Nonlinear dynamics near equilibrium points for a Solar Sail

Ariadna Farrés
ari@maia.ub.es

Àngel Jorba
angel@maia.ub.es

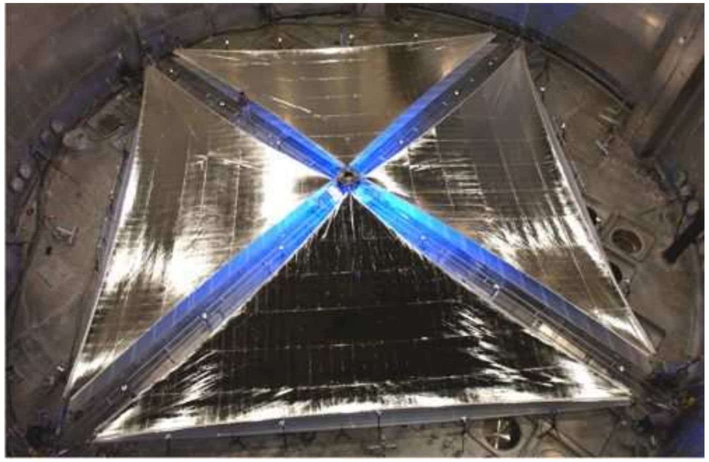
Universitat de Barcelona
Departament de Matemàtica Aplicada i Anàlisi

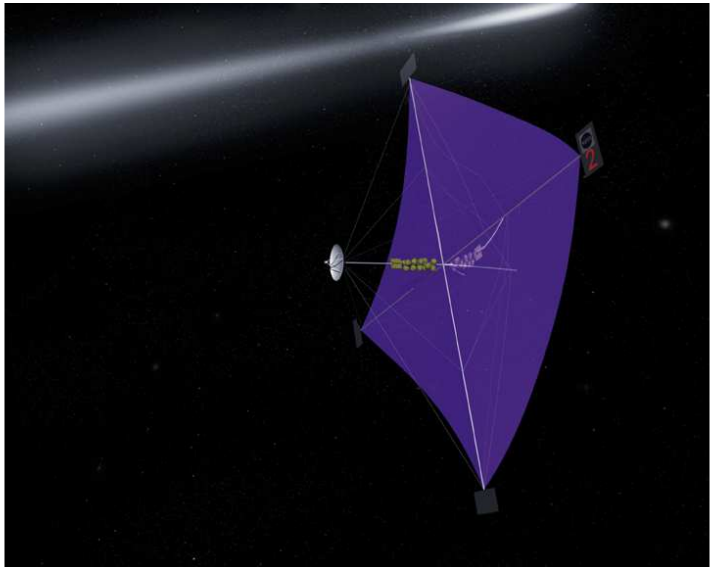
Contents

- Introduction to Solar Sails.
- Families of Equilibria.
- Periodic Motion around Equilibria.
- Reduction to the Centre Manifold.

What is a Solar Sail ?

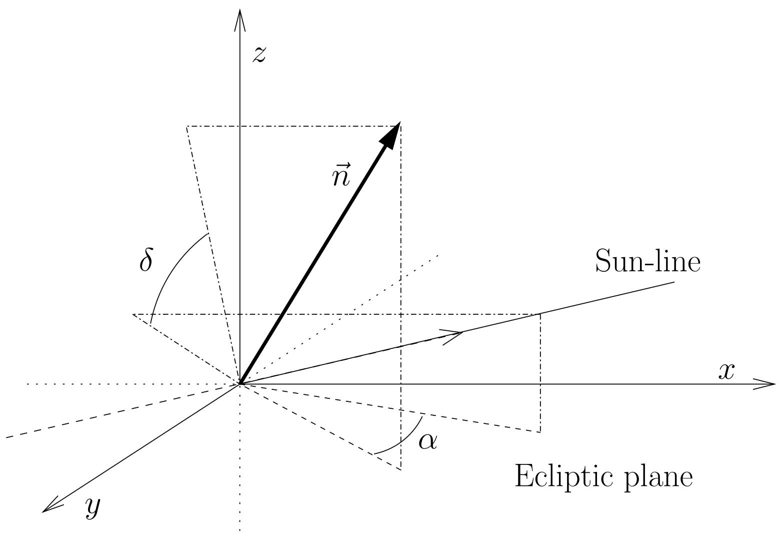
- It is a proposed form of spacecraft propulsion that uses large membrane mirrors.
- The impact of the photons emitted by the Sun onto the surface of the sail and its further reflection produce momentum.
- Solar Sails open a new wide range of possible mission that are not accessible for a traditional spacecraft.





Some Definitions

- The effectiveness of the sail is given by the dimensionless parameter β, the **lightness number**.
- The **sail orientation** is given by the normal vector to the surface of the sail ($\vec{n}$), parametrised by two angles, α and δ, where $\alpha \in [-\pi/2, \pi/2]$ and $\delta \in [-\pi/2, \pi/2]$.

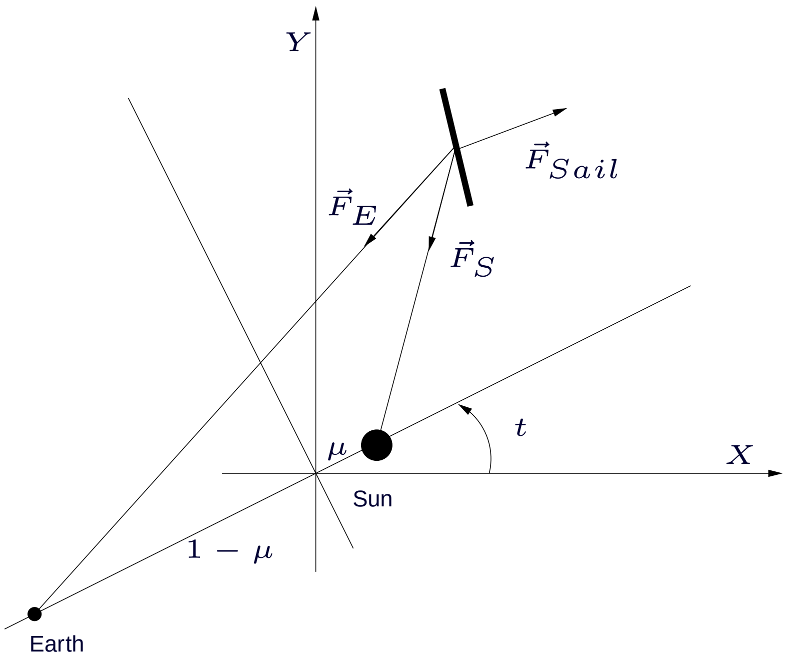


Equations of Motion (RTBPS)

- We consider that the sail is perfectly reflecting. So the force due to the sail is in the normal direction to the surface of the sail $\vec{n}$.

$$\vec{F}_{sail} = \beta \frac{m_s}{r_{ps}^2} \langle \vec{r}_s, \vec{n} \rangle^2 \vec{n}.$$

- We consider the gravitational attraction of Sun and Earth: we use the RTBP adding the radiation pressure to model the motion of the sail.



Equations of Motion (RTBPS)

The equations of motion are:

$$
\begin{aligned}
\ddot{x} &= 2\dot{y} + x - (1-\mu)\frac{x-\mu}{r_{ps}^3} - \mu\frac{x+1-\mu}{r_{pe}^3} + \beta\frac{1-\mu}{r_{ps}^2}\langle\vec{r}_s,\vec{n}\rangle^2 n_x, \\
\ddot{y} &= -2\dot{x} + y - \left(\frac{1-\mu}{r_{ps}^3} + \frac{\mu}{r_{pe}^3}\right) y + \beta\frac{1-\mu}{r_{ps}^2}\langle\vec{r}_s,\vec{n}\rangle^2 n_y, \\
\ddot{z} &= -\left(\frac{1-\mu}{r_{ps}^3} + \frac{\mu}{r_{pe}^3}\right) z + \beta\frac{1-\mu}{r_{ps}^2}\langle\vec{r}_s,\vec{n}\rangle^2 n_z,
\end{aligned}
$$

where,

$$
\begin{aligned}
n_x &= \cos(\phi(x,y)+\alpha)\cos(\psi(x,y,z)+\delta), \\
n_y &= \sin(\phi(x,y,z)+\alpha)\cos(\psi(x,y,z)+\delta), \\
n_z &= \sin(\psi(x,y,z)+\delta),
\end{aligned}
$$

with $\phi(x,y)$ and $\psi(x,y,z)$ defining the Sun - Sail direction in spherical coordinates ($\vec{r}_s = \vec{r}_{ps}/r_{ps}$).

