Personal notes on renormalization

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1 Dynamics of the logistic map

This will be a fast (and selective) review of the dynamics of the logistic map. Let us consider the logistic map in the multiplicative form,

$$\bar{x} = \ell_\alpha(x) = \alpha x(1-x). \tag{1}$$

It can be easily checked that for the parameter $\alpha \in [0,4]$, any point $x_0 \in [0,1]$ will remain in the interval $[0,1]$ under iteration by (1). This property can be used to do the following numerical computation which sketches the dynamics of the logistic map. For each value of the parameter α, we take an initial point $x_0 \in (0,1)$ and we iterate it N_1 times by the map, if N_1 is big this will bring the iterated point near to the attractor of the map for the given value of α. Finally we have plot the next N_2 iterations, which will be an approximation of the attractor of the map. In the figure 1 are shown the results of this computation.

It can be easily checked that for $\alpha \in (0,1)$ the point $x = 0$ is an attractor of the map. When the parameter α crosses the value 1 a bifurcation occurs, the fixed point $x = 0$ becomes unstable and the point $x = 1 - 1/\alpha$ becomes stable. When the value of the parameter α continues increasing, at $\alpha = s_0 = 2$ the attracting orbit of the map crosses the value $x = 1/2$ (at this value we have $D_x l_\alpha(x) = 0$, therefore the orbit is superattractor.). Let the parameter α increase again, when we reach the value $\alpha = f_1 = 3$ a new bifurcation occurs, but this one is a period doubling bifurcation, where the fixed point becomes unstable and appears a 2 periodic orbit. Increasing α again, at certain value s_1 one of the points of the periodic orbit crosses the value $x = 1/2$, and at certain value f_2, the periodic orbit doubles his period again. When α continues increasing this phenomena is repeated infinitely many times, accumulating to a certain parameter value $F_1 \approx 3.569945672...$, the phenomena described is known as the period doubling bifurcation cascade and the parameter value F_1 is known as the Feigenbaum critical value. This cascade can be seen in the figure 1.

In other words, consider f_n the parameter value where the "attracting" orbit of the logistic map doubles from period 2^{n-1} to 2^n, and s_n the parameter value where the attracting 2^n periodic orbit of the logistic map has zero differential, then we have that

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} f_n = F_1.$$

In fact, from the Feigenbaum conjectures follow that there exist a "universal" rate δ, such that

$$\lim_{n\to\infty} \frac{s_{n+1} - s_n}{s_n - s_{n-1}} = \lim_{n\to\infty} \frac{f_{n+1} - f_n}{f_n - f_{n-1}} = \frac{1}{\delta}.$$

