Advanced Course on Long Time Integrations

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Universitat de Barcelona, Institut de Matemàtica (IMUB)
Barcelona, September 3-7, 2007

Outline

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Newton's method

- The numerical methods to be presented for the computation of invariant objects end up solving a **non-linear system of equations**:

$$\boldsymbol{G}(\boldsymbol{x}) = 0, \qquad \boldsymbol{G} : \mathbb{R}^n \to \mathbb{R}^n.$$

- Given a good initial seed (which we will always have), a good (quadratically convergent) general strategy to solve it is **Newton's method**:

$$\begin{array}{l} \boldsymbol{x}_0 \text{ initial approximation} \\ \forall n \geq 0 \\ \qquad \boldsymbol{x}_{n+1} = \boldsymbol{x}_n - D\boldsymbol{G}(\boldsymbol{x}_n)^{-1}\boldsymbol{G}(\boldsymbol{x}_n) \end{array}$$

- $D\boldsymbol{G}(\boldsymbol{x}_n)$, is **never** inverted. Instead, the following linear system is solved:

$$D\boldsymbol{G}(\boldsymbol{x}_n)\left(\boldsymbol{x}_{n+1} - \boldsymbol{x}_n\right) = -\boldsymbol{G}(\boldsymbol{x}_n).$$

Newton's method

An algorithm:

input: $\boldsymbol{p}_0$, $\boldsymbol{G}$, tol, maxit

do: $\boldsymbol{p}$:=$\boldsymbol{p}_0$

for it from 1 to maxit do

if ($|\boldsymbol{G}(\boldsymbol{p})|$<tol) return $\boldsymbol{p}$

solve $D\boldsymbol{G}(\boldsymbol{p})\Delta\boldsymbol{p} = \boldsymbol{G}(\boldsymbol{p})$ for $\Delta\boldsymbol{p}$

$\boldsymbol{p} := \boldsymbol{p} - \Delta\boldsymbol{p}$

error (maxit exceeded)

output: $\boldsymbol{p}$ (if OK)

Newton's method

- In what follows, it will be convenient to be able to solve **non-square** linear systems.
- A way to handle non–square systems in Newton's method is to find the **least–squares minimum–norm** solution of the linear system for the correction.
- Assuming that
  - the system of equations has solution (may be non unique), and
  - the initial guess is **close to a solution**

  this strategy will converge to a nearby solution using minimum–norm corrections.

Solving non–square linear systems

- Assume A $m \times n$ matrix, with arbitrary m, n rank $A =: r \leq \min(m, n)$.
- For arbitrary m, n, a **least–squares solution** of $A\boldsymbol{x} = \boldsymbol{b}$,

$$\boldsymbol{x}^* : \quad \|\boldsymbol{b} - A\boldsymbol{x}^*\|_2 = \min_{\boldsymbol{x} \in \mathbb{R}^n} \|\boldsymbol{b} - A\boldsymbol{x}\|_2.$$

  always exists.
- If rank $A = n$, there is an unique least–squares solution.
- If rank $A < n$, there is a $(n - \operatorname{rank} A)$–dimensional space of least–squares solutions.
- We want to find the **minimum–norm least–squares solution**, that is

$$\boldsymbol{x}_{LS} : \quad \|\boldsymbol{x}_{LS}\| = \min\left\{ \|\boldsymbol{x}^*\|_2 : \ \|\boldsymbol{b} - A\boldsymbol{x}^*\|_2 = \min_{\boldsymbol{x} \in \mathbb{R}^n} \|\boldsymbol{b} - A\boldsymbol{x}\|_2 \right\}$$

Solving non–square linear systems

By applying Householder transformations with column pivoting[5], we obtain a decomposition

$$Q^\top A P = \begin{array}{cc} \begin{array}{cc} r & n-r \end{array} & \\ \begin{pmatrix} R_{11} & R_{12} \\ 0 & 0 \end{pmatrix} & \begin{array}{c} r \\ m-r \end{array} \end{array}$$

with R_{11} an $r \times r$ upper–triangular matrix with non–zero diagonal elements. If we denote

$$P^\top \boldsymbol{x} = \begin{pmatrix} \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix} \begin{array}{c} r \\ n-r \end{array}, \qquad Q^\top \boldsymbol{b} = \begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{d} \end{pmatrix} \begin{array}{c} r \\ m-r \end{array},$$

then the least–squares solutions are

$$P^\top x = \left\{ \begin{pmatrix} R_{11}^{-1} \boldsymbol{c} \\ 0 \end{pmatrix} + \begin{pmatrix} -R_{11}^{-1} R_{12} \\ I_{n-r} \end{pmatrix} \boldsymbol{z} \right\}_{z \in \mathbb{R}^{n-r}}$$

To find the minimum–norm element of the previous set is an **standard full–rank least–squares problem**.

Solving non–square linear systems

We can write a routine that, for a general $m \times n$ linear system of equations, finds

- the minimum–norm least–squares solution, and
- optionally, a basis of the kernel.

$$\left\{ \begin{array}{c} r \\ n-r \end{array} \; P \begin{array}{c} n-r \\ \begin{pmatrix} -R_{11}^{-1}R_{12} \\ I_{n-r} \end{pmatrix} \end{array} z \right\}_{z\in\mathbb{R}^{n-r}}$$

We can do this by using LAPACK[2] routines:

- DGEQPF: QR factorization with columnn pivoting,
- DGEQRF: standard QR factorization,
- DORMQR: to apply Householder transformations.
- DTRTRS: to solve upper-triangular linear systems.

