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BIPARTITE MEAN FIELD SPIN SYSTEMS. EXISTENCE AND SOLUTION

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Abstract

A mean field spin system consisting two interacting groups each with homogeneous interaction coefficients is introduced and studied. Existence of the thermodynamic limit is shown by an asymptotic sub-addittivity method and factorization of correlation functions is proved almost everywhere. The free energy solution of the model is obtained by upper and lower bounds and by showing that their difference vanishes for large volumes.

1 Introduction

In this work we consider the problem of characterizing the equilibrium statistical mechanics of an interacting system of two set of spins. We aim to tackle the most general case of a two population system in the mean field approximation.

Mean field two-population models have been useful since the study of metamagnets which started in the '70s (see [1, 2]), and have been encountered very recently in the study of loss of gibbsianness for a system whose evolution is described by Glauber dynamics (see [3]). In Refs [1, 2] a two-population mean-field model is used as an approximation to a bipartite lattice assumed to describe an antiferromagnetic system, and is found to reproduce qualitatively the expected phase transitions, which are then studied at criticality. In Ref. [3], instead, particles are subject to a time-evolving random field which acts on particles by partioning them in two groups, leading to a mean field model mathematically analogous to the former. In this case a characterization of the whole phase diagram is provided.

The systems considered by both works can be seen as a restriction of a more general model, which is presented here, and which arises naturally as an interacting mixture of two systems of Curie-Weiss type.

Our results can be summarised as follows. After introducing the model we show in section 3 that it is well posed by showing that its thermodynamic limit exists. The result is non trivial because sub-additivity is not met at finite volume. In section 4 we show that the system fulfills a factorization property for the correlation functions which reduces the equilibrium state to have only two degrees of freedom. The method is conceptually similar to the one developed by Guerra in [5] to derive identities for the overlap distributions in the Sherrington and Kirkpatrick model.

We also derive the pressure of the model by rigorous methods developed in the recent study of mean field spin glasses (see [10] and [9] for a review). It is interesting to notice that though very simple, our model encompasses a range of regimes that do not admit solution by the elegant interpolation method used in the celebrated existence result of the Sherrington and Kirkpatrick model [11]. This is due to the lack of positivity of the quadratic form describing the considered interaction. Nevertheless we are able to solve the model exactly, section 5, using the lower bound provided by the Gibbs variational principle, and thanks to a further bound given by a partitioning of the configuration space, itself originally devised in the study of spin glasses (see [10, 8]).

The results presented here could very possibly be obtained also by using standard methods developed for the study of the Curie-Weiss model (see [6] or [7]). The scheme that we propose, though, has several reasons to be considered: besides its direct physical significance that comes from the thermodynamic variational principle, and the mathematical robustness given by the explicit evaluation of upper and lower bounds, as well as by finite volume corrections, it is the only good candidate to the study of some natural generalizations involving random interactions, as mentioned at the end of the comments section.

As in the classical Curie-Weiss model, the exact solution is provided in an implicit form; for our system, however, we find two equations of state, which are coupled as well as trascendental, and this makes the full characterization of all the possible regimes highly non-trivial: a robust numerical analysis becomes essential and can be found in [12], where an application to social sciences is considered.

Some aspects of the regimes can nonetheless be studied analytically, and this is done in section 6, while a global study of the phase diagram for our model is left to be carried on in a future work.

2 The Model

Our model is defined by the Hamiltonian

$$H(\sigma) = -\frac{1}{2N}\sum_{i,j=1}^{N} J_{ij}\sigma_i\sigma_j - \sum_i h_i\sigma_i \ . \tag{2.1}$$

We consider Ising spins, $\sigma_i = \pm 1$, and symmetric interactions $J_{i,j}$. We divide the particles $P = \{1, 2, 3, ..., N\}$ into 2 types A and B with $A \cup B = P$, $A \cap B = \emptyset$, and sizes $N_1 = |A|$ and $N_2 = |B|$, where $N_1 + N_2 = N$. Given two particles i and j, their mutual interaction parameter J_{ij} depends on the subset they belong to, as specified by the matrix

$$\begin{array}{cc} & \begin{array}{cc} \overbrace{\qquad}^{N_1} & \overbrace{\qquad\quad}^{N_2} \end{array} \\ \begin{array}{r} N_1 \left\{ \vphantom{\mathbf{J_{11}}} \right. \\ N_2 \left\{ \vphantom{\begin{array}{c} a \\ \mathbf{J_{22}} \\ a \end{array}} \right. \end{array} & \left(\begin{array}{c|c} \mathbf{J_{11}} & \mathbf{J_{12}} \\ \hline & \\ \mathbf{J_{12}^{*}} & \mathbf{J_{22}} \\ & \end{array} \right) \end{array}$$

where each matrix block has constant elements: J_{11} and J_{22} tune the interactions within each of the two types, and J_{12} controls the interaction between two particles of different types. In view of the applications considered in the introduction, we assume $J_{11} > 0$ and $J_{22} > 0$, whereas J_{12} can be either positive or negative.

Analogously, the field h_i takes two values h_1 and h_2, depending on the type of i, as described by the following vector:

$$\begin{array}{l} N_1 \left\{ \vphantom{\mathbf{h_1}} \right. \\ \\ N_2 \left\{ \vphantom{\begin{array}{c} \\ \mathbf{h_2} \\ \\ \end{array}} \right. \end{array} \left(\begin{array}{c} \mathbf{h_1} \\ \hline \\ \mathbf{h_2} \\ \\ \end{array} \right)$$

By introducing the magnetization of a subset S as

$$m_S(\sigma) \;=\; \frac{1}{|S|}\sum_{i\in S}\sigma_i$$

and indicating by m_1 and m_2 the magnetizations within the subsets A and B and by $\alpha = \dfrac{N_1}{N}$ the relative size of subset A on the whole, we may easily express the Hamiltonian per particle as

$$\frac{H(\sigma)}{N} = -\frac{1}{2}\left[J_{11}\alpha^2 m_1^2 + 2J_{12}\alpha(1-\alpha)m_1m_2 + J_{22}(1-\alpha)^2 m_2^2\right] - h_1\alpha m_1 - h_2(1-\alpha)m_2 \qquad (2.2)$$

