Manifolds on the verge of a hyperbolicity breakdown

Àlex Haro, Rafael de la Llave

UNIVERSITAT DE BARCELONA

THE UNIVERSITY OF TEXAS AT AUSTIN

1 Introduction

The long term behavior of dynamical systems is organized by the invariant objects. Hence, it is important to understand which invariant objects persist under modifications of the system.

The persistence of an invariant object is related to the exponential rate of growth of the perturbations. An invariant manifold persists under perturbations if [HirschP69][Fenichel71] and only if [Mane78] it is normally hyperbolic.

There are spectral characterizations of hyperbolicity [Mather68] [HirschPS77][Swanson83].

We study a mechanism of destruction of invariant manifolds. The mechanism consists in the fact that the attracting directions merge with repelling directions.

This geometric mechanism has spectral implications.

We show that this mechanism satisfies scaling properties.

2 Set up

2.1 Quasi-periodic maps and invariant tori

Quasi-periodic forced systems are non-autonomous systems in which the external forcing is quasi-periodic.

A quasi-periodic map with frequency vector $\omega \in \mathbb{R}^d$ is a skew product in $\mathbb{R}^n \times \mathbb{T}^d$

$$\begin{cases} \bar{x} = F(x, \theta) \,, \\ \bar{\theta} = \theta + \omega \pmod 1 \,. \end{cases} \tag{1}$$

An invariant torus whose dynamics is the rotation ω is given as a solution $K : \mathbb{T}^d \to \mathbb{R}^n$ of the invariance equation

$$F(K(\theta - \omega), \theta - \omega) - K(\theta) = 0 \,. \tag{2}$$

2.2 Cocycles and transfer operators

The cocycle is the linearization around the torus, that is

$$\begin{cases} \bar{v} = M(\theta)v \, , \\ \bar{\theta} = \theta + \omega \, , \end{cases} \tag{3}$$

where $M(\theta) = \mathrm{D}F(K(\theta), \theta)$.

The transfer operator $\mathcal{M}$ acting on bounded vector fields $v : \mathbb{T}^d \to \mathbb{R}^n$ is given by

$$(\mathcal{M} v)(\theta) = M(\theta - \omega) v(\theta - \omega) \, . \tag{4}$$

Important: Dynamics (3) / Functional analisys (4)

[Mather 68][Sacker, Sell 74,76,78,80][Hirsch,Pugh,Shub 77][Mañé 78] [Chicone,Swanson 80][Johnson 80][Johnson,Sell 81][Swanson 81,83] [Latushkin,Stëpin 90,91][Latushkin 96][de la Llave 93]...

2.3 Invariant bundles

The invariant torus has an spectral gap when we can find numbers $0 < \lambda_- < \lambda_+$ and a splitting $\mathbb{R}^n = E_\theta^- \oplus E_\theta^+$ characterized by

$$v \in E_\theta^\pm \Leftrightarrow |M^{\mp m}(\theta)v| \leq C(\lambda_\pm)^{\mp m}|v|,\ m \geq 0. \tag{5}$$

The bundles $E_\theta^\pm$ are continuous and invariant: $M(\theta)E_\theta^\pm = E_{\theta+\omega}^\pm$.

Fixed $\theta \in \mathbb{T}^d$:

- The directions of the future iterates $M^m(\theta)v$ of a vector $v \notin E_\theta^-$ converge to the directions of $E_{\theta+m\omega}^+$.

  (The directions of E_θ^+ are attracting).

- The directions of the past iterates $M^{-m}(\theta)v$ of a vector $v \notin E_\theta^+$ converge to the directions of $E_{\theta-m\omega}^-$.

  (The directions of E_θ^- are repelling).

3 Bundle merging scenarios

3.1 Mechanism

For the sake of simplicity: $n = 2$, $d = 1$.

We consider a system (1) depending on a parameter ε: $F = F_\varepsilon$.

We assume that:

- The system posses a smooth invariant torus K_ε for $0 \leq \varepsilon < \varepsilon_c$;
- In these ranges of ε, the torus has an invariant splitting as in (5);
- As ε approaches ε_c, the distance Δ_ε between the invariant bundles goes to zero, but the Lyapunov multipliers $\Lambda_\varepsilon^\pm$ remain different from 1 and from each other.

The splitting becomes zero in a complicated collision set, and the collapse of the spectral subbundles implies the sudden growth of the spectrum.

3.2 SNAs in the projective linearization

For a fixed θ, the 1-D space $E_\theta^\pm$ is described by an angle $\alpha_\theta^\pm$ in $[0, \pi]$.

The invariant bundles $E_\theta^\pm$ are represented by curves in the projective bundle.

The mechanism described above corresponds to the collision of those curves forming a non-smooth object. These are commonly called SNAs. [Glendinning et al 00][Prasad et al 01][Osinga et al 01]

3.3 Two bundle-merging scenarios

a) The splitting corresponds to two stable bundles (slow and fast) (the Lyapunov multipliers do not straddle 1)

$$\Downarrow$$

the attracting-node torus is not destroyed and it can be continued beyond ε_c.

b) The splitting corresponds to one unstable and one stable bundle (the Lyapunov multipliers straddle 1).

$$\Downarrow$$

the saddle torus presumably breaks down at $\varepsilon = \varepsilon_c$.

3.4 Scalings

Assertion 1 *The observables* Δ *and* $\Lambda^{\pm}$ *satisfy*

$$\Delta_{\varepsilon} \approx \alpha(\varepsilon_c - \varepsilon)^{\beta} \ , \ \Lambda_{\varepsilon}^{\pm} \approx \Lambda_c^{\pm} + A^{\pm}(\varepsilon_c - \varepsilon)^{B}, \tag{6}$$

for $\varepsilon \lesssim \varepsilon_c$*, where* $\alpha, \beta, A^{\pm}, B$ *(and* $\Lambda_c^{\pm}$*) are parameters.*

Moreover:

a) *If the Lyapunov multipliers do not straddle* 1*,* $\beta \simeq 1, B \simeq 0.5$ *and* $\Lambda_{\varepsilon}^{\pm} \approx \Lambda_c^{\pm} + \bar{A}^{\pm}(\varepsilon - \varepsilon_c)^{\bar{B}}$ *for* $\varepsilon \gtrsim \varepsilon_c$*, with* $\bar{B} \simeq 1$*.*

b) *If the Lyapunov multipliers do straddle* 1*,* $\beta \simeq 1, B \simeq 1$*.*

3.5 Consequences

1) For $\varepsilon = \varepsilon_c$, the collision set is dense, and has zero measure.

   The invariant bundles are discontinuous in $\varepsilon = \varepsilon_c$, but they are defined is set of full measure and are measurable.

   [Oseledec 68]

2) The spectrum of the transfer operator $\mathcal{M}$ is a set of annuli centered in the origin of the complex plane, and each spectral annulus has associated an invariant subbundle characterized by growth rates.

   [Mather 68][Haro,de la Llave]

   If the invariant bundles are 1D, the spectrum is a union of circles.

   In the present case ($n = 2$):

   For $\varepsilon < \varepsilon_c$ the spectrum is just two circles of radii $\Lambda_\varepsilon^\pm$, but for $\varepsilon = \varepsilon_c$, the spectrum has to be the full annulus enclosed by these circles.