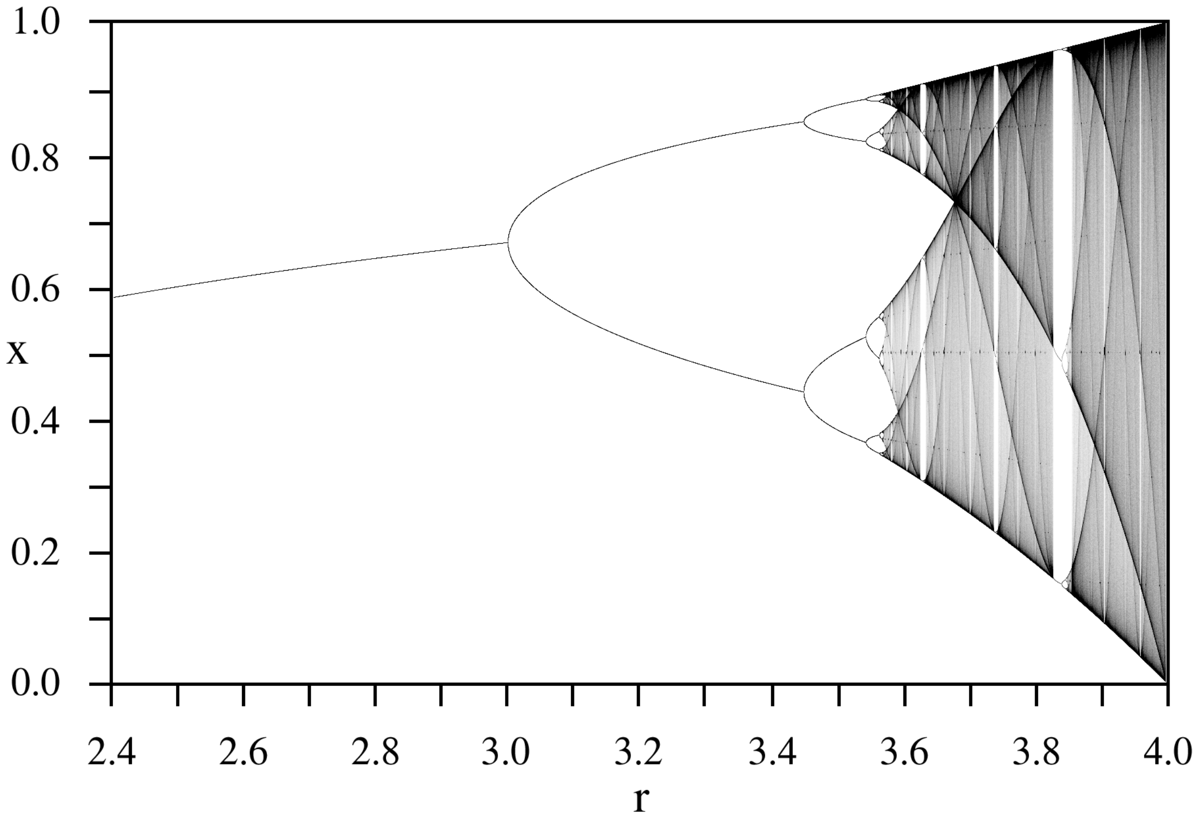

Figure 1: Bifurcations diagram of the logistic map (1), the vertical axis corresponds to the x value of the attractor and the horizontal axis corresponds the parameter α (although it says r in the picture).

The adjective "universal" is used in the sense that the accumulation rate δ does not depend on the family of maps considered.

1.1 Renormalization operator for one dimensional maps

1.1.1 Setting up the problem

Although the problem of the renormalization for one dimensional maps can be set up in a more general context ([5]), we will consider it in a more restrictive case, which will make it easier to understand.

Consider $\mathcal{M}$ the space of analytic even maps ψ from the interval $[-1,1]$ into itself such that

1. $\psi(0) = 1$,
2. $x\psi'(x) < 0$ for $x \neq 0$.

The condition 2) means that ψ is strictly increasing in $[-1,0)$ and strictly decreasing in $(0,1]$. Note that any map in $\mathcal{M}$ is unimodal, indeed 0 is the turning point of the map and the unimodal interval is $[\psi^2(0), \psi(0)]$, which makes this set up a particular case of the considered in [5].