Outline

Continuation methods[1]

In order to go from $\{\boldsymbol{F}(\boldsymbol{x}) = 0\}$ to $\{\boldsymbol{G}(\boldsymbol{x}) = 0\}$, we can consider a one–parametric family of intermediate problems $\boldsymbol{H}(\lambda, \boldsymbol{x})$, such that

$$\boldsymbol{H}(0, \boldsymbol{x}) = \boldsymbol{F}(x), \qquad \boldsymbol{H}(1, \boldsymbol{x}) = \boldsymbol{G}(x).$$

For instance,

$$\boldsymbol{H}(\lambda, \boldsymbol{x}) = (1 - \lambda)\boldsymbol{F}(\boldsymbol{x}) + \lambda \boldsymbol{G}(\boldsymbol{x}),$$

We can try to continue a solution $\boldsymbol{x}_0$ of $\boldsymbol{F}(\boldsymbol{x}) = 0$ to a solution of $\boldsymbol{G}(\boldsymbol{x}) = 0$ as

input: $\boldsymbol{x}_0 \in \mathbb{R}^n$ such that $\boldsymbol{H}(0, \boldsymbol{x}_0) = 0$
do: $\Delta\lambda := 1/m$
$\forall i = 1 \div m$
  $\lambda := i\Delta\lambda$
  solve $\boldsymbol{H}(\lambda, \boldsymbol{y}) = 0$ iteratively for $\boldsymbol{y}$ taking $\boldsymbol{x}$ as starting value
  $\boldsymbol{x} := \boldsymbol{y}$
output: $\boldsymbol{x}$

This procedure breaks down in the case of a turning point.

The predictor–corrector or pseudo–arclength method

$\boldsymbol{H}(\boldsymbol{y}) := \boldsymbol{H}(\lambda, \boldsymbol{x})$ defines implicitly a curve in $\mathbb{R}^{n+1}$.
We can continue this curve as follows:

input: $\boldsymbol{y} \in \mathbb{R}^n$ such that $\boldsymbol{H}(\boldsymbol{y}) = 0$

do: while $(\lambda = \Pi_0 \boldsymbol{y} < 1)$
  let $\boldsymbol{v} \in \ker D\boldsymbol{H}(\boldsymbol{y})$, $\|\boldsymbol{v}\|_2 = 1$, pointing in the right direction
  take $\boldsymbol{z} := \boldsymbol{y} + \gamma \boldsymbol{v}$, for suitable γ
  if $(\Pi_0 \boldsymbol{y} < 1)$
    solve $\boldsymbol{H}(\boldsymbol{z}) = 0$ iteratively for $\boldsymbol{z}$ by Newton's method taking minimum–norm corrections
  else
    $\gamma := (1 - \Pi_0 \boldsymbol{y}) / \Pi_0 \boldsymbol{v}$
    $\boldsymbol{z} := \boldsymbol{y} + \gamma \boldsymbol{v}$
    solve $\boldsymbol{H}(\boldsymbol{z}) = 0$ by Newton keeping $\Pi_0 \boldsymbol{z}$ constant
  $\boldsymbol{y} := \boldsymbol{z}$

output: $\boldsymbol{y}$

The predictor–corrector or pseudo–arclength method

- In this algorithm, γ should be chosen in order to keep (more or less) constant the number of Newton iterates in the refinement phase.

  A simple rule to do that is to assume that the number of Newton iterates is a linear function of the steplength chosen:

  $$\gamma = \frac{n_{des}}{n_{old}} \gamma_{old}.$$

- Note that in the pseudo–arclength method there is no distinguished coordinate to be thought as a parameter. We can therefore apply it to any system of non–linear equations $\boldsymbol{H}(\boldsymbol{y}) = 0$, as long as its solution is a curve.

- Note that the method works as long as $\dim \ker \boldsymbol{H}(\boldsymbol{y}) = 1$ on the solution curve. It is not necessary that $\boldsymbol{H} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ with $m = n + 1$.

The predictor–corrector or pseudo–arclength method

- The condition $\dim \ker \boldsymbol{H}(\boldsymbol{y}) = 1$ fails at bifurcation points (where $\dim \ker \boldsymbol{H}(\boldsymbol{y}) = 2$).

  Since it is difficult to exactly "fall over" a bifurcation point, the method usually "jumps over" them.

- A way to avoid "jumping" into bifurcated branches is to control the angle between successive iterates $\boldsymbol{y}_{n-1}, \boldsymbol{y}_n, \boldsymbol{y}_{n+1}$, that is, reduce steplength if

$$\langle \boldsymbol{y}_n - \boldsymbol{y}_{n-1}, \boldsymbol{y}_{n+1} - \boldsymbol{y}_n \rangle \leq 1 - \text{tol}.$$

- This last strategy is also useful to plot nice continuation curves.