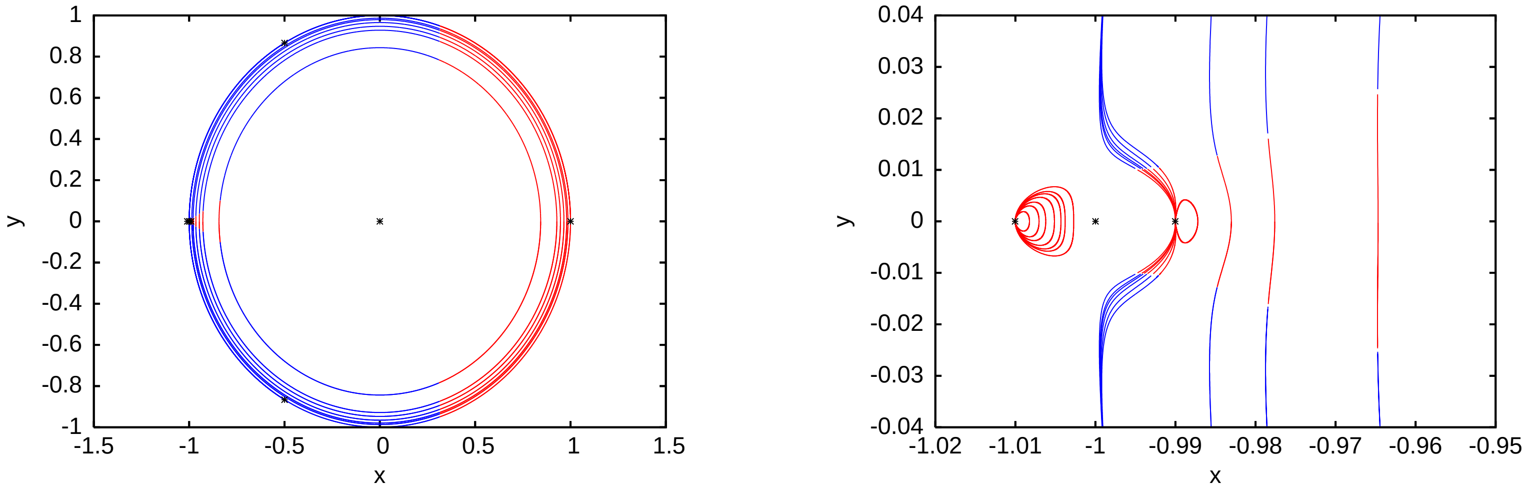
Equilibrium Points

- The RTBP has 5 equilibrium points (L_i). For small β, these 5 points are replaced by 5 continuous families of equilibria, parametrised by α and δ.

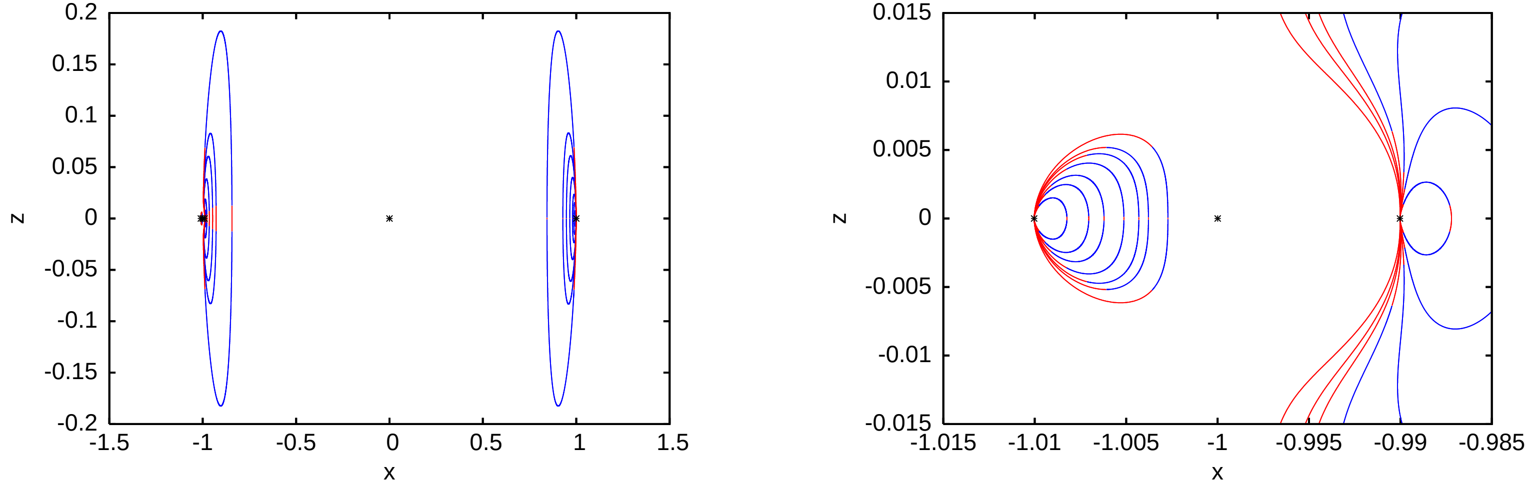
- For a fixed and small β, these families have two disconnected surfaces, S_1 and S_2. It can be seen that S_1 is diffeomorphic to a sphere and S_2 is diffeomorphic to a torus around the Sun.

- All these families can be computed numerically by means of a continuation method.

Equilibrium Points



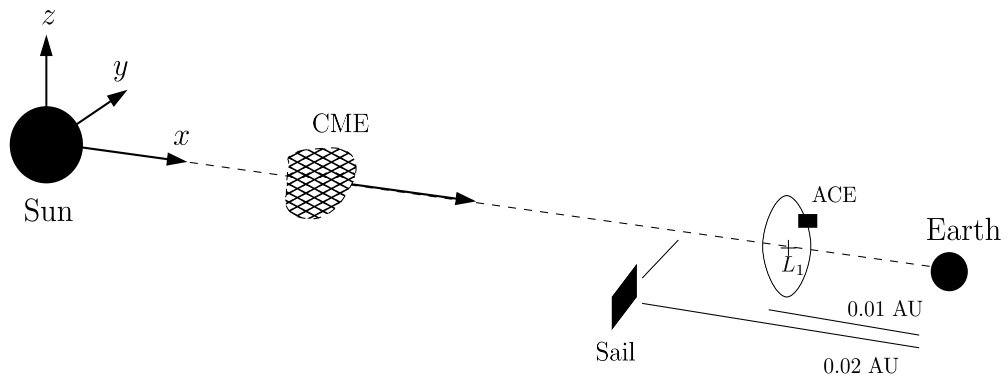
Equilibrium points in the $\{x, y\}$*- plane*



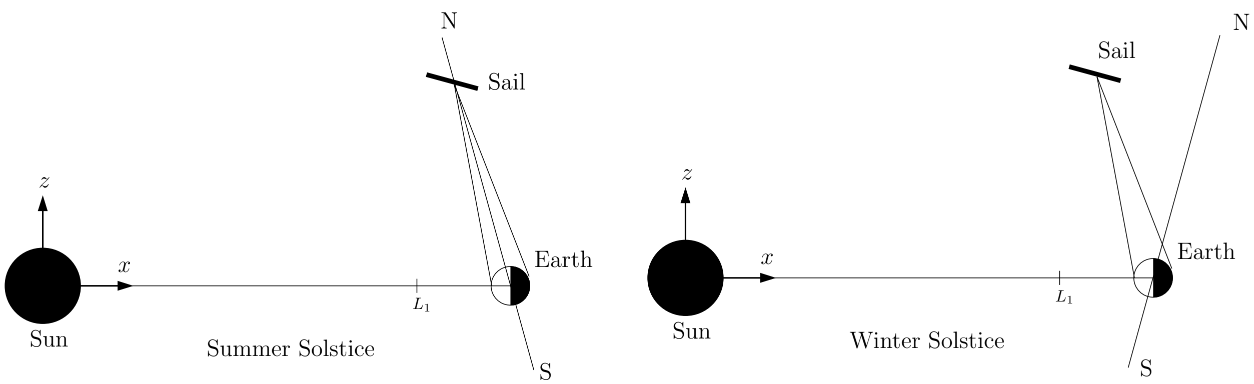
Equilibrium points in the $\{x, z\}$*- plane*

Some Interesting Missions

- Observations of the Sun provide information of the geomagnetic storms, as in the Geostorm Warning Mission.



- Observations of the Earth's poles, as in the Polar Observer.



- C. McInnes, “ Solar Sail: Technology, Dynamics and Mission Applications.”, *Springer-Praxis*, 1999.

- D. Lawrence and S. Piggott, “ Solar Sailing trajectory control for Sub-L1 stationkeeping”, *AIAA 2005-6173*.