The usual statistical mechanics framework defines the equilibrium value of an observable $f(\sigma)$ as the average with respect to the *Gibbs distribution* defined by the Hamiltonian. We call this average the *Gibbs state* for $f(\sigma)$, and write it explicitly as:

$$\langle\, f\,\rangle \;=\; \frac{\sum_\sigma f(\sigma)\, e^{-H(\sigma)}}{\sum_\sigma e^{-H(\sigma)}}\ .$$

The main observable for our model is the average of a spin configuration, i.e. the *magnetization*, $m(\sigma)$, which explicitly reads:

$$m(\sigma) = \frac{1}{N}\sum_{i=1}^{N}\sigma_i.$$

Our quantity of interest is therefore $\langle m\rangle$: to find it, as well as the moments of many other observables, statistical mechanics leads us to consider the pressure function:

$$p_N \;=\; \frac{1}{N}\log\sum_\sigma e^{-H(\sigma)}\ .$$

It is easy to verify that, once it's been derived exactly, the pressure is capable of generating the Gibbs state for the magnetization as

$$\langle m\rangle = \alpha\frac{\partial p_N}{\partial h_1} + (1-\alpha)\frac{\partial p_N}{\partial h_2}.$$

In order to simplify the analytical study of the model it is useful to observe that the Hamiltonian is left invariant by the action of a group of transformations, so that only a subspace of parameter space needs to be considered.

The symmetry group is described by $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

We can represent a point in our parameter space as $(\mathbf{m}, \mathbf{J}, \mathbf{h}, \hat{\alpha})$, where

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} J_{11} & J_{12} \\ J_{12} & J_{22} \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \quad \hat{\alpha} = \begin{pmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{pmatrix}.$$

Therefore, given the limitations on the values of our parameters, the whole parameter space is given by $S = [-1,1]^2 \times \mathbb{R}^2 \times \mathbb{R}_+ \times \mathbb{R}^2 \times [0,1]$.

If we consider the representation of G given by the 8 matrices

$$\begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}, \quad \epsilon_i = +1 \text{ or } -1 \quad \text{and} \quad \begin{pmatrix} 0 & \eta_1 \\ \eta_2 & 0 \end{pmatrix}, \quad \eta_i = +1 \text{ or } -1$$

we can immediately realize that G is a symmetry from the Hamiltonian representation:

$$\frac{H(\mathbf{m}, \mathbf{J}, \mathbf{h}, \hat{\alpha})}{N} = -\frac{1}{2}\langle \hat{\alpha}\,\mathbf{m}, \mathbf{J}\hat{\alpha}\,\mathbf{m}\rangle - \langle \mathbf{h}, \hat{\alpha}\,\mathbf{m}\rangle.$$

3 Existence of the thermodynamic limit

We shall prove that our model admits a thermodynamic limit by exploiting an existence theorem provided for mean field models in [13]: the result states that the existence of the pressure per particle for large volumes is guaranteed by a monotonicity condition on the equilibrium state of the Hamiltonian. Such a result proves to be quite useful when the condition of convexity introduced by the interpolation method [11, 10] doesn't apply due to lack of positivity of the quadratic form representing the interactions. We therefore prove the existence of the thermodynamic limit independently of an exact solution. Such a line of enquiry is pursued in view of further refinements of our model, that shall possibly involve random interactions of spin glass or random graph type, and that might or might not come with an exact expression for the pressure.

Proposition 3.1 *There exists a function p of the parameters $(\alpha, J_{1,1}, J_{1,2}, J_{2,2}, h_1, h_2)$ such that*

$$\lim_{N\to\infty} p_N = p\ .$$

The previous proposition is proved with a series of lemmas. Theorem 1 in [13] states that given an Hamiltonian H_N and its associated equilibrium state ω_N the model admits a thermodynamic limit whenever the physical condition

$$\omega_N(H_N) \geqslant \omega_N(H_{K_1}) + \omega_N(H_{K_2}), \qquad K_1 + K_2 = N, \tag{3.3}$$

is verified.

We proceed by first verifying this condition for a working Hamiltonian $\tilde{H}_N$, and then showing that its pressure $\tilde{p}_N$ tends to our original pressure p_N as N increases. We choose $\tilde{H}_N$ in such a way that the condition (3.3) is verified as an equality.

Our working Hamiltonian $\tilde{H}_N$ is defined as follows:

$$\tilde{H}_N = \tilde{H}_N^{(1)} + \tilde{H}_N^{(12)} + \tilde{H}_N^{(2)},$$

where

$$\tilde{H}_N^{(1)} = \alpha J_{11} \frac{1}{\alpha N - 1} \sum_{i \neq j = 1 \dots N_1} \xi_i \xi_j, \qquad \tilde{H}_N^{(2)} = (1-\alpha) J_{22} \frac{1}{(1-\alpha)N - 1} \sum_{i \neq j = 1 \dots N_2} \eta_i \eta_j,$$

$$\tilde{H}_N^{(12)} = \frac{1}{N} J_{12} \sum_{\substack{i=1\dots N_1 \\ j=1\dots N_2}} \xi_i \eta_j.$$

Lemma 3.1 *There exists a function $\tilde{p}$ such that*

$$\lim_{N\to\infty} \tilde{p}_N = \tilde{p}$$

Proof: By definition of $H_N^{(1)}$ and by the invariance of ω_N with respect to spin permutations,

$$\omega_N(\tilde{H}_N^{(1)}) = \omega(\alpha J_1 \frac{1}{\alpha N - 1} \sum_{i \neq j = 1 \dots N_1} \xi_i \xi_j) = \alpha J_1 \frac{(\alpha N - 1)\alpha N}{\alpha N - 1} \omega_N(\xi_1 \xi_2) = N \alpha^2 J_1 \omega(\xi_1 \xi_2).$$

We can find a similar form for $\tilde{H}^{(2)}$ and $\tilde{H}^{(12)}$, which implies that for any two positive integers $K_1 + K_2 = N$ we have

$$\omega(\tilde{H}_N) = \omega(\tilde{H}_{K_1} + \tilde{H}_{K_2}),$$

which verifies (3.3) and proves our Lemma. $\square$

The following two Lemmas show that the difference between H_N and $\tilde{H}_N$ is thermodynamically negligible and as a consequence their pressures coincide in the thermodynamic limit.

