Hamiltonian formulation of reduced Vlasov-Maxwell equations

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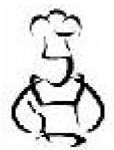
➢ importance of stability vs instability in devices involving a large number of charged particles interacting with fields: plasma physics (tokamaks), free electron lasers

➢ Here: reduced models of such systems (easier simulation, better understanding of the dynamics)

Outline

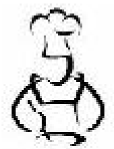
- Hamiltonian description of charges particles and electromagnetic fields

- Reduction of Vlasov-Maxwell equations using Lie transforms



Alain J. BRIZARD (Saint Michael's College, Vermont, USA)

- Reduced Hamiltonian model for the Free Electron Laser



Romain BACHELARD (Synchrotron Soleil, Paris)
Michel VITTOT (CPT, Marseille)

Motion of a charged particle in electromagnetic fields

> in canonical form

$$H\left(\mathbf{p},\mathbf{q},t\right)=\frac{\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\left(\mathbf{q},t\right)\right)^2}{2m}+eV\left(\mathbf{q},t\right)\quad \text{with } \left\{f,g\right\}=\frac{\partial f}{\partial \mathbf{q}}\cdot\frac{\partial g}{\partial \mathbf{p}}-\frac{\partial f}{\partial \mathbf{p}}\cdot\frac{\partial g}{\partial \mathbf{q}}$$

$$\text{equations of motion :}\begin{cases}\dot{\mathbf{p}}=\left\{\mathbf{p},H\right\}=-\dfrac{\partial H}{\partial \mathbf{q}}\\ \dot{\mathbf{q}}=\left\{\mathbf{q},H\right\}=\dfrac{\partial H}{\partial \mathbf{p}}\end{cases}$$

> in non-canonical form: *physical variables*

$$\boxed{\begin{aligned}\mathbf{v}&=\frac{1}{m}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\left(\mathbf{q},t\right)\right)\\ \mathbf{x}&=\mathbf{q}\end{aligned}}$$

$$H\left(\mathbf{v},\mathbf{x},t\right)=\frac{1}{2}m\mathbf{v}^2+eV\left(\mathbf{x},t\right)\quad \text{with } \left\{f,g\right\}=\frac{1}{m}\left(\frac{\partial f}{\partial \mathbf{x}}\cdot\frac{\partial g}{\partial \mathbf{v}}-\frac{\partial f}{\partial \mathbf{v}}\cdot\frac{\partial g}{\partial \mathbf{x}}\right)+\frac{e\mathbf{B}}{m^2c}\cdot\left(\frac{\partial f}{\partial \mathbf{v}}\times\frac{\partial g}{\partial \mathbf{v}}\right)$$

gyroscopic bracket

$$\text{equations of motion :}\begin{cases}\dot{\mathbf{x}}=\left\{\mathbf{x},H\right\}=\mathbf{v}\\ \dot{\mathbf{v}}=\left\{\mathbf{v},H\right\}=\dfrac{e}{m}\left(\mathbf{E}+\dfrac{\mathbf{v}\times\mathbf{B}}{c}\right)\end{cases}$$

Definition: Hamiltonian system

- a scalar function H, the Hamiltonian

- a Poisson bracket $\{F,G\}$ with the properties

  antisymmetric $\{F,G\} = -\{G,F\}$

  Leibnitz law $\{F,GK\} = \{F,G\}K + G\{F,K\}$

  Jacobi identity $\{\{F,G\},K\} + \{\{K,F\},G\} + \{\{G,K\},F\} = 0$

- equations of motion

$$\frac{dF}{dt} = \{F,H\}$$

- a conserved quantity

$$\{F,H\} = 0$$

Eulerian version: case of a density of charged particles

- density of particles in phase space $f(\mathbf{x}, \mathbf{v}, t)$

  example: $f(\mathbf{x}, \mathbf{v}, t) = \dfrac{1}{N} \sum_i \delta(\mathbf{x} - \mathbf{x}_i(t)) \delta(\mathbf{v} - \mathbf{v}_i(t))$ Klimontovitch distribution

- evolution given by the Vlasov equation

$$\frac{\partial f}{\partial t} = -\mathbf{v} \cdot \nabla f - \frac{e}{m}\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \cdot \frac{\partial f}{\partial \mathbf{v}}$$

- Eulerian, not Lagrangian:

  for any observable $\mathcal{F}[f]$, we have $\dfrac{d\mathcal{F}}{dt} = \dfrac{\partial \mathcal{F}}{\partial t} = [\mathcal{F}, \mathcal{H}]$

