NORMAL FORMS:

THE CENTRE × CENTRE × SADDLE CASE.

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WORKING SEMINAR IN CELESTIAL MECHANICS

Normal Forms: centre x centre x saddle

- Phase space $\mathbb{R}^{2n} = \mathbb{R}^6$.
- $H$ smooth Hamiltonian, equilibrium pt at zero.

**Definition** The eq. point is centre x centre x saddle if there exists a canonical syst. of coords. $(x_1, y_1)$ in which the Hamiltonian takes the form

\[
H(x_1y_1) = H_2(x_1y_1) + H_P(x_1y_1)
\]

where

\[
H_2 = \frac{\omega_1}{2} (x_1^2 + y_1^2) + \frac{\omega_2}{2} (x_2^2 + y_2^2) + \lambda x_3 y_3
\]

and $H_P$ has a zero of order 3 at the origin.

**Remark** Equations of motion. Linear approximation

\[
\begin{align*}
\dot{x}_1 &= \omega_1 y_1 + \frac{\partial H_P}{\partial y_1} \\
\dot{y}_1 &= -\omega_1 x_1 - \frac{\partial H_P}{\partial x_1} \\
\dot{x}_3 &= \lambda x_3 + \frac{\partial H_P}{\partial y_3} \\
\dot{y}_3 &= -\lambda y_3 - \frac{\partial H_P}{\partial x_3}
\end{align*}
\]

\[
\begin{align*}
\dot{x}_1 &= -\omega_1^2 x_1 \\
x_3(t) &= x_3^0 e^{\lambda t} \\
y_3(t) &= y_3^0 e^{-\lambda t}
\end{align*}
\]
Definition (semi-simple NF)

A system with an eq. point of type $c_x c_x s$ is in (semi-simple) normal form up to order $r \geq 3$ if

$$H^{(r)} = H_2 + Z^{(r)} + R^{(r)}$$

where

- $H_2 = \frac{w_1}{2} (x^2 + y_1^2) + \frac{w_2}{2} (x^2 + y_2^2) + \lambda x_3 y_3$

- $Z^{(r)}$ is a polyn. of degree $r$: $[H_2, Z^{(r)}] \equiv 0$.

- $R^{(r)}$ is a small reminder:

$$|R^{(r)}(z)| \leq C_r \|z\|^{r+1} \quad \forall z \in U^{(r)}$$

small neighborhood of the origin.
Theorem. For any integer $r \geq 2$, there exists a neighbourhood $U^{(r)}$ of the origin and a canonical transf.

$$T_r : \mathbb{R}^6 \to U^{(r)} \to \mathbb{R}^6$$

which puts the system (1) in NF up to order $r$

$$H^{(r)} := H \circ T_r = H_2 + Z^{(r)} + R^{(r)}.$$ 

Moreover, $T_r$ (and $T_r^{-1}$) is close to the identity

$$|z - T_r(z)| \leq C_r \|z\|^2 \quad \forall z \in U^{(r)}.$$ 

If the frequencies $\omega$ and $\omega_2$ are non-resonant to order $r$, the function $Z^{(r)}$ depends only on the basic invariants

$$I_1 = \frac{x^2 + y_1^2}{2}, \quad I_2 = \frac{x_2^2 + y_2^2}{2} \quad \text{actions}$$

$$I_3 = x_3 y_3.$$
Proof. (Lie transform method)

Idea: construct a canonical transf. $\Phi_r$ that puts the system in a form that is as simple as possible.

More precisely, construct a canonical transf. $\Phi_3$ that normalizes terms of order 3, followed by a canonical transf. $\Phi_4$ that normalizes order 4, etc.

- Each transf. $\Phi_j$ is constructed as the time-1 flow of a suitable auxiliary Hamiltonian $G_j$ (generating Hamiltonian).

**Definition** $\mathcal{H}_j$ - the set of real-valued homogeneous polynomials of degree $j$.

**Remark**

\[ f \in \mathcal{H}_i, g \in \mathcal{H}_j \implies \{fg\} \in \mathcal{H}_{i+j-2}. \]
Lie transform

Given \( G \in U_j \), let \( \dot{z} = X_G(z) \) be the associated Hamiltonian eq, and define

\[ \phi := \phi^t \big|_{t=1} \] the time-1 map

(\underline{the Lie transform generated by} \( G \)).

It is well-known that \( \phi \) is \underline{canonical}.

Lemma: Let \( H \) be a polynomial, and \( G \in U_j \) (\( j \geq 3 \)), \( \phi \) as before.