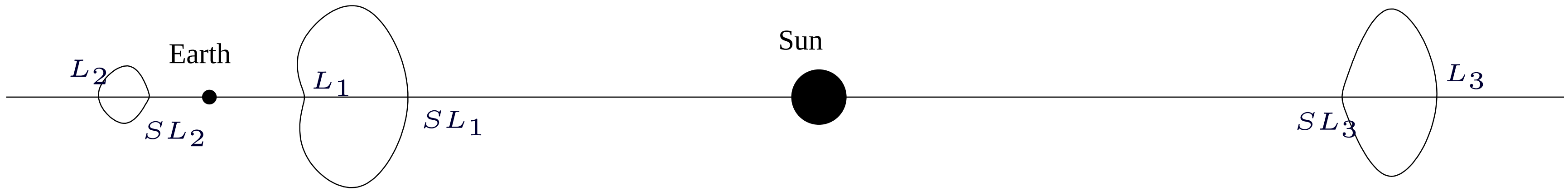
- J. Bookless and C. McInnes, “Control of Lagrange point orbits using Solar Sail propulsion.”, *56th Astronautical Conference 2005.*

- A. Farrés and À. Jorba, “Solar Sail surfing along familes of equilibrium points.”, *Acta Astronautica* Volume 63, Issues 1-4, July-August 2008, Pages 249-257.

- A. Farrés and À. Jorba, “A dynamical System Approach for the Station Keeping of a Solar Sail.”, *Journal of Astronautical Science*. (to apear in 2008)

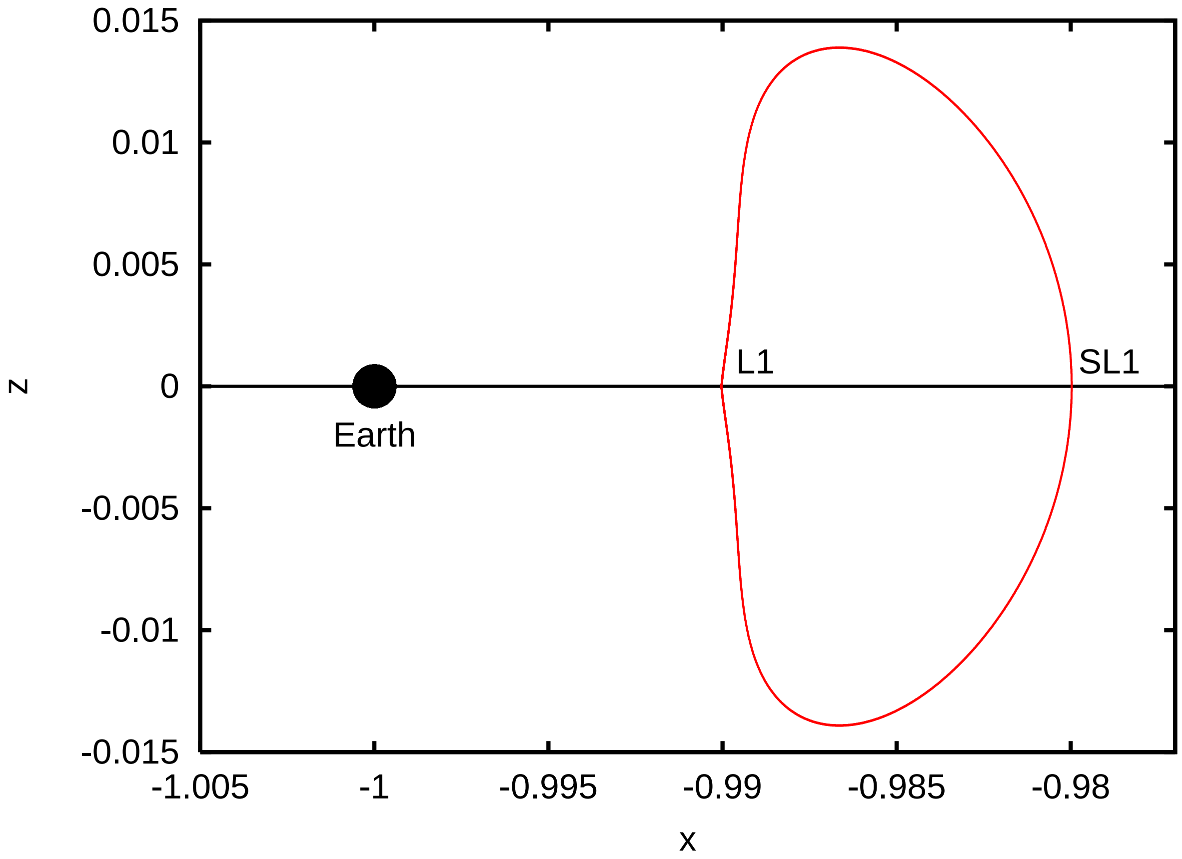
From now on ...

We fix $\alpha = 0$ and $\beta = 0.051689$.



- Here, we have 3 families of equilibrium points on the $\{x, z\}$ - plane parametrised by the angle δ.
- The linear behaviour for all these equilibrium points is of the type centre$\times$centre$\times$saddle.
- We want to study the families of periodic orbits that appear around these equilibrium points for a fixed δ.
- For practical reasons we focus on the region around SL_1.

Family of equilibrium points around SL_1 for $\alpha = 0$ and $\beta = 0.051689$



Motion around the equilibrium points

- As we have said, the linear behaviour around the fixed point is centre$\times$centre$\times$saddle.
- So up to first order the solutions around the fixed point are:

$$
\begin{aligned}
\phi(t) &= A_0[\cos(\omega_1 t + \psi_1)\vec{v}_1 + \sin(\omega_1 t + \psi_1)\vec{u}_1] \\
&+ B_0[\cos(\omega_2 t + \psi_2)\vec{v}_2 + \sin(\omega_2 t + \psi_2)\vec{u}_2] \\
&+ C_0 e^{\lambda t}\vec{v}_\lambda + D_0 e^{-\lambda t}\vec{v}_{-\lambda}
\end{aligned}
$$

Where,

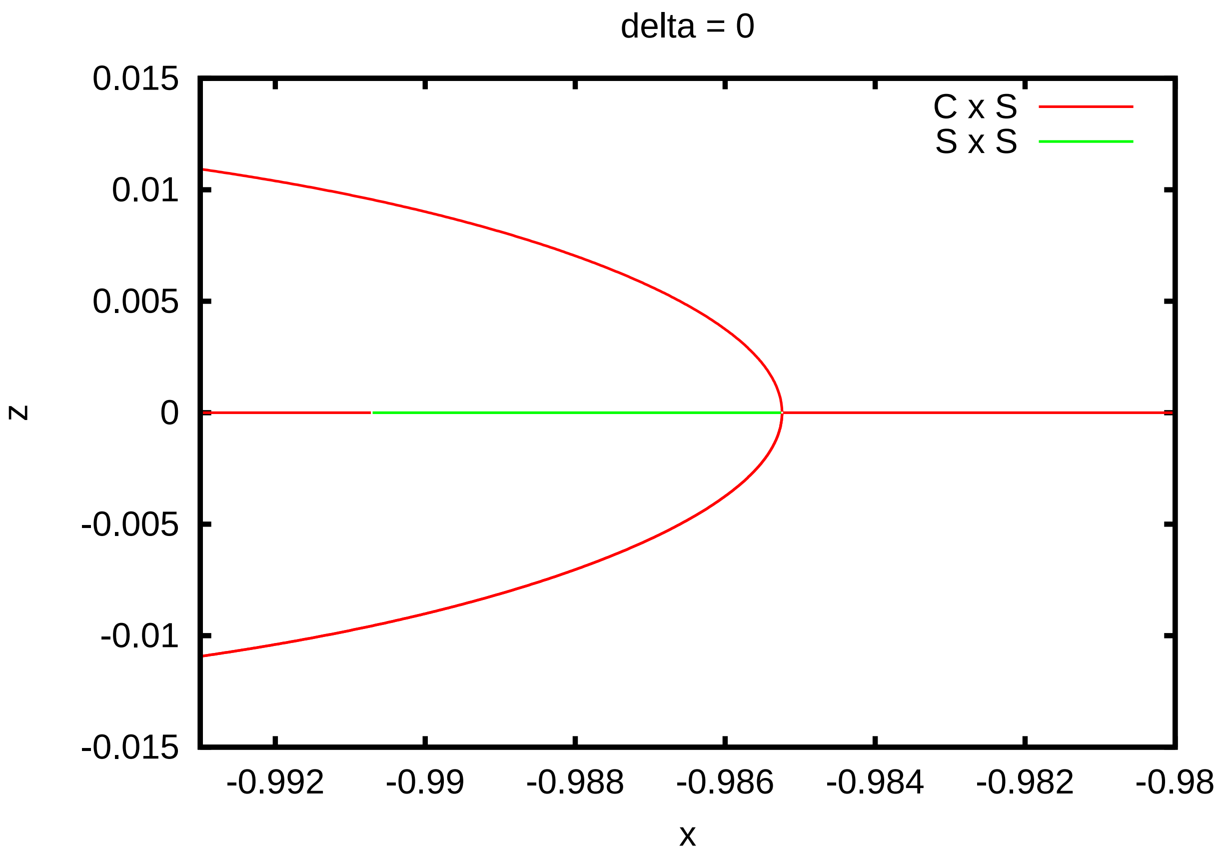
- ○ $\pm \mathrm{i}\omega_1$ eigenvalues with $\vec{v}_1 \pm \mathrm{i}\vec{u}_1$ as eigenvectors.
- ○ $\pm \mathrm{i}\omega_2$ eigenvalues with $\vec{v}_2 \pm \mathrm{i}\vec{u}_2$ as eigenvectors.
- ○ $\pm\lambda$ eigenvalues with $\vec{v}_\lambda$, $\vec{v}_{-\lambda}$ as eigenvectors.

