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# *Numerical Integration Methods Applied to Astrodynamics and Astronomy (III)*

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# Outline

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- A Taylor N-body integrator.
- JPL model.
- Restricted  $(N + 1)$ -body problems: propagation of a region of the phase space.

## Applications:

- Apophis;
- probe accelerated by a constant low-thrust.

# *N*-body model

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Motion of the Solar System described by the  $N$ -body model:

$$\ddot{X}_i = \sum_{j=1, j \neq i}^{11} \frac{Gm_j(X_j - X_i)}{r_{ij}^3}, \quad i = 1, \dots, 11. \quad (1)$$

- We consider the mutual gravitational attraction of 9 planets, Sun and Moon.
- Units adopted: AU, kg, day.
- Are we neglecting effects which may be relevant?

# *N-body integrator*

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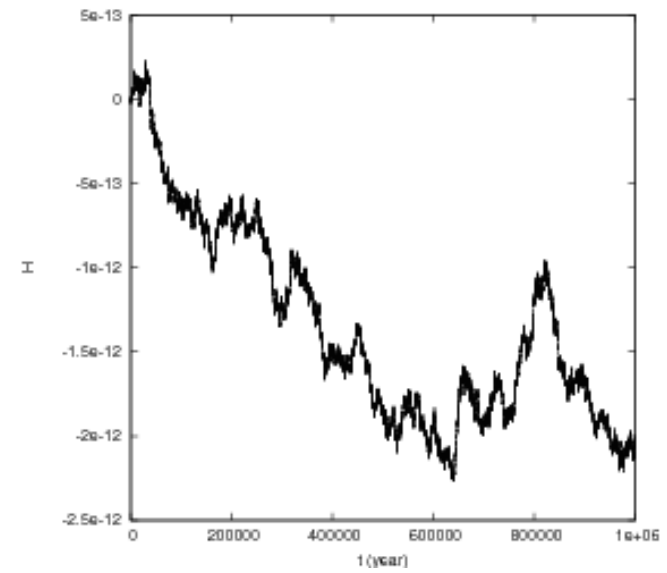
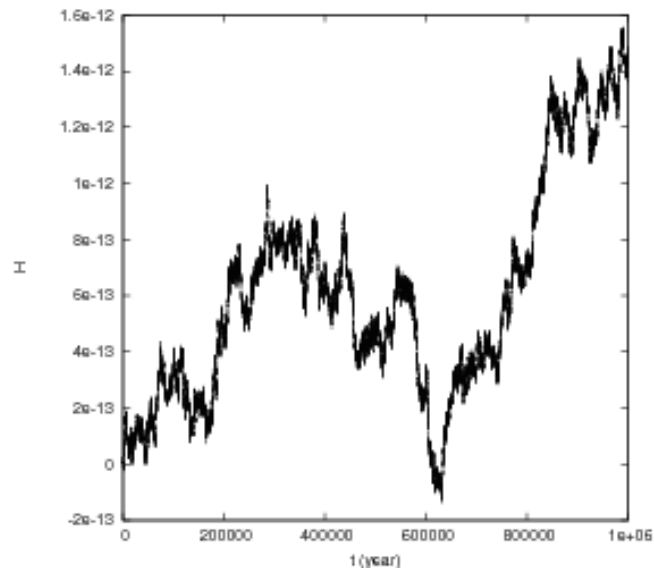
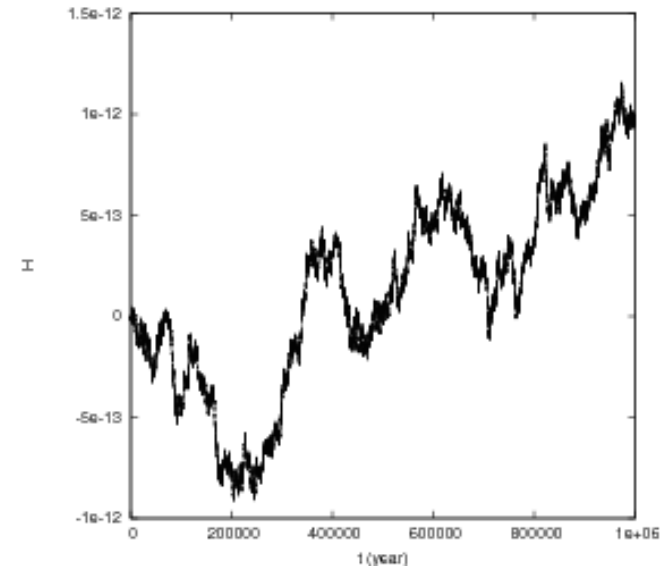
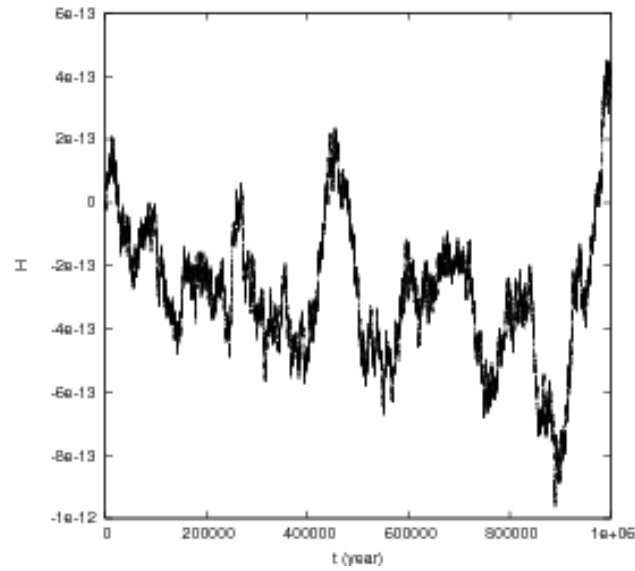
We take advantage of:

- the conservation of the centre of mass to compute the orbit of the Sun and thus keep fixed the centre of mass at the origin;
- the symmetry of the problem to save computational time.

