Seminar: RTBP III Libration Points Motion
(by A. Ferrés)

Remember eq. motion:

\[
\begin{align*}
\ddot{x} - 2\dot{y} &= 2x, \\
\ddot{y} + 2\dot{x} &= 2y, \\
\dddot{z} &= 2z.
\end{align*}
\]

Eq. Points:

There are five: \( L_1, L_2, L_3, L_4, L_5 \).

In this talk we will focus on the motion around the collinear points: \( L_1, L_2, L_3 \).

\( L_4 \) and \( L_5 \) have been very studied and are very useful for mission applications.

We want to understand the periodic and quasi-periodic motion around these points.

As we saw in RTBP Part II these fixed points are of the type centre-centre-saddle.
Hohon around the collinear of points:

A change of variables by bring the fixed point to the origin:

\[
\begin{align*}
  x &= \xi_1 x + \mu + \xi_1 \\
  y &= \xi_2 y \\
  z &= \xi_3 z
\end{align*}
\]

on \( \xi_j \) is distance to the closest prime or solution of the Euler quintic

\[
\begin{align*}
  a_1 &= -1 + \xi_1 \\
  a_2 &= -1 - \xi_2 \\
  a_3 &= \xi_3
\end{align*}
\]

(Note: Scaling is added here so that the expansion around the fixed point is has good numerical properties.)

Using Legendre Polynomials:

\[
\left[\sqrt{(x-A)^2 + (y-B)^2 + (z-C)^2} \right]^2 = \frac{1}{8} \sum_{n=0}^{10} \left( \frac{1}{6} \right)^n \frac{\partial^n (Ax + By + Cz)}{\partial \rho^n}
\]

on \( D = A^2 + B^2 + C^2 \), \( \rho^2 = x^2 + y^2 + z^2 \), \( P_n \) polynomial legendre

of degree \( n \).

Eq. expand:

\[
\begin{align*}
  x - 2y - (1+2c_2) x &= \frac{2}{\partial x} \sum_{n=3} c_n (\rho) \rho^n P_n(\rho) \\
  y + 2x + (\epsilon+1) y &= \frac{2}{\partial y} \sum_{n=3} c_n (\rho) \rho^n P_n(\rho) \\
  z + (c_2 + 1) z &= \frac{2}{\partial z} \sum_{n=3} c_n (\rho) \rho^n P_n(\rho)
\end{align*}
\]
\[
\begin{align*}
C_n(\mu) &= \frac{A}{\xi_3^n} \left[ (1 + \xi_3^n) \mu + (1 - \mu) \frac{\xi_3^{n+1}}{1 + \xi_3^{n+1}} \right] \quad \text{for } j=1,2, \quad i,j=1,2, \\
C_n(\mu) &= \frac{(-1)^n}{\xi_3^n} \left[ (1 - \mu) + \mu \frac{\xi_3^{n+1}}{1 + \xi_3^{n+1}} \right] \quad \text{for } j=3.
\end{align*}
\]

Stability of the fixed Points (rep's & rep's):

\[
J_{(03)} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 - \xi_2 & 0 & 0 & 0 & 2 & 0 \\
0 & 1 - \xi_2 & 0 & -2 & 0 & 0 \\
0 & 0 & -\xi_2 & 0 & 0 & 0
\end{bmatrix}
\]

\(2)\) Notice that the \(z_2, z_3\) plane motion is decoupled from the rest \(x, y, x, y\).

\[
\rho_\infty(1) = \left[ (1 + (2 - C_2))^2 + (1 + (2 - 2C_2^2)) \right] / (1 + C_2).
\]
Taking $V = I^2$ we can see that the roots are:

$$V_1 = \frac{c_2 - 2 - \sqrt{9c_2^2 - 8c_2}}{2}, \quad V_2 = \frac{c_2 - 2 + \sqrt{9c_2^2 - 8c_2}}{2}$$

$$\text{As } c_2 > 1 \quad \forall \mu \in [0, 1/2]$$

$$\Rightarrow V_1 > 0, \quad V_2 < 0 \Rightarrow W_1 = \sqrt{-V_1}, \text{ complex eigenvalue}$$

$$\lambda = \sqrt{V_2}, \text{ real eigenvalue}$$

$$W_2 = \sqrt{-(c_2)}, \text{ complex eigenvalue}$$

$$\Rightarrow$$ fixed points are **unstable**

centre x centre x saddle.

One can easily compute the eigenvectors and see:

$$M_\lambda = (2\lambda, \lambda^2 - 2c_2 - 1, 0, 2\lambda^2, \lambda(\lambda^2 - (1 + 2c_2)), 0)$$

$$M_{-\lambda} = (-2\lambda, \lambda^2 - 2c_2 - 1, 0, 2\lambda^2, -\lambda(\lambda^2 - (1 + 2c_2)), 0)$$

$$M_{w_1} = (0, -w_2, -1 - 2c_2, 0, -2w_2, 0, 0)$$

$$V_{w_1} = 2w_1, \quad 0, 0, 0, -w_1, (w_2^2 + 1 + 2c_2), 0)$$

$$M_{w_2} = (0, 0, 1, 0, 0, 0, 0)$$

$$V_{w_2} = (0, 0, 0, 0, 0, 0, 1)$$
- We have 2 control directions + a saddle plane.