Bifurcation points

- If $\dim \ker D\boldsymbol{G}(\boldsymbol{y}) \geq 2$, $\boldsymbol{y}$ is a bifurcation point.
- Rigorous analysis of a bifurcation point requires the evaluation of the second derivatives of $\boldsymbol{G}$ at $\boldsymbol{y}$, that is

$$D^2\boldsymbol{G}(\boldsymbol{y}) = \left(\partial_{y_i}\partial_{y_j}\boldsymbol{G}(\boldsymbol{y})\right)_{i,j=0\div n},$$

  where the matrix is symmetric and each of its components is an n–dimensional vector.
- They can be computed from the second variational equations.
- Often one can find what happens in a bifurcation point in terms of the dynamics around it, whithout the need of a rigorous analysis of bifurcations (for a rigorous analysis see e.g. [13])

Outline

Continuous Dynamical Systems

Defined by a system of autonomous (i.e. time–independent) ODE.

$$\begin{cases} \dot{x}_1 & = & f_1(x_1, x_2, \ldots, x_n), \\ \dot{x}_2 & = & f_2(x_1, x_2, \ldots, x_n), \\ & \vdots & \\ \dot{x}_n & = & f_n(x_1, x_2, \ldots, x_n), \end{cases}$$

In short,

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \text{for} \quad \boldsymbol{x} \in \mathbb{R}^n, \quad \boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n.$$

with

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n, \quad \boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} f_1(\boldsymbol{x}) \\ \vdots \\ f_n(\boldsymbol{x}) \end{pmatrix} \in \mathbb{R}^n$$

Continuous Dynamical Systems

- An autonomous system of ODE
$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \text{for} \quad \boldsymbol{x} \in \mathbb{R}^n, \quad \boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n.$$
allows to define the **flow**,
$$\phi_t(\boldsymbol{x}), \quad t \in \mathbb{R}, \quad \boldsymbol{x} \in \mathbb{R}^n,$$
that satisfies
$$\begin{cases} \frac{d}{dt}\phi_t(\boldsymbol{x}) & = & \boldsymbol{f}\big(\phi_t(\boldsymbol{x})\big), \\ \phi_0(\boldsymbol{x}) & = & \boldsymbol{x}, \end{cases}$$
and
$$\phi_{s+t}(\boldsymbol{x}) = \phi_s\big(\phi_t(\boldsymbol{x})\big).$$
- Examples: RTBP, Hill's problem (Gerard's session).
- We will always assume a Hamiltonian dynamical system:
$$H : \mathbb{R}^n \longrightarrow \mathbb{R}^n \qquad (n \text{ even})$$
$$\dot{x} = J\nabla H(x), \qquad J = \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right)$$

Poincaré maps

- Let Σ be a hypersurface of $\mathbb{R}^n$, and assume it is transversal to the vectorfield, that is, the vectorfield is not tangent Σ in any point of Σ,

$$\forall \boldsymbol{x} \in \Sigma \quad \boldsymbol{f}(\boldsymbol{x}) \notin T_{\boldsymbol{x}}(\Sigma).$$

- Let $\boldsymbol{x}_0$ be such that $\phi_{T_0} \in \Sigma$ for some $T_0 > 0$, and assume that T_0 is minimum with this property.
- Under suitable hypothesis, there exists a neighborhood $U \ni x_0$ and a map $\tau : U \to \mathbb{R}^n$, known as **time–return map**, such that

$$\phi_{\tau(\boldsymbol{x})}(\boldsymbol{x}) \in \Sigma \qquad \forall \boldsymbol{x} \in U.$$

- The map

$$\boldsymbol{P}(\boldsymbol{x}) = \phi_{\tau(\boldsymbol{x})}(\boldsymbol{x})$$

  is called **Poincaré map** or **first–return map** corresponding to Σ.
- The restriction of $\boldsymbol{P}$ to $V := \Sigma \cap U$ defines a discrete dynamical system.

Poincaré maps: numerical computation

We are given:

- A Poincaré map, $\boldsymbol{P}(x) = \phi_{\tau(x)}(\boldsymbol{x})$.
- A surface of section, $\Sigma = \{g(\boldsymbol{x}) = 0\}$, to be traversed from $\{g(\boldsymbol{x}) < 0\}$ to $\{g(\boldsymbol{x}) > 0\}$

We can numerically evaluate $\boldsymbol{P}(\boldsymbol{x})$ by the following algorithm:

input: $\boldsymbol{x}$, g, $\boldsymbol{f}$, tol

do: t:=0, $\boldsymbol{y}$:=$\boldsymbol{x}$, h:=tol

while $(g(\boldsymbol{y}) \geq 0)$

  $(t,\boldsymbol{y},h)$:=IntStep$(t,\boldsymbol{y},h,\boldsymbol{f},tol)$

while $(g(\boldsymbol{y}) < 0)$

  $(t,\boldsymbol{y},h)$:=IntStep$(t,\boldsymbol{y},h,\boldsymbol{f},tol)$

while $(|g(\boldsymbol{y})| <$ tol$)$

  $\delta := -g(\boldsymbol{y})/\big(Dg(\boldsymbol{y})\boldsymbol{f}(\boldsymbol{y})\big)$

  $(t,\boldsymbol{y},h)$:=Flow$(t, t+\delta, \boldsymbol{y}, h, \boldsymbol{f},tol)$

output: t, $\boldsymbol{y}$.

Poincaré maps: differential

In order to differentiate $\boldsymbol{P}(\boldsymbol{x})$, we need

- $D\phi_{\tau(\boldsymbol{x})}(\boldsymbol{x})$, from variational equations.
- $D\tau(\boldsymbol{x})$.