4 Numerical evidence

4.1 Bundle merging causing growth of the spectrum

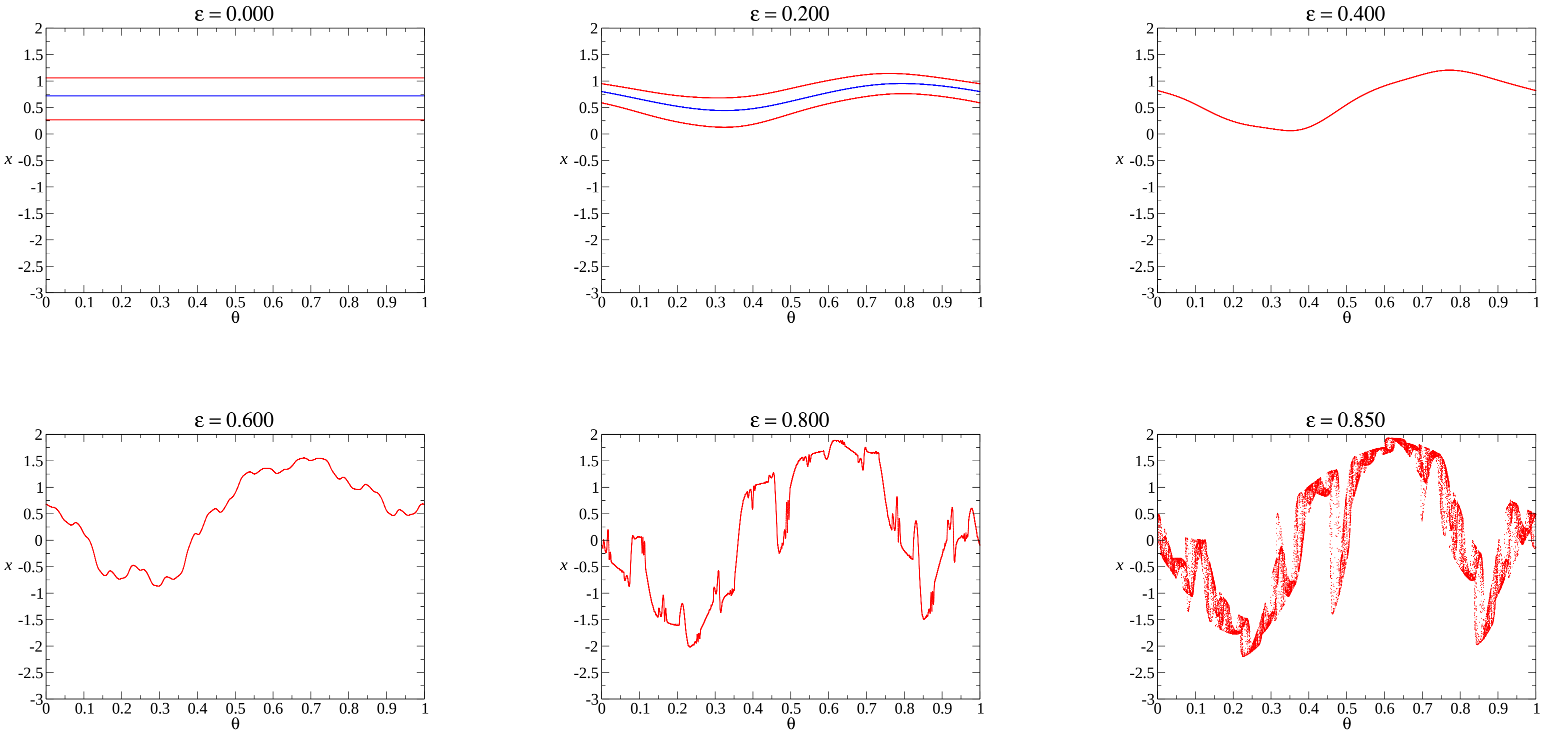
Model (dissipative): rotating Hénon map

$$\begin{cases} \bar{x} = 1 + y - a\,x^2 + \varepsilon\cos(2\pi\theta) \\ \bar{y} = bx \\ \bar{\theta} = \theta + \omega \pmod 1 \end{cases}$$

- $a = 0.68$, $b = 0.1$ are the parameters of the Hénon map.
- ε is the forcing parameter;
- $\omega = \frac{1}{2}(\sqrt{5} - 1)$ is the frequency of the quasi-periodic forcing.

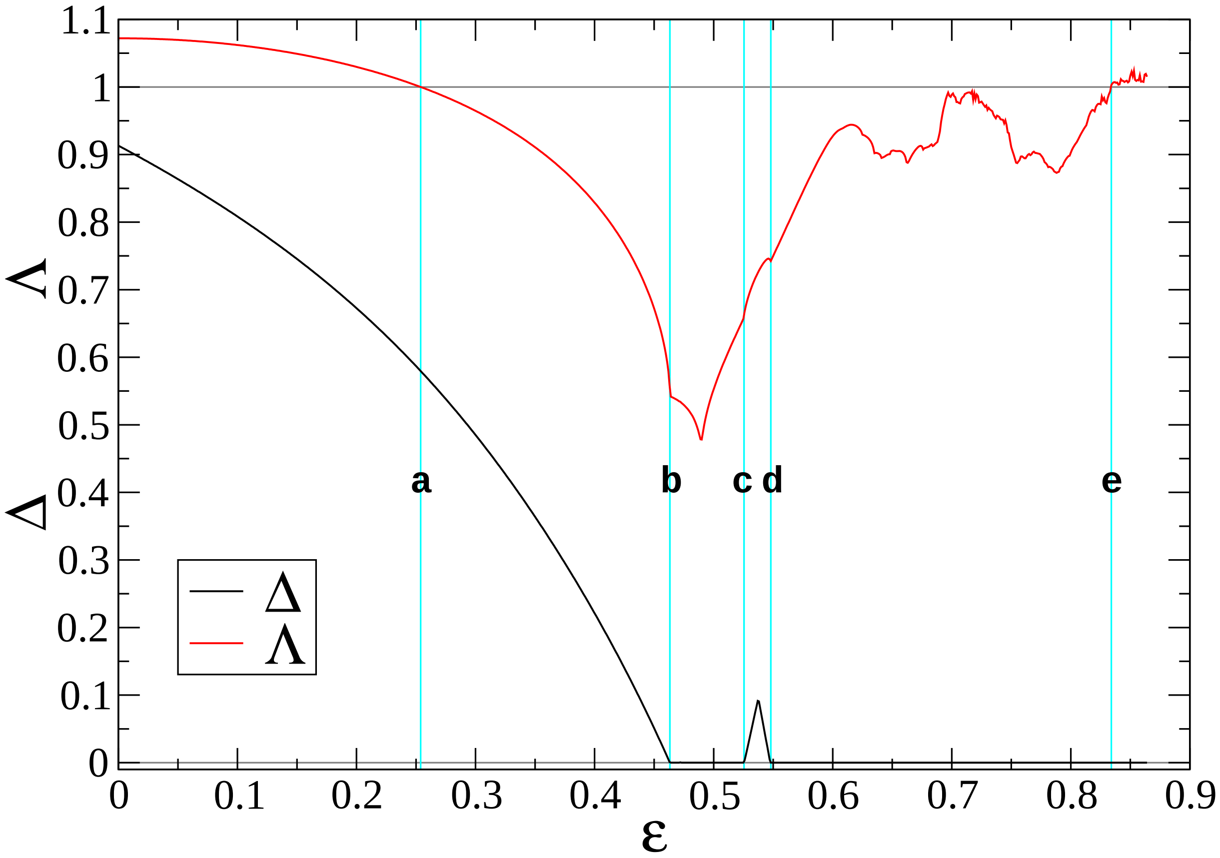
[Krauskopf,Osinga 98][Feudel,Osinga 00]

Continuation of an invariant torus



... and generation of an strange attractor

Continuation of observables



Distance between bundles (Δ) and maximal Lyapunov multiplier (Λ).

