\{i \mid 0 \le i < k^s,\, n' + \sum_{j=0}^{i-1} a_j^s \equiv r \bmod 3\} = \{i \mid 1 \le i \le k^s,\, n' + \sum_{j=0}^{i-1} a_j^s \equiv r \bmod 3\}$, and it is clear that $\gamma(n', r) = \tilde{\gamma}(r - n')$ and

$$\Gamma(n', r) = \gamma(n', r+1) + \gamma(n', r+2) = \tilde{\gamma}(r + 1 - n') + \tilde{\gamma}(r + 2 - n')\,, \tag{8.11}$$

where, of course, here, and in what follows, we reduce the sums modulo 2. Furthermore, $\Gamma(n', r) = \Gamma(n' \bmod 3, r)$. This means that we need only examine the three residue classes modulo 3. Moreover we have $\Gamma(n', r) = \Gamma(n' + 1, r + 1)$.

For an inactive orbit we have $\Gamma(n', r) = 0$, since $\tilde{\gamma}(u) + \tilde{\gamma}(u') = 0$ for all choices of u, $u' \in \mathbb{Z}_3$. (This follows because $\tilde{\gamma} \equiv 0$ or $\tilde{\gamma} \equiv 1$.) However if the orbit is active then as n' runs from 0 to 2 precisely two

$\Gamma(n', r)$ will be 1 and the third 0. We show the possibilities for $r = 0$ (i.e., for P_0^s) in the following table. We have $\Gamma(n', 0) = \tilde{\gamma}(1 - n') + \tilde{\gamma}(2 - n')$, so that

$$\Gamma(0,0) = \tilde{\gamma}(1) + \tilde{\gamma}(2) \tag{8.12}$$
$$\Gamma(1,0) = \tilde{\gamma}(0) + \tilde{\gamma}(1) \tag{8.13}$$
$$\Gamma(2,0) = \tilde{\gamma}(2) + \tilde{\gamma}(0)\,. \tag{8.14}$$

$\tilde{\gamma}(0)$	$\tilde{\gamma}(1)$	$\tilde{\gamma}(2)$	active	$\Gamma(0,0)$	$\Gamma(1,0)$	$\Gamma(2,0)$
0	0	0	no	0	0	0
1	0	0	yes	0	1	1
0	1	0	yes	1	1	0
0	0	1	yes	1	0	1
1	1	0	yes	1	0	1
0	1	1	yes	0	1	1
1	0	1	yes	1	1	0
1	1	1	no	0	0	0

The parity of the residue class n' in P_1^s is

$$\Gamma(n', 1) = \Gamma(n' - 1, 0)\,. \tag{8.15}$$

Thus the residue classes occurring an odd number of times in P_1^s are those occurring an odd number of times in P_0^s increased by 1 modulo 3. It follows that one residue class occurs in both P_0^s and P_1^s and the other two in one each. We establish the following notation. Let c_0^s, c_1^s be the residue classes occurring in P_0^s and let c_1^s, c_2^s the residue classes occurring in P_1^s. Then

$$P_0^s = \prod_{\substack{n \equiv c_0^s \bmod 3 \\ 0 \le n < m^s}} X_n^s \prod_{\substack{n \equiv c_1^s \bmod 3 \\ 0 \le n < m^s}} X_n^s \tag{8.16}$$
$$P_1^s = \prod_{\substack{n \equiv c_1^s \bmod 3 \\ 0 \le n < m^s}} X_n^s \prod_{\substack{n \equiv c_2^s \bmod 3 \\ 0 \le n < m^s}} X_n^s\,, \tag{8.17}$$

a product of two residue classes modulo 3.