Lyapunov Centre Theorem: Assume we have a system. If we have a non-degenerate 1st integral and an equilibrium point with \( \pm \omega_i, \lambda_i, \ldots \) then \( \lambda_i \neq 0 \) and every solution of \( \forall i, \lambda_i \neq 0 \), \( n \to \infty \) then a family of periodic orbits emerging from the equilibrium.

Assume that as \( \lambda_1, \lambda_2 \neq 0 \) we have two families of periodic orbits; one for each frequency \( \lambda_1 \) and \( \lambda_2 \):

- \( \lambda_1 \) gives rise to the planar family (Lyapunov orbits).
- \( \lambda_2 \) gives rise to the vertical family (Vertical Lyap. orbits).

We can use a continuation method to follow these families. (Orbit: very unreliable. Suitable to use Zabel. Shooting; other option Liustof. Bifurc.)
**Linear Approximation of Motion Around Equilibrium:**

\[ x(t) = A e^{\lambda t} \dot{x}_0 + B e^{-\lambda t} \dot{x}_0 + \]

\[ + \left[ \frac{\alpha}{\omega} \sin \left( \omega t + \phi_0 \right) \dot{x}_0 \right] + \]

\[ + D \left[ \frac{\alpha}{\omega^2} \sin \left( \omega^2 t + \phi_0 \right) \dot{x}_0 \right] \]

on \( A, B, C, D, \phi_0, \phi_0 \) given by the initial condition at \( t = t_0 \).

- **Taking:** \( A = B = C = 0 \) \( \rightarrow \) **Vertical family**

  \[ A = B = D = 0 \] \( \rightarrow \) **Planar family**

- **Stability Around Periodic Orbit:**

  - **Periodic orbit** \( \rightarrow \) \( \Phi_T(x) \) = flow around trajectory

  and \( \Phi_T(x) \) = monodromy matrix

  for the periodic orbit \( x_0 \).
As the system is Hamiltonian it has a 1st Integral

\[ \text{Spect } \mathcal{M} = \lambda_1, \lambda_2, \lambda_1^{-1}, \lambda_2^{-1} \]

- We define the stability parameters \( S_j = \lambda_j + \lambda_j^{-1} \), \( j = 1, 2 \).

- Hyperbolic: \( S_j \in \mathbb{R}, |S_j| > 2 \) (\( \lambda_j \in \mathbb{R} \setminus \{1, -1\} \))

- Elliptic: \( S_j \in \mathbb{C}, |S_j| < 2 \) (\( \lambda_j = e^{i\theta} \))

- We have followed the Planar and Vertical Lyapunov families for \( \lambda_1 \) and \( \lambda_2 \) for \( \mu = 1 \) for Sun-Earth model.

- Special attention to 1st bifurcations + where they die.

Classification: Periodic Orbits.
Planar families:

- \( L_1 \) plane family aligns with a colinear with one of the two primaries.
- At the beginning, \( L_1 \) plane family bifurcates.

\[ C \times S \rightarrow S \times S \] and the Halo orbits no longer.

(Show/See present OP librarc pdf)
**Vertical Families:**

- \( L_1 \) terminates at: orbit surrounding massive body & \( L_1, L_3 \)
- \( L_2 \) "": orbit surrounding two bodies & \( L_1, L_3 \)
- \( L_3 \) "": two bodies & \( L_1, L_2, L_3 \)

*Note: bifurcation space at the beginning*

(See point OP-libro_poly)
They appear when planar orbit bifurcates.

- Symmetric w.r.t. $y=0$ and $z=0$. 
- With this we can see the family of invariant hypersurfaces around the eq. point.

- \[ H = \lambda (x_1 y_1 + i w, \frac{(x_1^2 + y_1^2)}{2} + i w_2 \frac{(x_2^2 + y_2^2)}{2}) \]

  as \( w, w_2 > 0 \) \( \Rightarrow \) for fixed \( H = \hbar \) energy level, the motion is bounded by an ellipsoid on the centre manifold (taking \( x = y = 0 \)).

- Look at pictures of the centre manifold for different \( \hbar \).

- As \( \hbar \) increases \( \Rightarrow \) planar \( |y| \) motion and \( \text{Halo orbits} \) appear.

- We see all that can be here.

- Plots:

  \[ x_2 = 0 \quad 2(h) \]
  \[ y = 0 \quad \text{both of } n^{th} \text{ order} \]

  (two different sections): \( x_3 = 0 \quad x_5(h) \quad x_1 = 0 \quad x_2(h) \)
To have a more complete understanding of the motion it is interesting to do the so-called reduction to the centre manifold.

Due to the big intractability of the eq. point, Poincaré sections & numerical integration gives problems as the trajectories quickly escape from the vicinity of the eq. point.

By reducing to the centre manifold:
(1) we decouple the elliptic directions from the hyperbolic ones up to high order.
(2) use the high order approximation of the centre manifold to do numerical integration.
(3) how to do this? (see REF)