The last quantity can be obtained by implicit differentiation:

$$\begin{aligned} 0 &\equiv g(\boldsymbol{P}(\boldsymbol{x})) \\ \Longrightarrow D\tau(\boldsymbol{x}) &= -\frac{Dg\big(\boldsymbol{P}(x)\big)D\phi_{\tau(\boldsymbol{x})}(x)}{Dg\big(\boldsymbol{P}(x)\big)\boldsymbol{f}\big(\boldsymbol{P}(x)\big)}. \end{aligned}$$

From the chain rule,

$$D\boldsymbol{P}(x) = -\underbrace{\boldsymbol{f}\big(\boldsymbol{P}(x)\big)}_{n\times 1}\frac{\overbrace{Dg\big(\boldsymbol{P}(x)\big)}^{1\times n}\,\overbrace{D\phi_{\tau(\boldsymbol{x})}(\boldsymbol{x})}^{n\times n}}{Dg\big(\boldsymbol{P}(x)\big)\boldsymbol{f}\big(\boldsymbol{P}(x)\big)} + \underbrace{D\phi_{\tau(\boldsymbol{x})}(\boldsymbol{x})}_{n\times n}.$$

It can be avoided in some situations, as we will see.

Outline

Computation of fixed points

- A fixed point of
$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}),$$
is a point $p \in \mathbb{R}^n$ such that $\boldsymbol{f}(p) = 0$.
- For simple models, fixed points can be found analitically.
- Wen it is not possible, Newton's method can be used in order to find a zero of

$$\begin{array}{rcl} \boldsymbol{G} : \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ \boldsymbol{x} & \longmapsto & \boldsymbol{f}(\boldsymbol{x}) \end{array}$$

Linear behavior around a fixed point

Interesting in order to:

- Understand the dynamics around a fixed point.
- Obtain good initial guesses for the objects that originate around a fixed point.

Linear behavior around a fixed point: flows

Consider a flow

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}),$$

with a fixed point $\boldsymbol{p}$,

$$\boldsymbol{f}(\boldsymbol{p}) = 0,$$

The Taylor expansion of $\boldsymbol{f}$ around $\boldsymbol{p}$ up to order one is

$$\boldsymbol{f}(\boldsymbol{x}) = \underbrace{\boldsymbol{f}(\boldsymbol{p})}_{=0} + \underbrace{D\boldsymbol{f}(\boldsymbol{p})}_{=:A}(\boldsymbol{x} - \boldsymbol{p}) + O(\|\boldsymbol{x} - \boldsymbol{p}\|^2),$$

so that the linearized flow around $\boldsymbol{p}$ is

$$\dot{\boldsymbol{x}} = A(\boldsymbol{x} - \boldsymbol{p}).$$

The eigenvalues of A are known as the **exponents** of the fixed point $\boldsymbol{p}$.
For Hamiltonian systems,

$$\lambda \in \operatorname{Spec} A \Longrightarrow -\lambda \in \operatorname{Spec} A.$$

Linear behavior around a fixed point: flows

Assume $\lambda \in \operatorname{Spec} A$, $\lambda \neq 0$, $A\boldsymbol{v} = \lambda \boldsymbol{v}$, $\boldsymbol{v} \neq 0$.

- If $\lambda \in \mathbb{R}$, consider $\varphi(t) = \boldsymbol{p} + e^{\lambda t}\boldsymbol{v}$. Then:
  - $\varphi(t)$ satisfies the system of ODE of the linearized flow,

$$\begin{aligned} \varphi'(t) &= \lambda e^{\lambda t}\boldsymbol{v} = e^{\lambda t}(\lambda \boldsymbol{v}) = e^{\lambda t}(A\boldsymbol{v}) = A(e^{\lambda t}\boldsymbol{v}) \\ &= A\big(\varphi(t) - \boldsymbol{p}\big). \end{aligned}$$

  - If $\lambda > 0$, $\varphi(t) \xrightarrow{t\to+\infty} \boldsymbol{p}$, so that it gives a **stable manifold** of the linearized flow.
  - If $\lambda < 0$, $\varphi(t) \xrightarrow{t\to-\infty} \boldsymbol{p}$, so that it gives an **unstable manifold** of the linearized flow.

The extistence of a stable or unstable manifold of the full (nonlinear) dynamical system is ensured by the **stable manifold theorem** for flows.

Linear behavior around a fixed point: flows

Assume $\lambda \in \operatorname{Spec} A$, $\lambda \neq 0$, $A\boldsymbol{v} = \lambda \boldsymbol{v}$, $\boldsymbol{v} \neq 0$.

- If $\lambda = i\omega$, for $\omega \in \mathbb{R}$, let $\boldsymbol{v}_1 + i\boldsymbol{v}_2$ be a corresponding eigenvector, with $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^n$. Then

$$A\boldsymbol{v}_1 + iA\boldsymbol{v}_2 = A(\boldsymbol{v}_1 + i\boldsymbol{v}_2) = -\omega \boldsymbol{v}_2 + i\omega \boldsymbol{v}_1 .$$

Therefore, if we define

$$\varphi_\gamma(t) = \boldsymbol{p} + \gamma\big((\cos \omega t)\boldsymbol{v}_1 - (\sin \omega t)\boldsymbol{v}_2\big),$$

we have

$$\varphi'_\gamma(t) = A\big(\varphi(t) - p\big),$$

so that $\varphi_\gamma(t)$ satifies the linearized system of ODE
Under non–resonance conditions with respect to the remaining eigenvalues, the existence of a family of periodic orbits for the full nonlinear system, with limiting period $2\pi/\omega$, is ensured by **Liapunov's center theorem**.