- still a Hamiltonian system

  Hamiltonian : $\mathcal{H}[f] = \iint d^3x d^3v\, f\left(\dfrac{m}{2}\mathbf{v}^2 + eV\right)$

  with $[\mathcal{F}, \mathcal{G}] = \iint d^3x d^3v\, f\left\{\dfrac{\delta \mathcal{F}}{\delta f}, \dfrac{\delta \mathcal{G}}{\delta f}\right\}$

Eulerian version: case of a density of charged particles

- an example: $\rho\left[f\right]\left(\mathbf{x}_0\right) = e\int d^3v\, f(\mathbf{x}_0, \mathbf{v})$

$$\frac{\partial\rho}{\partial t} = \left[\rho, \mathcal{H}\right] = \int d^3x d^3v\, f \left\{\frac{\delta\rho}{\delta f}, \frac{\delta\mathcal{H}}{\delta f}\right\}$$

- functional derivatives

$$\mathcal{F}\left[f + \phi\right] = \mathcal{F}\left[f\right] + \int d^3x d^3v\ \frac{\delta\mathcal{F}}{\delta f}\phi + O\left(\phi^2\right)$$

- here: $\frac{\delta\rho}{\delta f} = e\,\delta\left(\mathbf{x} - \mathbf{x}_0\right)$ and $\frac{\delta\mathcal{H}}{\delta f} = \frac{1}{2}m\mathbf{v}^2 + eV$

- therefore: $\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial\mathbf{x}}\cdot\mathbf{J}$ with $\mathbf{J}\left[f\right] = e\int d^3v\, f\,\mathbf{v}$

Vlasov-Maxwell equations: self-consistent dynamics

> description of the dynamics of a collisionless plasma (low density)

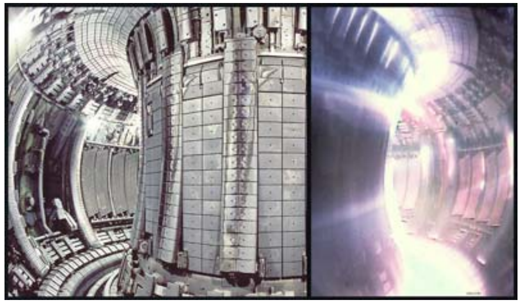
Variables: particle density $f(\mathbf{x},\mathbf{v},t)$, electric field $\mathbf{E}(\mathbf{x},t)$, magnetic field $\mathbf{B}(\mathbf{x},t)$

$$\frac{\partial f}{\partial t} = -\mathbf{v}\cdot\nabla f - \frac{e}{m}\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right)\cdot\frac{\partial f}{\partial \mathbf{v}}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla\times\mathbf{E}$$

$$\frac{\partial \mathbf{E}}{\partial t} = c\nabla\times\mathbf{B} - 4\pi\mathbf{J}$$

where $\nabla\cdot\mathbf{E} = 4\pi\rho$ and $\nabla\cdot\mathbf{B} = 0$



Vlasov-Maxwell equations... still a Hamiltonian system

$$\text{Hamiltonian}: \mathcal{H}\left[\mathbf{E},\mathbf{B},f\right] = \iint d^3x d^3v\, f\, \frac{m}{2}\mathbf{v}^2 + \int d^3x\, \frac{\left|\mathbf{E}\right|^2 + \left|\mathbf{B}\right|^2}{8\pi}$$

$$\text{with } \left[\mathcal{F},\mathcal{G}\right] = \iint d^3x d^3v\, f\left\{\frac{\delta\mathcal{F}}{\delta f}, \frac{\delta\mathcal{G}}{\delta f}\right\}$$

$$+\frac{4\pi e}{m}\iint d^3x d^3v\, \frac{\partial f}{\partial \mathbf{v}}\cdot\left[\frac{\delta\mathcal{F}}{\delta\mathbf{E}}\frac{\delta\mathcal{G}}{\delta f} - \frac{\delta\mathcal{F}}{\delta f}\frac{\delta\mathcal{G}}{\delta\mathbf{E}}\right]$$

$$+4\pi c\int d^3x\left[\frac{\delta\mathcal{F}}{\delta\mathbf{E}}\cdot\nabla\times\frac{\delta\mathcal{G}}{\delta\mathbf{B}} - \nabla\times\frac{\delta\mathcal{F}}{\delta\mathbf{B}}\cdot\frac{\delta\mathcal{G}}{\delta\mathbf{E}}\right]$$

Equation of motion for $\mathcal{F}\left[\mathbf{E},\mathbf{B},f\right]$: $\boxed{\frac{d\mathcal{F}}{dt} = \frac{\partial\mathcal{F}}{\partial t} = \left[\mathcal{F},\mathcal{H}\right]}$