Description of transitions

a) $\varepsilon \simeq 0.2549$: period “halving” bifurcation, the unstable bundle becomes a slow stable bundle;

b) $\varepsilon \simeq 0.4633$: the slow directions merge with the fast directions causing a loss of reducibility;

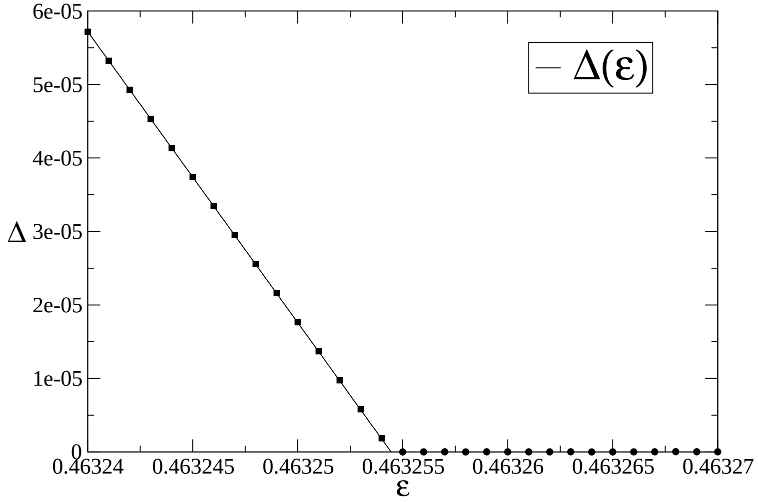
c) $\varepsilon \simeq 0.5256$: a gap in the spectrum appears again, so there are slow and fast directions.

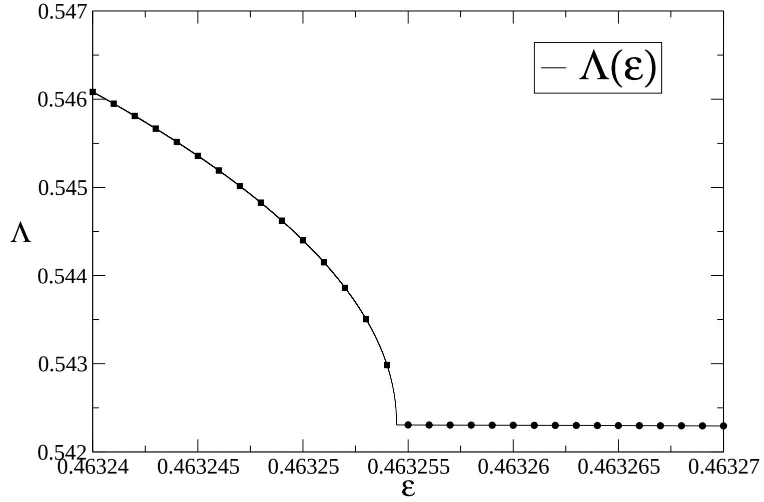
d) $\varepsilon \simeq 0.5475$: the gap disappears, and the spectrum is again a full annulus.

e) $\varepsilon \simeq 0.8337$: the external radius of the spectral annulus crosses the unit circle, and the attracting torus is destroyed.

Remark. Along the continuation, the Lyapunov multipliers are different, because $\Lambda > \sqrt{|b|} \simeq 0.3162$.

Transition b: quantitative estimates (fits of Δ and Λ)





$\Lambda_c = 0.5423122 \pm 5 \cdot 10^{-7}$

$\varepsilon_c = 0.46325447112 \pm 1 \cdot 10^{-11}$

$\alpha = 3.94933 \pm 7 \cdot 10^{-5}$

$\beta = 0.999979 \pm 2 \cdot 10^{-6}$

$A = 1.015 \pm 1 \cdot 10^{-3}$

$B = 0.5020 \pm 1 \cdot 10^{-4}$

$\bar{A} = -0.7409 \pm 6 \cdot 10^{-4}$

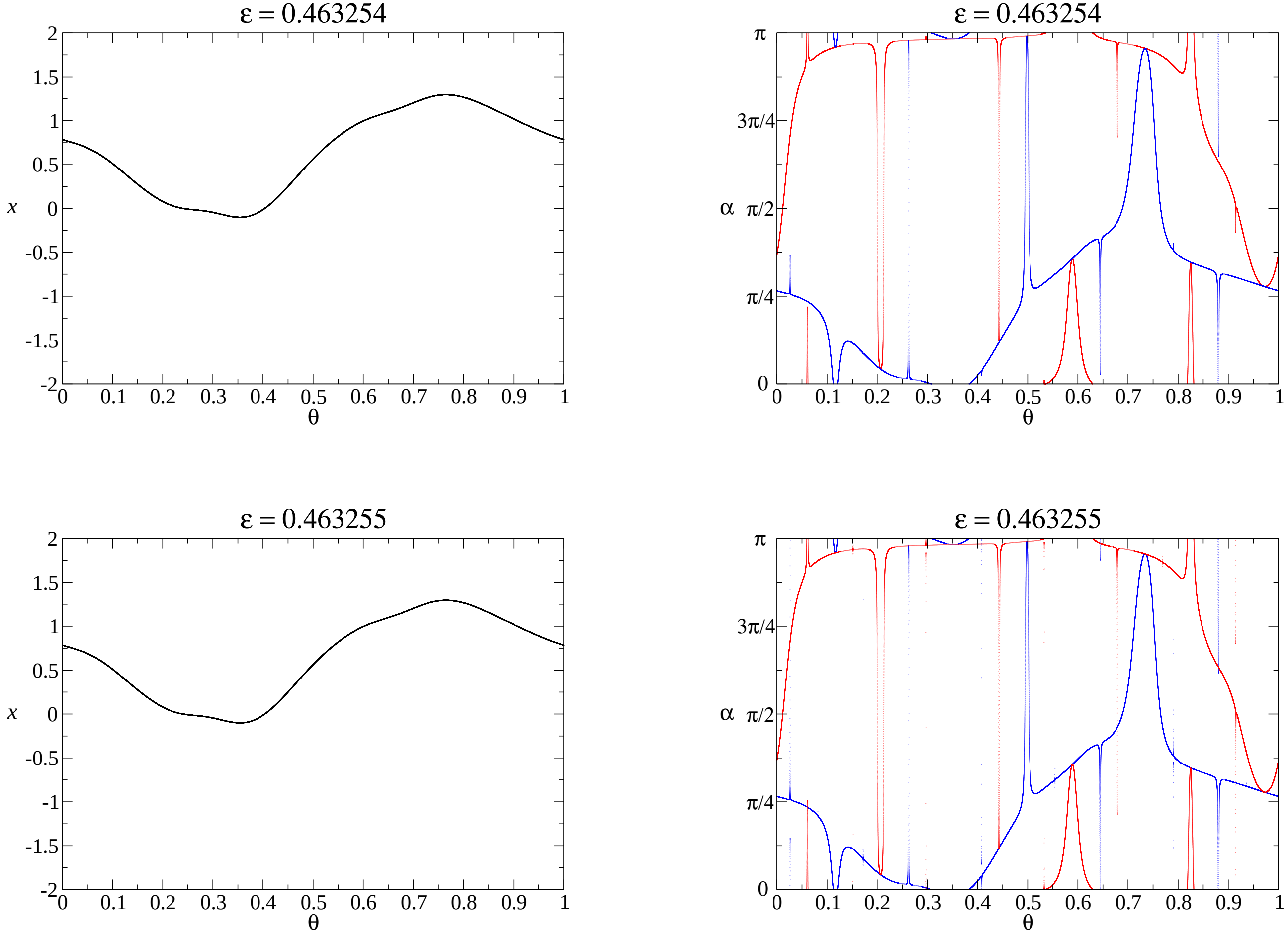
$\bar{B} = 1.00035 \pm 8 \cdot 10^{-5}$

Note that the computations of invariant tori are always done in the region when we have reliable error bounds.