Linear behavior around a fixed point: flows

Assume $\lambda \in \operatorname{Spec} A$, $\lambda \neq 0$, $A\boldsymbol{v} = \lambda \boldsymbol{v}$, $\boldsymbol{v} \neq 0$.

- The case $\lambda = a + i\omega$ for $a, \omega \in \mathbb{R}$, $a, \omega \neq 0$ corresponds to a **sink** or a **source**, depending on wether $\operatorname{Re} \lambda < 0$ or $\operatorname{Re} \lambda > 0$, respectively.
- It is an impossible case in a Hamiltonian system, and will not be considered here.

Outline

Computation of p.o.: autonomous case

Consider an autonomous system of ODE,

$$\dot{x} = \boldsymbol{f}(\boldsymbol{x}),$$

Assume we look for an o.p. as a fixed point of

$$\boldsymbol{F}(\boldsymbol{x}) = \phi_T(\boldsymbol{x}).$$

Then we would look for a zero of

$$\boldsymbol{G}(\boldsymbol{x}) := \boldsymbol{F}(\boldsymbol{x}) - \boldsymbol{x} = \phi_T(\boldsymbol{x}) - \boldsymbol{x}.$$

by Newton's method.

But, for $\boldsymbol{x}_0$ in the o.p., $D\boldsymbol{G}(\boldsymbol{x}_0)$ is **singular**, because $\{\boldsymbol{G}(\boldsymbol{x}) = 0\}$ has the whole o.p. as solution.

Computation of p.o.: autonomous case

- We look for a fixed point of a Poincaré map corresponding to a section that intersects transversally the p.o. at $\boldsymbol{x}_0$.

  $$\boldsymbol{P}(\boldsymbol{x}) = \phi_{\tau(\boldsymbol{x})}(\boldsymbol{x}),$$

  Then $\boldsymbol{x}_0$ is the only point of the p.o. that is also a fixed point of the Poincaré map.
  We look for a zero of

  $$\boldsymbol{G}(\boldsymbol{x}) := \boldsymbol{P}(\boldsymbol{x}) - \boldsymbol{x} = D\boldsymbol{P}(\boldsymbol{x}) - \boldsymbol{x},$$

  by Newton's method. Its differential is

  $$D\boldsymbol{G}(\boldsymbol{x}) = D\boldsymbol{P}(\boldsymbol{x}) - I_n.$$

- Alternatively, we can consider T an additional unknown and solve

  $$\left.\begin{array}{rcl} g(\boldsymbol{x}) & = & 0 \\ \phi_T(\boldsymbol{x}) & = & \boldsymbol{x} \end{array}\right\}.$$

Computation of p.o.: autonomous case

The previous approach works in oder to find an isolated p.o., but

- In autonomous Hamiltonian systems (like the RTBP or Hill's problem), periodic orbits are not isolated but embedded in families.
- This gives a curve of fixed points in the Poincaré section.
- This curve is solution $\boldsymbol{G}(\boldsymbol{x}) = 0$, so $D\boldsymbol{G}(\boldsymbol{x})$ is singular at any of this points.

The solution to this problem is to add an additional constraint in order to have an unique o.p. as solution.

It can be, either

- To prescribe a certain period.
- To prescribe an energy level.

Practical implementation

Consider the RTBP (or any autonomous Hamiltonian system).

We can consider the following system of equations:

$$\left.\begin{array}{rcl} H(\boldsymbol{x})-h & = & 0 \\ \tau(\boldsymbol{x})-T & = & 0 \\ \phi_{\tau(\boldsymbol{x})}(\boldsymbol{x})-\boldsymbol{x} & = & 0 \end{array}\right\},$$

with unknowns $(h,T,\boldsymbol{x})=(h,T,x,y,z,p_x,p_y,p_z)$.

This system, as is, does not need to be compatible, because both the energy and the period (locally) determine a unique p.o. in the family.

What we can do is to eliminate equations and unknows in the previous system in order to obtain que equations for a particular approach:

- By “eliminating” an equation, we mean exactly this.
- By “eliminating” an unknown, we mean to keep it constant in Newton’s method, as if it were a parameter.

Practical implementation

Consider the RTBP (or any autonomous Hamiltonian system).

We can consider the following system of equations:

$$\left.\begin{array}{rcl} H(\boldsymbol{x}) - h & = & 0 \\ \tau(\boldsymbol{x}) - T & = & 0 \\ \phi_{\tau(\boldsymbol{x})}(\boldsymbol{x}) - \boldsymbol{x} & = & 0 \end{array}\right\},$$

with unknowns $(h, T, \boldsymbol{x}) = (h, T, x, y, z, p_x, p_y, p_z)$.
In this way, for instance:

- To compute a p.o. of a given energy level, we eliminate equation 2 and unknowns h, T.
- To compute a p.o. of a given period, we eliminate equation 1 and unknowns h, T.
- To compute a p.o. of a given energy level an a prescribed value of a coordinate, we eliminate the second equation and the unknowns h and the prescribed coordinate.

Practical implementation

Consider the RTBP (or any autonomous Hamiltonian system).

We can consider the following system of equations:

$$\left.\begin{array}{rcl} H(\boldsymbol{x}) - h & = & 0 \\ \tau(\boldsymbol{x}) - T & = & 0 \\ \phi_{\tau(\boldsymbol{x})}(\boldsymbol{x}) - \boldsymbol{x} & = & 0 \end{array}\right\},$$

with unknowns $(h, T, \boldsymbol{x}) = (h, T, x, y, z, p_x, p_y, p_z)$.