We now return to example 1, which is active with respect to $m = 30$. Recall that the code is 1122, so $\ell^s = 6$ and $m^s = 6$. For convenience we drop the index s. The matrix M for this orbit is

$$M = \begin{pmatrix} X_0 & X_5 & X_4 & X_2 \\ X_1 & X_0 & X_5 & X_3 \\ X_2 & X_1 & X_0 & X_4 \\ X_3 & X_2 & X_1 & X_5 \\ X_4 & X_3 & X_2 & X_0 \\ X_5 & X_4 & X_3 & X_1 \end{pmatrix}\,. \tag{8.18}$$

Then, in view of lemma 2 (1),

$$P_0 = \prod_{\substack{n \not\equiv 0 \bmod 3 \\ 0 \le n < 6}} D_n \tag{8.19}$$

$$= \prod_{n=1,2,4,5} \prod_{i=0}^{3} M_{n,i} \tag{8.20}$$

$$= (X_1 X_0 X_5 X_3)(X_2 X_1 X_0 X_4)(X_4 X_3 X_2 X_0)(X_5 X_4 X_3 X_1) \tag{8.21}$$

$$= X_0 X_1 X_3 X_4 \,. \tag{8.22}$$

Similarly $P_1 = X_1 X_2 X_4 X_5$. Now $a_0, a_0 + a_1, a_0 + a_1 + a_2, a_0 + a_1 + a_2 + a_3 = 1, 2, 4, 6 \equiv 1, 2, 1, 0 \bmod 3$. Hence $\tilde{\gamma}(0) = 1$, $\tilde{\gamma}(1) = 0$, $\tilde{\gamma}(2) = 1$, and so $\Gamma(0,0) = \tilde{\gamma}(1) + \tilde{\gamma}(2) = 1$, $\Gamma(1,0) = \tilde{\gamma}(0) + \tilde{\gamma}(1) = 1$, $\Gamma(2,0) = \tilde{\gamma}(2) + \tilde{\gamma}(0) = 0$. Thus $c_0 = 0$, $c_1 = 1$, $c_2 = 2$. We have therefore that

$$P_0 = \prod_{\substack{n \equiv 0 \bmod 3 \\ 0 \le n < 6}} X_n \prod_{\substack{n \equiv 1 \bmod 3 \\ 0 \le n < 6}} X_n \tag{8.23}$$

$$= (X_0 X_3)(X_1 X_4) \,, \tag{8.24}$$

and

$$P_1 = \prod_{\substack{n \equiv 1 \bmod 3 \\ 0 \le n < 6}} X_n \prod_{\substack{n \equiv 2 \bmod 3 \\ 0 \le n < 6}} X_n \tag{8.25}$$

$$= (X_1 X_4)(X_2 X_5) \,, \tag{8.26}$$

which indeed agree with the direct calculation above.

We next consider example 3, which is inactive with respect to $m = 30$. Since $m = 30$, and $\ell^s = 9$, we have $m^s = 3$. We have (again dropping the index s) that the matrix M for this orbit is

$$M = \begin{pmatrix} X_0 & X_2 & X_1 \\ X_1 & X_0 & X_2 \\ X_2 & X_1 & X_0 \end{pmatrix} . \tag{8.27}$$

We have

$$\prod_{\substack{n \equiv 1 \bmod 3 \\ 0 \le n < 3}} M_{n,i} \prod_{\substack{n \equiv 2 \bmod 3 \\ 0 \le n < 3}} M_{n,i} = 1 \,, \tag{8.28}$$

as is easily seen (using $X_n^2 = 1$ for all n). Hence $P_0 = 1$ and similarly $P_1 = 1$.