For convenience we shall re-express our Hamiltonian in the following way:

$$H_N = H_N^{(1)} + H_N^{(12)} + H_N^{(2)}$$

where we define

$$H_N^{(1)} = \frac{1}{N} J_{11} \sum_{i,j=1\ldots N_1} \xi_i \xi_j, \qquad H_N^{(2)} = \frac{1}{N} J_{22} \sum_{i,j=1\ldots N_2} \eta_i \eta_j,$$

$$H_N^{(12)} = \frac{1}{N} J_{12} \sum_{\substack{i=1\ldots N_1 \\ j=1\ldots N_2}} \xi_i \eta_j \ .$$

Lemma 3.2

$$H_N = \tilde{H}_N + O(1) \tag{3.4}$$

i.e.

$$\lim_{N\to\infty} \frac{H_N}{N} = \lim_{N\to\infty} \frac{\tilde{H}_N}{N}$$

Proof:

$$H_N^{(1)} = \frac{1}{N} J_{11} \sum_{i,j=1\ldots N_1} \xi_i \xi_j = \frac{N_1 - 1}{N} J_{11} \frac{1}{N_1 - 1} \sum_{i\neq j=1\ldots N_1} \xi_i \xi_j + \frac{1}{N} J_{11} \sum_{i=1\ldots N_1} \xi_i \xi_i,$$

and since $\alpha = \dfrac{N_1}{N}$

$$\begin{aligned} &= \frac{\alpha N - 1}{\alpha N} \alpha J_{11} \frac{1}{\alpha N - 1} \sum_{i\neq j=1\ldots N_1} \xi_i \xi_j + \alpha J_{11} = \\ &= \alpha J_{11} \frac{1}{\alpha N - 1} \sum_{i\neq j=1\ldots N_1} \xi_i \xi_j - \alpha J_{11} \frac{1}{\alpha N(\alpha N - 1)} \sum_{i\neq j=1\ldots N_1} \xi_i \xi_j + \alpha J_{11}, \end{aligned}$$

and so

$$H_N^{(1)} = \tilde{H}_N^{(1)} + O(1).$$

We can similarly get estimates for $H_N^{(1)}$ and $H_N^{(12)}$ in terms of $\tilde{H}_N^{(1)}$ and $\tilde{H}_N^{(12)}$, which implies

$$H_N = \tilde{H}_N + O(1).$$

□

Lemma 3.3 *Say* $p_N = \dfrac{1}{N} \ln Z_N$, *and say* $h_N(\sigma) = \dfrac{H_N(\sigma)}{N}$. *Define* $\tilde{Z}$, $\tilde{p}_N$ *and* $\tilde{h}_N$ *in an analogous way.*

Define

$$k_N = \|h_N - \tilde{h}_N\| = \sup_{\sigma\in\{-1,+1\}^N} \{|h_N(\sigma) - \tilde{h}_N(\sigma)|\} < \infty. \tag{3.5}$$

Then

$$|p_N - \tilde{p}_N| \leqslant \|h_N - \tilde{h}_N\| \; .$$

Proof:

$$\begin{aligned} p_N - \tilde{p}_N &= \frac{1}{N}\ln Z_N - \frac{1}{N}\ln \tilde{Z}_N = \frac{1}{N}\ln\frac{Z_N}{\tilde{Z}_N} \\ &= \frac{1}{N}\ln\frac{\sum_\sigma e^{-H_N(\sigma)}}{\sum_\sigma e^{-\tilde{H}_N(\sigma)}} \leqslant \frac{1}{N}\ln\frac{\sum_\sigma e^{-H_N(\sigma)}}{\sum_\sigma e^{-N(h_N(\sigma)+k_N)}} = \\ &= \frac{1}{N}\ln\frac{\sum_\sigma e^{-H_N(\sigma)}}{e^{-Nk_N}\sum_\sigma e^{-Nh_N(\sigma)}} = \frac{1}{N}\ln e^{Nk_N} = k_N = \|h_N - \tilde{h}_N\| \end{aligned}$$

where the inequality follows from the definition of k_N in (3.5) and from monotonicity of the exponential and logarithmic functions. The inequality for $\tilde{p}_N - p_N$ is obtained in a similar fashion. □

We are now ready to prove the main result for this section:

Proof of Proposition 3.1: The existence of the thermodynamic limit follows from our Lemmas. Indeed, since by Lemma 3.1 the limit for $\tilde{p}_N$ exists, Lemma 3.3 and Lemma 3.2 tell us that

$$\lim_{N\to\infty} |p_N - \tilde{p}_N| \leqslant \lim_{N\to\infty} \|h_N - \tilde{h}_N\| = 0,$$

implying our result. □

4 Factorization properties

In this section we shall prove that the correlation functions of our model factorize completely in the thermodynamic limit, for almost every choice of parameters. This implies that all the thermodynamic properties of the system can be described by the magnetizations m_1 and m_2 of the two subsets A and B defined in Section 2. Indeed, the exact solution of the model, to be derived in the next section, comes as two coupled equations of state for m_1 and m_2.