With any of the previous choices, we end up with an $(n+2) \times (n+1)$ (nonlinear) system with unique solution.

Multiple shooting

The neighborhood of the collinear libration points of the RTBP is highly unstable.

- The monodromy matrices of p.o. have eigenvalues as large as 2000, or even more.
- This means that any error in the initial condition is amplified by this factor.
- This is also true for the numerical truncation error.

We can reduce these amplification factors by making use of **multiple shooting**.

Idea: introduce additional objects and matching conditions in order to reduce integratin time.

Multiple shooting

The neighborhood of the collinear libration points of the RTBP is highly unstable.

Instead of loking for

$$h, T, \boldsymbol{x},$$

we look for

$$h, T, \boldsymbol{x}_0, \dots, \boldsymbol{x}_{m-1},$$

for $m > 1$, satisfying

$$\left.\begin{array}{rcll} H(\boldsymbol{x}_0) - h & = & 0 & \\ \tau(\boldsymbol{x}_{m-1}) - \frac{T}{m} & = & 0 & \\ \phi_{T/m}(\boldsymbol{x}_i) - \boldsymbol{x}_{i+1} & = & 0, & i = 0 \div m-2 \\ \phi_{\tau(\boldsymbol{x}_{m-1})}(\boldsymbol{x}_{m-1}) - \boldsymbol{x}_0 & = & 0 & \end{array}\right\}.$$

Whis this approach, one obtains amplification factors that are, tipically, the m-th root of the starting ones.

Linear behavior around a p.o.

Consider an autonomous Hamiltonian system.

- An initial condition $\boldsymbol{x}_0$ of a T–periodic orbit is also a fixed point of ϕ_T.
- Consider the RTBP (or any autonomous Hamiltonian system), and let $\boldsymbol{x}_0$ be an initial condition of a T–periodic orbit. Then, its monodromy matrix,

$$M := D\phi_T(\boldsymbol{x}_0)$$

  has 1 as double eigenvalue.
- Moreover, $M := D\phi_T(\boldsymbol{x}_0)$ is a **symplectic matrix**, which implies: if λ is an eigenvalue of M, then $1/\lambda$ is also eigenvalue.
  Then,

$$\operatorname{Spec} M = \{1, 1, \lambda_1, \lambda_1^{-1}, \lambda_2, \lambda_2^{-1}\},$$

  and we will assume that $|\lambda_i| \leq |\lambda_i^{-1}|$.

Linear behavior around a p.o.

Consider an autonomous Hamiltonian system.
Let $\boldsymbol{x}_0$ s.t. $\phi_T(\boldsymbol{x}_0) = \boldsymbol{x}_0$, $M := D\phi_T(\boldsymbol{x}_0)$.
$\operatorname{Spec} M = \{1, 1, \lambda_1, \lambda_1^{-1}, \lambda_2, \lambda_2^{-1}\}$.

The linear behaviour around a p.o. is better studied in terms of its **stability parameters**, s_1 and s_2, which are defined as

$$s_1 = \lambda_1 + 1/\lambda_1, \quad s_2 = \lambda_2 + 1/\lambda_2.$$

It is easy to check that

$$\begin{aligned}
s_i \in \mathbb{R}, \quad |s_i| > 2 \quad &\Longleftrightarrow \quad \lambda_i \in \mathbb{R}\backslash\{\pm 1\}, \\
s_i \in \mathbb{R}, \quad |s_i| \leq 2 \quad &\Longleftrightarrow \quad \lambda_i \in \mathbb{C}, \quad |\lambda_i| = 1, \\
s_i \in \mathbb{C}\backslash\mathbb{R} \quad &\Longleftrightarrow \quad \lambda_i \in \mathbb{C}\backslash\mathbb{R}, \quad |\lambda_i| \neq 1.
\end{aligned}$$

Linear behavior around a p.o.

Consider an autonomous Hamiltonian system.
Let $\boldsymbol{x}_0$ s.t. $\phi_T(\boldsymbol{x}_0) = \boldsymbol{x}_0$, $M := D\phi_T(\boldsymbol{x}_0)$.
$\operatorname{Spec} M = \{1, 1, \lambda_1, \lambda_1^{-1}, \lambda_2, \lambda_2^{-1}\}$.
$s_1 = \lambda_1 + 1/\lambda_1$, $s_2 = \lambda_2 + 1/\lambda_2$.

- If $s_i \in \mathbb{R}$, $|s_i| > 2$ **(hyperbolic case)** $\Rightarrow \lambda_i \in \mathbb{R}\backslash\{\pm 1\}$.
  - There is a stable manifold of the fixed point of ϕ_T, tangent to the λ_i–eigendirection at $\boldsymbol{x}_0$.
  - There is an unstable manifold of the fixed point ϕ_T, tangent to the λ_i^{-1}–eigendirection at $\boldsymbol{x}_0$.

  In terms of the p.o.:
  - There is a stable manifold of the p.o. whose section through the $\lambda_i, \lambda_i^{-1}$–eigenplane is tangent to the λ_i–eigendirection.
  - There is an unstable manifold of teh p.o. whose section through the $\lambda_i, \lambda_i^{-1}$–eigenplane is tangent to the λ_i^{-1}–eigendirection.

Linear behavior around a p.o.