Motion around the equilibrium points

- We take the linear approximation to compute an initial periodic orbit for each family. We then use a continuation method to compute the rest of the family.
  - Planar family: $A_0 = \gamma$ and $B_0 = D_0 = E_0 = 0$.
  - Vertical family : $B_0 = \gamma$ and $A_0 = D_0 = E_0 = 0$.
- We use a parallel shooting method to compute the periodic orbits.
- We have done this for different values of δ.

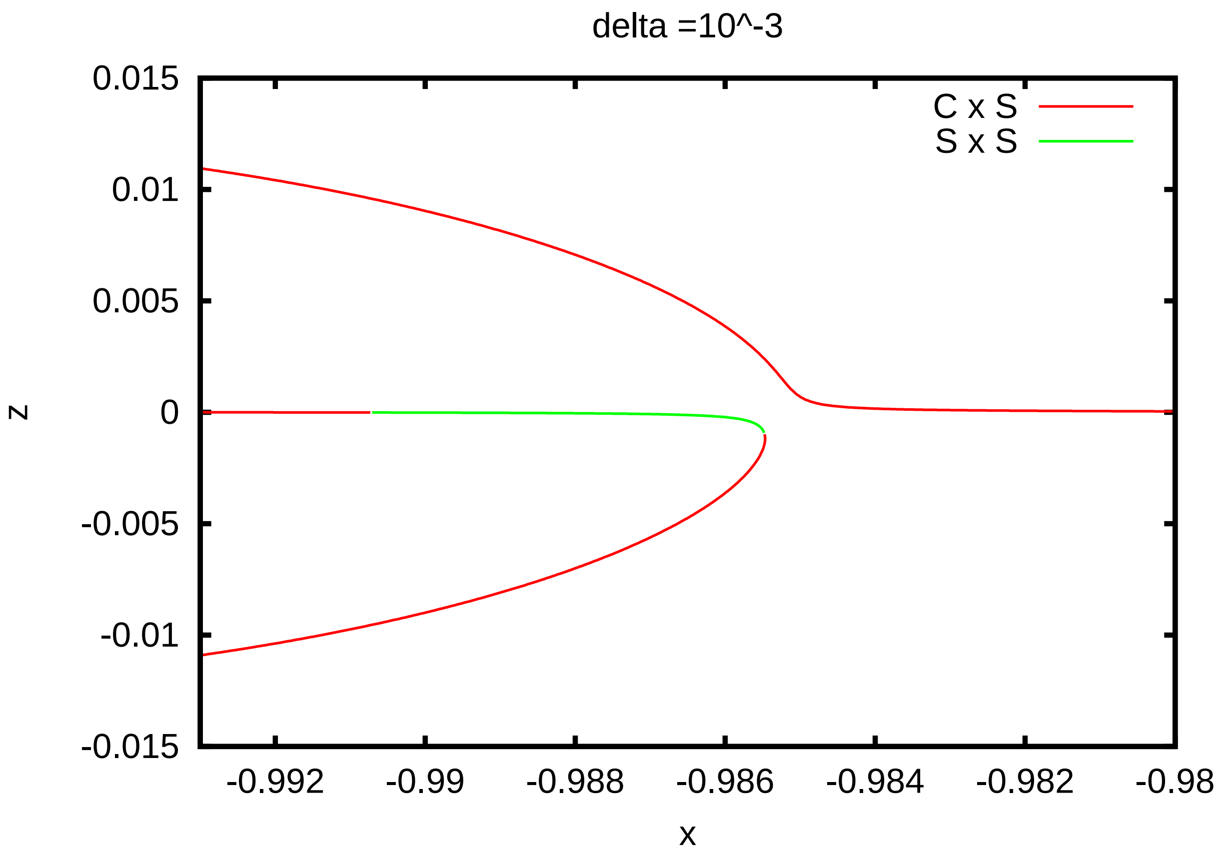
Planar Family of Periodic Orbits

- We have computed the planar family for $\delta = 0$. (i.e. sail perpendicular to Sun).

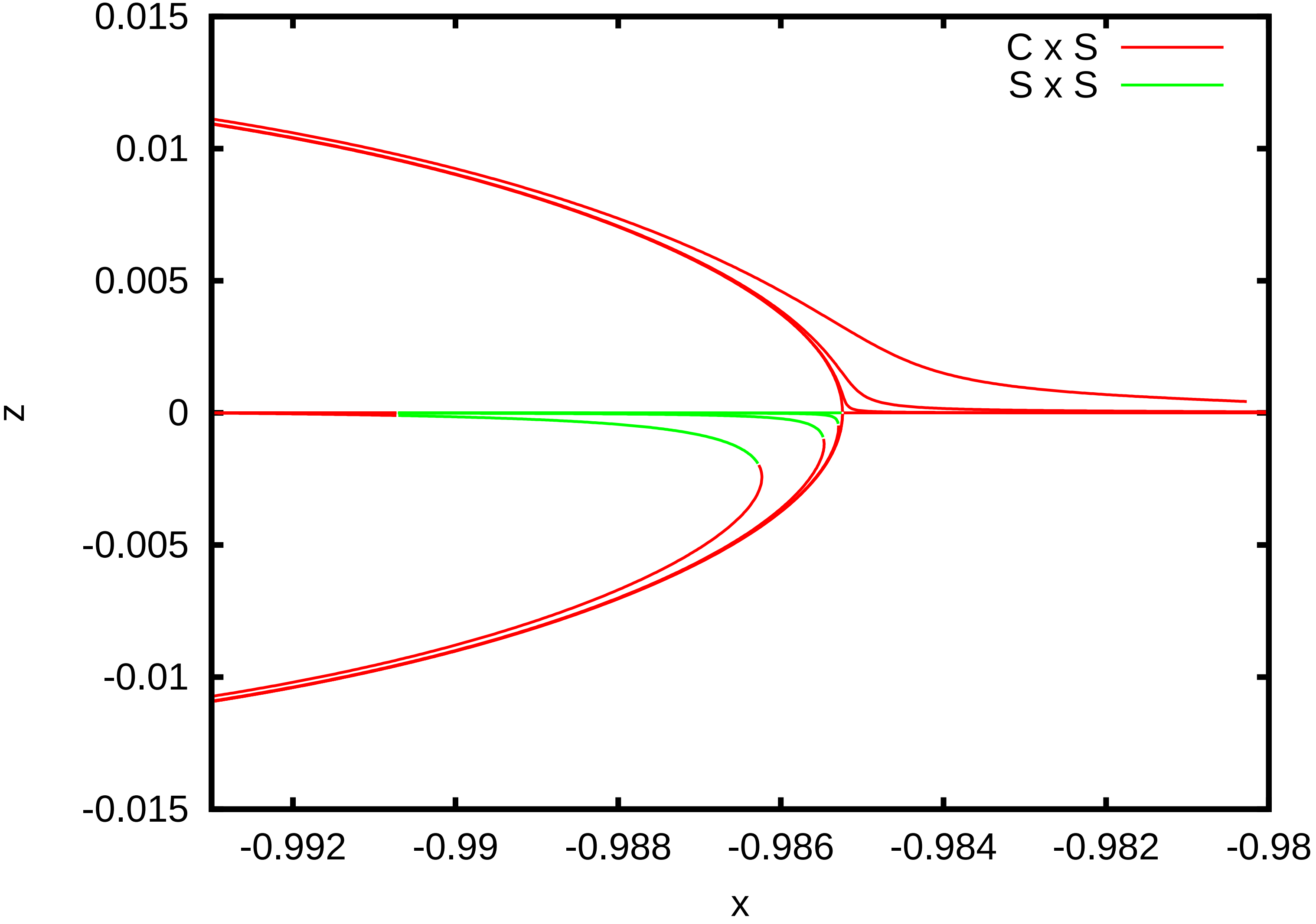


Continuation of the Planar Family

- We have computed the planar family for $\delta = 0.001$.