Proposition 4.2

$$\lim_{N\to\infty} \big(\omega_N(\sigma_i\sigma_j) - \omega_N(\sigma_i)\omega_N(\sigma_j)\big) = 0$$

for almost every choice of parameters $(\alpha, J_{11}, J_{12}, J_{22}, h_1, h_2)$*, where* σ_i*,* σ_j *are spins of any two distinct particles in the system.*

Proof: We recall the definition of the Hamiltonian per particle

$$\frac{H_N(\sigma)}{N} = -\frac{1}{2}\left[J_{11}\alpha^2 m_1^2 + 2J_{12}\alpha(1-\alpha)m_1 m_2 + J_{22}(1-\alpha)^2 m_2^2\right] - h_1\alpha m_1 - h_2(1-\alpha)m_2,$$

and of the pressure per particle

$$p_N = \frac{1}{N}\ln\sum_{\sigma} e^{-H_N(\sigma)}.$$

By taking first and second partial derivatives of p_N with respect to h_1 we get

$$\frac{\partial p_N}{\partial h_1} = \frac{1}{N}\sum_{\sigma}\alpha N m_1(\sigma)\frac{e^{-H(\sigma)}}{Z_N} = \alpha\omega_N(m_1), \qquad \frac{\partial^2 p_N}{\partial\, h_1^2} = N\alpha^2(\omega_N(m_1^2) - \omega_N(m_1)^2).$$

By using these relations we can bound above the integral with respect to h_1 of the fluctuations of m_1 in the Gibbs state:

$$\begin{aligned}\left|\int_{h_1^{(1)}}^{h_1^{(2)}}(\omega_N(m_1^2) - \omega_N(m_1)^2)\, dh_1\right| &= \frac{1}{N\alpha^2}\left|\int_{h_1^{(1)}}^{h_1^{(2)}}\frac{\partial^2 p_N}{\partial h_1^2}\, dh_1\right| = \frac{1}{N\alpha^2}\left|\frac{\partial p_N}{\partial\, h_1}\Big|_{h_1^{(1)}}^{h_1^{(2)}}\right| \leqslant \\ &\leqslant \frac{1}{N\alpha}\left(\left|\omega_N(m_1)|_{h_1^{(2)}}\right| + \left|\omega_N(m_1)|_{h_1^{(1)}}\right|\right) = O\left(\frac{1}{N}\right).\end{aligned} \tag{4.6}$$

On the other hand, given any $\alpha \in (0,1)$ we have that

$$\omega_N(m_1) = \frac{1}{\alpha}\frac{\partial p_N}{\partial h_1},$$

and

$$\omega_N(m_1^2) = \frac{2}{\alpha^2}\frac{\partial p_N}{\partial J_{11}},$$

so, by convexity of the thermodynamic pressure $p = \lim_{N\to\infty} p_N$, both quantities $\frac{\partial p_N}{\partial h_1}$ and $\frac{\partial p_N}{\partial J_{11}}$ have well defined thermodynamic limits almost everywhere. This together with (4.6) implies that

$$\lim_{N\to\infty}(\omega_N(m_1^2) - \omega_N(m_1)^2) = 0 \quad \text{a.e. in } h_1,\ J_{11}. \tag{4.7}$$

In order to prove our statement we first consider spins of particles of type A, which we shall call ξ_i. Translation invariance of the Gibbs measure tells us that

$$\begin{aligned}
\omega_N(m_1) &= \omega_N(\frac{1}{N}\sum_{i=1}^{N_1}\xi_i) = \alpha\omega_N(\xi_1), \\
\omega_N(m_1^2) &= \omega_N(\frac{1}{N^2}\sum_{i,j=1}^{N_1}\xi_i\xi_j) = \omega_N(\frac{1}{N^2}\sum_{i\neq j=1}^{N_1}\xi_i\xi_j) + \omega_N(\frac{1}{N^2}\sum_{i=j=1}^{N_1}\xi_i\xi_j) = \\
&= \alpha\frac{N_1-1}{N}\omega_N(\xi_1\xi_2) + \frac{\alpha}{N}.
\end{aligned} \tag{4.8}$$

We have that (4.8) and (4.7) imply

$$\lim_{N\to\infty}\omega_N(\xi_i\xi_j) - \omega_N(\xi_i)\omega_N(\xi_j) = 0, \tag{4.9}$$

which verifies our statement for all couples of distinct spins $i \neq j$ of type A as defined in section 2.

Working in strict analogy as above we also get

$$\lim_{N\to\infty}(\omega_N(m_2^2) - \omega_N(m_2)^2) = 0 \quad \text{a.e. in } h_2,\ J_{22}. \tag{4.10}$$

Furthermore, by defining $\text{Var}_N(m_1) = \big(\omega_N(m_1^2) - \omega_N(m_1)^2\big)$, and analogously for m_2, we exploit (4.7) and (4.10), and use the Cauchy-Schwartz inequality to get

$$|\omega_N(m_1m_2) - \omega_N(m_1)\omega_N(m_2)| \leqslant \sqrt{\text{Var}(m_1)\text{Var}(m_2)} \underset{N\to\infty}{\longrightarrow} 0 \quad \text{a.e. in } J_{11},\ J_{12},\ J_{22},\ h_1,\ h_2 \tag{4.11}$$

By using (4.10) and (4.11) we can therefore verify statements which are analogous to 4.9, but which concern $\omega_N(\xi_i\eta_j)$ and $\omega_N(\eta_i\eta_j)$ where ξ are spins of type A and η are spins of type B.

We have thus proved our claim for any couple of spins in the global system.

□

5 Solution of the model

We shall derive upper and lower bounds for the thermodynamic limit of the pressure. The lower bound is obtained through the standard entropic variational principle, while the upper bound is derived by a decoupling strategy.

5.1 Upper bound

In order to find an upper bound for the pressure we shall divide the configuration space into a partition of microstates of equal magnetization, following [10, 8]. Since subset A consists of N_1 spins, its magnetization can take exactly N_1+1 values, which are the elements of the set

$$R_{N_1} = \Big\{ -1, -1 + \frac{1}{2N_1}, \dots, 1 - \frac{1}{2N_1}, 1\Big\}.$$

Clearly for every $m_1(\sigma)$ we have that

$$\sum_{\mu_1 \in R_{N_1}} \delta_{m_1,\mu_1} = 1,$$

where $\delta_{x,y}$ is a Kronecker delta. We can similarly define a set R_{N_2}, so we have that

$$\begin{aligned} Z_N &= \sum_{\sigma} \exp\big\{\frac{N}{2}(J_{11}\alpha^2 m_1^2 + 2J_{12}\alpha(1-\alpha)m_1m_2 + J_{22}(1-\alpha)^2 m_2^2) + h_1N_1m_1 + h_2N_2m_2\big\} = \\ &= \sum_{\sigma} \sum_{\substack{\mu_1 \in R_{N_1} \\ \mu_2 \in R_{N_2}}} \delta_{m_1,\mu_1}\delta_{m_2,\mu_2} \exp\big\{\frac{N}{2}(J_{11}\alpha^2 m_1^2 + 2J_{12}\alpha(1-\alpha)m_1m_2 + J_{22}(1-\alpha)^2 m_2^2) + \\ &\quad + h_1N_1m_1 + h_2N_2m_2\big\}. \end{aligned} \tag{5.12}$$

Thanks to the Kronecker delta symbols, we can substitute m_1 (the average of the spins within a configuration) with the parameter μ_1 (which is not coupled to the spin configurations) in any convenient fashion, and the same holds for m_2 and μ_2.