Consider an autonomous Hamiltonian system.
Let $\boldsymbol{x}_0$ s.t. $\phi_T(\boldsymbol{x}_0) = \boldsymbol{x}_0$, $M := D\phi_T(\boldsymbol{x}_0)$.
$\operatorname{Spec} M = \{1, 1, \lambda_1, \lambda_1^{-1}, \lambda_2, \lambda_2^{-1}\}$.
$s_1 = \lambda_1 + 1/\lambda_1,\ s_2 = \lambda_2 + 1/\lambda_2$.

- If $s_i \in \mathbb{R}$, $|s_i| \leq 2$ (**elliptic case**), let $\lambda_i = \cos\rho + i\sin\rho$ ($\Rightarrow s_i = 2\cos\rho$), and let $\boldsymbol{v}$ be s.t. $M\boldsymbol{v} = \lambda_i \boldsymbol{v}$, $\boldsymbol{v} \neq 0$, $\boldsymbol{v} = \boldsymbol{v}_1 + i\boldsymbol{v}_2$.
  - There is a continuous, one–parametric family of closed curves invariant by the linearization of ϕ_T around $\boldsymbol{x}_0$ in the $\{\boldsymbol{x}_0 + \alpha_1 \operatorname{Re} \boldsymbol{v}_1 + \alpha_2 \operatorname{Im} \boldsymbol{v}_2\}_{\alpha_1, \alpha_2 \in \mathbb{R}}$ plane, with rotation number ρ.
  - Under generic non–degeneracy conditions, there is a Cantorian family of invariant curves around $\boldsymbol{x}_0$, with limiting rotation number ρ. When transported by the flow, these invariant curves generate two–dimensional invariant tori.
  - Rational values (times 2π) for ρ also give rise to bifurcated p.o., with period $2\pi/\rho$. The particular values $\rho = 2\pi$ ($s_i = 2$) and $\rho = \pi$ ($s_i = -2$), are known as the **parabolic case**.

Outline

Strategy

We will consider two cases:

- Continuation with respect to the energy with multiple shooting. The equations to continue are

$$\left\{ \begin{array}{rcll} H(\boldsymbol{x}_0) - h & = & 0, & \\ \phi_{T/m}(\boldsymbol{x}_i) - \boldsymbol{x}_{i+1} & = & 0, & i = 0 \div m-2, \\ \phi_{\tau(\boldsymbol{x}_{m-1})}(\boldsymbol{x}_{m-1}) - \boldsymbol{x}_0 & = & 0, & \end{array} \right.$$

  with unknowns $h, \boldsymbol{x}_0, \ldots, \boldsymbol{x}_{m-1}$.

  Note this in this case T is a parameter, but should be close to the period of the p.o. It is conveinent to set $T := \tau(\boldsymbol{x}_0)$ at every continuation step, and recompute $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n-1}$, in order to have them equally spaced in time along the p.o.

Strategy

We will consider two cases:

- Continuation with respect to the period with multiple shooting. The equations to consider are

$$\left\{ \begin{array}{rcll} \tau(\boldsymbol{x}_{m-1}) - \frac{T}{m} & = & 0 & \\ \phi_{T/m}(\boldsymbol{x}_i) - \boldsymbol{x}_{i+1} & = & 0, & i = 0 \div m-2, \\ \phi_{\tau(\boldsymbol{x}_{m-1})}(\boldsymbol{x}_{m-1}) - \boldsymbol{x}_0 & = & 0, & \end{array} \right.$$

  with unknows $T, \boldsymbol{x}_0, \ldots, \boldsymbol{x}_{m-1}$.

Note that the two systems of equations just considered can be evaluated by the same routine by eliminating suitable equations and unknowns.

Example: the Lyapunov families around L_1 in the RTBP

Linear behavior around L_1

Denote the system of ODE of the RTBP for the Earth–Moon mass parameter as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}).$$

Then

$$\operatorname{Spec}(D\boldsymbol{f}(L_1)) = \{\lambda, -\lambda, i\omega_v, -i\omega_v, i\omega_p, -i\omega_p\},$$

with $\lambda, \omega_p, \omega_v > 0$.
Then,

- the eigenvalues $\pm\lambda$ give rise to stable and unstable manifodls.
- the eigenvalues $\pm i\omega_p$, $\pm i\omega_v$ give rise to a center manifold, on which
  - the eigenvalues $\pm i\omega_p$ give rise to the Lyapunov planar family of p.o.,
  - the eigenvalues $\pm i\omega_v$ give rise to the Lypaunov vertical family of p.o.

Example: the Lyapunov families around L_1 in the RTBP

Linear behavior around L_1

Denote the system of ODE of the RTBP for the Earth–Moon mass parameter as

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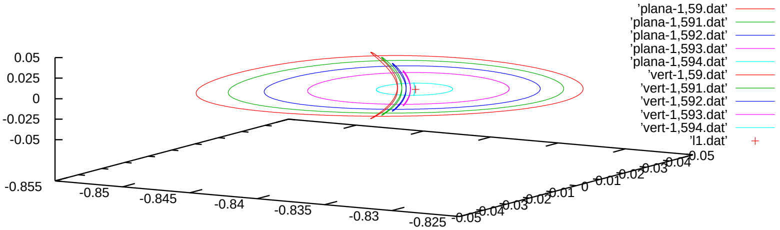
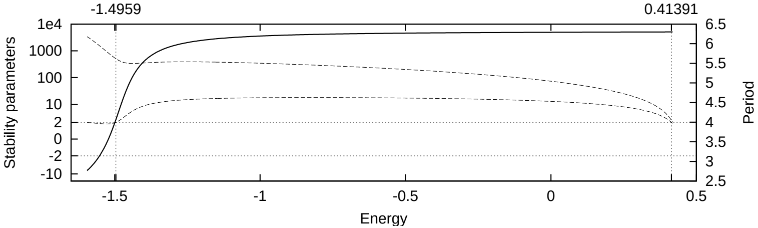
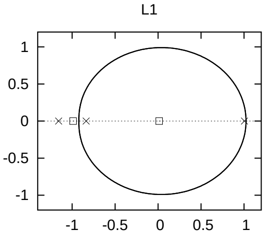
with $\lambda, \omega_p, \omega_v > 0$.