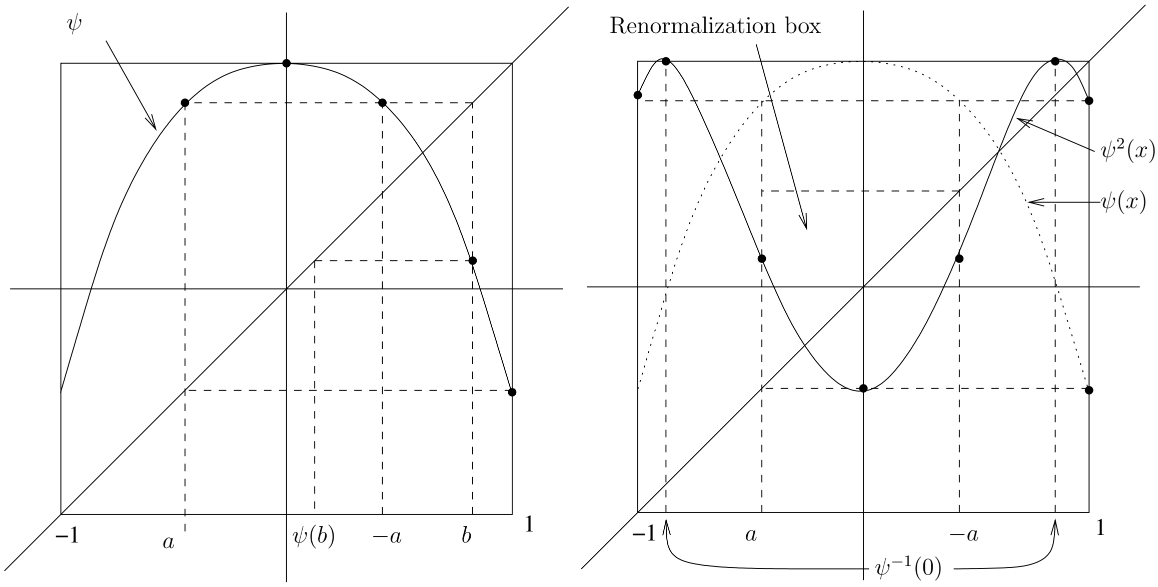

Figure 2: In the left we have a function $\psi \in \mathcal{D}(\mathcal{R})$ and are shown the geometric meaning of the values a, b and $\psi(b)$. In the right we have the function ψ as before and the function ϕ^2. Notice the resemblance between ψ^2 restricted in the renormalization box $[a, -a]$ and ψ in $[-1, 1]$.

Set $a = \psi(1) = \psi^2(0)$, and $b = \psi(a)$. Now, we define $\mathcal{D}(\mathcal{R})$ as the set of $\phi \in \mathcal{M}$ such that

1. $a < 0$,
2. $b > -a$,
3. $\phi(b) \leq -a$.

In the figure 2 is plotted a map of $\mathcal{D}(\mathcal{R})$. Consider the intervals $I_0 = [a, -a]$ and $I_1 = [b, 1]$. For any $\psi \in \mathcal{D}(\mathcal{R})$, it maps each of these two intervals to the other. Actually, if we consider $\psi^2_{I_0}$ and we apply the change of variables $x \to ax$ to ψ^2, this turns out to be in $\mathcal{M}$. Actually this operation we have just described is known as renormalization (the doubling case), and the set $\mathcal{D}(\mathcal{R})$ is the domain of the renormalizable functions. The formal definition is as follows.

Definition 1.1. *We define the* ***renormalization operator*** $\mathcal{R} : \mathcal{D}(\mathcal{R}) \to \mathcal{M}$ *as*

$$\mathcal{R}(\psi)(x) = \frac{1}{a}\psi \circ \psi(ax). \tag{2}$$

where $a = \psi(1)$.

This definition is a concrete case, where we have only considered the doubling case, for a concrete coordinates. It can be defined in a more general context, see for instance [5]. On the other hand the form of the maps considered allows us to write down the affine transformation explicitly.

1.1.2 The Feigenbaum Conjectures

The renormalization operator was first introduced by Feigenbaum when he was studying the logistic map ([2] and [3]). He also proposed some conjectures on the operator, which explained the cascades of period-doubling bifurcation that exhibits the logistic map.