Therefore we can use the following relations in order to linearize all quadratic terms appearing in the Hamiltonian

$$\begin{aligned} (m_1-\mu_1)^2 &= 0, \\ (m_2-\mu_2)^2 &= 0, \\ (m_1-\mu_1)(m_2-\mu_2) &= 0. \end{aligned}$$

Once we've carried out these substitutions into (5.12) we are left with a function which depends only linearly on m_1 and m_2:

$$\begin{aligned} Z_N &= \sum_{\sigma} \sum_{\substack{\mu_1 \in R_{N_1} \\ \mu_2 \in R_{N_2}}} \delta_{m_1,\mu_1}\delta_{m_2,\mu_2} \exp\big\{\frac{N}{2}(J_{11}\alpha^2(2m_1\mu_1 - \mu_1^2) + 2J_{12}\alpha(1-\alpha)m_1\mu_2 + \\ &\quad + J_{22}(1-\alpha)^2(2m_2\mu_2 - \mu_2^2)) + 2J_{12}\alpha(1-\alpha)m_2\mu_1 + 2J_{12}\alpha(1-\alpha)\mu_1\mu_2 + \\ &\quad + h_1N_1m_1 + h_2N_2m_2\big\}, \end{aligned}$$

and bounding above the Kronecker deltas by 1 we get

$$\begin{aligned} Z_N \;\leqslant\; & \sum_{\sigma} \sum_{\substack{\mu_1 \in R_{N_1} \\ \mu_2 \in R_{N_2}}} \exp\Big\{ -\frac{N}{2}(J_{11}\alpha^2\mu_1^2 + 2J_{12}\alpha(1-\alpha)\mu_1\mu_2 + J_{22}(1-\alpha)^2\mu_2^2) + \\ & +\big((J_{11}\alpha\mu_1 + J_{12}(1-\alpha)\mu_2 + h_1)N_1 m_1 + (J_{12}\alpha\mu_1 + J_{22}(1-\alpha)\mu_2 + h_2)N_2 m_2\big)\Big\}. \end{aligned} \tag{5.13}$$

Since both sums are taken over finitely many terms, it is possible to exchange the order of the two summation symbols, in order to carry out the sum over the spin configurations, which now factorizes, thanks to the linearity of the interaction with respect to the ms. This way we get:

$$Z_N \leqslant \sum_{\substack{\mu_1 \in R_{N_1} \\ \mu_2 \in R_{N_2}}} G(\mu_1, \mu_2).$$

where

$$\begin{aligned} G(\mu_1,\mu_2) \;=\; & \exp\Big\{ -\frac{N}{2}(J_{11}\alpha^2\mu_1^2 + 2J_{12}\alpha(1-\alpha)\mu_1\mu_2 + J_{22}(1-\alpha)^2\mu_2^2)\Big\} \cdot \\ & \cdot 2^{N_1}\big(\cosh\big(J_{11}\alpha\mu_1 + J_{12}(1-\alpha)\mu_2 + h_1\big)\big)^{N_1} \\ & \cdot 2^{N_2}\big(\cosh\big(J_{12}\alpha\mu_1 + J_{22}(1-\alpha)\mu_2 + h_2\big)\big)^{N_2} \end{aligned} \tag{5.14}$$

Since the summation is taken over the ranges R_{N_1} and R_{N_2}, of cardinality N_1+1 and N_2+1, we get that the total number of terms is $(N_1+1)(N_2+1)$. Therefore

$$Z_N \;\leqslant\; (N_1+1)(N_2+1) \sup_{\mu_1,\mu_2} G, \tag{5.15}$$

which leads to the following upper bound for p_N:

$$\begin{aligned} p_N \;=\; & \frac{1}{N}\ln Z_N \leqslant \ln\Big((N_1+1)(N_2+1)\sup_{\mu_1,\mu_2} G\Big) \\ \leqslant\; & \frac{1}{N}\ln(N_1+1) + \frac{1}{N}\ln(N_2+1) + \frac{1}{N}\sup_{\mu_1,\mu_2}\ln G\,. \end{aligned} \tag{5.16}$$

where the last line follows from monotonicity of the logarithm. Now defining the N independent function

$$\begin{aligned} p_{UP}(\mu_1,\mu_2) = \frac{1}{N}\ln G \;=\; & \ln 2 - \frac{1}{2}(J_{11}\alpha^2\mu_1^2 + 2J_{12}\alpha(1-\alpha)\mu_1\mu_2 + J_{22}(1-\alpha)^2\mu_2^2) + \\ & +\alpha\ln\cosh\big(J_{11}\alpha\mu_1 + J_{12}(1-\alpha)\mu_2 + h_1\big) + \\ & (1-\alpha)\ln\cosh\big(J_{12}\alpha\mu_1 + J_{22}(1-\alpha)\mu_2 + h_2\big), \end{aligned} \tag{5.17}$$

the thermodynamic limit gives:

$$\limsup_{N\to\infty} p_N \leqslant \sup_{\mu_1,\ \mu_2} p_{UP}(\mu_1,\mu_2). \tag{5.18}$$

We can summarize the previous computation into the following:

Lemma 5.4 *Given a Hamiltonian as defined in (2.2), and defining the pressure per particle as* $p_N = \frac{1}{N}\ln Z$, *given parameters* $J_{11}, J_{12}, J_{22}, h_1, h_2$ *and* α, *the following inequality holds:*

$$\limsup_{N\to\infty} p_N \leqslant \sup_{\mu_1,\mu_2} p_{UP}$$

where

$$\begin{aligned} p_{UP} \quad = \quad & \ln 2 - \frac{1}{2}(J_{11}\alpha^2\mu_1^2 + 2J_{12}\alpha(1-\alpha)\mu_1\mu_2 + J_{22}(1-\alpha)^2\mu_2^2) + \\ & +\alpha \ln\cosh\big(J_{11}\alpha\mu_1 + J_{12}(1-\alpha)\mu_2 + h_1\big) + \\ & (1-\alpha)\ln\cosh\big(J_{12}\alpha\mu_1 + J_{22}(1-\alpha)\mu_2 + h_2\big), \end{aligned} \tag{5.19}$$

and $(\mu_1,\mu_2) \in [-1,1]^2$.

5.2 Lower bound

The lower bound is provided by exploiting the well-known Gibbs entropic variational principle (see [14], pag. 188). In our case, instead of considering the whole space of *ansatz* probability distributions considered in [14], we shall restrict to a much smaller one, and use the upper bound derived in the last section in order to show that the lower bound corresponding to the restricted space is sharp in the thermodynamic limit.

The mean-field nature of our Hamiltonian allows us to restrict the variational problem to a two-degrees of freedom product measures represented through the non-interacting Hamiltonian:

$$\tilde{H} = -r_1\sum_{i=1}^{N_1}\xi_i - r_2\sum_{i=1}^{N_2}\eta_i,$$

and so, given a Hamiltonian $\tilde{H}$, we define the ansatz Gibbs state corresponding to it as $f(\sigma)$ as:

$$\tilde{\omega}(f) = \frac{\sum_\sigma f(\sigma)e^{-\tilde{H}(\sigma)}}{\sum_\sigma e^{-\tilde{H}(\sigma)}}$$

In order to facilitate our task, we shall express the variational principle of [14] in the following simple form:

Proposition 5.3 *Let a Hamiltonian H, and its associated partition function $Z = \sum_\sigma e^{-H}$ be given. Consider an arbitrary trial Hamiltonian $\tilde{H}$ and its associated partition function $\tilde{Z}$. The following inequality holds:*

$$\ln Z \geqslant \ln \tilde{Z} - \tilde{\omega}(H) + \tilde{\omega}(\tilde{H}) \ . \tag{5.20}$$

Given a Hamiltonian as defined in (2.2) and its associated pressure per particle $p_N = \frac{1}{N} \ln Z$, the following inequality follows from (5.20)*:*

$$\liminf_{N\to\infty} p_N \geqslant \sup_{\mu_1,\mu_2} p_{LOW} \tag{5.21}$$

where

$$\begin{aligned} p_{LOW}(\mu_1,\mu_2) \quad = \quad & \frac{1}{2}(J_{11}\alpha^2\mu_1^2 + J_{22}(1-\alpha)^2\mu_2^2 + 2J_{12}\alpha(1-\alpha)\mu_1\mu_2) + \\ & +\alpha h_1\mu_1 + (1-\alpha)h_2\mu_2 + \\ & +\alpha(-\frac{1+\mu_1}{2}\ln(\frac{1+\mu_1}{2}) - \frac{1-\mu_1}{2}\ln(\frac{1-\mu_1}{2})) + \\ & +(1-\alpha)(-\frac{1+\mu_2}{2}\ln(\frac{1+\mu_2}{2}) - \frac{1-\mu_2}{2}\ln(\frac{1-\mu_2}{2})). \end{aligned} \tag{5.22}$$

and $(\mu_1,\mu_2) \in [-1,1]^2$.

Proof: The (5.20) follows straightforwardly from Jensen's inequality:

$$e^{\tilde{\omega}(-H+\tilde{H})} \leq \tilde{\omega}(e^{-H+\tilde{H}}) \ . \tag{5.23}$$

It is convenient to express the Hamiltonian using the simbol ξ for the spins of type A and η for those of type B as:

$$H(\sigma) = -\frac{1}{2N}(J_{11}\sum_{i,j}\xi_i\xi_j + 2J_{12}\sum_{i,j}\xi_i\eta_j + J_{22}\sum_{i,j}\eta_i\eta_j) - h_1\sum_i \xi_i - h_2\sum_i \eta_i \ ; \tag{5.24}$$

indeed its expectation on the trial state is

$$\tilde{\omega}(H) = -\frac{1}{2N}(J_{11}\sum_{i,j}\tilde{\omega}(\xi_i\xi_j) + 2J_{12}\sum_{i,j}\tilde{\omega}(\xi_i\eta_j) + J_{22}\sum_{i,j}\tilde{\omega}(\eta_i\eta_j)) - h_1\sum_i\tilde{\omega}(\xi_i) - h_2\sum_i\tilde{\omega}(\eta_i) \tag{5.25}$$

and a standard computation for the moments leads to

$$\begin{aligned} \tilde{\omega}(H) \quad = \quad & -\frac{N}{2}(J_{11}(\alpha^2-\alpha/N)(\tanh r_1)^2 + J_{11}\alpha/N + J_{22}((1-\alpha)^2-(1-\alpha)/N)(\tanh r_2)^2 + \\ & +J_{22}(1-\alpha)/N + 2J_{12}\alpha(1-\alpha)\tanh r_1 \tanh r_2) - N\alpha h_1 \tanh r_1 - N(1-\alpha)h_2\tanh r_2. \end{aligned} \tag{5.26}$$

Analogously, the Gibbs state of $\tilde{H}$ is:

$$\tilde{\omega}(\tilde{H}) = -N\alpha r_1 \tanh r_1 - N(1-\alpha) r_2 \tanh r_2,$$

and the non interacting partition function is:

$$\tilde{Z}_N = \sum_\sigma e^{-\tilde{H}(\sigma)} = (e^{r_2} + e^{-r_2}) = 2^{N_1}(\cosh r_1)^{N_1} + 2^{N_2}(\cosh r_2)^{N_2}$$

which implies that the non-interacting pressure gives

$$\tilde{p}_N = \frac{1}{N} \ln \tilde{Z}_N = \ln 2 + \alpha \ln \cosh r_1 + (1-\alpha) \ln \cosh r_2$$