Remark: $\operatorname{div}\mathbf{B}$ et $\operatorname{div}\mathbf{E} - \int d^3p\, f$ are conserved quantities

antisymmetry, Leibnitz, Jacobi

Morrison, PLA (1980)
Marsden, Weinstein, Physica D (1982)

From microscopic to macroscopic Vlasov-Maxwell equations

- Elimination (or decoupling) of fast time and small spatial scales for a better understanding of complex plasma phenomena

- reduced Maxwell equations in terms of **D** and **H**

$$\nabla \cdot \mathbf{D} = 4\pi\rho_R$$

$$\frac{\partial \mathbf{D}}{\partial t} = c\nabla \times \mathbf{H} - 4\pi\mathbf{J}_R$$

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \qquad \mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$$

where

$$\rho_R = \rho + \nabla \cdot \mathbf{P}, \quad \mathbf{J}_R = \mathbf{J} - c\nabla \times \mathbf{M} - \frac{\partial \mathbf{P}}{\partial t}$$

reduced polarization density / magnetization current / polarization current density

- Can we represent the reduced Vlasov-Maxwell equations as a Hamiltonian system?
  Hint: use of Lie transforms

- *Deliverables*: Expressions of the polarization **P** and magnetization **M** vectors

Reduced fields as Lie transforms of f, $\mathbf{E}$ and $\mathbf{B}$

Given a functional $\mathcal{S}\left[\mathbf{E}(\mathbf{x},t),\mathbf{B}(\mathbf{x},t),f(\mathbf{x},\mathbf{v},t)\right]$, we define some new fields as

$$\begin{pmatrix}\mathbf{D}\\ \mathbf{H}\\ F\end{pmatrix} = \mathrm{e}^{-\mathcal{L}_{\mathcal{S}}}\begin{pmatrix}\mathbf{E}\\ \mathbf{B}\\ f\end{pmatrix} = \begin{pmatrix}\mathbf{E}-\left[\mathcal{S},\mathbf{E}\right]+\frac{1}{2}\left[\mathcal{S},\left[\mathcal{S},\mathbf{E}\right]\right]+\cdots\\ \mathbf{B}-\left[\mathcal{S},\mathbf{B}\right]+\frac{1}{2}\left[\mathcal{S},\left[\mathcal{S},\mathbf{B}\right]\right]+\cdots\\ f-\left[\mathcal{S},f\right]+\frac{1}{2}\left[\mathcal{S},\left[\mathcal{S},f\right]\right]+\cdots\end{pmatrix}$$

Remark: If the variable χ is only a function of $\mathbf{x}$

then $\mathrm{e}^{-\mathcal{L}_{\mathcal{S}}}\chi$ is only a function of $\mathbf{x}$

The functionals transforms into $\bar{\mathcal{F}} = \mathrm{e}^{-\mathcal{L}_{\mathcal{S}}}\mathcal{F}$,

resulting in a new Hamiltonian and a new Poisson bracket...

Polarization, magnetization, reduced density, etc...

$$\mathbf{P} = \frac{1}{4\pi}\left(\mathrm{e}^{-\mathcal{L}_S} - 1\right)\mathbf{E} = c\nabla\times\frac{\delta\mathcal{S}}{\delta\mathbf{B}} - \frac{e}{m}\int d^3v f\,\frac{\partial}{\partial\mathbf{v}}\left(\frac{\delta\mathcal{S}}{\delta f}\right) + \cdots$$

$$\mathbf{M} = \frac{1}{4\pi}\left(1 - \mathrm{e}^{-\mathcal{L}_S}\right)\mathbf{B} = c\nabla\times\frac{\delta\mathcal{S}}{\delta\mathbf{E}} + \cdots$$

so that $\begin{cases} \mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} \\ \mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} \end{cases}$

Reduced evolution operator $\dfrac{\partial\bar{\mathcal{F}}}{\partial\bar{t}} \equiv \left(\mathrm{e}^{-\mathcal{L}_S}\dfrac{\partial}{\partial t}\mathrm{e}^{\mathcal{L}_S}\right)\bar{\mathcal{F}}$

$$= \mathrm{e}^{-\mathcal{L}_S}\left[\mathrm{e}^{\mathcal{L}_S}\bar{\mathcal{F}}, \mathrm{e}^{\mathcal{L}_S}\bar{\mathcal{H}}\right] = \left[\bar{\mathcal{F}}, \bar{\mathcal{H}}\right]$$