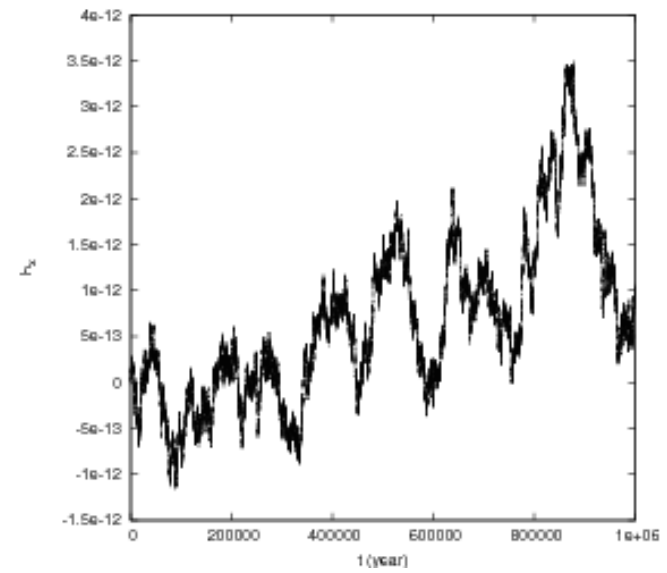
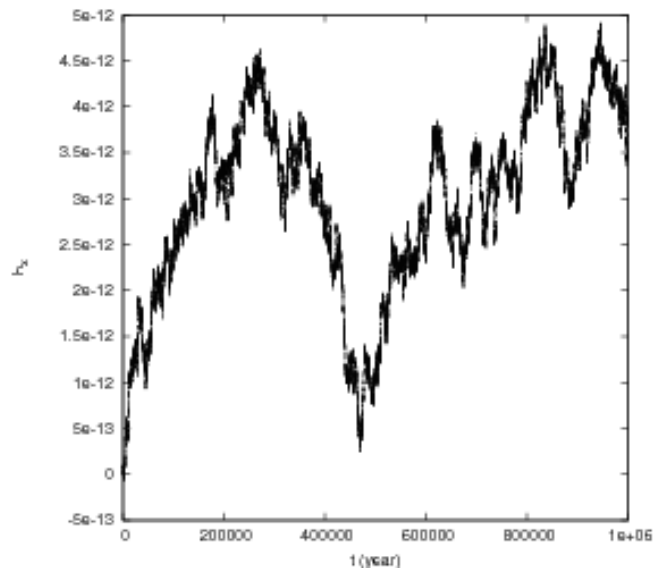
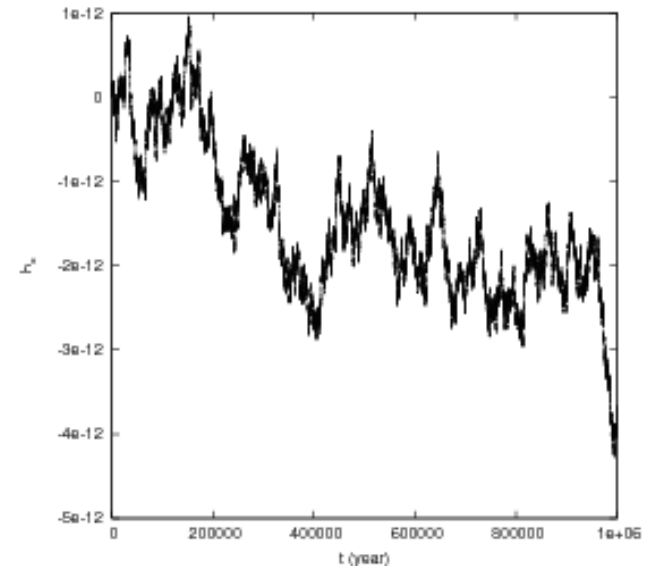
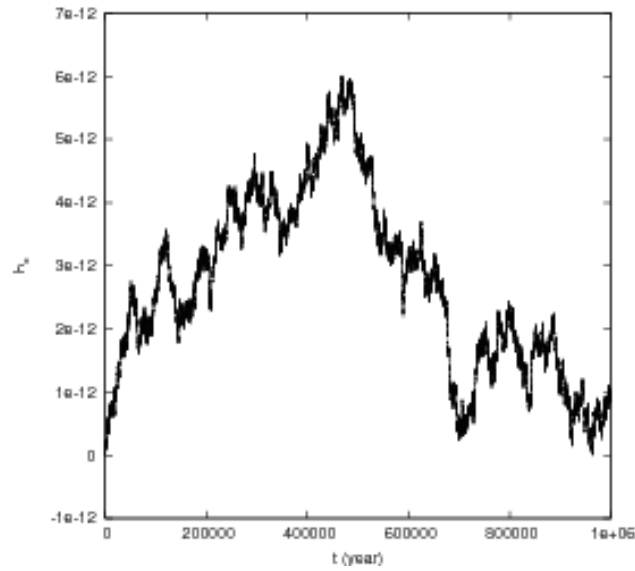
We performed simulations up to 1000000 years to check the accuracy obtained:

- we cannot distinguish between the variation obtained for the first integrals and a random walk;
- only source of error is the round off.