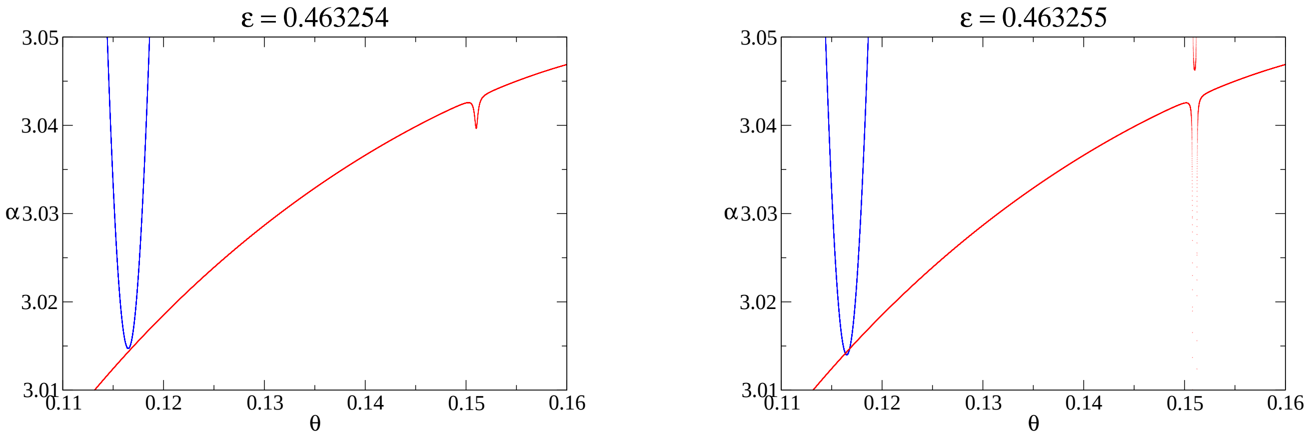
The reliable computations fit at straight line in log-log variables.

Many figures of ε_c are obtained from the fit.

Transition b: visual verification



x-curve of an attracting torus and α-curves of its slow and fast stable bundles.



Zooms of the α-curves

An analytical justification of bundle collapse

- For $\varepsilon = 0.463$, the torus is attracting and the cocycle is reducible to a constant diagonal matrix

$$\mathrm{diag}(-0.60343049952943990398 9, 0.16571916745670102293 3).$$

- For $\varepsilon = 0.530$, the torus is attracting and the cocycle is reducible to a constant diagonal matrix

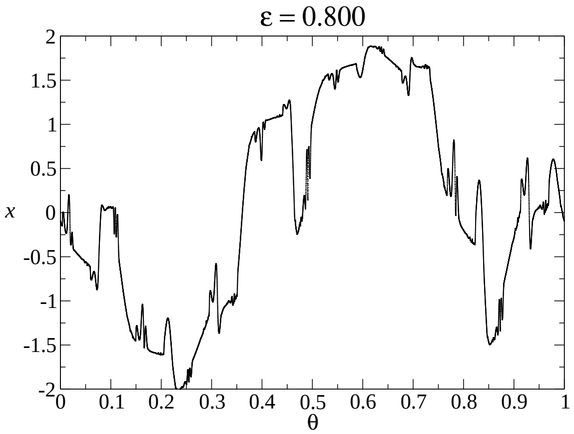
$$\mathrm{diag}(0.69454675004648078136 3, -0.14397878903516250496 6).$$

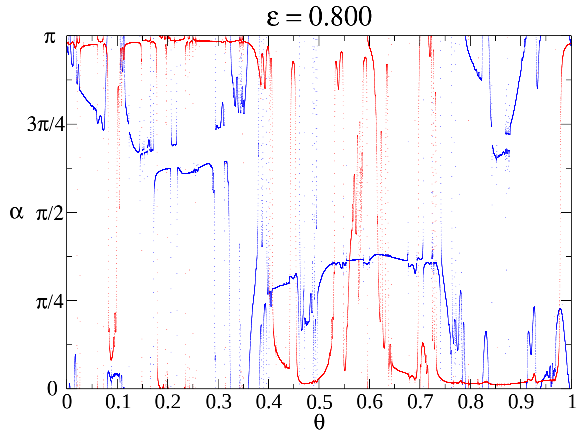
The Lyapunov multipliers are different during the continuation	$\Rightarrow$	The cocycle can not be reducible during the whole continuation!

The bundles have a collapse, and the spectrum grows suddenly.

Remark. The justification can be made purely topological.

The transition e

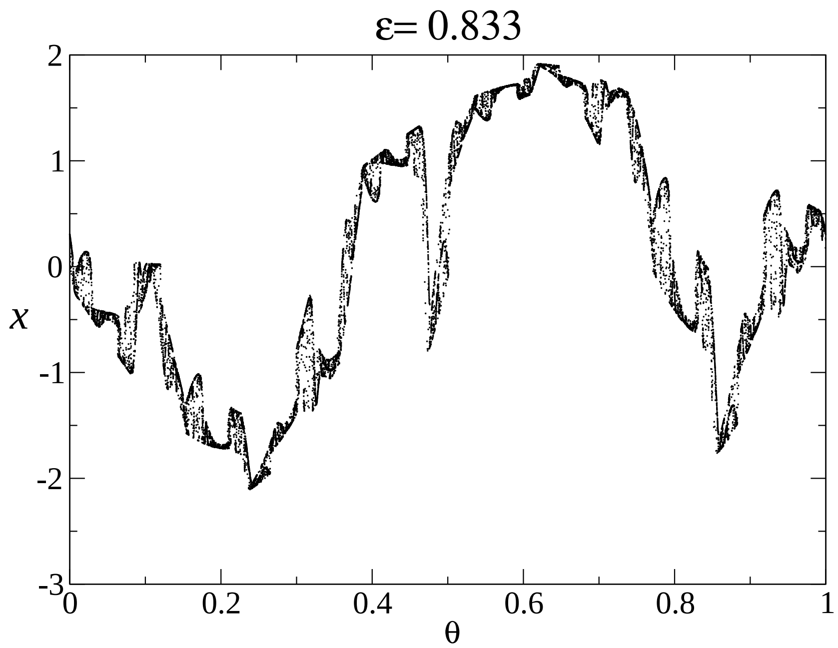




x-curve of an attracting torus close to breakdown, and

α-curves of its slow and fast directions, before the transition e.