In order to simplify the presentation of the next section, we introduce a relabeling of the X_n^s for active orbits. Specifically, we define, for $n \ge 0$

$$Y_n^s = X_{n + c_0^s}^s \tag{8.29}$$

where we reduce modulo m^s. Then $Y_0^s = X_{c_0^s}^s$, $Y_1^s = X_{c_1^s}^s$ and $Y_2^s = X_{c_2^s}^s$, and the column $(Y_0^s, \ldots, Y_{m^s-1}^s)$ is simply $(X_0^s, \ldots, X_{m^s-1}^s)$ relabeled, and, in particular, the two columns have the same period as sequences of ± 1. Then, in terms of the Y_n^s we have

$$P_0^s = \prod_{\substack{n\equiv 0 \bmod 3\\ 0\le n<m^s}} Y_n^s \prod_{\substack{n\equiv 1 \bmod 3\\ 0\le n<m^s}} Y_n^s \tag{8.30}$$

$$P_1^s = \prod_{\substack{n\equiv 1 \bmod 3\\ 0\le n<m^s}} Y_n^s \prod_{\substack{n\equiv 2 \bmod 3\\ 0\le n<m^s}} Y_n^s\,. \tag{8.31}$$

In summary, we have the following proposition.

Proposition 9. *Let $m \equiv 0 \bmod 3$. Then the products P_0^s, P_1^s for the orbit s satisfy $P_0^s = P_1^s = 1$ irrespective of the values of $X_0^s, \dots, X_{m^s-1}^s$ if and only if the orbit s is not active with respect to m. If, on the other hand, the orbit s is active with respect to m, then P_0^s and P_1^s are given by (8.16) and (8.17) where $\Gamma(c_0^s, 0) = \Gamma(c_1^s, 0) = 1$ and $\Gamma(c_1^s, 1) = \Gamma(c_2^s, 1) = 1$. In terms of the relabeled variables $Y_0^s, \dots, Y_{m^s-1}^s$, P_0^s and P_1^s are given by (8.30) and (8.31).*

In summary, for orbits active with respect to m, we have a systematic method of calculating the products P_0^s, P_1^s given by (8.16–8.17) in terms of the unknowns $X_0^s, X_1^s, \dots, X_{m^s-1}^s$ or, equivalently, $Y_0^s, Y_1^s, \dots, Y_{m^s-1}^s$.

8.3 Realisation of the possible values of p for $m \equiv 0 \bmod 3$

We know from theorem 2 that either $p = m$ or $p = 2m$ when $m \equiv 0 \bmod 3$. We now wish to see which of these cases can occur. We recall that if $m \not\equiv 0 \bmod 3$ then both $p = m$ and $p = 3m$ can occur for suitable choices of Q_0^{1+}, Q_1^{1+}. In the case $m \equiv 0 \bmod 3$ these quantities do not determine Q_m^{1+}/Q_0^{1+}, Q_{m+1}^{1+}/Q_1^{1+} and the period p is determined completely by the columns of discontinuities $(X_0^s, \dots, X_{k^s-1}^s)$.

However, unless the restricted discontinuity set D contains some periodic orbits that are active with respect to m, then the period $p = m$, since we have $P_0 = P_1 = 1$ in this case.

Now suppose that there are some periodic orbits in D which are active with respect to m. We shall now study whether, by appropriate choice of the X_n^s for the active orbits, we may ensure both that R_n has period $m^s = \gcd(\ell^s, m)$ on the orbit s and that either $P_0 = P_1 = 1$ (in which case $p = m$) or not (in which case $p = 2m$). We divide the analysis into three distinct subcases which we treat separately. Our constructions serve only as examples; they are by no means unique. Other choices of the X_n^s can be made to achieve the same result.

8.3.1 Case (i): D contains at least two periodic orbits active with respect to m

Let there be v such orbits labeled $s_0, \dots, s_{v-1}$ for $1 \le v \le t-1$. Referring to subsection 8.2, we recall that (with respect to the $Y_n^{s_i}$), 0 and 1 are the residue classes occurring to an odd power in $P_0(s_i)$, and 1 and 2 are the residue classes occurring to an odd power in $P_1(s_i)$. Then, by assigning Y_n^s appropriately, we may ensure that that $p = m$ or $p = 2m$.

We first of all assign those X_n^s for $s \ne s_0, \dots, s_{v-1}$ so that the period of the column $(X_0^s, \dots, X_{m^s-1}^s)$ is m^s, but otherwise arbitrarily.