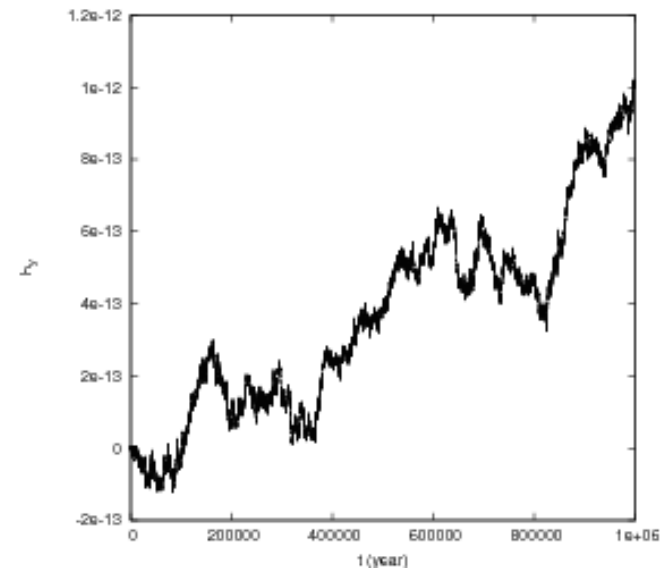
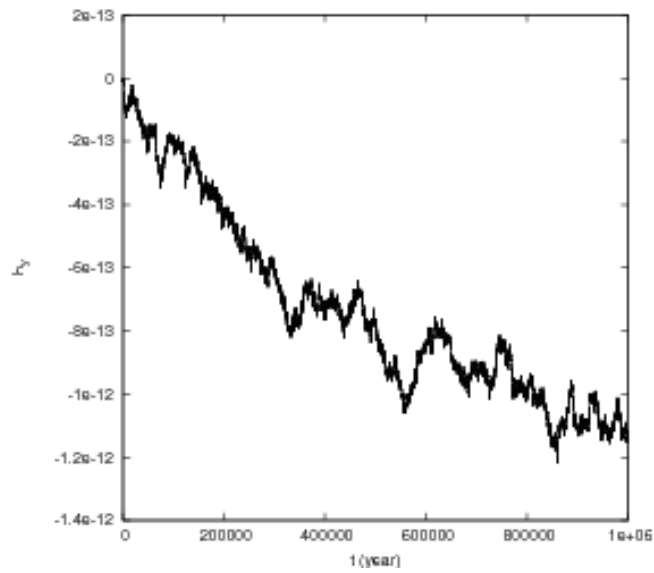
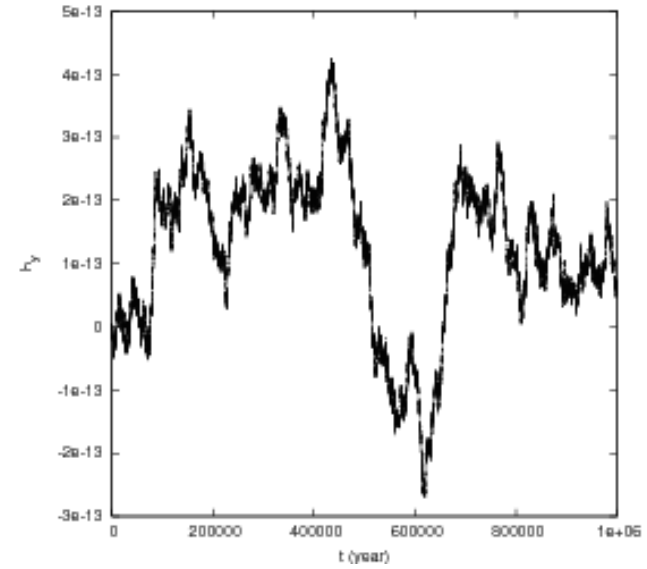
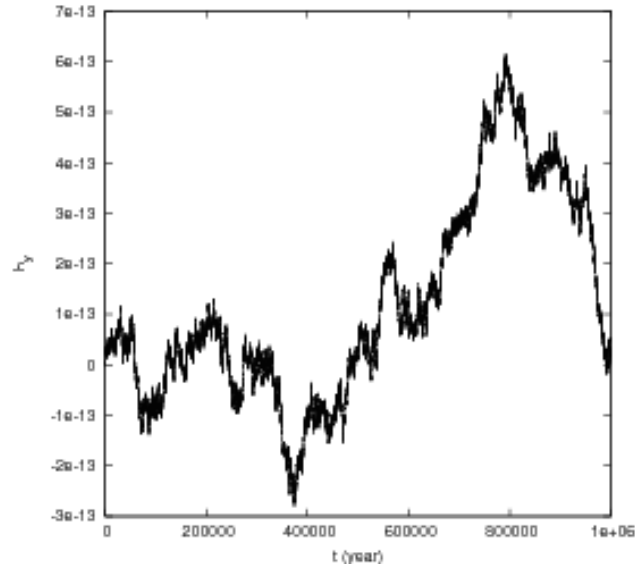
# Variation of $H$



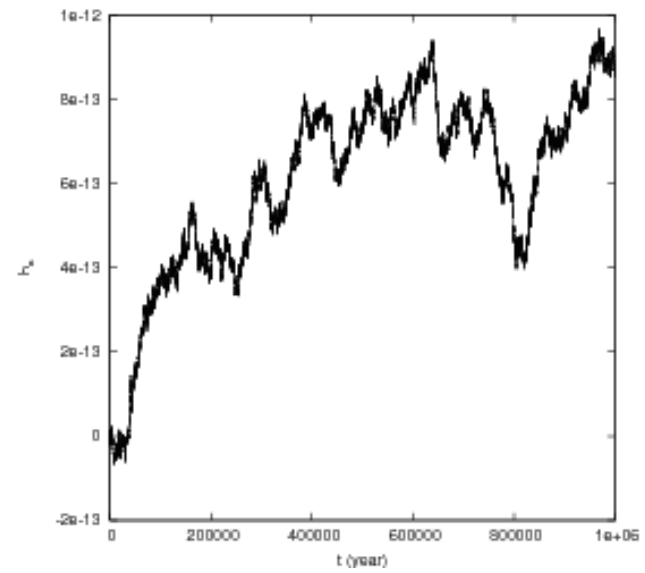
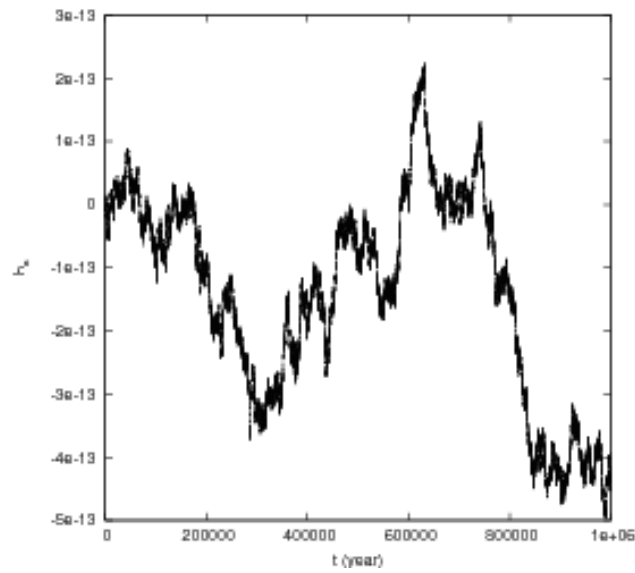
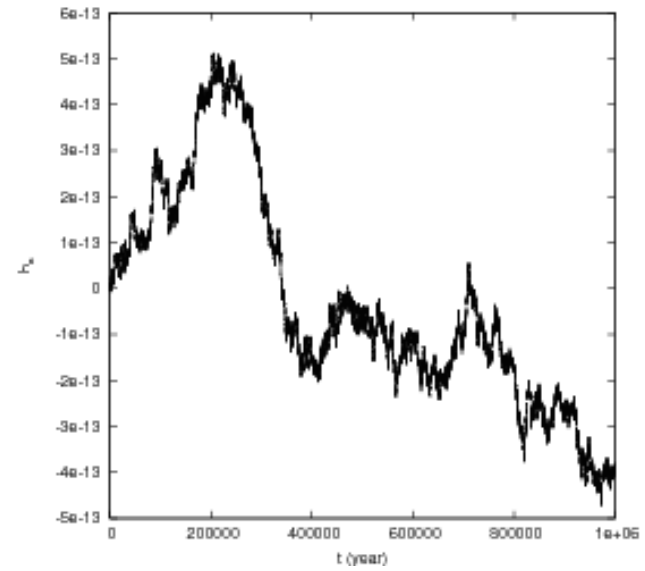
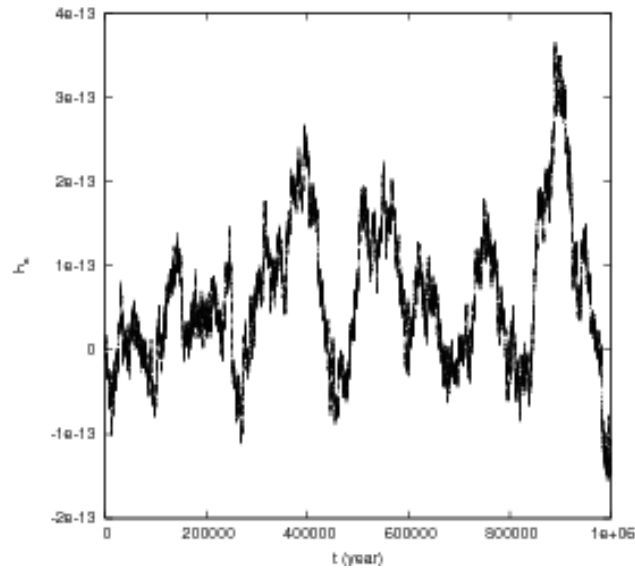
# Variation of $h_x$