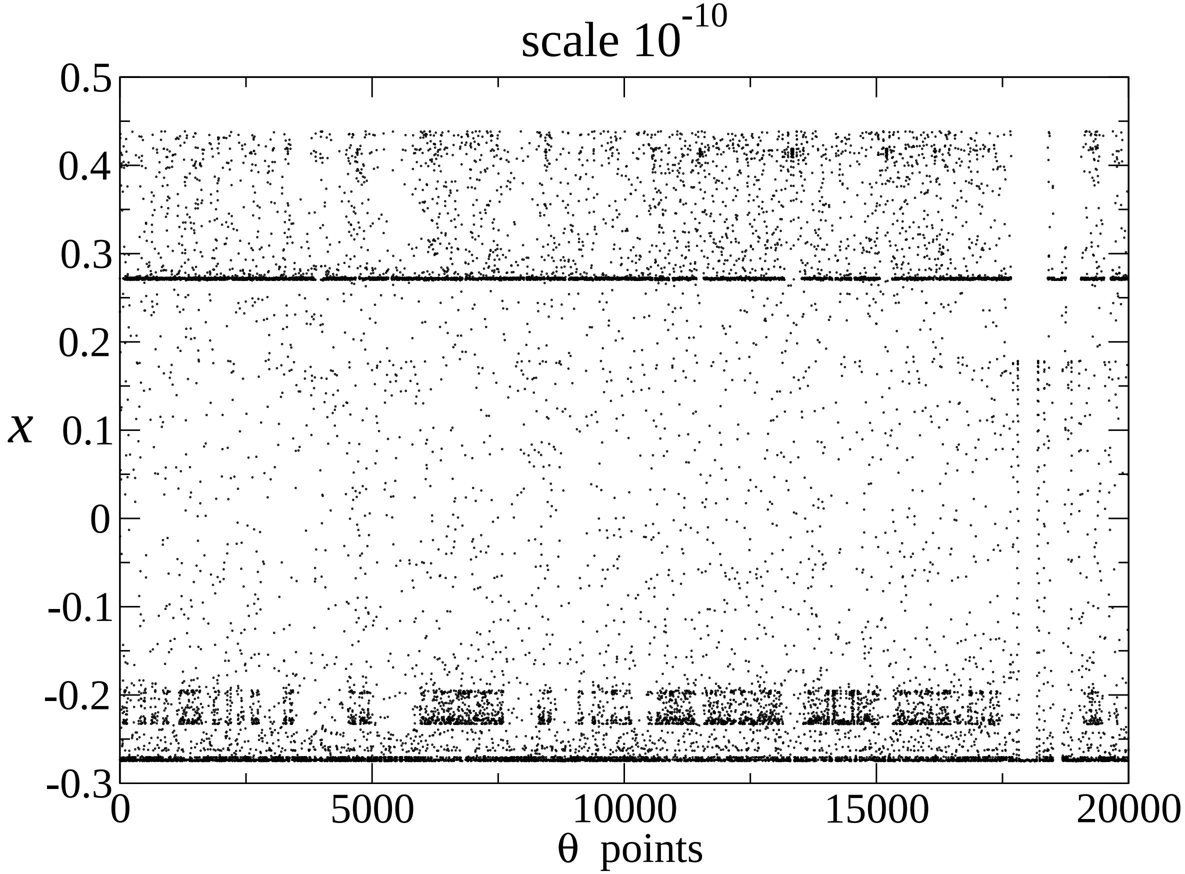
A word of caution:

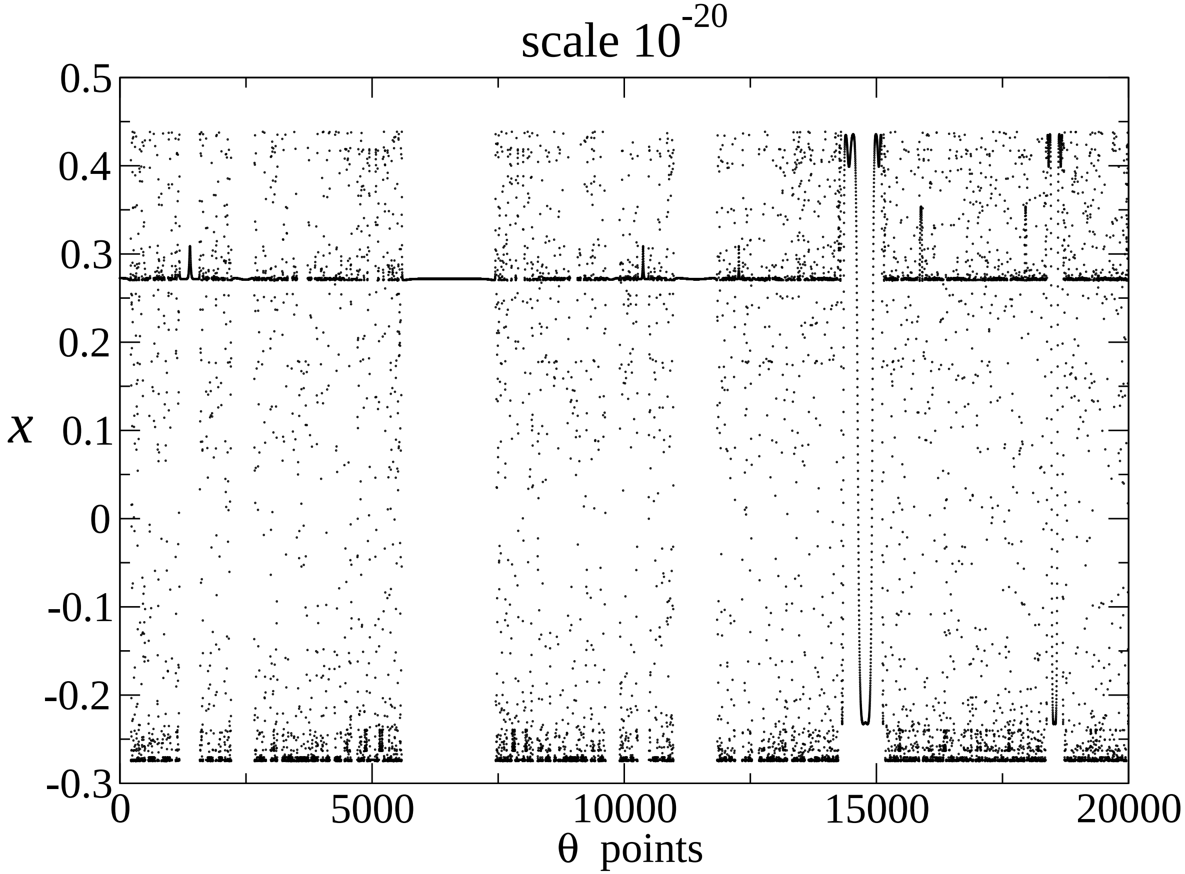


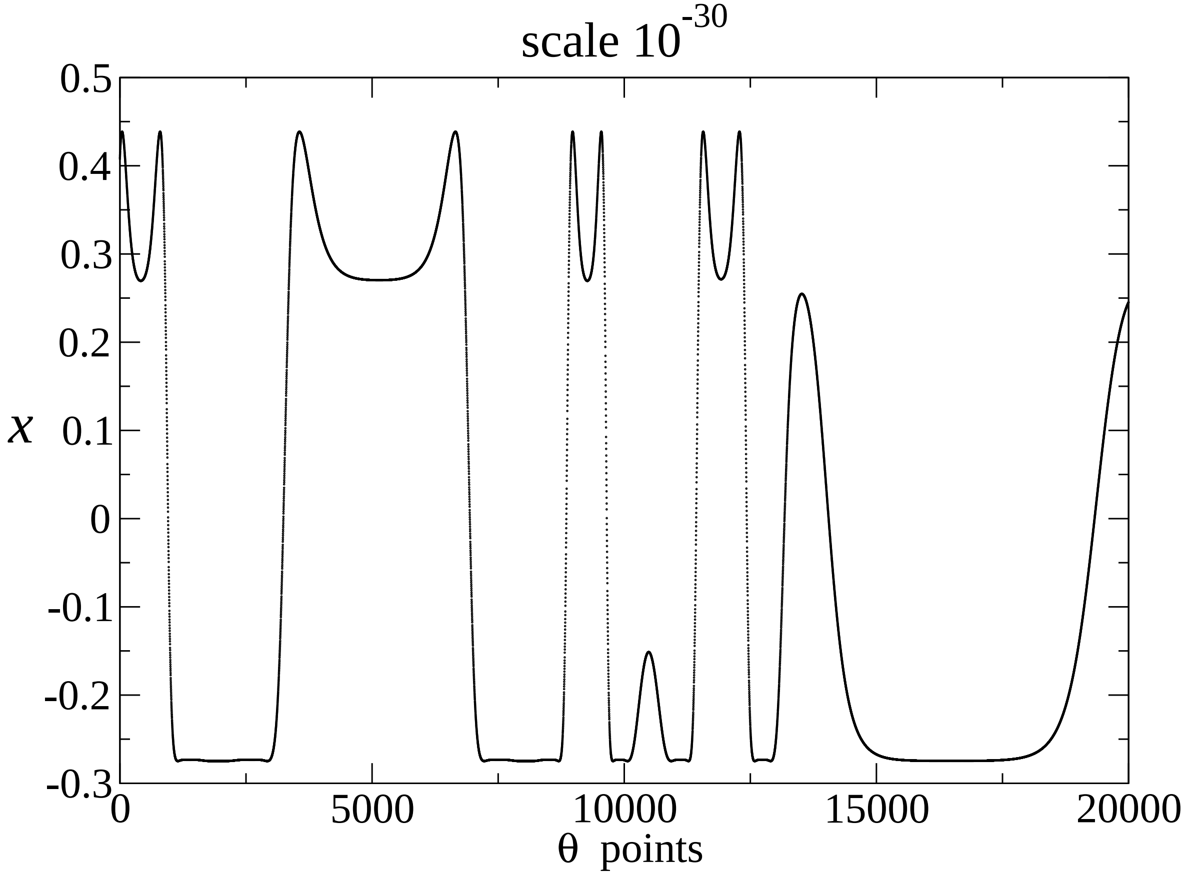
An SNA?

Zooming up using ≈ 50 decimal figures!

(See [Broer,Simó,Vitolo 05][Jorba,Tatjer 05] for rotating logistic map)


scale 10^{-10}
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
x
0
5000
10000
15000
20000
θ points


scale 10^-20
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
x
0
5000
10000
15000
20000
θ points


scale 10^-30
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
x
0
5000
10000
15000
20000
θ points

In summary, the formation of an strange non chaotic attractor for the linearized dynamics of an attracting torus produces a sudden growth of the spectrum and it is the prelude of the destruction of the torus and the formation of an strange chaotic attractor for the non linear dynamics.

4.2 Bundle merging causing breakdown

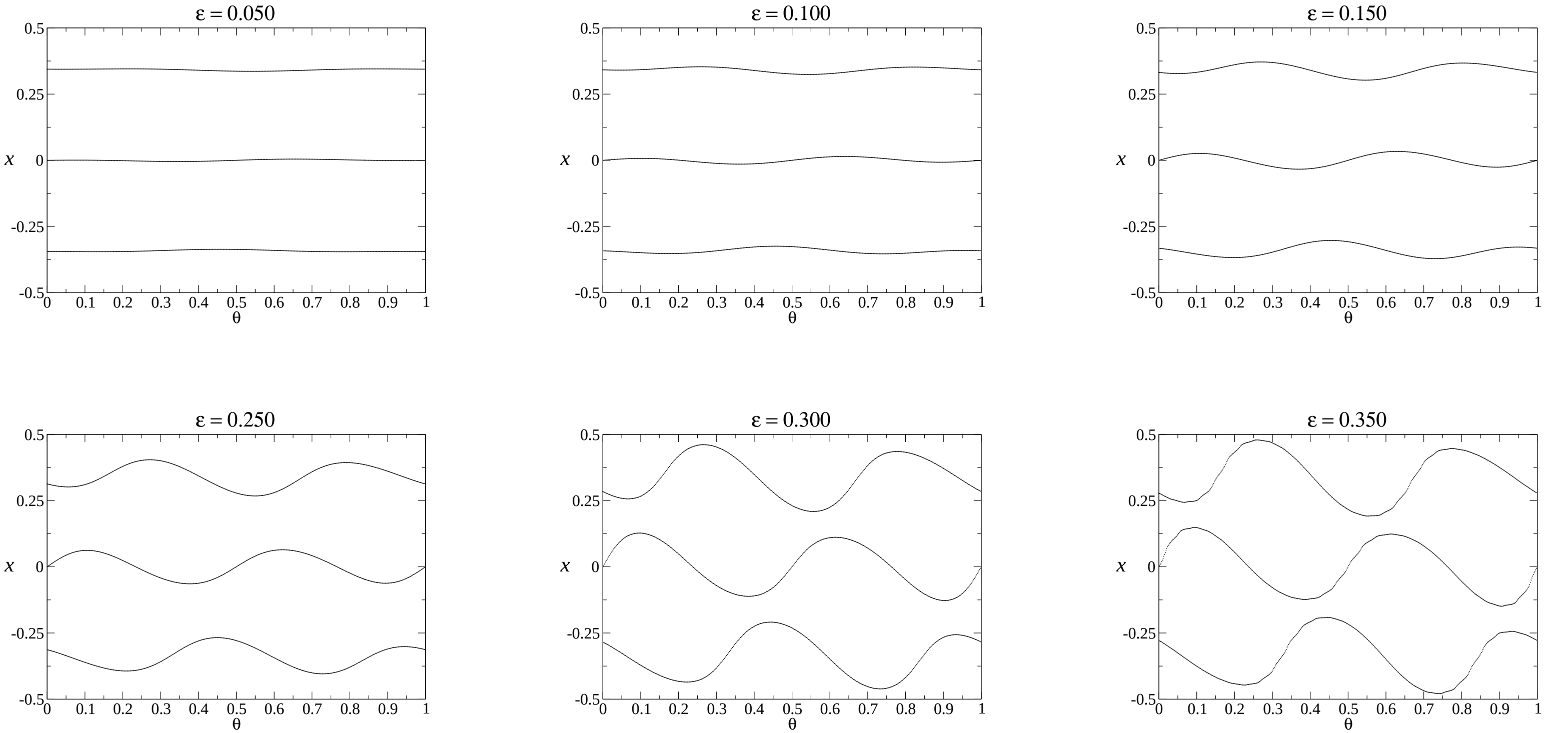
Model (conservative): rotating standard map

$$\begin{cases} \bar{x} = x + \bar{y} \pmod 1 \\ \bar{y} = y - \dfrac{\sin(2\pi x)}{2\pi}(\kappa + \varepsilon\cos(2\pi\theta)) \\ \bar{\theta} = \theta + \omega \pmod 1 \end{cases}$$