Reduced Vlasov-Maxwell equations

$$\frac{\partial \mathbf{D}}{\partial \overline{t}} = c\nabla\times\mathbf{H} - 4\pi\overline{\mathbf{J}}$$
$$\frac{\partial \mathbf{H}}{\partial \overline{t}} = -c\nabla\times\mathbf{D}$$

$$\frac{\partial \mathbf{D}}{\partial t} = \frac{\partial \mathbf{D}}{\partial \overline{t}} + \left[\mathbf{D}, \mathcal{H} - \overline{\mathcal{H}}\right] = c\nabla\times\mathbf{H} - 4\pi\mathbf{J}_{R}$$
$$\frac{\partial \mathbf{H}}{\partial t} = \frac{\partial \mathbf{H}}{\partial \overline{t}} + \left[\mathbf{H}, \mathcal{H} - \overline{\mathcal{H}}\right] = -c\nabla\times\mathbf{D} - 4\pi\frac{\partial \mathbf{M}}{\partial t} + 4\pi c\nabla\times\mathbf{P}$$

Reduced Vlasov equation $\quad \dfrac{\partial F}{\partial \overline{t}} = -\mathbf{v}\cdot\nabla F - \dfrac{e}{m}\left(\mathbf{D} + \dfrac{\mathbf{v}}{c}\times\mathbf{H}\right)\cdot\dfrac{\partial F}{\partial \mathbf{v}}$

$$F = f - \left\{f, \frac{\delta\mathcal{S}}{\delta f}\right\} - \frac{4\pi e}{m}\frac{\partial f}{\partial \mathbf{v}}\cdot\frac{\delta\mathcal{S}}{\delta\mathbf{E}} + \cdots$$

guiding center theory / gyrokinetics

What $\mathcal{S}$?

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{H} \\ F \end{pmatrix} = \mathrm{e}^{-\mathcal{L}_{\mathcal{S}}} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \\ f \end{pmatrix}$$

- Elimination of small spatial and fast time (averaging) scales of $f(\mathbf{x},\mathbf{v},t)$:

  guiding-center, gyrokinetics } collisionless plasmas (low frequency phenomena)

  reduced models for free electron lasers

-Use of KAM algorithms (at least one step process)

$$\text{For} \quad \mathrm{F} = f + \delta f, \ \text{homological equation } \mathbb{P}\left(\delta f + \left[\mathcal{S}, f\right]\right) = 0$$

- Advantages: preserve the structure of the equations, invertible, symbolic calculus

Brizard, Hahm, Rev. Mod. Phys. (2007)

Strategies to reduce Vlasov-Maxwell equations

> rigorous: $\bar{\mathcal{H}} = \mathrm{e}^{-\mathcal{L}_S}\mathcal{H}$

> non-rigorous: truncate the Hamiltonian system

- the equations of motion
- the Hamiltonian and the Poisson bracket

> the canonical version provides a way out...

Reduced model for the Free Electron Laser

From: Vlasov-Maxwell Hamiltonian

$$\mathcal{H}\left[\mathbf{E},\mathbf{B},f\right]=\iint d^3x d^3p\ f\left(\mathbf{x},\mathbf{p}\right)\sqrt{1+\mathbf{p}^2}+\int d^3x\,\frac{\left|\mathbf{E}\right|^2+\left|\mathbf{B}\right|^2}{2}$$

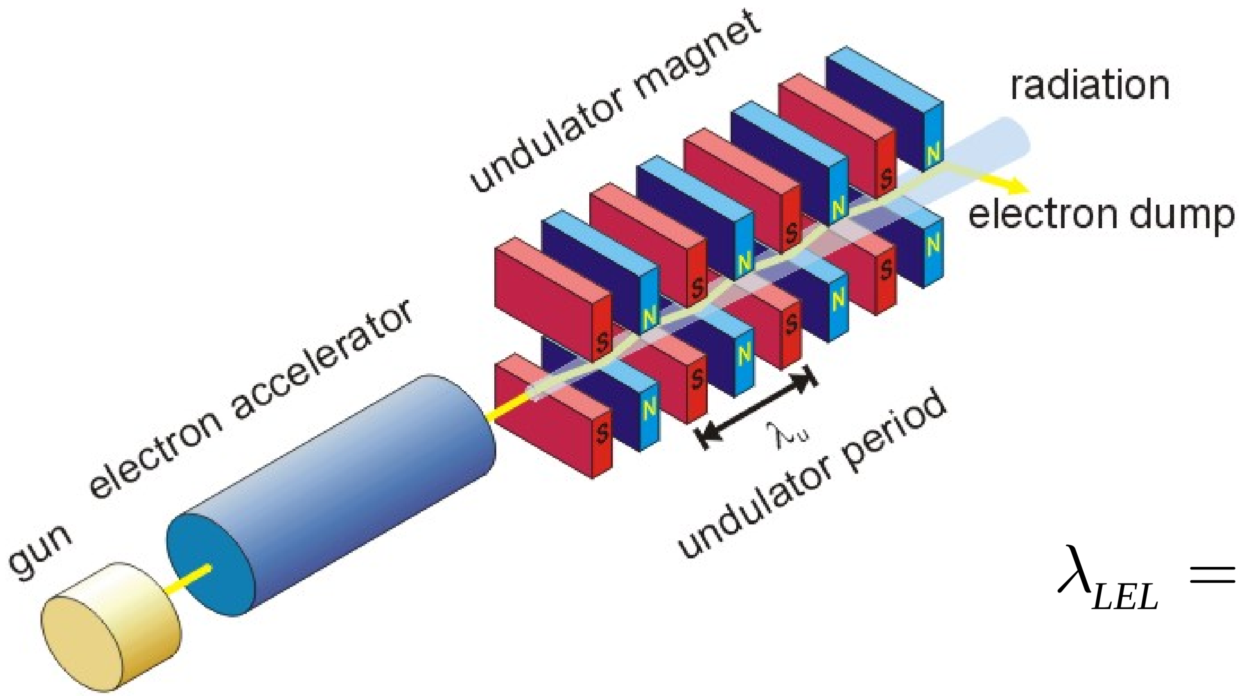
To: Bonifacio's reduced FEL Hamiltonian model