So we can finally apply Proposition (5.20) in order to find a lower bound for the pressure $p_N = \frac{1}{N} \ln Z_N$:

$$p_N = \frac{1}{N} \ln Z_N \geqslant \frac{1}{N} \left(\ln \tilde{Z}_N - \tilde{\omega}(H) + \tilde{\omega}(\tilde{H}) \right) \tag{5.27}$$

which explicitly reads:

$$\begin{aligned} p_N = \frac{1}{N} \ln Z_N \; \geqslant \; & \ln 2 + \alpha \ln \cosh r_1 + (1-\alpha) \ln \cosh r_2 + \\ & + \frac{1}{2} \Big(J_{11} \alpha^2 (\tanh r_1)^2 + J_{22} (1-\alpha)^2 (\tanh r_2)^2 + 2 J_{12} \alpha (1-\alpha) \tanh r_1 \tanh r_2 \Big) + \\ & + \alpha h_1 \tanh r_1 + (1-\alpha) h_2 \tanh r_2 - \alpha r_1 \tanh r_1 - (1-\alpha) r_2 \tanh r_2 \\ & + J_{11}\alpha/2N + J_{22}(1-\alpha)/2N - J_{11}\alpha(\tanh r_1)^2/N - J_{22}(1-\alpha)(\tanh r_2)^2/N. \end{aligned} \tag{5.28}$$

Taking the lim inf over N and the supremum in r_1 and r_2 of the left hand side we get the (5.21) after performing the change of variables $\mu_1 = \tanh r_1$ and $\mu_2 = \tanh r_2$.

□

5.3 Exact solution of the model

Though the functions p_{LOW} and p_{UP} are different, it is easily checked that they share the same local suprema. Indeed, if we differentiate both functions with respect to parameters μ_1 and μ_2, we see that the extremality conditions are given in both cases by the Mean Field Equations:

$$\begin{cases} \mu_1 = \tanh(J_{11}\alpha\mu_1 + J_{12}(1-\alpha)\mu_2 + h_1) \\ \mu_2 = \tanh(J_{12}\alpha\mu_1 + J_{22}(1-\alpha)\mu_2 + h_2) \end{cases} \tag{5.29}$$

If we now use these equations to express $\tanh^{-1}\mu_i$ as a function of μ_i and we substitute back into p_{UP} and p_{LOW} we get the same function:

$$p(\mu_1,\mu_2) = -\frac{1}{2}(J_{11}\alpha^2\mu_1^2+2J_{12}\alpha(1-\alpha)\mu_1\mu_2+J_{22}(1-\alpha)^2\mu_2^2)+-\frac{1}{2}\alpha\ln\frac{1-\mu_1^2}{4}-\frac{1}{2}(1-\alpha)\ln\frac{1-\mu_2^2}{4}. \tag{5.30}$$

Since this function returns the value of the pressure when the couple $(\mu_1,\ \mu_2)$ corresponds to an extremum, and this is the same both for p_{LOW} and p_{UP}, we have proved the following:

Theorem 1 *Given a hamiltonian as defined in (5.24), and defining the pressure per particle as* $p_N = \frac{1}{N}\ln Z$*, given parameters* $J_{11}, J_{12}, J_{22}, h_1, h_2$ *and* α*, the thermodynamic limit*

$$\lim_{N\to\infty} p_N = p$$

of the pressure exists, and can be expressed in one of the following equivalent forms:

a) $p = \sup_{\mu_1,\mu_2} p_{LOW}(\mu_1,\mu_2)$

b) $p = \sup_{\mu_1,\mu_2} p_{UP}(\mu_1,\mu_2)$

6 Preliminary analytic result

Though analysis cannot solve our problem exactly, it can tell us a what to expect when we solve it numerically. In particular, in this section we shall prove that, for any choice of the parameters, the total number of local maxima for the function $p(\mu_1,\mu_2)$ is less or equal to five.

We recall that the mean field equations for our two-population model are:

$$\begin{cases} \mu_1 &= \tanh(J_{11}\alpha\mu_1 + J_{12}(1-\alpha)\mu_2 + h_1) \\ \mu_2 &= \tanh(J_{12}\alpha\mu_1 + J_{22}(1-\alpha)\mu_2 + h_2) \end{cases},$$

and correspond to the stationarity conditions of $p(\mu_1,\mu_2)$. So, a subset of solutions to this system of equations are local maxima, and some among them correspond to the thermodynamic equilibrium.

These equations give a two-dimensional generalization of the Curie-Weiss mean field equation. Solutions of the classic Curie-Weiss model can be analysed by elementary geometry: in our case, however, the geometry is that of 2 dimensional maps, and it pays to recall that Henon's map, a simingly harmless 2 dimensional diffeomorhism of $\mathbb{R}^2$, is known to exhibit full-fledged chaos. Therefore, the parametric dependence of solutions, and in particular the number of solutions corresponding to local maxima of $p(\mu_1, \mu_2)$, is in no way apparent from the equations themselves.

We can, nevertheless, recover some geometric features from the analogy with one-dimensional picture. For the classic Curie-Weiss equation, continuity and the Intermediate Value Theorem from elementary calculus assure the existence of at least one solution. In higher dimensions we can resort to the analogous result, *Brouwer's Fixed Point Theorem*, which states that any continuous map on a topological closed ball has at least one fixed point. This theorem, applied to the smooth map R on the square $[-1,1]^2$, given by

$$\begin{cases} R_1(\mu_1,\ \mu_2) &= \tanh(J_{11}\alpha\mu_1 + J_{12}(1-\alpha)\mu_2 + h_1) \\ R_2(\mu_1,\ \mu_2) &= \tanh(J_{12}\alpha\mu_1 + J_{22}(1-\alpha)\mu_2 + h_2) \end{cases}$$

establishes the existence of at least one point of thermodynamic equilibrium.