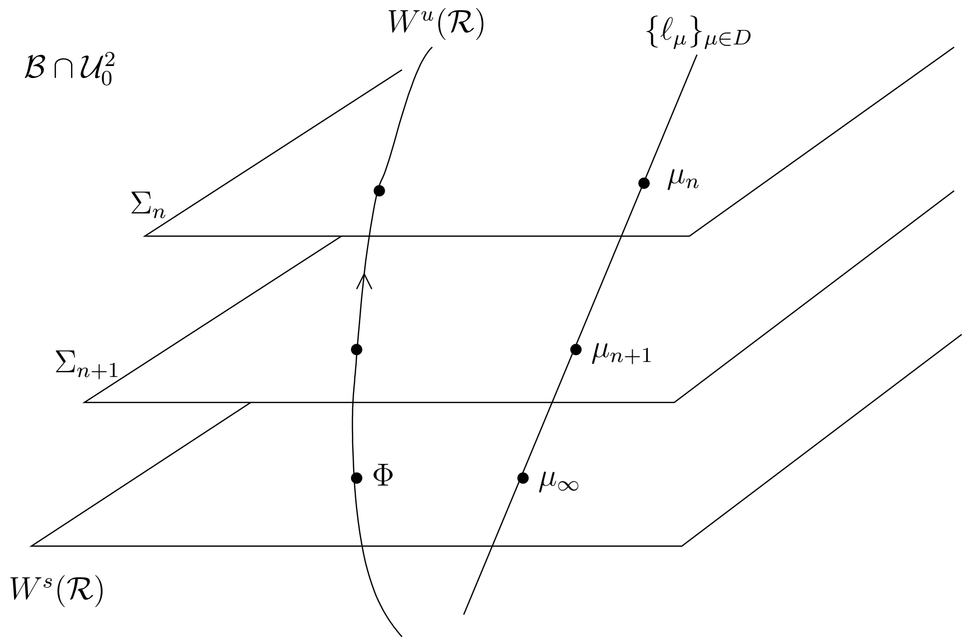

Figure 3: Representation of the dynamics of $\mathcal{R}$. In the figure are plotted Φ the fixed point of $\mathcal{R}$, the stable and the unstable manifold ($W^s(\mathcal{R})$ and $W^u(\mathcal{R})$) associated to this point and the different "bifurcation" manifold Σ_n. It is also represented a one parameter family of maps such that crosses the stable manifold transversally and the parameters values μ_n, for which the family of maps intersects the different surfaces Σ_n.

Those conjectures were:

1. There exists a Banach space $\mathcal{B}$ of analytic functions such that $\mathcal{R}_{|\mathcal{B}}$ is a bounded C^2 operator, which has a fixed point Φ (referred as the Feigenbaum fixed point).
2. The derivative $D\mathcal{R}$ is a compact operator whose spectrum has a unique eigenvalue $\delta = 4.66920...$ outside the unit circle and all the other eigenvalues are in the interior of the unit disc.
3. Let $\Sigma_n \subset \mathcal{B}$ be the set of maps in the neighborhood of Φ for which the critical point is periodic of period 2^n. Then Σ_n, which is a codimension one Banach submanifold, intersects the local unstable manifold of $\mathcal{R}$ transversally for n large enough.

The first proofs obtained of the conjectures were computer assisted, due to Lanford([4]) and to Eckmann and Wittwer ([1]). For the time being we follow with the consequences of the conjectures and in the next subsection we will do some comments on the computer assisted proofs.

The conjecture 1) and 2) imply that exists a local unstable manifold W^u of dimension one, associated to the unstable eigenvalue δ and tangent to the corresponding eigenvector. The unstable and the stable manifolds intersect transversally in Φ. See the figure 3 for a schematic representation of the dynamics.

Note that for a given $\psi_0 \in W^u$, for each $j > 0$ there exist a unique $\psi_j \in W^u$ such that $\mathcal{R}^j(\psi_j) = \psi_0$. Moreover the sequence ψ_j converge geometrically to Φ, with rate δ, i.e. $\|\Phi - \psi_j\| \approx \delta^{-j}$. On the other hand can be checked that $\mathcal{R}(\Sigma_{n+1}) \subset \Sigma_n$, if they are regular (no critical point), then we will have equality in a small neighborhood of Φ (because they have the same codimension).