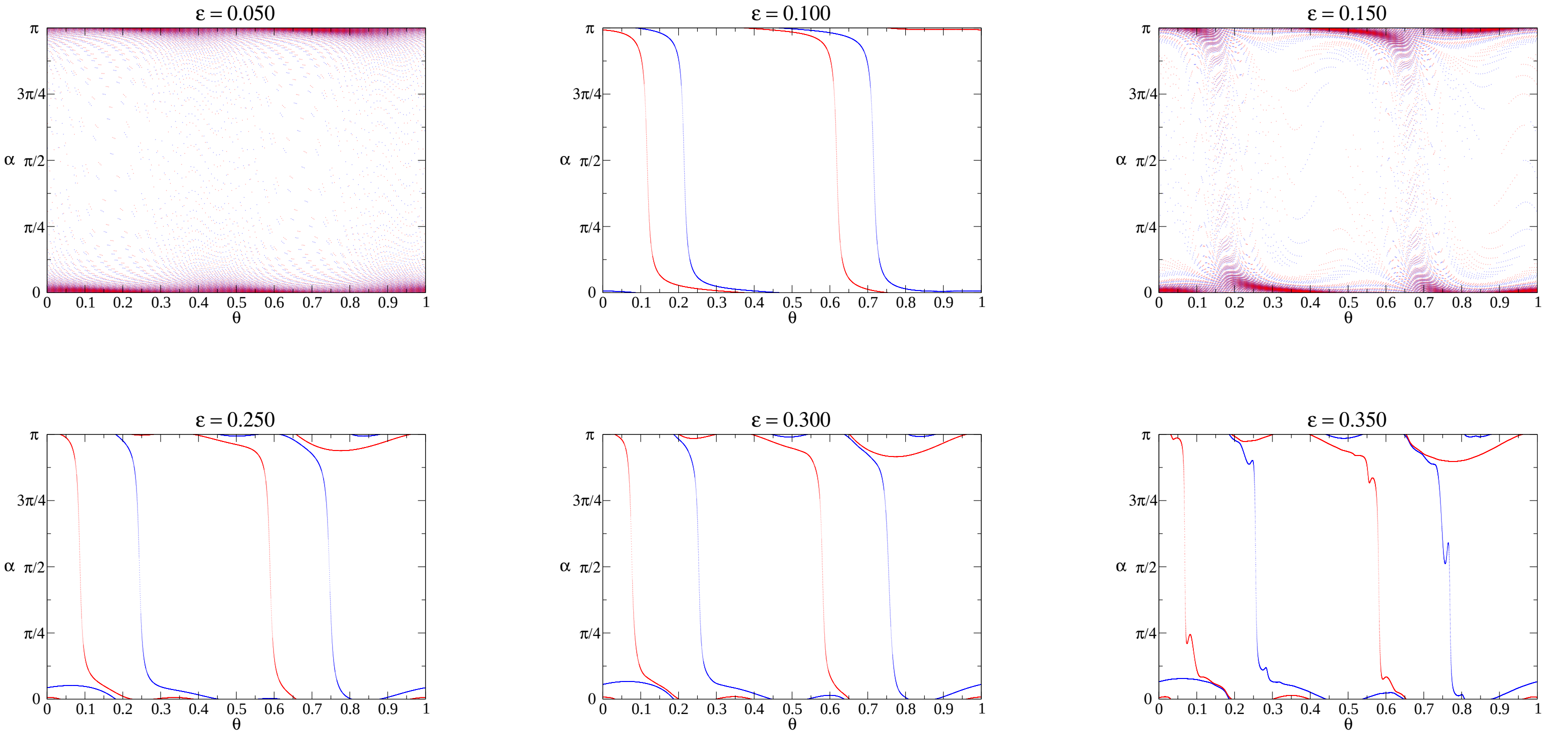
- $\kappa = 0.2$ is the parameter of the standard map;
- ε is the quasi-periodic parameter;
- $\omega = \sqrt[3]{\frac{19}{27} + \sqrt{\frac{11}{27}}} + \sqrt[3]{\frac{19}{27} - \sqrt{\frac{11}{27}}} - \frac{2}{3}$.

[Tompaidis 96][Haro 98]

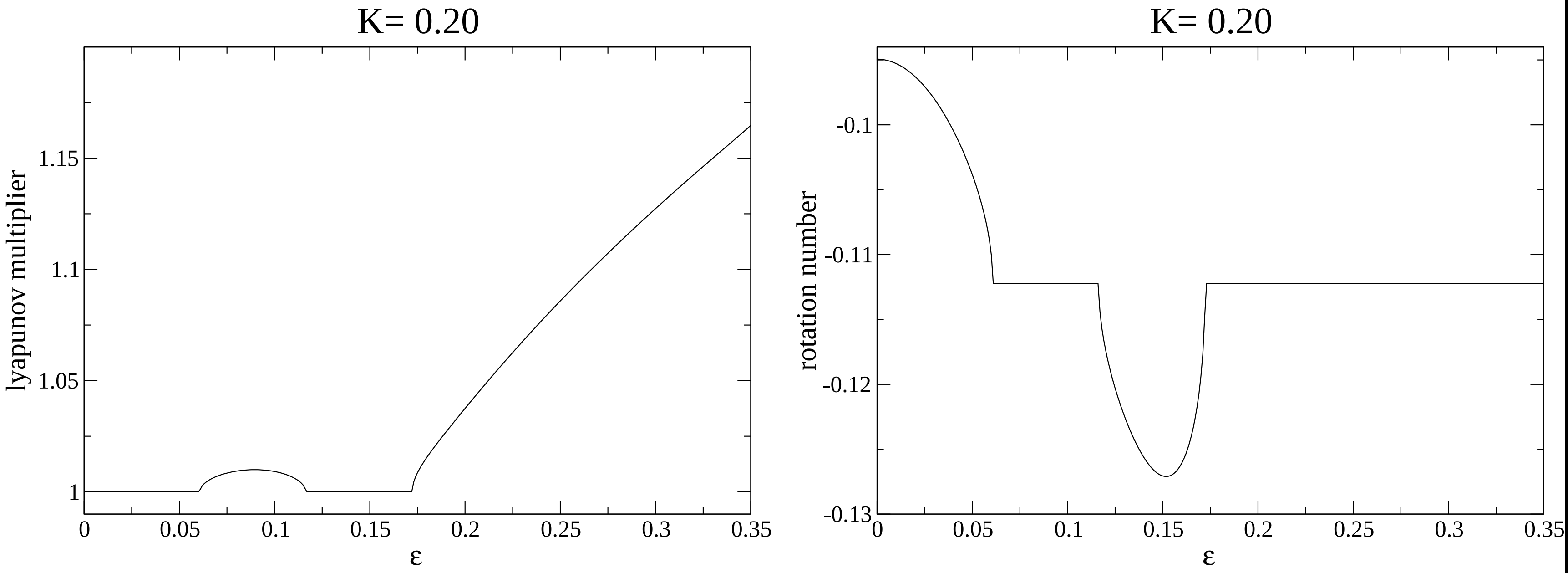
Continuation of a 3-periodic torus ...