$$H\left[f,I,\varphi\right]=\iint d\theta dp\, f\left(\theta,p\right)\left(\frac{p^2}{2}+2\sqrt{I}\sin\left(\theta-\varphi\right)\right)$$

with $\left(I,\varphi\right)$ intensity and phase of the electromagnetic wave

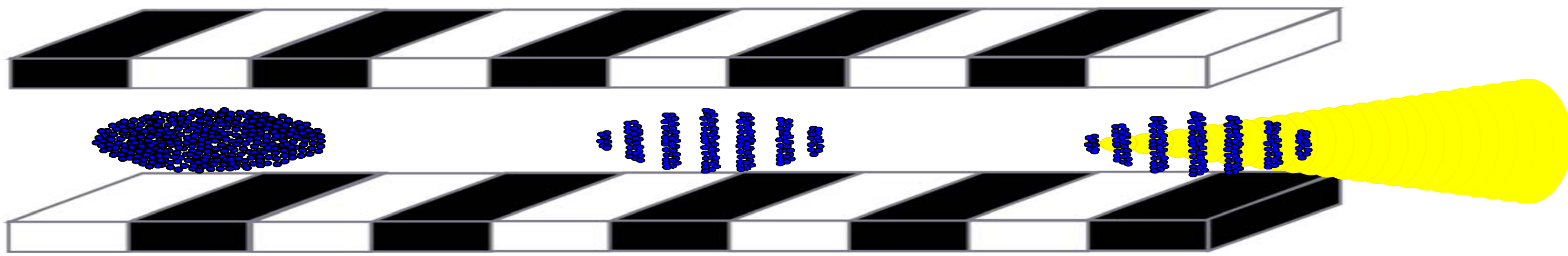
... in a Hamiltonian way

A Free Electron… what?



General layout of free-electron laser

$$\lambda_{LEL} = \frac{\lambda_u}{2\gamma^2}(1 + K_{rms}^2)$$



Vlasov-Maxwell: canonical version

① Change of variables: $\left(f,\mathbf{E},\mathbf{B}\right)\mapsto(f_{\mathrm{m}},\mathbf{Y},\mathbf{A})$

$$f\left(\mathbf{x},\mathbf{p}\right)=f_{\mathrm{m}}\left(\mathbf{x},\mathbf{p}+\mathbf{A}\left(\mathbf{x}\right)\right)$$

$$\mathbf{E}=-\mathbf{Y}$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$

Bracket: canonical

$$\left[\mathcal{F},\mathcal{G}\right]=\iint d^3x d^3p\, f_{\mathrm{m}}\left[\nabla\frac{\delta\mathcal{F}}{\delta f_{\mathrm{m}}}\cdot\frac{\partial}{\partial\mathbf{p}}\frac{\delta\mathcal{G}}{\delta f_{\mathrm{m}}}-\frac{\partial}{\partial\mathbf{p}}\frac{\delta\mathcal{F}}{\delta f_{\mathrm{m}}}\cdot\nabla\frac{\delta\mathcal{G}}{\delta f_{\mathrm{m}}}\right]+\int d^3x\left[\frac{\delta\mathcal{F}}{\delta\mathbf{A}}\cdot\frac{\delta\mathcal{G}}{\delta\mathbf{Y}}-\frac{\delta\mathcal{F}}{\delta\mathbf{Y}}\cdot\frac{\delta\mathcal{G}}{\delta\mathbf{A}}\right]$$