The Taylor expansion of \( H \circ \phi \) is

\[ K := H \circ \phi = H + \frac{1}{1!} \{ H, G \} + \frac{1}{2!} \{ \{ H, G \}, G \} + \cdots \]
Suppose that our Hamiltonian $H$ has already been normalized up to order $j-1$, and we hit it with the Lie transform $\phi_j$ generated by $G_j \in \mathcal{H}_j$ (unknown).

The new Hamiltonian will temporarily be called

$$K = K_2 + K_3 + K_4 + \cdots, \quad K_i \in \mathcal{H}_i.$$  

Then we have

$$K_i = H_i, \quad i < j$$

$$K_j = H_j + \frac{1}{j!} H_2 \cdot G_j, \quad i = j$$

$$K_i = H_i + \frac{1}{i!} H_2 \cdot G_j + \frac{1}{i!} H_3 \cdot G_j + \cdots, \quad i > j.$$
Homological equation

We want to construct $G_j$ such that

$$K_j = H_j + \gamma H_2, G_j \iff \{G_j, H_2\} = H_j - K_j,$$

is as simple as possible.

Definition (Homological operator)

$$\mathcal{L} : H_j \to H_j$$

$$G \to \mathcal{L}G := \gamma [G, H_2].$$

We want to solve

$$\mathcal{L} G_j = H_j - K_j$$

with $K_j$ as simple as possible.

For instance, if $H_j \in \text{im} \mathcal{L}$ then we can find $G_j$ such that

$$\mathcal{L} G_j = H_j \Rightarrow K_j = 0.$$
In general $H_j \notin \text{im } \mathcal{E}$, but $\text{im } \mathcal{E}
abla$
given a complement $N_j$ to the space $\mathcal{H}_j$
$\mathcal{H}_j = \text{im } \mathcal{E} \oplus N_j$
We can make $K_j$ lie in this complement:

$L_6_j = H_j - K_j$

$\uparrow \quad \uparrow \quad \uparrow$
$\text{im } \mathcal{E} \quad H_j \quad N_j$

$N_j$ is the normal form space.

It is easy to see that the homological op. $L$
diagonalizes in any one of the spaces $H_j$. 
To this end, introduce complex variables

\[ q_1 = \frac{1}{\sqrt{2}} (x_1 + iy_1), \quad p_1 = \frac{1}{\sqrt{2}} (x_1^* - iy_1) \]
\[ q_2 = \frac{1}{\sqrt{2}} (x_2 + iy_2), \quad p_2 = \frac{1}{\sqrt{2}} (x_2^* - iy_2) \]
\[ q_3 = x_3, \quad p_3 = y_3. \]

Remark: This change is not the same as [Jorba].

The symplectic form is modified to

\[ i (dq_1 \wedge dp_1 + dq_2 \wedge dp_2) + dq_3 \wedge dp_3. \]

In these complex variables, the invariants become

\[ I_1 = \frac{x_1^2 + y_1^2}{2} = q_1 p_1, \quad I_2 = \frac{x_2^2 + y_2^2}{2} = q_2 p_2, \quad I_3 = x_3 y_3 = q_3 p_3, \]

and

\[ H_2(q_p) = w_1 q_1 p_1 + w_2 q_2 p_2 + \lambda q_3 p_3. \]

Reality condition: \( H(q_p) = \sum h_{\alpha \beta} q^\alpha p^\beta \)

\[ h_{\alpha \beta} = \overline{h_{\beta \alpha}}. \]
Lemma

$$\varepsilon q^L_p = \left[ i(l_1-m_1)\omega_1 + i(l_2-m_2)\omega_2 + (l_5-m_5) \right] q^L_p.$$ 

Corollary The homological op. \( \varepsilon \) is semisimple \( \Rightarrow \)

$$H_j = \text{im} \varepsilon \oplus \ker \varepsilon.$$ 

Hence we can choose

$$N_j = \ker \varepsilon$$

as the normal form spaces. \( \square \).

Notice that

\[ K_j \in \ker \varepsilon \iff \varepsilon K_j = 0 \iff \langle K_j, H_2 \rangle = 0. \]
Corollary 2. (Description problem)

The space \( \ker F \) is spanned by the monomials \( \ell^m \) such that

\[
(l_1 - m_1)\omega_1 + (l_2 - m_2)\omega_2 = 0 \quad \text{and} \quad l_3 = m_3. \quad \Box.
\]

Thus if the frequencies \( \omega_1, \omega_2 \) are nonresonant to degree \( r \),

\[
k_1\omega_1 + k_2\omega_2 \neq 0 \quad \forall (k_1, k_2) \in \mathbb{Z}^2, \quad 0 < |k_1 + k_2| \leq r,
\]

the normal form space is spanned by the monomials

\[
(q_1 \ell_1)^{l_1} (q_2 \ell_2)^{l_2} (q_3 \ell_3)^{l_3} \Rightarrow
\]

the normal form is a polynomial of deg. \( r \) that depends only on \( I_1, I_2, I_3 \).