Consider the assignments given by the following table:

	s_0	s_1	s_2	$\dots$	s_{v-1}
Y_0^s	±1	1	1	$\dots$	1
Y_1^s	-1	-1	-1	$\dots$	-1
Y_2^s	1	±1	1	$\dots$	1
Y_3^s	1	1	1	$\dots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$
$Y_{m^s-1}^s$	1	1	1	$\dots$	1

For each of $s = s_0, \dots, s_{v-1}$ we set $Y_n^s = 1$ for $n \neq 0, 1, 2$, so that

$$P_0 = \prod_{i=0}^{v-1} Y_0^{s_i} \prod_{i=0}^{v-1} Y_1^{s_i}\,, \qquad P_1 = \prod_{i=0}^{v-1} Y_1^{s_i} \prod_{i=0}^{v-1} Y_2^{s_i}\,. \tag{8.32}$$

Then from (8.32) we have that $P_0 = (\pm1)(-1)^v$, $P_1 = (-1)^v(\pm1)$.

The two ±1 entries can be chosen to obtain either $p = m$ or $p = 2m$, as required. To achieve $p = m$ we choose both the signs to be $(-1)^v$, and to achieve $p = 2m$ we choose (at least one of) the signs to be $-(-1)^v$.

8.3.2 Case (ii): D contains only one periodic orbit active with respect to m, but $m^s \neq 3$

Let the active periodic orbit be s_0. Firstly, the Y_n^s for $s \neq s_0$ are set to have period m^s but are otherwise arbitrary. We assign $Y_n^{s_0}$ according to the following table.

	s_0
Y_0^s	±1
Y_1^s	-1
Y_2^s	±1
Y_3^s	1
$\vdots$	$\vdots$
$Y_{m^s-1}^s$	1

Then $P_0 = P_1 = \mp1$ and so the two ±1 entries can be chosen to obtain either $p = m$ or $p = 2m$, as required. We note that since $m^{s_0} \geq 6$ we have the column $(Y_0^{s_0}, \dots, Y_{m^{s_0}-1}^{s_0})$ has period m^s, even when $Y_0^{s_0} = Y_1^{s_0} = Y_2^{s_0} = -1$.

8.3.3 Case (iii): D contains only one periodic orbit active with respect to m, and $m^s = 3$

Let the active periodic orbit be s_0. Then $m^{s_0} = 3$, and $P_0 = Y_0^{s_0} Y_1^{s_0}$, $P_1 = Y_1^{s_0} Y_2^{s_0}$. It is not possible to choose $Y_0^{s_0}$, $Y_1^{s_0}$, $Y_2^{s_0}$ so that $P_0 = P_1 = 1$ and $Y_0^{s_0}$, $Y_1^{s_0}$, $Y_2^{s_0}$ are not all equal. (Note that if they are all equal then the period is 1 not 3.) Thus it is not possible to obtain $p = m$ in this case and $p = 2m$ for any choice of $Y_0^{s_0}$, $Y_1^{s_0}$, $Y_2^{s_0}$ of period 3.

8.4 Theorem 3

In summary, combining the results of section 8, we have proved the following theorem:

Theorem 3. *Let D be a set of t periodic orbits (y_i^s), $0 \leq s \leq t-1$, $0 \leq i \leq k^s - 1$ of F, and let ℓ^s, ℓ be as in section 5.4. Then, for any $m \mid \ell$, setting $m^s = gcd(m, \ell^s)$, the variables $X_0^s, \ldots, X_{m^s-1}^s$ for $0 \leq s \leq t-1$ may be chosen so that:*

1. *R_n is periodic with period m^s on y_i^s;*
2. *R_n is periodic with period m;*
3. *the restricted discontinuity set of Q_n is D.*

The period p of Q_n is related to m as follows:

4. *if $m \not\equiv 0 \bmod 3$ then there are choices of Q_0^{1+}, Q_1^{1+} so that $p = m$ or $p = 3m$;*
5. *if $m \equiv 0 \bmod 3$ and there are no periodic orbits in D active with respect to m, then $p = m$;*
6. *if $m \equiv 0 \bmod 3$ and there are at least two periodic orbits in D active with respect to m then there are choices of X_i^s such that $p = m$ or $p = 2m$;*
7. *similarly, if $m \equiv 0 \bmod 3$ and there is only one periodic orbit in D active with respect to m, and $m^s > 3$, then there are choices of X_i^s such that $p = m$ or $p = 2m$;*
8. *finally, if $m \equiv 0 \bmod 3$ and there is exactly one periodic orbit in D active with respect to m, and, for that orbit, $m^s = 3$, then $p = 2m$ for all choices of X_i^s of period m^s.*

The theorem has an important corollary (theorem 1).

Corollary 1. *For every $p \geq 1$ there is a periodic orbit Q_n of (1.5) of period p.*

Proof. Let $p \in \mathbb{N}$ and let D consist of the orbit with code equal to p copies of 111222. Then $\ell = 9p$ and the orbit is inactive (as can be easily checked). Let $m = p$. We may use part 4 or part 5 of theorem 3 to obtain an orbit of Q_n of period p depending on whether $p \not\equiv 0 \bmod 3$ or $p \equiv 0 \bmod 3$. □

As an illustration of the theory, we refer back to examples 1 – 3 with reference to theorem 3. Let D consist of the union of the three periodic orbits in examples 1 – 3. Recall that $\ell = 90$ in this case. Let us choose $m = 30$. Then we are in case 7 of theorem 3, since we have one active orbit with respect to m and $m^s = \gcd(30, 6) = 6$. We therefore are able to choose the X_n^s so that $p = 30$ or $p = 60$ in this case.

Finally, let us return to figure 1 and the periodic orbit found by Feudel *et al* ([2]). In this case D consists of the fixed point with code 1 and the period 2 orbit with code 21. Then $\ell = \mathrm{lcm}(1, 3) = 3$ and, choosing $m = 3$, we have that the period 2 orbit is active with respect to $m = 3$. We are therefore in case 8 of theorem 3 so that we have $p = 2m = 6$ in this case. In fact we have $P_0 = 1$ and $P_1 = -1$, as can be easily seen from figure 1.

9 Conclusion

Orbits of the renormalisation recursion (1.5) arise in the analysis of self-similarity in a variety of phenomena. The recursion, despite being multiplicative in nature, is also nontrivial from a mathematical point of view.

In this paper we have given the complete solution in the case of piecewise-constant functions taking values ± 1. A particular instance of such a solution was numerically calculated by Feudel *et al* [2] in their analysis of the autocorrelation function for a strange nonchaotic attractor.

In a previous paper [8] we have considered analytic solutions of the fixed point equation corresponding to (1.5). This solution helps explain the universality of the supercritical regime of the Harper equation, and is also directly of importance in the study of the onset of a strange nonchaotic attractor [6], [7]. We hope to be able to combine the ideas on periodic orbit structure developed in this paper with the analysis of our previous work [8] to understand the universal strange attractor found in a generalised Harper equation [6].

We remark that an additive version of the renormalisation recursion (1.5) is derived in [3] in the analysis of the self-similarity of the autocorrelation of a quasiperiodically forced two-level system. Again, the piecewise-constant periodic orbits are important in determining the precise nature of the autocorrelation. Much of our work in this paper is also applicable to this additive case, but there are also some subtle differences which we shall explore in the near future.

Finally we remark that the fact that ω in (1.1) is the golden mean is essential for our analysis. It seems likely that a similar study could be undertaken for other quadratic irrationals (which have eventually periodic continued fraction expansions). However it is not clear how to extend our work to more general rotation numbers.

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