Hamiltonian: $\mathcal{H}=\iint d^3x d^3p\, f_{\mathrm{m}}\sqrt{1+\left(\mathbf{p}-\mathbf{A}\right)^2}+\int d^3x\,\frac{\left|\mathbf{Y}\right|^2+\left|\nabla\times\mathbf{A}\right|^2}{2}$

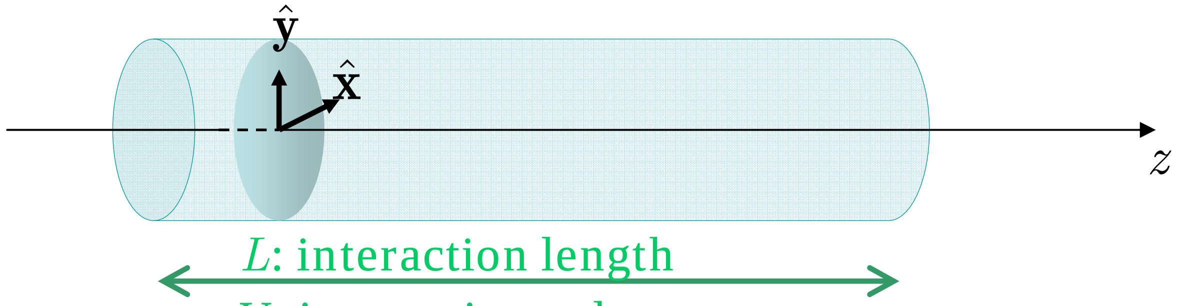
② Translation of **A** by a constant function (external field-undulator): $\mathbf{A}\left(\mathbf{x},t\right)\mapsto\mathbf{A}_w\left(\mathbf{x}\right)+\mathbf{A}\left(\mathbf{x},t\right)$

Bracket: canonical (canonical transformation)

Hamiltonian: $\mathcal{H}=\iint d^3x d^3p\, f_{\mathrm{m}}\sqrt{1+\left(\mathbf{p}-\mathbf{A}_w-\mathbf{A}\right)^2}+\int d^3x\,\frac{\left|\mathbf{Y}\right|^2+2\nabla\times\mathbf{A}_w\cdot\nabla\times\mathbf{A}+\left|\nabla\times\mathbf{A}\right|^2}{2}$

$\mathbf{A}_w$ helicoidal undulator: $\mathbf{A}_w=\frac{a_w}{\sqrt{2}}\left(\mathrm{e}^{-ik_w z}\,\hat{\mathbf{e}}+\mathrm{e}^{ik_w z}\,\hat{\mathbf{e}}^*\right)$

One mode for the radiated field



③ Paraxial approximation and circularly polarized radiated field

$$\mathbf{A} = -\frac{i}{\sqrt{2}}\left(a\mathrm{e}^{ikz}\hat{\mathbf{e}} - a^{*}\mathrm{e}^{-ikz}\hat{\mathbf{e}}^{*}\right) \quad \text{where } \hat{\mathbf{e}} = \frac{\hat{\mathbf{x}} + i\hat{\mathbf{y}}}{\sqrt{2}}$$

$$\mathbf{Y} = \frac{k}{\sqrt{2}}\left(a\mathrm{e}^{ikz}\hat{\mathbf{e}} + a^{*}\mathrm{e}^{-ikz}\hat{\mathbf{e}}^{*}\right)$$

Remark: a does not depend on x and y but depends on time (dynamical variable)

Bracket: $$\left[\mathcal{F},\mathcal{G}\right] = \iint d^3x d^3p\, f_{\mathrm{m}} \left[\nabla \frac{\delta\mathcal{F}}{\delta f_{\mathrm{m}}} \cdot \frac{\partial}{\partial \mathbf{p}} \frac{\delta\mathcal{G}}{\delta f_{\mathrm{m}}} - \frac{\partial}{\partial \mathbf{p}} \frac{\delta\mathcal{F}}{\delta f_{\mathrm{m}}} \cdot \nabla \frac{\delta\mathcal{G}}{\delta f_{\mathrm{m}}}\right] + \frac{ik}{V}\left(\frac{\partial\mathcal{F}}{\partial a}\frac{\partial\mathcal{G}}{\partial a^{*}} - \frac{\partial\mathcal{F}}{\partial a^{*}}\frac{\partial\mathcal{G}}{\partial a}\right)$$