Remark: The space normal form space is a ring of invariant

\[
\frac{d}{dt} f(e^{At}) = 0 \iff \{ f, H_2 \} = 0.
\]
In the literature there are other algorithms to compute the NF, see e.g. the book [Murdock].

1. 'Direct' algorithms: Consider the system of diff. eqs.

\[ \dot{x} = X_{4}(x) = A_{4}x + q_{2}(x) + q_{3}(x) + \ldots, \quad a_{j} \in V_{j}^{2n} \]

and construct the canonical transformations \( \phi_{j} \) directly.

- Homological operator

\[ L : V_{j}^{2n} \rightarrow V_{j}^{2n} \]

\[ \psi(x) \rightarrow \psi^{-1}(x)A_{j} - A_{j}\psi(x) \]

- \( N_{j} = \ker L \) is a module over the ring of invariants:

\[ \dot{x} = A_{j}x + Q_{1}(I_{1}, I_{2}, I_{3})u_{1} + Q_{2}(I_{1}, I_{2}, I_{3})u_{2} + \ldots + Q_{6}(I_{1}, I_{2}, I_{3})u_{6} \]

real invariant \( \rightarrow \) basic equivariants.

2. 'Recursive' algorithms [Deprit, Giorgilli-Galgani, etc.]

Construct only one canonical transf. \( \mathcal{T} \) in a recursive way.
System centered at libration point $L_1$:

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + q_p - x p_y - \sum_{n \geq 2} c_n(x) p_n(x),$$

One can find symplectic coords. in which

$$H = H_2 + H_P,$$

where

$$H_2 = \frac{\omega_1}{2} (x_i^2 + y_i^2) + \frac{\omega_2}{2} (x_2^2 + y_2^2) + \lambda x_3 y_3.$$

By our theorem,

- there exists a neigh. $U^{(r)}$ and a canonical tr. $T_r$ which puts the system in NF of order $r$:

$$H^{(r)} = H_0 T_r = H_2 + Z^{(r)} + R^{(r)}.$$

- If the pugs $\omega_1, \omega_2$ are non-resonant to order $r$ (this is checked numerically), the func. $Z^{(r)}$ depends only on $I_1, I_2, I_3$. 

Spatial circular RTBP
Geometrical structures in NF's [Murdock, ch. 5].

- What does the NF tell us about dynamics?

Consider \( \dot{x} = a(x) \) - full system

\( \dot{x} = \hat{a}(x) \) - truncated system in NF.

**Theorem** Assume \( \dot{x} = \hat{a}(x) \) is in truncated NF.

Then the linear subspaces

\[ E^s, E^u, E^c, E^{cs}, E^{cu}, E^{ss}, E^{su} \]

defined in the usual way are invariant by the flow of the truncated system.

**Theorem** Assume \( \dot{x} = \hat{a}(x) \) is in truncated NF.

Then the stable/unstable fibrations of \( E^c, E^u \)

are preserved by the flow of the truncated system.

Other structures: periodic orbits, invariant tori.
Definition: A local manifold $M \subset \mathbb{R}^n$ is a graph over a linear subspace $E \subset \mathbb{R}^n$ if:

$M = \text{graph}(\tau)$

- $M$ has $k$-th order contact with $E$ if

$\tau(0) = D\tau(0) = \ldots = D^k\tau(0) = 0.$

- $M$ is invariant under a flow if
Theorem (Stable, unstable, center, etc. theorem)

In the full system $x = a(x)$,

- There exist unique smooth local invariant stable/unstable manifolds $W^s, W^u$, expressible as graphs over $E^s, E^u$,
  having $k$-th order contact with $E^s, E^u$.

- For each $k$: $k \leq k < \infty$, there exists a (not necessarily unique) local invariant $C^k$ center manifold $W^c$,
  expressible as a graph over $E^c$,
  having $k$-th order contact with $E^c$. 
**Remark**

The usual versions of the st, unst, centre theorem do not assume that the system is in NF. In this case, the local manifolds have only 1st order contact (tangent) with $E^s, E^u, E^c$.

**Remark**

Computing NF to degree r automatically computes approximations to these manifolds:

\[
W^u, W^s, \quad \text{r-th order NF}
\]

\[
X = \hat{a}(X)
\]