# Variation of $h_y$



# Variation of $h_z$





# JPL Solar System Ephemerides

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We compare the results given by our  $N$  – body integrator with the JPL ephemerides DE405.

*Ephemeris*: tabulation of computed positions and velocities (and/or derived quantities) of an orbiting body at specific times.

JPL Ephemerides DE405:

- Files which store information to derive position of Sun, Earth, Moon and the planets.
- Information stored as coefficients of interpolatory polynomials.
- DE405 defined from December 9, 1599 to February 1, 2200.
- J2000 coordinates.

# J2000 coordinates

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J2000 Reference system:

- origin at the Solar System barycentre;
- $XY$  plane parallel to the mean Earth Equatorial plane;
- $Z$  axis orthogonal to this plane;
- $X$  axis points to the vernal point;
- $Y$  axis to have a positive oriented reference system.

All these references are taken at 2000.0 (Jan 1, 2000, at 12:00 UT).

# *The JPL model*

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Most significant effects considered in DE405:

- point mass interactions among Moon, planets and Sun;
- general relativity;
- Newtonian perturbations of selected asteroids;
- action upon the shape of the Earth from Moon and Sun;



# *The JPL model*

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- action upon the shape of the Moon from Earth and Sun;
- physical libration of the Moon, modelled as a solid body with tidal and rotational distortion, including both elastic and dissipational effects;
- the effect upon the Moon's motion caused by tides raised upon the Earth by Moon and Sun.

# *The JPL Horizons System*

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JPL Horizons system to find initial conditions for any body in the Solar System:

<http://ssd.jpl.nasa.gov/?horizons>

# *N* – body model vs JPL ephemerides

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Main discrepancies due to:

- *N*–body model does not consider relativistic correction for the orbit of Mercury;
- *N*–body model does not consider non–sphericity of the Earth for the motion of the Moon.

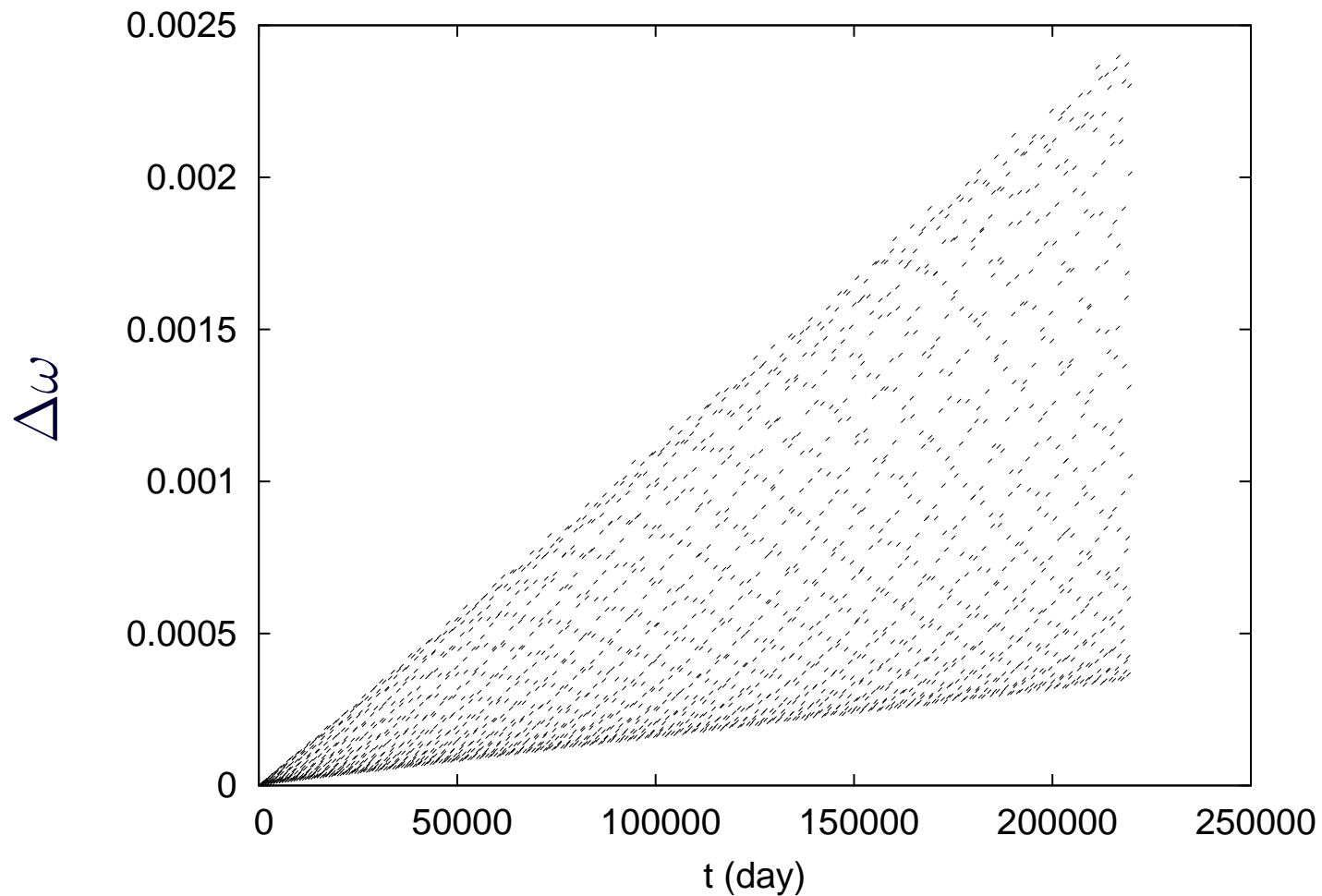
Effects:

- relativistic correction to the precession of perihelion of Mercury,  
 $\Delta\omega \approx 42.978''$  per century.
- $J_2$  correction to the precession of perigee of the Moon,  
 $\Delta\omega \approx -6 \times 10^{-5}$  rad per year.