Hamiltonian: $$\mathcal{H} = \iint d^3x d^3p\, f_{\mathrm{m}} \sqrt{1 + \mathbf{p}^2 + aa^{*} - i\sqrt{2}\left(a\,\mathrm{e}^{ikz}\,\hat{\mathbf{e}} - a^{*}\,\mathrm{e}^{-ikz}\,\hat{\mathbf{e}}^{*}\right)\cdot \mathbf{A}_w + \left|\mathbf{A}_w\right|^2}$$

$$+ k^2 V a a^{*} - \frac{ikS}{\sqrt{2}} \int dz \left(a\,\mathrm{e}^{ikz}\,\hat{\mathbf{e}} - a^{*}\,\mathrm{e}^{-ikz}\,\hat{\mathbf{e}}^{*}\right)\cdot \nabla \times \mathbf{A}_w$$

Dimensional reduction

④ The fields do not depend on x and $y \Rightarrow$ no transverse velocity dispersion

$$f\left(\mathbf{x},\mathbf{p}\right)=\widehat{f}\left(\mathbf{x},p_{\parallel}\right)\delta\left(\mathbf{p}_{\perp}\right)\quad \text{if}\ \ \mathbf{p}_{\perp}\left(t=0\right)=0$$

If $\mathbf{p}_{\perp}\left(t\right)=0$ then no modification of the $\left(x,y\right)$ distribution

$$f\left(\mathbf{x},\mathbf{p}\right)=\tilde{f}\left(z,p_{\parallel}\right)\delta\left(\mathbf{p}_{\perp}\right)\delta\left(x\right)\delta\left(y\right)\quad \text{if}\ \ x\left(t=0\right)=y\left(t=0\right)=0\quad \text{(injection at the center)}$$

Bracket: $$\left[\mathcal{F},\mathcal{G}\right]=\iint dz dp_{\parallel}\,\tilde{f}\left[\frac{\partial}{\partial p_{\parallel}}\frac{\delta\mathcal{F}}{\delta\tilde{f}}\frac{\partial}{\partial z}\frac{\delta\mathcal{G}}{\delta\tilde{f}}-\frac{\partial}{\partial z}\frac{\delta\mathcal{F}}{\delta\tilde{f}}\frac{\partial}{\partial p_{\parallel}}\frac{\delta\mathcal{G}}{\delta\tilde{f}}\right]+\frac{ik}{V}\left(\frac{\partial\mathcal{F}}{\partial a}\frac{\partial\mathcal{G}}{\partial a^{*}}-\frac{\partial\mathcal{F}}{\partial a^{*}}\frac{\partial\mathcal{G}}{\partial a}\right)$$

Hamiltonian: $$\mathcal{H}=\iint dz dp_{\parallel}\,\tilde{f}\sqrt{1+p_{\parallel}^{2}+aa^{*}-i\sqrt{2}\left(a\,\mathrm{e}^{ikz}\,\hat{\mathbf{e}}-a^{*}\,\mathrm{e}^{-ikz}\,\hat{\mathbf{e}}^{*}\right)\cdot\mathbf{A}_{w}+\left|\mathbf{A}_{w}\right|^{2}}$$
$$+k^{2}Vaa^{*}-\frac{ikS}{\sqrt{2}}\int dz\left(a\,\mathrm{e}^{ikz}\,\hat{\mathbf{e}}-a^{*}\,\mathrm{e}^{-ikz}\,\hat{\mathbf{e}}^{*}\right)\cdot\nabla\times\mathbf{A}_{w}$$

vanishing

⑤ Autonomization: $$\left[\mathcal{F},\mathcal{G}\right]_{\mathrm{a}}=\left[\mathcal{F},\mathcal{G}\right]+\frac{\partial\mathcal{F}}{\partial t}\frac{\partial\mathcal{G}}{\partial E}-\frac{\partial\mathcal{F}}{\partial E}\frac{\partial\mathcal{G}}{\partial t}\quad \text{with}\ \ \mathcal{H}_{\mathrm{a}}=\mathcal{H}+E$$

Time dependent transformation (canonical):

$$\hat{f}\left(\theta,p_{\parallel}\right)=\tilde{f}\left(z,p_{\parallel}\right)\quad \text{with}\ \ \theta=\left(k+k_{w}\right)z-kt$$

$$\hat{a}=a\,\mathrm{e}^{ikt}$$

$$\hat{E}=E+Vk^{2}aa^{*}\ \ \text{and}\ \ \hat{t}=t$$

Bracket: canonical

Hamiltonian: $$\mathcal{H}=\iint d\theta dp_{\parallel}\,\hat{f}\left(\sqrt{1+p_{\parallel}^{2}+\hat{a}\hat{a}^{*}-ia_{w}\left(\hat{a}\,\mathrm{e}^{i\theta}-\hat{a}^{*}\,\mathrm{e}^{-i\theta}\right)+a_{w}^{\ 2}}-\frac{k}{k+k_{w}}p_{\parallel}\right)$$