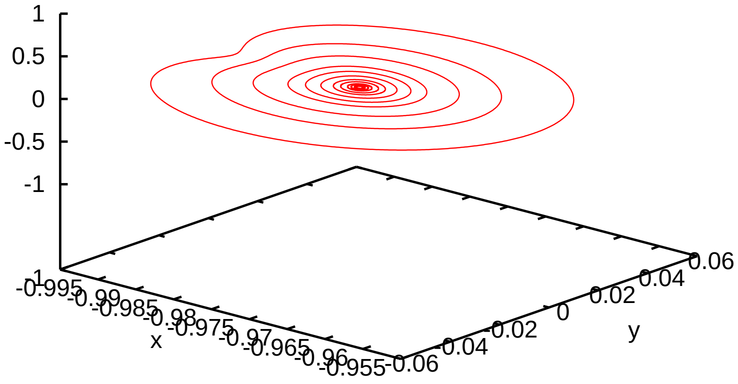


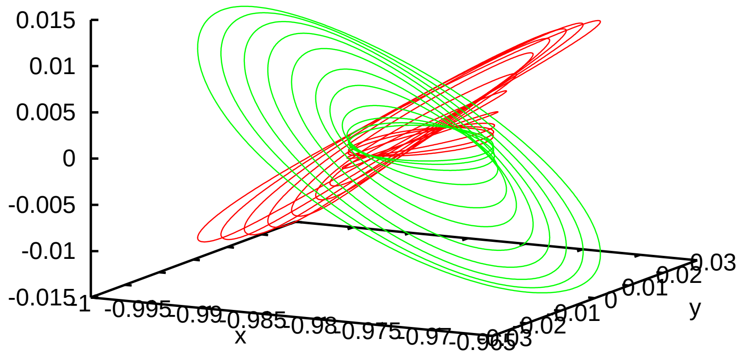
Continuation of the Planar Family



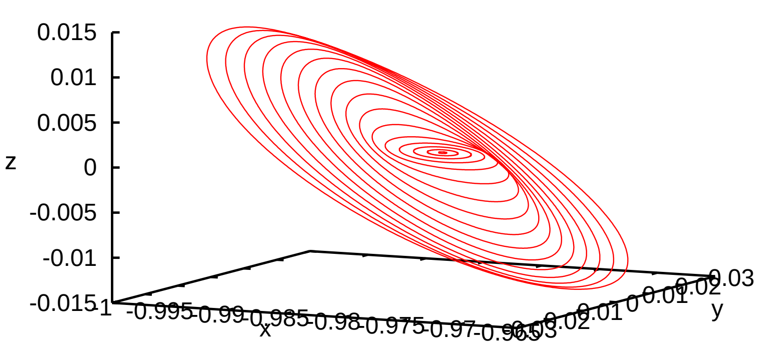
Planar Family of Periodic Orbits

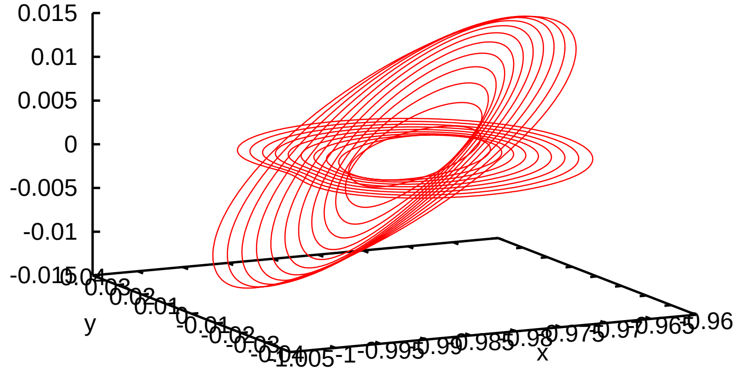
Periodic Orbits for $\delta = 0$.





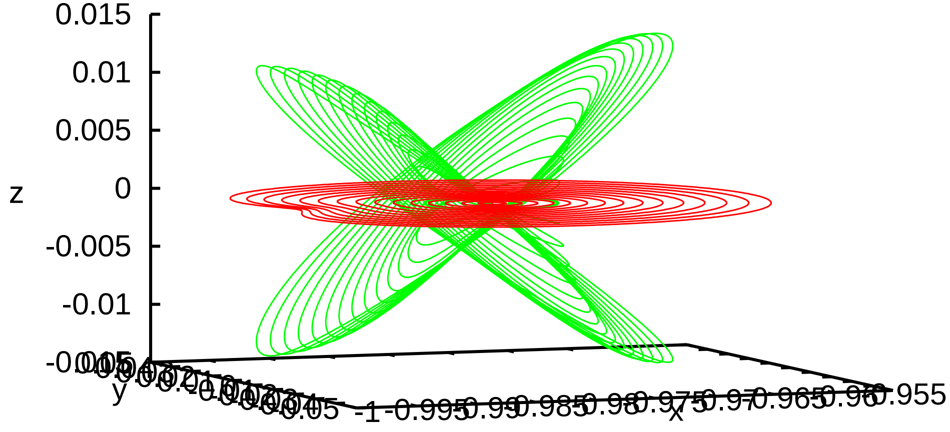
Periodic Orbits for $\delta = 0.01$.



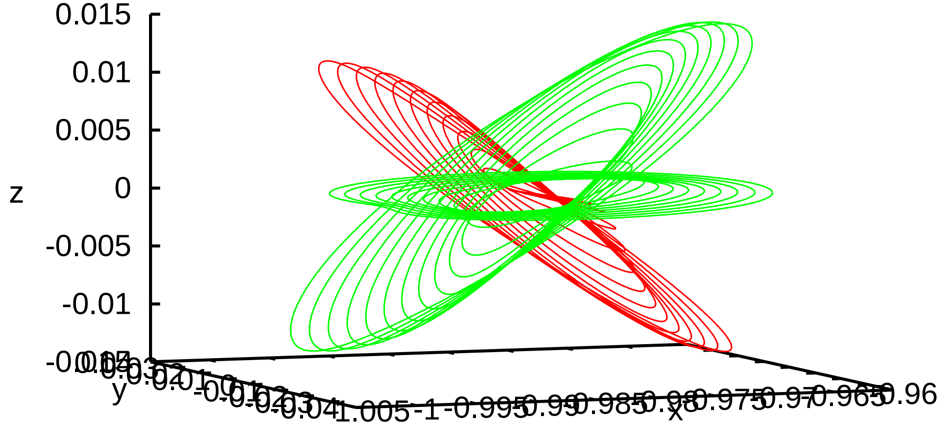


Planar Family of Periodic Orbits

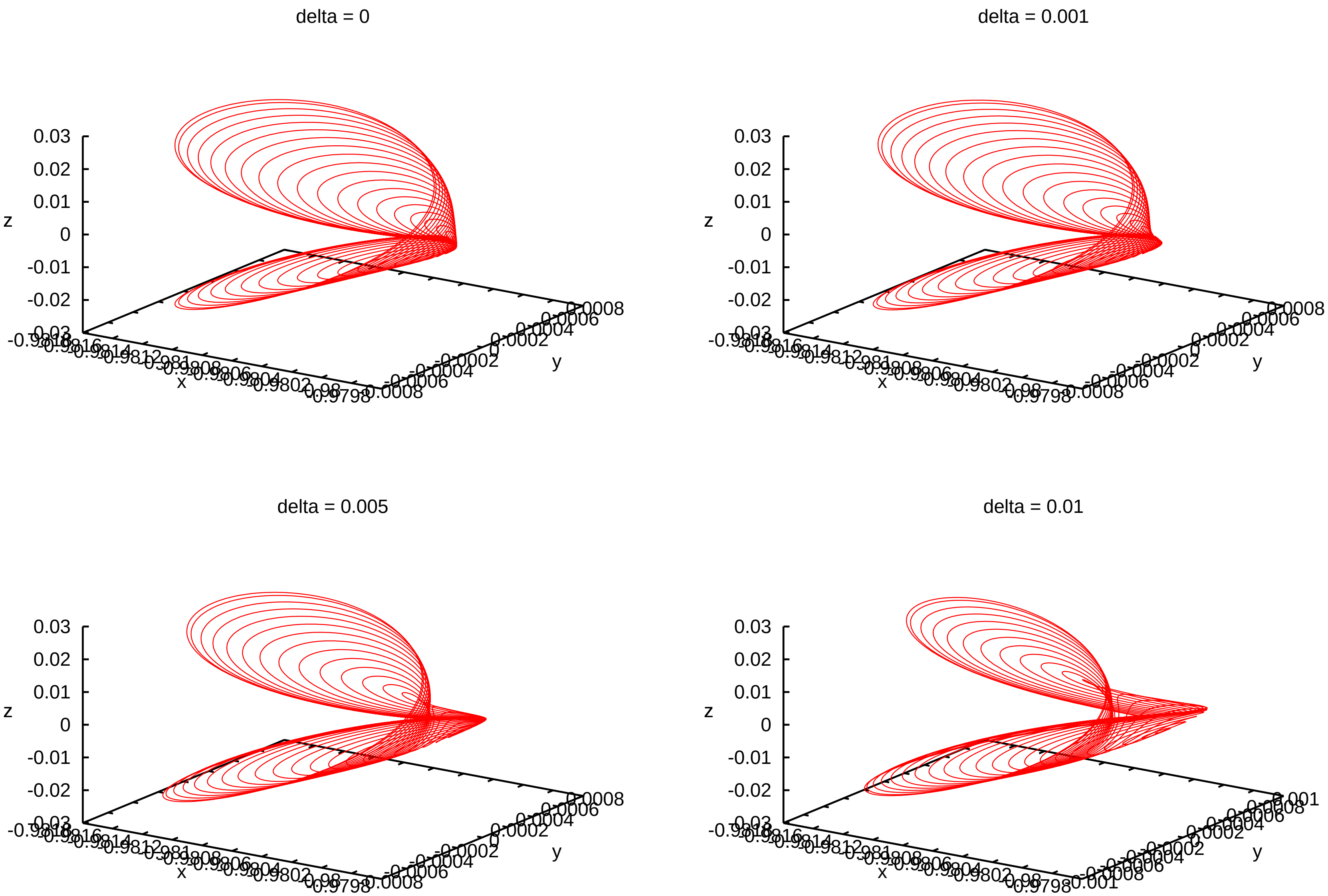
Familly for $\delta = 0$



Familly for $\delta = 0.01$



Vertical Family of Periodic Orbits



Reduction to the Centre Manifold

Using an appropriate linear transformation, the equations around the fixed point can be written as,

$$\begin{aligned} \dot{x} &= Ax + f(x,y), \quad x \in \mathbb{R}^4, \\ \dot{y} &= By + g(x,y), \quad y \in \mathbb{R}^2, \end{aligned}$$

where A is an elliptic matrix and B an hyperbolic one, and $f(0,0) = g(0,0) = 0$ and $Df(0,0) = Dg(0,0) = 0$.