# *N* – body model vs JPL ephemerides

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Mercury:



# Apophis

Restricted  $(N + 1)$ -body problem: massless particle affected by 9 planets, Sun and Moon.

Apophis: NEO exhibiting a close approach with the Earth on April 13, 2029.

<b>e</b>	0.191
<b>i</b>	3.331 deg
<b>a</b>	0.922 AU
<b>P</b>	323.5d





# *Apophis: numerical integration*

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Apophis' equation of motion:

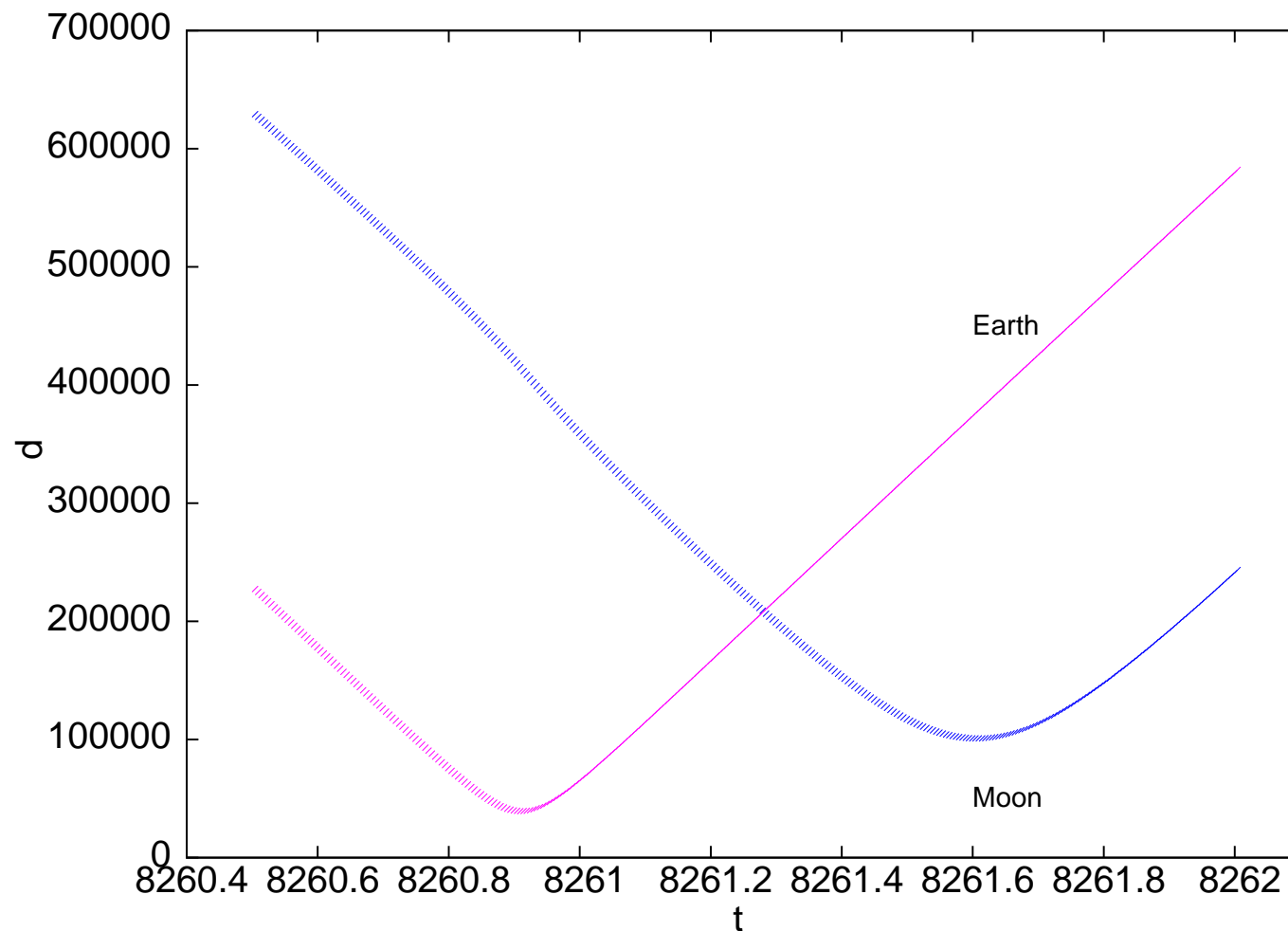
$$\ddot{X}_a = \sum_{j=1}^{11} \frac{Gm_j(X_j - X_a)}{r_{ja}^3}. \quad (2)$$

We propose an alternative method to integrate it:

- JPL ephemerides for the main bodies.
- Taylor integrator for Apophis: the step of integration is small enough to compute the jet for the planets as affected just by the mutual gravitational attraction.

# *Apophis: numerical integration*

Earth–Apophis and Moon–Apophis distance (km) as a function of time (day), starting from September 1, 2006.



# *Apophis: numerical integration*

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video

# *Apophis: propagation of a mesh of points*

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First way to propagate a region of the phase space is to propagate a mesh of points (box).

- Initial condition: uncertainty only in position; 7 km long on the tangent to the orbit direction and 3 km long on two other given orthogonal directions.
- The box stretches out along the direction of the orbit as time goes by.

# *Apophis: propagation of a mesh of points*

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Recent observations give:

- $\sigma_a \approx 9.6 \times 10^{-9}$  AU;
- $\sigma_M \approx 1.08 \times 10^{-6}$  degrees.

It follows:

- uncertainty of about 1.4 km for the position;

$$(9.6 \times 10^{-9} \times 1.5 \times 10^8)$$

- uncertainty of about 2.6 km along the velocity's direction.

$$(1.08 \times 10^{-6} \times \frac{\pi}{180} \times 1.5 \times 10^8)$$

# Apophis: propagation of a mesh of points

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First approximation of the motion of Apophis given by the two-body problem approximation.