Bonifacio's FEL model

⑥ Resonance condition: $p_{\parallel} = p_R + p$ with $\left|p\right| \ll p_R$

weak radiated field: $\left|\hat{a}\right| \ll \gamma_R \equiv \sqrt{1 + p_R^{\ 2}}$

Bracket: $$\left[\mathcal{F},\mathcal{G}\right] = \left(k + k_w\right)\iint d\theta dp\, \hat{f}\left[\frac{\partial}{\partial\theta}\frac{\delta\mathcal{F}}{\delta\hat{f}}\frac{\partial}{\partial p}\frac{\delta\mathcal{G}}{\delta\hat{f}} - \frac{\partial}{\partial p}\frac{\delta\mathcal{F}}{\delta\hat{f}}\frac{\partial}{\partial\theta}\frac{\delta\mathcal{G}}{\delta\hat{f}}\right] + \frac{i}{kV}\left(\frac{\partial\mathcal{F}}{\partial\hat{a}^*}\frac{\partial\mathcal{G}}{\partial\hat{a}} - \frac{\partial\mathcal{F}}{\partial\hat{a}}\frac{\partial\mathcal{G}}{\partial\hat{a}^*}\right)$$

Hamiltonian: $$\mathcal{H} = \iint d\theta dp\, \hat{f}\left(\theta, p\right)\left(\frac{1 + a_w^{\ 2}}{\gamma_R^{\ 3}}\frac{p^2}{2} - \frac{ia_w}{\gamma_R}\left(\hat{a}\,\mathrm{e}^{i\theta} - \hat{a}^*\,\mathrm{e}^{-i\theta}\right)\right)$$

⑦ Normalization

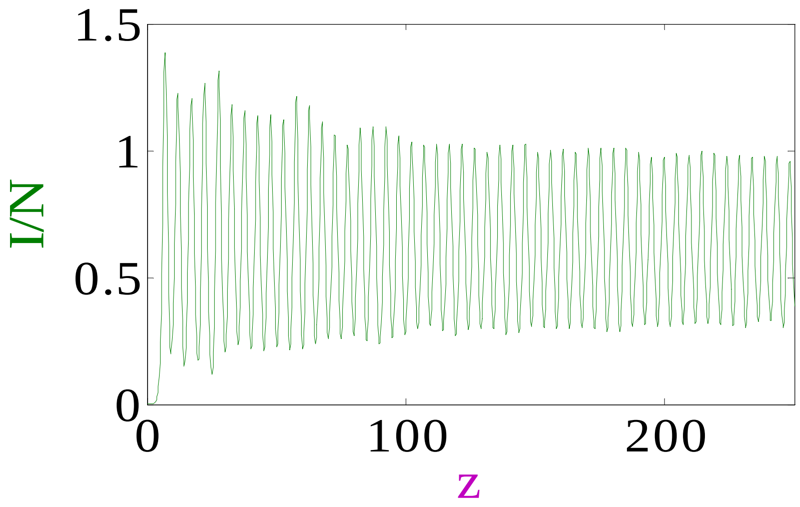
⑧ Transformation (canonical) into intensity/phase $a = i\sqrt{I}\mathrm{e}^{-i\varphi}$

Bracket: canonical $$\left[\mathcal{F},\mathcal{G}\right] = \iint d\theta dp\, f\left[\frac{\partial}{\partial\theta}\frac{\delta\mathcal{F}}{\delta f}\frac{\partial}{\partial p}\frac{\delta\mathcal{G}}{\delta f} - \frac{\partial}{\partial p}\frac{\delta\mathcal{F}}{\delta f}\frac{\partial}{\partial\theta}\frac{\delta\mathcal{G}}{\delta f}\right] + \frac{\partial\mathcal{F}}{\partial\varphi}\frac{\partial\mathcal{G}}{\partial I} - \frac{\partial\mathcal{F}}{\partial I}\frac{\partial\mathcal{G}}{\partial\varphi}$$

Hamiltonian: $$\mathcal{H} = \iint d\theta dp\, f\left(\theta, p\right)\frac{p^2}{2} + 2\sqrt{I}\iint d\theta dp f\left(\theta, p\right)\cos\left(\theta - \varphi\right)$$

Outlook: on the use of reduced Hamiltonian models

$$H_N = \sum_{j=1}^{N} \frac{p_j^2}{2} + 2\sqrt{\frac{I}{N}} \sum_{j=1}^{N} \cos(\theta_j - \varphi)$$



Long-range interacting systems : QSS, transition to equilibrium,...

Gyrokinetics: understand plasma disruption, control,...

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References: Bachelard, Chandre, Vittot, PRE (2008)
Chandre, Brizard, in preparation.