If the conjecture 3) holds, i.e. the submanifolds Σ_n intersects transversally W^u, we can apply the λ-lemma and therefore will have that the submanifolds Σ_n accumulate at the local stable manifold geometrically, with rate δ^{-1}. Indeed, if we consider other codimension one "bifurcation manifolds" transversal to $W^u(\mathcal{R})$, we will have the same property of accumulation. For example consider

$$\hat{\Sigma}_0 = \left\{\psi \in \mathcal{B} \text{ sucht that } \psi'(x_0) = -1,\ (\psi \circ \psi)'''(x_0) < 0\right\}$$

where x_0 is a fixed point of ψ and consider

$$\hat{\Sigma}_n = \left\{\psi \in \mathcal{B} \text{ such that } \mathcal{R}^n(\psi) \in \hat{\Sigma}_0\right\}.$$

This sets $\hat{\Sigma}_n$ are a codimension one submanifold, and they follow the same asymptotic behavior as Σ_n.

Consider a one parametric family of renormalizable maps $\{\ell_\mu\}_{\mu\in D}$ with $D \subset \mathbb{R}$, for instance the logistic map. Suppose that the family crosses transversally $W^s(\mathcal{R})$ (for a certain parameter μ_∞). Then (for n big enough) this family must cross each Σ_n transversally in a single parameter value μ_n. Indeed this parameters μ_n will tend to μ_∞ in a geometric way (determined by δ) as n grows to infinity.

1.1.3 On the numeric assisted proof of the conjectures

In this subsection we will give an brief idea of how the Feigenbaum conjectures were proved with the use of computer. On a computer rigorous interval arithmetics are possible. The idea of interval arithmetics can be extended to arithmetics of balls in Banach spaces.

Suppose we have X a Banach space equipped with a given norm $\|\cdot\|_A$, and $F : X \to X$ and operator. To prove that F has a fixed point note that is enough to prove it for a subspace Y with a stronger norm $\|\cdot\|_B$. (We say that $\|\cdot\|_B$ is stronger that $\|\cdot\|_A$ in Y, if for all $y \in Y$ we have $\|y\|_A \leq \|y\|_B$.)

Let $\mathbb{D}_1$ be the unit disc in the complex plane and $\mathcal{RH}(\mathbb{D}_1)$ the Banach space of real analytic functions on $\mathbb{D}_1$ and continuous in $\overline{\mathbb{D}}_1$ equipped with the supreme norm ($\|\cdot\|_\infty$). Consider also the space ℓ^1 of absolutely convergent real sequences with the standard norm $\|\cdot\|_1$.

Note that we can consider an inclusion of ℓ^1 in $\mathcal{RH}(\mathbb{D}_1)$, given by

$$\begin{array}{cccc} i: & \ell^1 & \to & \mathcal{RH}(\mathbb{D}_1) \\ & f = (f_0, f_1, \dots, f_n, \dots) & \mapsto & f(z) = \displaystyle\sum_{i=0}^{\infty} f_i z^i \end{array} \tag{3}$$

Let X be the space defined by $i(l^1) \subset \mathcal{RH}(\mathbb{D}_1)$, and we can consider $\|\cdot\|_S$ the norm in X induced by i. In other words given $f \in X$ we define

$$\|f\|_S = \sum_{i=0}^{\infty} |f_i|$$

where $f(z) = \sum_{i=0}^{\infty} f_i z^i$.

Given a set of $n+1$ intervals $I_0, \dots, I_n$ and a positive number r we can define the closed ball $B = B(I_0, \dots, I_n, r)$ as the set of functions $f \in X$ such that $f(z) = \sum_{i=0}^{\infty} f_i z^i$, with $f_i \in I_i$ for

$i = 0, \ldots, n$, and $\sum_{i>n} |f_i| < r$. Given two different balls B_1 and B_2 we can construct a third ball B_3 such that $f_1 + f_2 \in B_3$ for all $f_1 \in B_1$ and $f_2 \in B_2$. The same way we can construct new balls for the pointwise multiplication, scalar multiplication, composition of functions and differentiation of functions. This defines a rigorous arithmetic of balls in the space X. The idea behind the computer assisted proofs of [4] and [1] is use this rigorous arithmetics to prove that a convenient operator is contractive in X with the norm $\|\cdot\|_S$, and therefore prove the Feigenbaum conjectures.

References

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