Kepler's third law  $n^2 a^3 = \mu$  gives an estimate of the propagation of an uncertainty in position:

$$2na^3 \Delta n + 3a^2 n^2 \Delta a = 0.$$

It follows that any initial condition associated with  $a_i = a_0 + \Delta a = 1 + \Delta a$  will be characterised by

$$n_i = n_0 + \Delta n = 1 - \frac{3}{2} \Delta a.$$

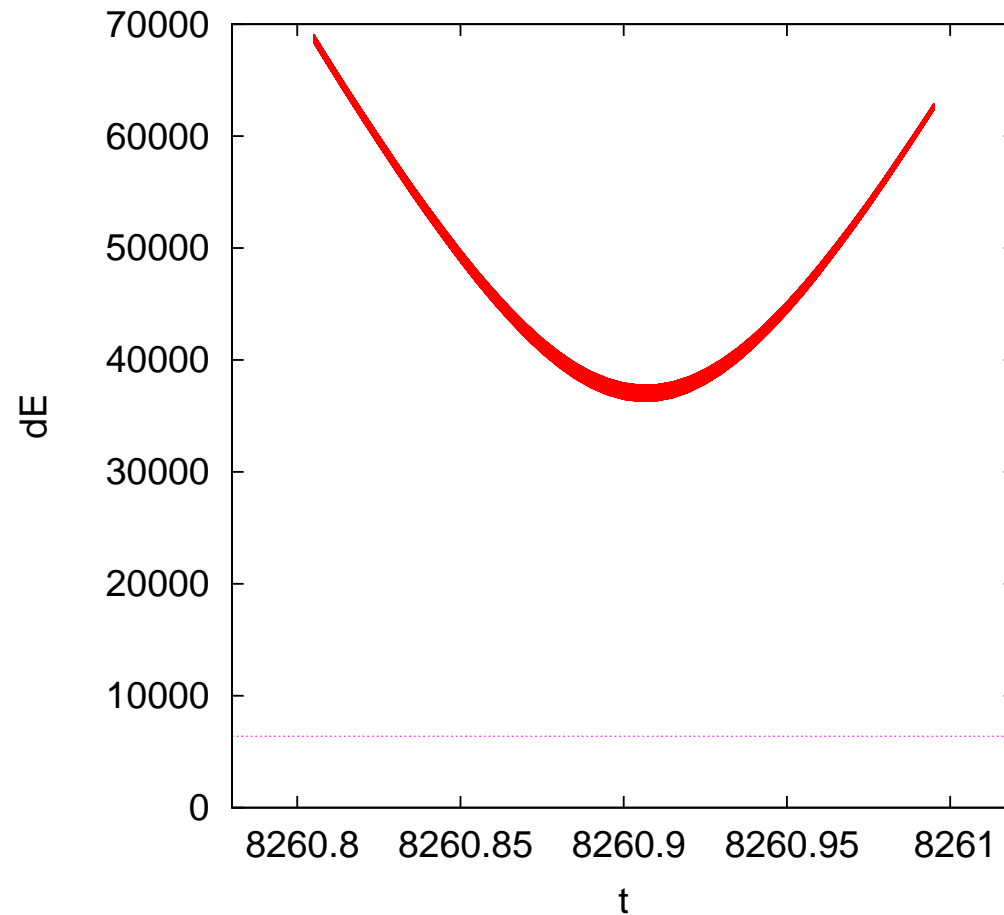
After  $m$  revolutions of the nominal orbit, the  $i$ -orbit will be displaced by an angle

$$\Delta\psi = -\frac{3}{2} m T_0 \Delta a = -3m\pi \Delta a \approx -10m \Delta a.$$

# *Apophis: propagation of a mesh of points*

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Earth–box distance (km) as a function of time (day) starting from September 1, 2006.



# *Apophis: variational equations*

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We considered first and second order variational equations to approximate the box along time.

- They provide accurate information on the evolution over time of close initial conditions with a lower computational effort.
- It would be possible to consider higher order variational equations to obtain a even better description of the dynamics.



# Apophis: variational equations

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Let us consider

$$\dot{x} = f(x), \quad x \in \mathcal{U} \subset \mathbb{R}^n, \quad (3)$$

and let  $\phi_t(x)$  be the solution of (??) with  $\phi_0(x) = x$ .

If  $f$  is of class  $\mathcal{C}^r$ , then  $\phi$  is also of class  $\mathcal{C}^r$  and thus:

$$\phi_t(x + h) = \phi_t(x) + \frac{\partial \phi_t(x)}{\partial x} h + \frac{1}{2} h^T \frac{\partial^2 \phi_t(x)}{\partial x^2} h + \dots \quad (4)$$

The variational eqns of order  $j$  are the differential eqns satisfied by  $\frac{\partial^j \phi_t(x)}{\partial x^j}$ .

# *Apothis: variational equations*

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First variational equations:

$$\frac{d}{dt} \frac{\partial \phi_t(x)}{\partial x} = Df(\phi_t(x)) \frac{\partial \phi_t(x)}{\partial x}. \quad (5)$$

Second variational equations:

$$\frac{d}{dt} \frac{\partial^2 \phi_t(x)}{\partial x^2} = \left( \frac{\partial \phi_t(x)}{\partial x} \right)^T D^2 f(\phi_t(x)) \left( \frac{\partial \phi_t(x)}{\partial x} \right) + Df(\phi_t(x)) \frac{\partial^2 \phi_t(x)}{\partial x^2}. \quad (6)$$

Initial conditions:

$$\left. \frac{\partial \phi_t(x)}{\partial x} \right|_{t=0} = I, \quad \left. \frac{\partial^2 \phi_t(x)}{\partial x^2} \right|_{t=0} = 0.$$

# *Apophis: variational equations*

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In components, second variational equations read:

$$\frac{d}{dt}w_{k,ij} = \sum_{q,p=1}^n \frac{\partial^2 f_{w_k}}{\partial w_q \partial w_p} w_{q,i} w_{p,j} + \sum_{q=1}^n \frac{\partial f_{w_k}}{\partial w_q} w_{q,ij},$$

where  $w_k$  denotes the  $k$ -component of  $\phi_t(x)$ .

In our case, we can reduce the number of equations:

- first variational eqns from 36 to 18;
- second variational eqns from 216 to 63.

# *Apophis: variational equations*

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We take an orthonormal basis composed by the tangent to the orbit vector and by two vectors orthogonal to this one and we consider how an uncertainty in position is reflected along these directions.

We apply the variational terms of first and second order to this orthonormal basis at each step of integration.