- We want to obtain $y = v(x)$, with $v(0) = 0$, $Dv(0) = 0$, the local expression of the centre manifold.
- The flow restricted to the invariant manifold is

$$\dot{x} = Ax + f(x, v(x)).$$

Approximating the Centre Manifold

To find $y = v(x)$ we substitute this expression on the differential equations. So $v(x)$ must satisfy,

$$Dv(x)Ax - Bv(x) = g(x, v(x)) - Dv(x)f(x, v(x)). \qquad (1)$$

We take,

$$v(x) = \left(\sum_{|k|\geq 2} v_{1,k} x^k, \sum_{|k|\geq 2} v_{2,k} x^k \right), \qquad k \in (\mathbb{N} \cup \{0\})^4,$$

its expansion as power series.

The left hand side is a linear operator w.r.t $v(x)$ and the right hand side is non-linear.

Approximating the Centre Manifold

The left hand side of equation (1),

$$L(x) = Dv(x)Ax - Bv(x),$$

diagonalizes if A and B are diagonal.

In particular, if $A = diag(\mathrm{i}\omega_1, -\mathrm{i}\omega_1, \mathrm{i}\omega_2, -\mathrm{i}\omega_2)$ and $B = diag(\lambda, -\lambda)$ then,

$$L(x) = \begin{pmatrix} \sum_{|k|\geq 2} (\mathrm{i}\omega_1 k_1 - \mathrm{i}\omega_1 k_2 + \mathrm{i}\omega_2 k_3 - \mathrm{i}\omega_2 k_4 - \lambda) v_{1,k} x^k \\ \sum_{|k|\geq 2} (\mathrm{i}\omega_1 k_1 - \mathrm{i}\omega_1 k_2 + \mathrm{i}\omega_2 k_3 - \mathrm{i}\omega_2 k_4 + \lambda) v_{2,k} x^k \end{pmatrix}.$$

Approximating the Centre Manifold

The right hand side of equation (1),

$$h(x) = g(x, v(x)) - Dv(x)f(x, v(x)),$$

can be expressed as,

$$h(x) = \left(\sum_{|k| \geq 2} h_{1,k}x^k \, , \, \sum_{|k| \geq 2} h_{2,k}x^k \right)^T ,$$

where $h_{i,k}$ depend on $v_{i,j}$ in a known way ($i = 1, 2$).

- It can be seen that for a fixed degree $|k| = n$, the $h_{i,k}$ depend only on the $v_{i,j}$ such that $|j| < n$.

Approximating the Centre Manifold

Now we can solve equation (1) in an iterative way, equalising the left and the right hand side degree by degree. We have to solve a diagonal system at each degree.

Notice:

- It is important to have a fast way to find the $h_{i,k}$ to get up to high degrees.
- We do not recommend to expand $f(x,y)$ y $g(x,y)$, and then compose with $y = v(x)$. One should find other alternative ways, faster in terms of computational time.
- The matrixes A and B don't have to be diagonal, but then one must solve a larger linear system at each degree.

On the efficient computation of $h_{i,j}$

We recall that the equations of motion for $\alpha = 0$ are,

$$\begin{aligned}
\ddot{x} &= 2\dot{y} + x - \kappa_s \frac{x-\mu}{r_{ps}^3} - \kappa_e \frac{x+1-\mu}{r_{pe}^3} + \kappa_{sail} \frac{z(x-\mu)}{r_{ps}^3 r_2}, \\
\ddot{y} &= -2\dot{x} + y - \left(\frac{\kappa_s}{r_{ps}^3} + \frac{\kappa_e}{r_{pe}^3} \right) y + \kappa_{sail} \frac{zy}{r_{ps}^3 r_2}, \\
\ddot{z} &= -\left(\frac{\kappa_s}{r_{ps}^3} + \frac{\kappa_e}{r_{pe}^3} \right) z - \kappa_{sail} \frac{r_2}{r_{ps}^3},
\end{aligned}$$

where $\kappa_s = (1-\mu)(1-\beta\cos^3\alpha)$, $\kappa_e = \mu$, $\kappa_{sail} = \beta(1-\mu)\cos^2\alpha\sin\alpha$.

- To expand the equations of motion we use the Legendre polynomials.

On the efficient computation of $h_{i,j}$

For example:

- $1/r_{ps}$ can be expanded as,

$$\sum_{n \geq 0} c_n T_n(x, y, z),$$

where the $T_n(x, y, z)$ are homogeneous polynomials of degree n that are computed in a recurrent way.

$$T_n = \frac{2n-1}{n} x T_{n-1} - \frac{n-1}{n}(x^2 + y^2 + z^2) T_{n-2},$$

with $T_0 = 1, \qquad T_1 = x.$

On the efficient computation of $h_{i,j}$

- The functions $f(\bar{x}, \bar{y})$ and $g(\bar{x}, \bar{y})$ can be computed in a reccurrent way as they are found after applying a linear transformation to the expansion of the system.
- Composing these recurrences with $v(\bar{x})$ we can compute the expansions of $f(\bar{x}, v(\bar{x}))$ and $g(\bar{x}, v(\bar{x}))$ in a recurrent way and so for the $h_{i,j}$.

For example:

$$T_0 = 1, \qquad T_1 = x(\bar{x}, v(\bar{x})),$$

$$T_n = \frac{2n-1}{n} x(\bar{x}, v(\bar{x})) T_{n-1} - \frac{n-1}{n} \left(x(\bar{x}, v(\bar{x}))^2 + y(\bar{x}, v(\bar{x}))^2 + z(\bar{x}, v(\bar{x}))^2 \right) T_{n-2}.$$

Validation Test

- Given an initial condition v_0, we denote v_1 and $\tilde{v}_1$ to the integration at time $t = 0.1$ of v_0 on the centre manifold and the complete system respectively.
- The error behaves as: $|\tilde{v}_1 - v_1| = ch^{n+1}$, where h is the distance to the origin of v_0.
- If we consider the centre manifold up to degree 8:

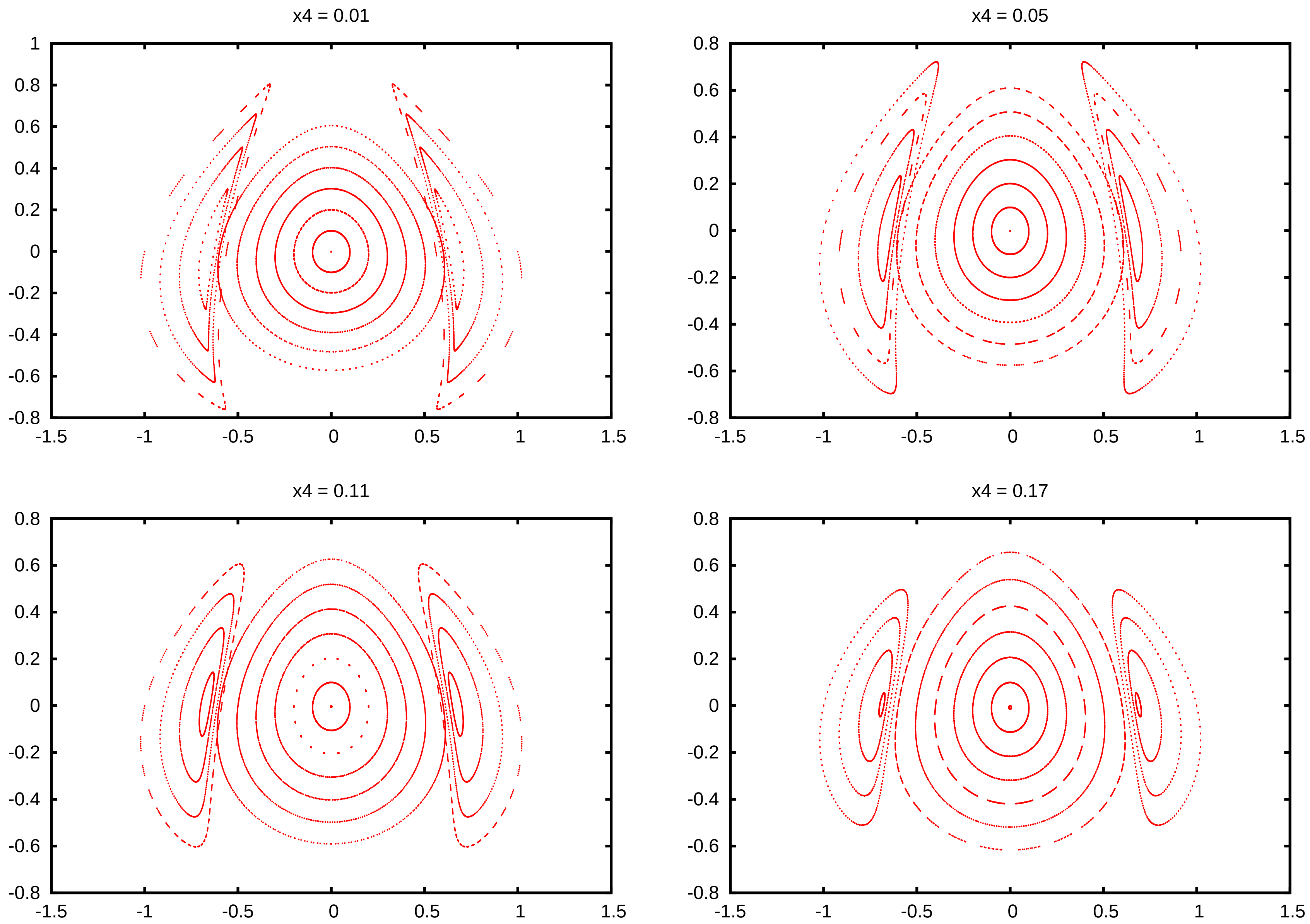
h	$\|\tilde{v}_1 - v_1\|$	$n+1$
0.04	$2.3643547906724647e-15$	
0.08	$1.2618898774811476e-12$	9.059923
0.16	$6.9534006796827247e-10$	9.105988
0.32	$3.9879163406855978e-07$	9.163700

Results for $\delta = 0$

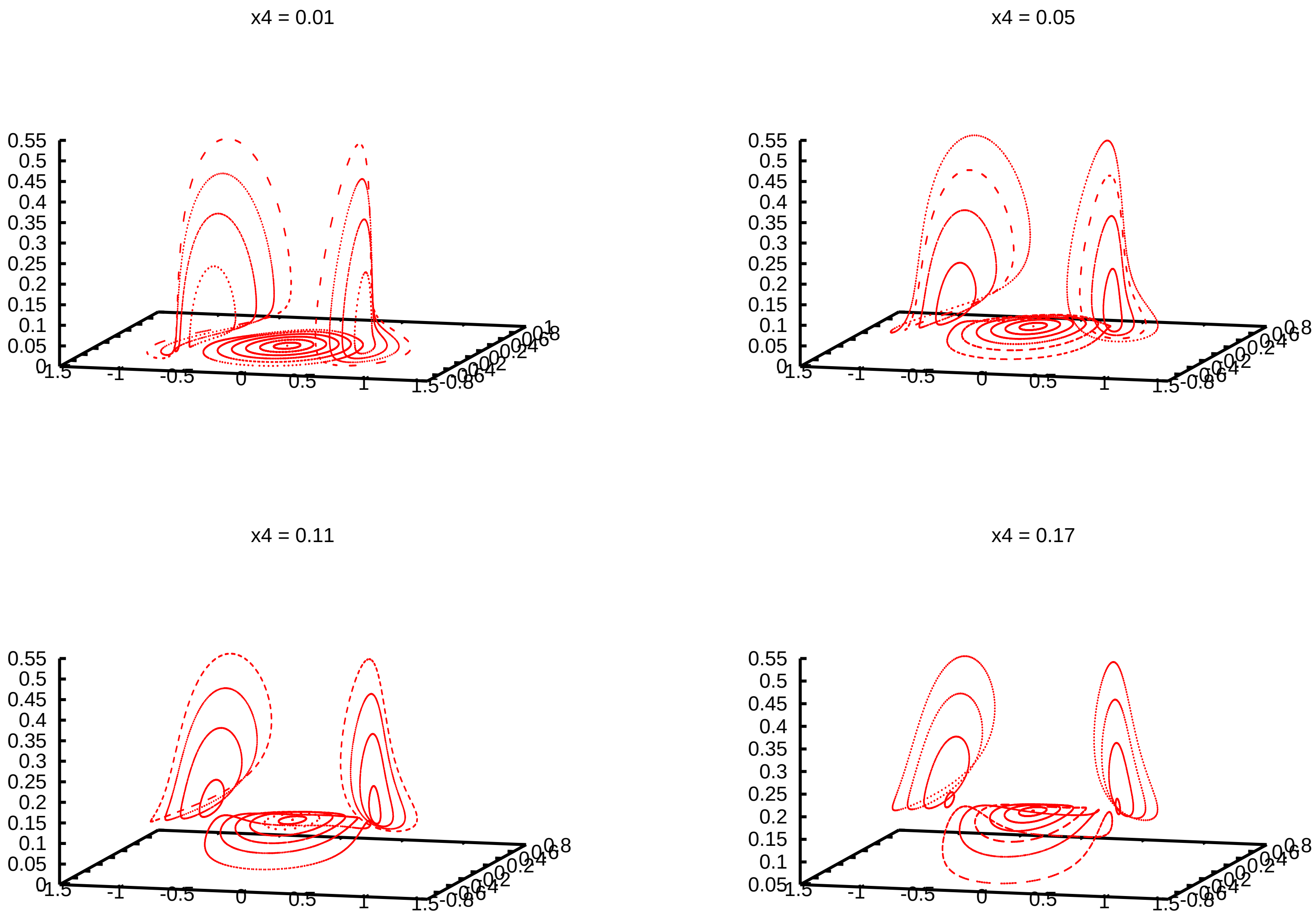
We have computed the reduction of to the centre manifold around Sub-L_1 up to degree 32. (it takes 17min of CPU time)

- After this reduction we are in a four dimensional phase space (x_1, x_2, x_3, x_4).
- We fix a Poincaré section $x_3 = 0$ to reduce the system to a three dimensional phase space.
- We have taken several initial conditions and computed their successive images on the Poincaré section.