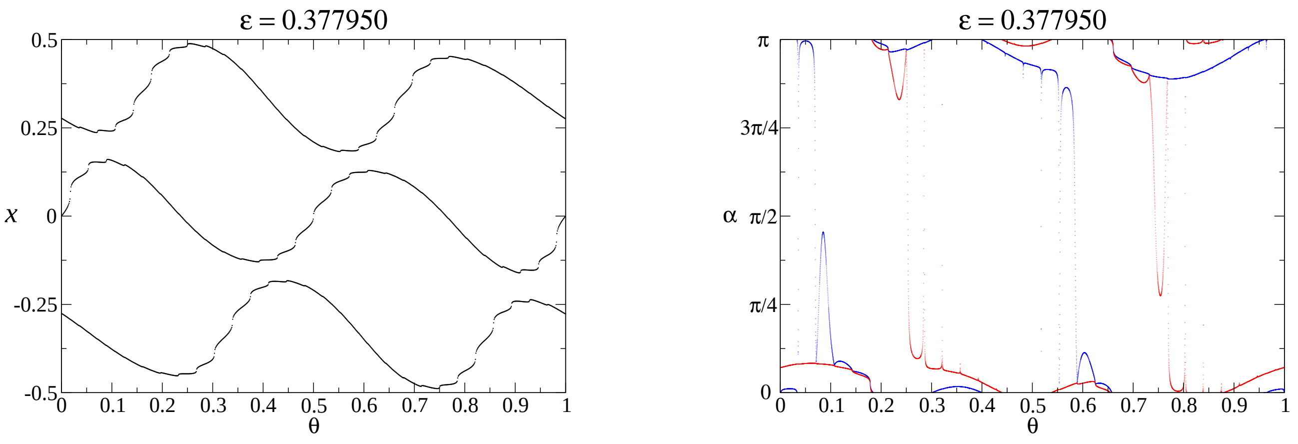
... and of its projectivized cocycle



Analysis of the cocycle

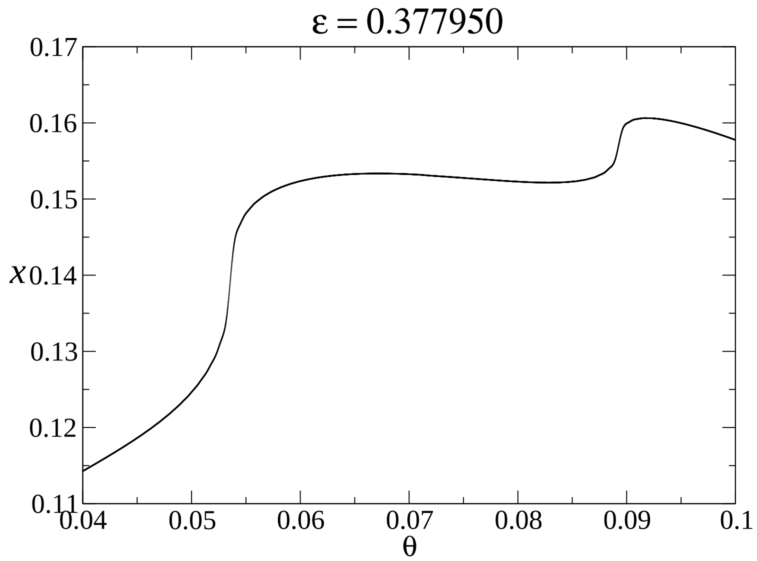


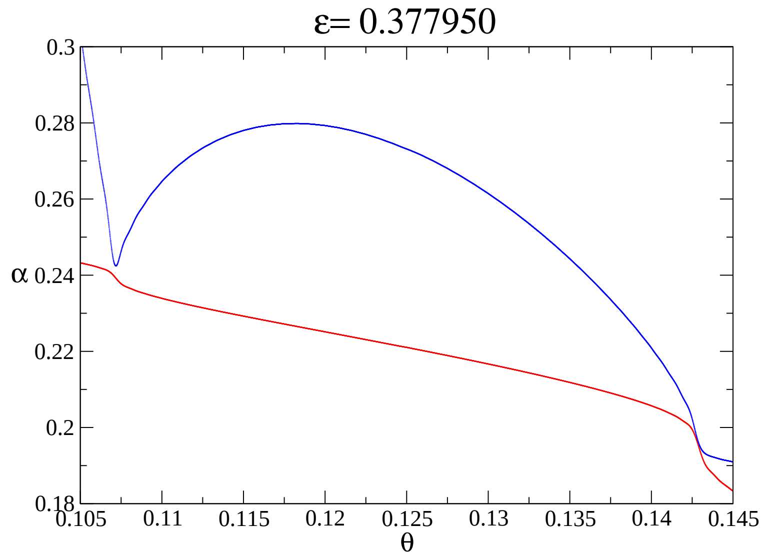
The bundle collapse and breakdown of the torus



x-curve of a saddle 3-torus close to breakdown, and

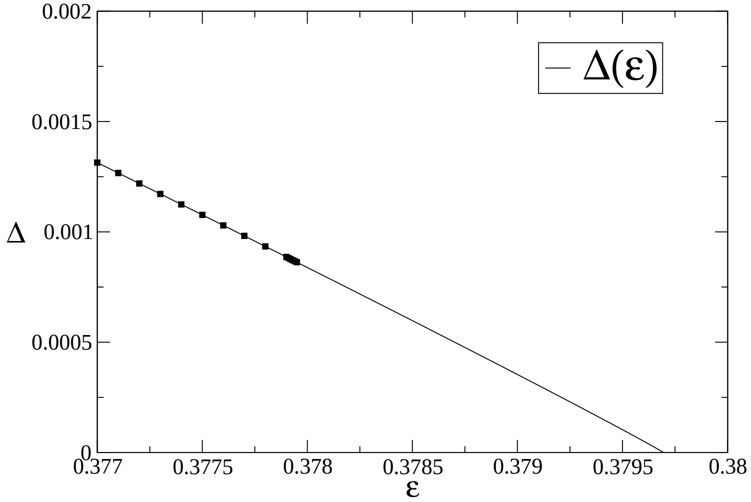
α-curves of its slow and fast directions.

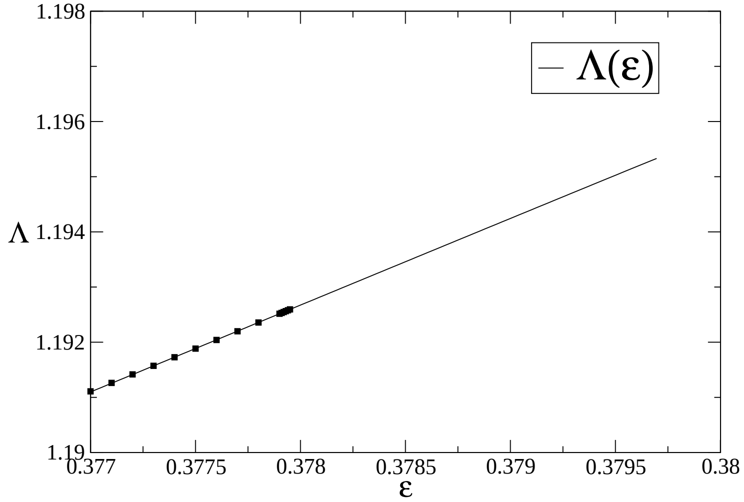




Zooms of the previous pictures

Quantitative estimates (fits of Δ and Λ)





$\varepsilon_c = 0.3796965 \pm 8 \cdot 10^{-7}$

$\alpha = 0.4063 \pm 7 \cdot 10^{-4}$

$\beta = 0.9693 \pm 4 \cdot 10^{-4}$

$\Lambda_c = 1.19533 \pm 9 \cdot 10^{-5}$

$A = -1.6 \pm 2 \cdot 10^{-1}$

$B = 1.00 \pm 3 \cdot 10^{-2}$

In summary, the formation of an strange non chaotic attractor for the linearized dynamics of a saddle type torus produces the sudden growth of the spectrum and the destruction of the torus.