Recall:

$$\phi_t(x + h) = \phi_t(x) + \frac{\partial \phi_t(x)}{\partial x} h + \frac{1}{2} h^T \frac{\partial^2 \phi_t(x)}{\partial x^2} h + \dots$$

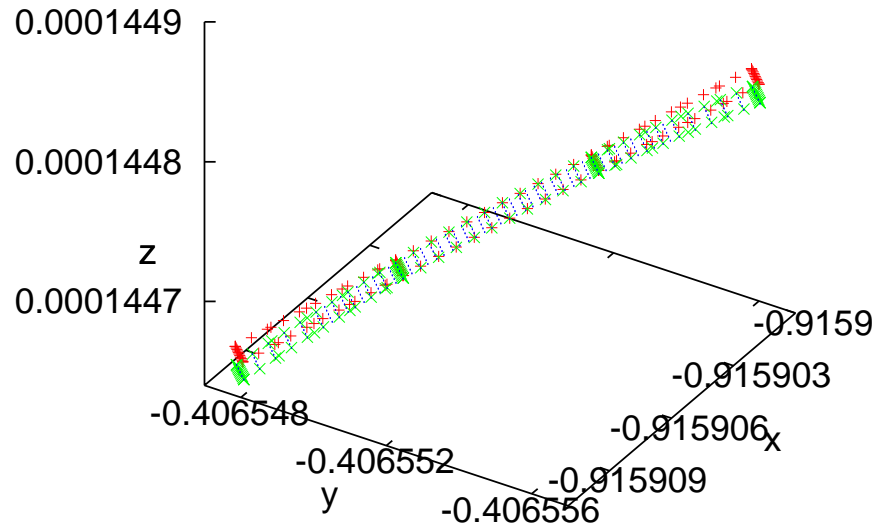
# *Apophis: variational equations*

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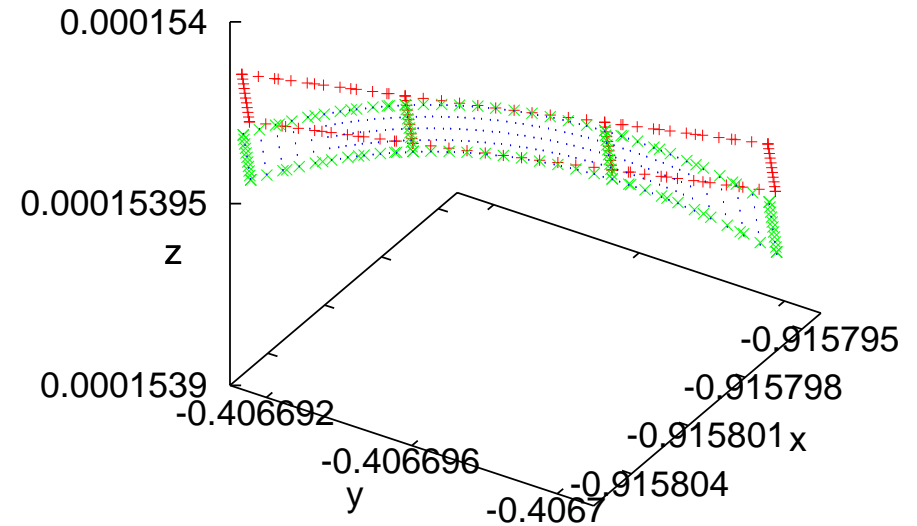
The results show that we need the quadratic approximation given by the second variational equations to describe the first estimated close approach of Apophis with the Earth (Friday 13 April 2029).

After that, the dynamics becomes very sensitive to the initial conditions and there exists an instant of time from which it is no longer possible to predict the behaviour of the box with this approach.

# Apophis: variational equations



April 13th 2029 22:48 h



April 13th 2029 23:02 h

Blue: cloud of points; red: linear approximation; green: quadratic approximation.

# Computational time

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- CPU computational time with Intel Xeon CPU 2.66GHz.
- Numerical integration from 1 September 2006 to 13 April 2029.

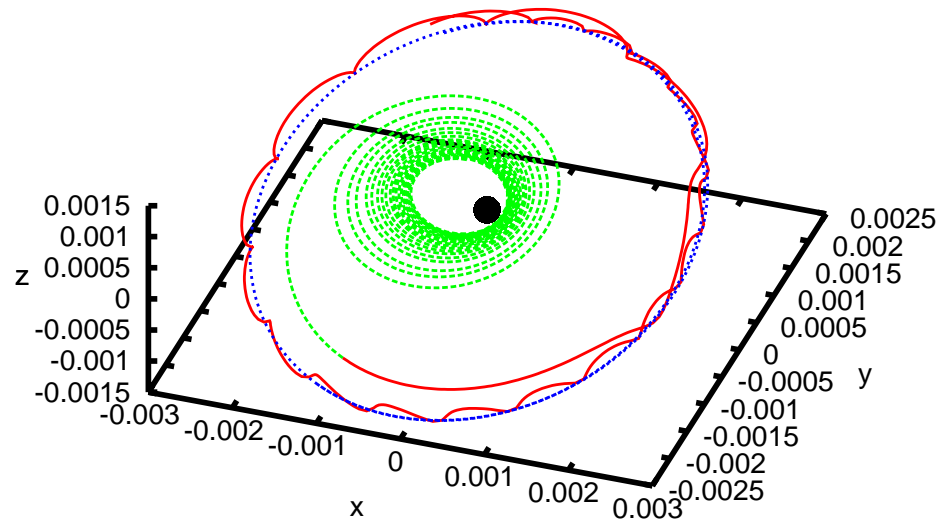
<b>Equations</b>	<b>CPU time (s)</b>
vector field	1.169
first variational	1.572
second variational	11.129

# Probe

Transfer of a probe from the Earth to the Moon.

Forces considered:

- gravitational attraction of 9 planets, Sun and Moon;
- constant low-thrust in a given direction.





# Low thrust: numerical integration

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$$\ddot{X}_{sat} = \sum_{j=1}^{11} \frac{Gm_j(X_j - X_{sat})}{r_{j sat}^3} + F_T \frac{V_{sat} - V_c}{\|V_{sat} - V_c\|}, \quad (7)$$

where  $V_c \in \mathbb{R}^3$  is the velocity of the body w.r.t. which we are accelerating/braking and  $F_T$  is the thrust magnitude.

Observed significant loss of digits:

- Initial condition w.r.t. Solar System Barycenter, far from Earth – Moon neighbourhood.
- Initial error grows in time because of the dynamics.