We can gain further information by considering the precise form of the equations: by inverting the hyperbolic tangent in the first equation, we can μ_2 as a function of μ_1, and vice-versa for the second equation. Therefore, when $J_{12} \neq 0$ we can rewrite the equations in the following fashion:

$$\begin{cases} \mu_2 &= \dfrac{1}{J_{12}(1-\alpha)}(\tanh^{-1}\mu_1 - J_{11}\alpha\mu_1 - h_1) \\ \mu_1 &= \dfrac{1}{J_{12}\alpha}(\tanh^{-1}\mu_2 - J_{22}(1-\alpha)\mu_2 - h_2) \end{cases} \tag{6.31}$$

Consider, for example, the first equation: this defines a function $\mu_2(\mu_1)$, and we shall call its graph *curve* γ_1. Let's consider the second derivative of this function:

$$\frac{\partial^2\mu_2}{\partial\mu_1^2} = -\frac{1}{J_{12}(1-\alpha)} \cdot \frac{2\mu_1}{(1-\mu_1^2)^2}.$$

We see immediately that this second derivative is strictly increasing, and that it changes sign exactly at zero. This implies that γ_1 can be divided into three monotonic pieces, each having strictly positive *third* derivative as a function of μ_1. The same thing holds for the second equation, which defines a function $\mu_1(\mu_2)$, and a corresponding *curve* γ_2. An analytical argument easily establishes that there exist at most 9 crossing points of γ_1 and γ_2 (for convenience we shall label the three monotonic pieces of γ_1 as I, II and III, from left to right): since γ_2, too, has a strictly positive third derivative, it follows that it intersects each of the three monotonic pieces of γ_1 at most three times, and this leaves the number of intersections between γ_1 and γ_2 bounded above by 9 (see an example of this in Figure 1).

By definition of the mean field equations, the stationary points of the pressure correspond to crossing points of γ_1 and γ_2. Furthermore, common sense tells us that not all of these stationary points can be local maxima. This is indeed true, and it is proved by the following:

Proposition 6.4 *The function $p(\mu_1, \mu_2)$ admits at most 5 maxima.*

To prove 6.4 we shall need the following:

Lemma 6.5 *Say P_1 and P_2 are two crossing points linked by a monotonic piece of one of the two functions considered above. Then at most one of them is a local maximum of the pressure $p(\mu_1, \mu_2)$.*

Proof of Lemma 6.5: The proof consists of a simple observation about the meaning of our curves. The mean field equations as stationarity conditions for the pressure, so each of γ_1 and γ_2 are made of points where one of the two components of the gradient of $p(\mu_1, \mu_2)$ vanishes. Without loss of generality assume that P_1 is a maximum, and that the component that vanishes on the piece of curve that links P_1 to P_2 is $\dfrac{\partial p}{\partial \mu_1}$.

Since P_1 is a local maximum, $p(\mu_1, \mu_2)$ locally increases on the piece of curve γ. On the other hand, the directional derivative of $p(\mu_1, \mu_2)$ along γ is given by

$$\hat{\mathbf{t}} \cdot \nabla p$$

where $\hat{\mathbf{t}}$ is the unit tangent to γ. Now we just need to notice that by the assumptions, for any point in γ, $\hat{\mathbf{t}}$ lies in the same quadrant, while ∇p is vertical with a definite verse. This implies that the scalar product giving directional derivative is strictly non-negative over all γ, which prevents P_2 form being a maximum. $\square$

Proof of Proposition 6.4: The proof considers two separate cases:

a) All crossing points can be joined in a chain by using monotonic pieces of curve such as the one defined in the lemma;

b) At least one crossing point is linked to the others only by non-monotonic pieces of curve.

In case a), all stationary can be joined in chain in which no two local maxima can be nearest neighbours, by the lemma. Since there are at most 9 stationary points, there can be at most 5 local maxima.

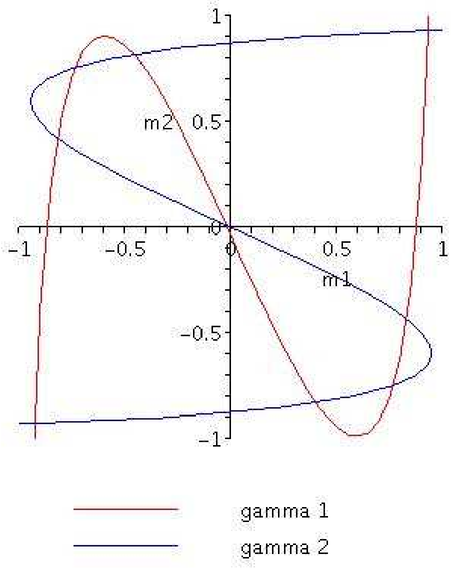

Figure 1: The crossing points correspond to solutions of the mean field equations

For case b) assume that there is a point, call it P, which is not linked to any other point by a monotonic piece of curve. Without loss of generality, say that P lies on I (which, we recall, is defined as the leftmost monotonic piece of γ_1). By assumption, I cannot contain other crossing points apart from P, for otherwise P would be monotonically linked to at least one of them, contradicting the assumption. On the other hand, each of II and III contain at most 3 stationary points, and, by Lemma 6.5, at most 2 of these are maxima. So we have at most 2 maxima on each of II and III, and and at most 1 maximum on I, which leaves the total bounded above by 5. The cases in which P lies on II, or on III, are proved analogously, giving the result.

□

7 Comments

The considered model generalises models which arise naturally as approximations of various problems in theoretical physics. Furthermore, the upcoming study of social phenomena by statistical mechanics methods provides another importance source of interest for a model describing long-range interactions between two homogeneous populations.

In [12] we show that it is possible to give a cultural-contact interpretation to the model presented here, and thanks to the mathematical results just derived, to provide non-trivial information about its regimes.

It is not known at present which is the mathematical structure underlying the social interaction network. However, it is well-accepted that the topology should be of the "small world" type predicted in [15], at least to some degree. For that reason the natural refinement to be considered for the proposed model will introduce random interaction and quenched measures in the spirit of spin glasses. We plan to return on those topics in future works.

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