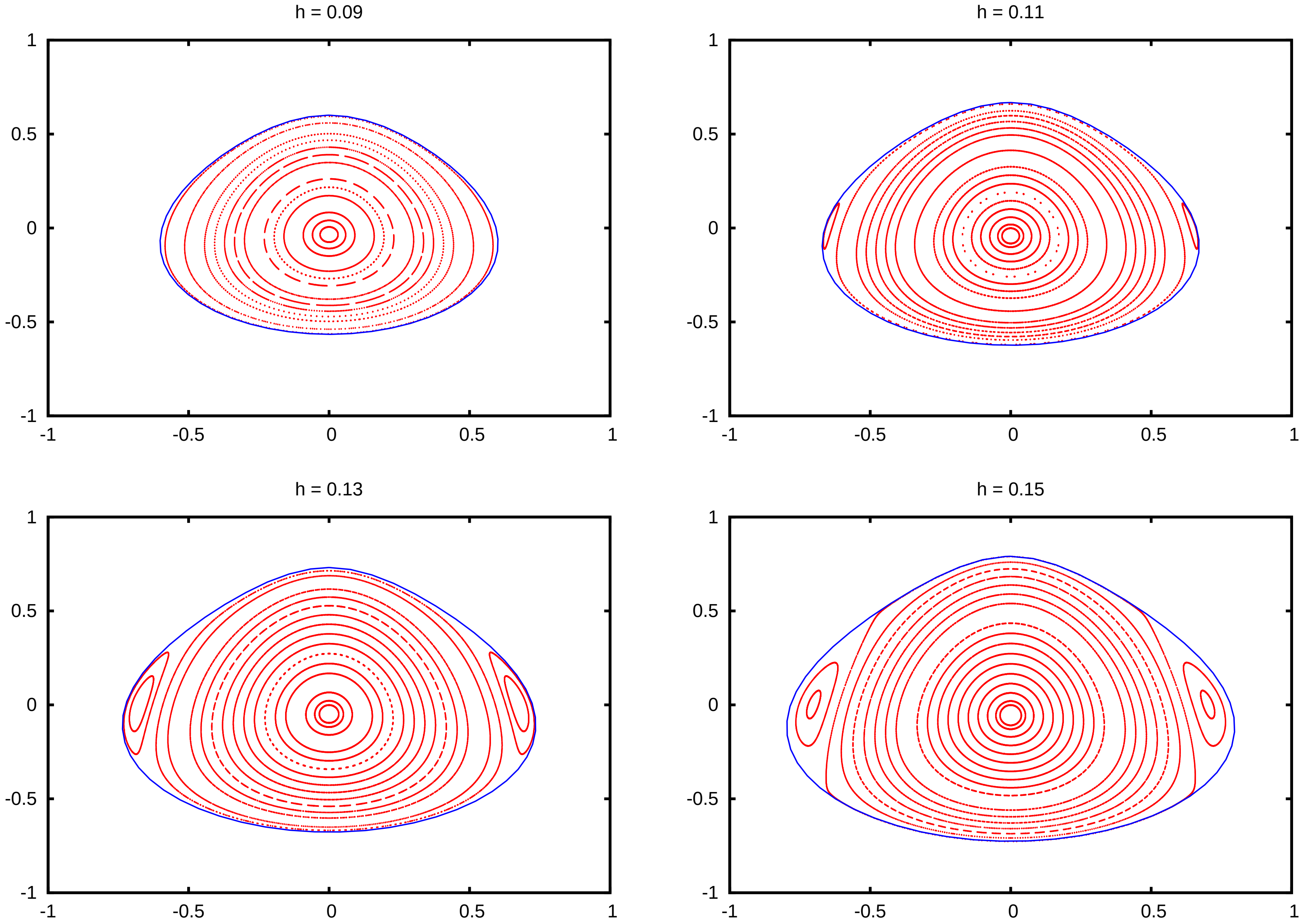
Results for $\delta = 0$



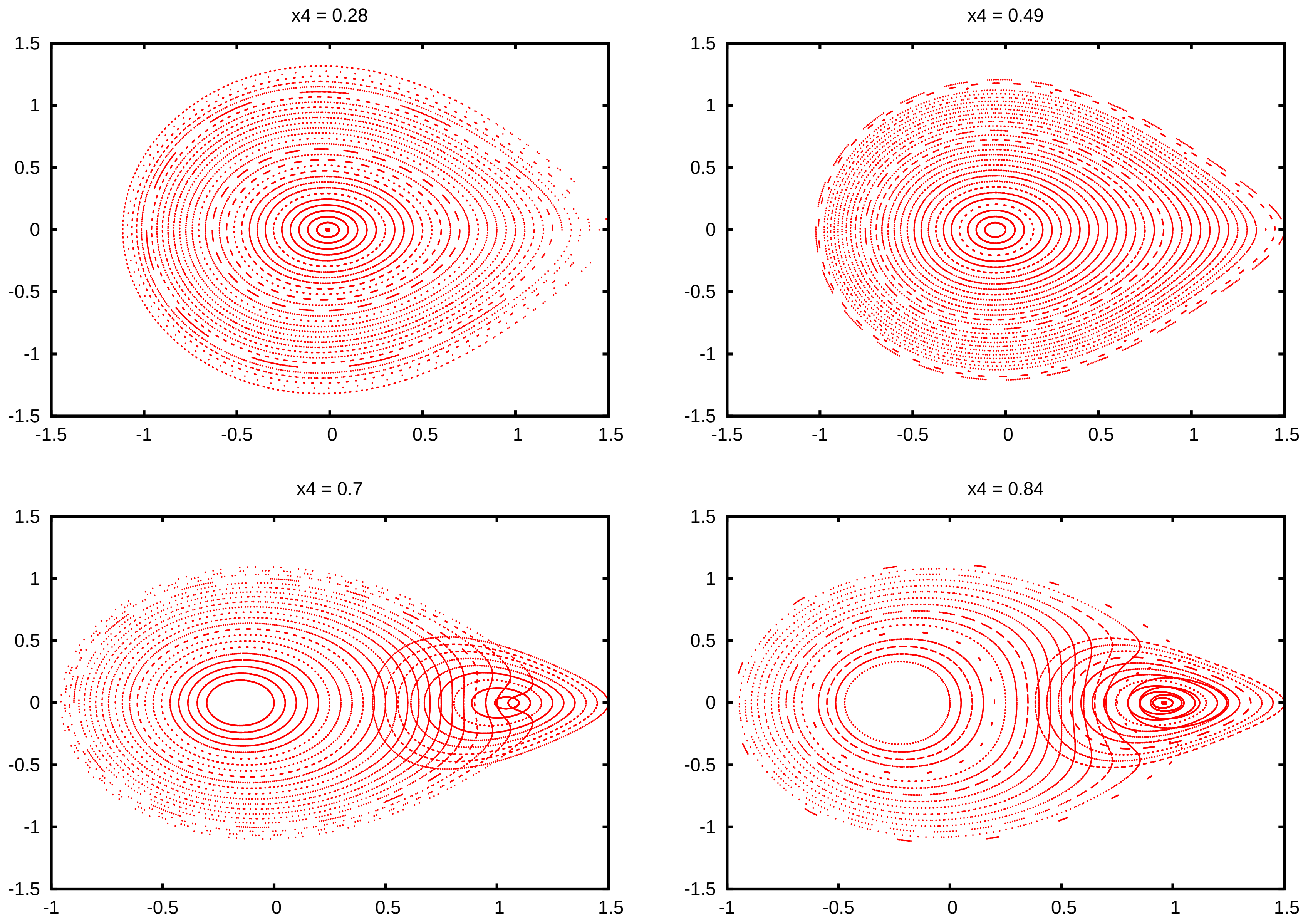
Results for $\delta = 0$



Results for $\delta = 0$ (for a fixed energy level)

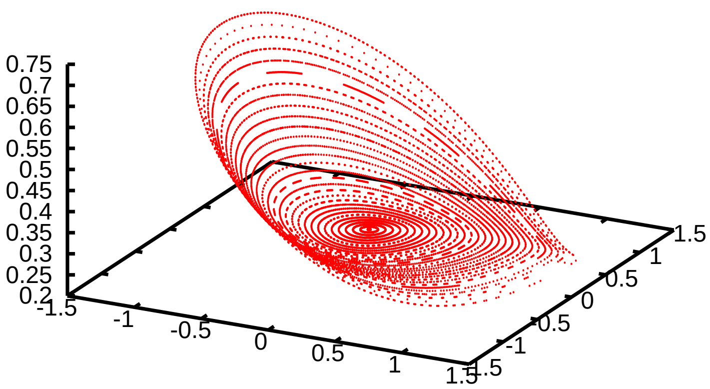


Results for $\delta = 0.05$

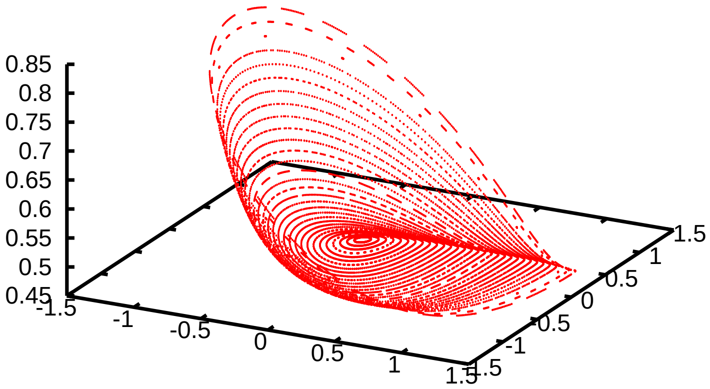


Results for $\delta = 0.05$

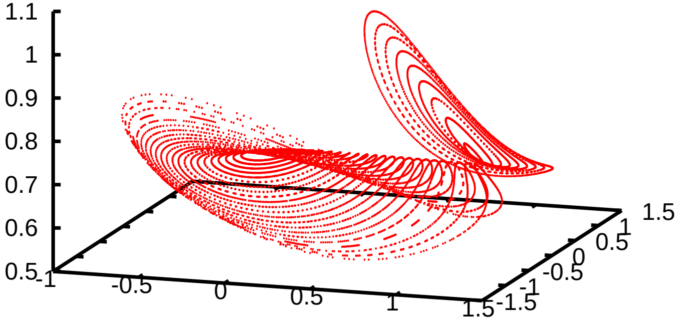
x4 = 0.28



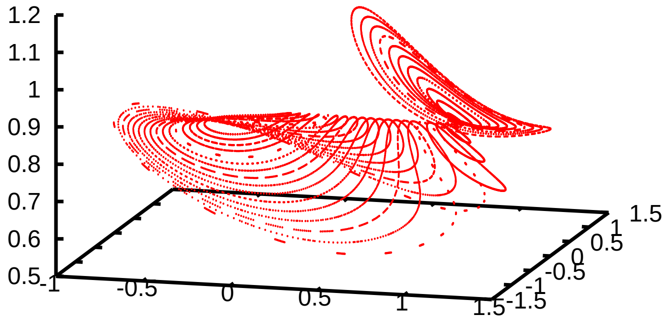
x4 = 0.49



x4 = 0.7



x4 = 0.84



The End

Thank You !!!