# *Low thrust: numerical integration*

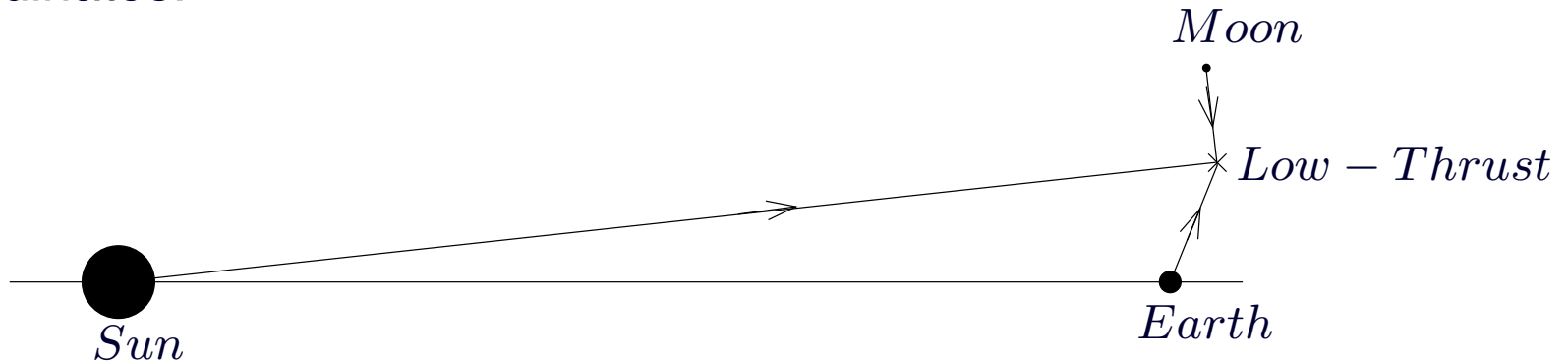
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(??) faces the problem in an inertial reference system with origin at the Solar System Barycentre, but the motion of the spacecraft takes place in the Earth–Moon neighbourhood. This means that we are dealing with an initial condition which keeps few information about the dynamics we are interested in.

On the other hand, the extra force introduced is not as big as to gain altitude soon and thus the spacecraft performs thousands of revolutions before escaping from the Earth.

# Low thrust: numerical integration

New coordinates:



If  $Y = X_{sat} - X_c$  and  $VY = V_{sat} - V_c$ , then

$$\ddot{Y} = \sum_{j=1}^{11} \frac{Gm_j(X_j - X_c - Y)}{r_{jcY}^3} - \ddot{X}_c + F_T \frac{VY}{\|VY\|}. \quad (8)$$

Planets' data still from JPL ephemerides.

# Low thrust: numerical integration

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Eq.	relative error	absolute error
(??)	1.56e-9	1.53e-9
(??)	7.16e-13	7.e-13

Eq.	relative error	absolute error
(??)	1.03e-5	1.41e-7
(??)	1.45e-9	2.e-11

Table 1: Relative and absolute error in position (up) and velocity (down) obtained by integrating equations (??) and (??) starting from a same initial condition up to 730.5 days. The errors refer to the results obtained in double precision and considering the solution obtained with quadruple precision as the exact one.

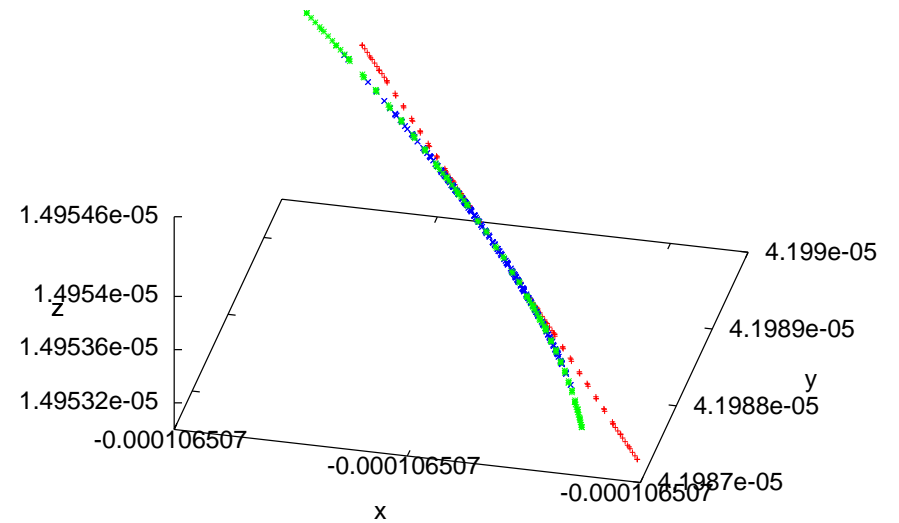
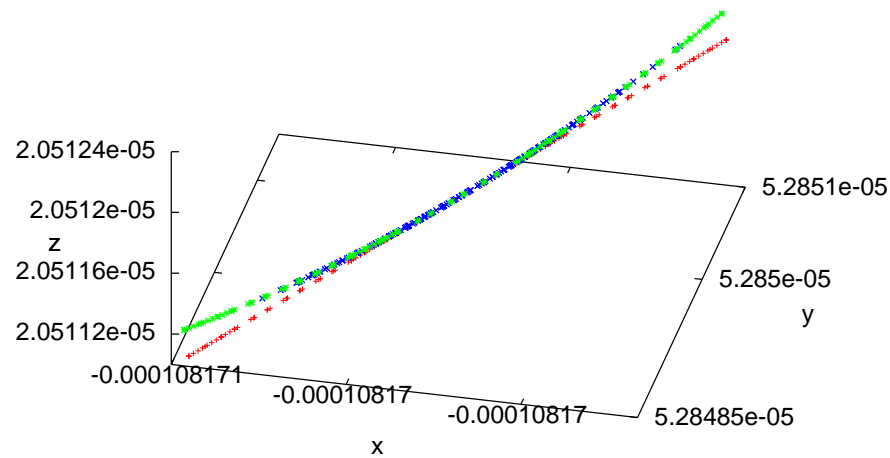
# *Low thrust: propagation of a region*

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We apply the same concepts explained for Apophis: propagation of a cloud of initial conditions and usage of first and second order variational equations.

- Initial condition: uncertainty only in position; 30 cm in the three directions.
- The box stretches out along the direction of the orbit as time goes by.
- Quadratic approximation needed according to the distance to a main body and to the time integrated.

# Low thrust: propagation of a region



Blue: cloud of points; red: linear approximation; green: quadratic approximation.

# *Low thrust: considerations*

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The instant of performing the manoeuvre becomes a critical aspect in the propagation of a region of the phase space.

- We can decide to fix the same value of time for each point in the box: the goal of the mission might not be achieved.
- We can decide to set another requirement to be fulfilled: it would mean to consider as many missions as the points in the box.

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