Continuous averaging

Heuristic arguments:

Let $\hat{a}(z)$ be an analytic vector field and consider

$$\dot{z} = \hat{a}(z) \quad z \in \mathcal{M} \quad (\text{in dim.})$$

We want a c.o.v. $z \mapsto w(z, \hat{s})$; $\hat{s} > 0$

defined as a shift along solutions of

$$\frac{\partial w}{\partial \hat{s}} = F(w, \hat{s}) \quad w(z; 0) = \Theta z$$

Then we get

$$\dot{w} = a(w, \hat{s})$$

and we want an easy way to get a simpler $\sim \hat{a}$

$$\Theta \frac{a}{\hat{s}} = -[F, a]$$
where \[ [v_1, v_2] = \partial_{v_2} v_1 - \partial_{v_1} v_2. \]

If the system we are considering is non-autonomous,

\[
\frac{\partial \mathbf{a}}{\partial s} = \frac{\partial \mathbf{F}}{\partial t} - [\mathbf{F}, \mathbf{a}]
\]

\( \mathbf{F} \) is "more or less" arbitrary, but a good choice (Trenchev's choice) is \( \mathbf{F} = \varepsilon \mathbf{a} \) with \( \varepsilon \) linear (operator).

As a particular case, and the one he considers, suppose \( \hat{\mathbf{a}} = \varepsilon \hat{\mathbf{v}} \), \( \mathbf{a} = \varepsilon \mathbf{v} \) with

\[
\mathbf{v}(z, t, \varepsilon, \delta) = \mathbf{v}^0 + \sum_{k \geq 0} \mathbf{v}^k(z, \varepsilon, \delta) e^{ik_t} + \sum_{k < 0} \mathbf{v}^k(z, \varepsilon, \delta) e^{ik_t}
\]

\[ = \mathbf{v}^0 + \mathbf{v}^+ + \mathbf{v}^-
\]

Then we could take choose \( \varepsilon \mathbf{v} = i(\mathbf{v}^+ - \mathbf{v}^-) \).
Why??

\[ V_\delta = (\xi V)_t + \varepsilon [\xi V, \mathcal{R}] \]

Expanding as Fourier series we get:

\[ V^k = -|k| V^k + i\varepsilon \text{sgn}(k)[V^0, V^k] + \varepsilon [\xi V, V^k], \quad k \in \mathbb{Z} \]

\[ V^k(z, \varepsilon, 0) = \hat{V}^k(z, \varepsilon), \quad k \in \mathbb{Z} \quad (\text{sgn}(0) = 0) \]

If we neglect \( O(\varepsilon) \) terms and the initial system as initial condition, we get

\[ V^k(z, \varepsilon, \delta) = e^{-|k|\delta} \hat{V}^k(z) \]

Now if \( \delta \) is increasing, high-freq. terms decrease.

If we just neglect \( \varepsilon [\xi V, V^k] \) we can also solve:

\[ V^k(z, \varepsilon, \delta) = e^{-|k|\delta} \hat{V}^k(z) \text{sgn}(k)(z) \]

g(t) flow of \( \dot{z} = V^0(z) \).
there we have to work with complex time (that's why we asked for analyticity).

\( \delta \) can be as big as \( \delta \sim \alpha/\varepsilon \) with

\( \alpha = \text{Im} \) (closest singularity of \( g^t \) to the real axis)

Trendel's theorem:

Under some non-singular, non-degenerate conditions, there is a cool \( v \), \( f(z,t,\varepsilon) \to v \) in \( \varepsilon \), s.t.:

1. \( f(z,t,\varepsilon) = z + O(\varepsilon) \)

2. \( \varepsilon \hat{v} \xrightarrow{\delta} \hat{v}^0(z) + \varepsilon^2 \hat{v}_x(z,t,\varepsilon) + \hat{v}(z,t,\varepsilon) \)

and there is a \( C_0 > 0 \) s.t.

\[ |\hat{v}(z,t,\varepsilon)| \leq C_0 e^{\alpha/\varepsilon} \]

\( \alpha \) is related to an acuity city domain
Some ideas of the proof

The "heuristic" part required the dropping of terms of high order.

For the complete proof:

1. Expand as Fourier series

2. You get an infinite system of coupled pols's

3. Use the method of majorants to prove existence of a solution for the whole system

4. **Moreover**, you should be able to sum the terms you get

5. **These bounds need to be very sharp** in very concrete domains

6. Finally you compare the final system with the original system.
Trescher's averaging and splitting of separatrices

Trescher studied the case of

\[ x = \frac{\partial \hat{H}}{\partial y}, \quad \dot{y} = -\frac{\partial \hat{H}}{\partial x} \]

\[ \hat{H} = \hat{H}^0(x,y) + e^{it/\varepsilon} \hat{H}^{(+)}(x,y) + e^{-it/\varepsilon} \hat{H}^{(-)}(x,y) \]

\[ \hat{H}^0 = y^{1/2} - \cos x - 1 \]

\[ \hat{H}^{+} = B_+ e^{ix} + B_- e^{-ix} \]

\[ (\bar{B}_+ = B_- = B_+ = B_-) \]

i.e. pendulum with rapidly oscillating point

System close to

\[ x = \frac{\partial \hat{H}^0}{\partial y}, \quad \dot{y} = -\frac{\partial \hat{H}^0}{\partial x} \]

PENDULUM.

Applying (i.e. solving the systems of PDE's) the averaging method we get

\[ A = 4 \sigma \varepsilon \left| P^+(\varepsilon) \right| + O(\sigma^2) \]

where \( P^+(0) \neq 0 \), \( P^+ \) is a "Melnikov integral" and

\[ \sigma \approx \frac{-\pi}{2\varepsilon} \]

\[ \varepsilon \approx \sigma \]
References: (D.V. Treshev) (Googk), preprints in his page

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  (preprint in English) (Russ. J. M. P. 1998)

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  (S. Agar, 1995)

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