Example: the Lyapunov families around L_1 in the RTBP

Liapunov vertical family

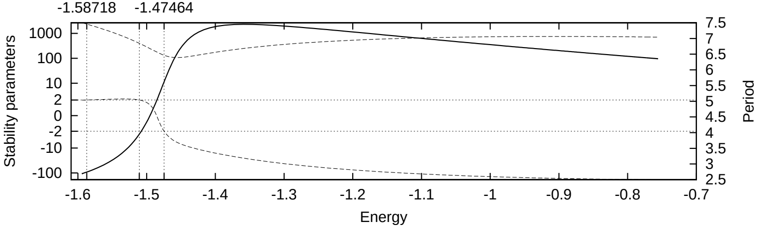




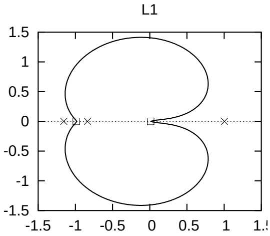
Example: the Lyapunov families around L_1 in the RTBP

Liapunov planar family

Go to collision with the Earth.



#bif.	Energy	Type
1	-1.58718	A
2	-1.51070	B
3	-1.47464	C

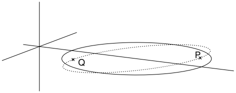


Example: the Lyapunov families around L_1 in the RTBP

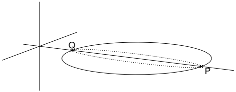
Liapunov planar family

#bif.	Energy	Type	
1	-1.58718	A	Halo family
2	-1.51070	B	Bridge to the vertical family
3	-1.47464	C	Not continuated

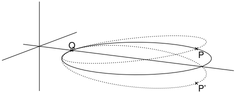
Type A



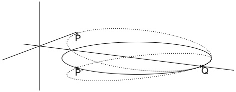
Type B



Type C



Type D



Outline

Numerical computation of invariant 2D tori

We develop the methodology [9, 12] for an autonomous Hamiltonian system.

- We could look for a parametrization of a 2D torus,

$$\begin{array}{rccc} \psi : & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^6 \\ & (\theta_1, \theta_2) & \mapsto & \psi(\theta_1, \theta_2), \end{array}$$

with ψ 2π–periodic function in θ_1, θ_2, by solving

$$\psi(\theta_1 + t\omega_1, \theta_2 + t\omega_2) = \phi_t\big(\psi(\theta_1, \theta_2)\big), \quad \forall t \in \mathbb{R}, \quad \forall \theta_1, \theta_2 \in [0, 2\pi],$$

where ω_1, ω_2 are the frequencies of the torus.

Numerical computation of invariant 2D tori

- We could look for a parametrization of a 2D torus by solving

$$\psi(\theta_1 + t\omega_1, \theta_2 + t\omega_2) = \phi_t\big(\psi(\theta_1, \theta_2)\big), \quad \forall t \in \mathbb{R}, \quad \forall \theta_1, \theta_2 \in [0, 2\pi],$$

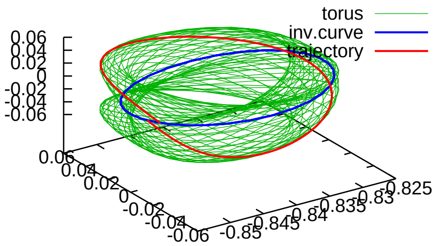
  where ω_1, ω_2 are the frequencies of the torus.
- In order to reduce the dimension of the problem, we observe that $\varphi(\xi) = \psi(\xi, 0)$ is a curve invariant by $\phi_{2\pi/\omega_2}$, and satisfies

$$\varphi(\xi + \rho) = \phi_{T_2}\big(\varphi(\xi)\big),$$

  for $\rho = 2\pi\omega_1/\omega_2$ and $T_2 = 2\pi/\omega_2$.
- Once we have φ, we can recover ψ by

$$\psi(\theta_1, \theta_2) = \phi_{\frac{\theta_2}{2\pi}T_2}\Big(\varphi(\theta_1 - \frac{\theta_2}{2\pi}\rho)\Big)$$

Numerical computation of invariant 2D tori



$$\phi_{T_2}\big(\varphi(\xi)\big) = \varphi(\xi + \rho)$$

Discretization

- Note that
$$\varphi(\xi+\rho) = \phi_{T_2}\big(\varphi(\xi)\big),$$
is a functional equation: we have “infinite equations” (one for each value of $\xi \in [0, 2\pi)$) and “infinite unknowns” (we cannot describe a general function φ by a finite number of parameters).
- We discretize function space by looking for φ as a truncated Fourier series,
$$\varphi(\xi) = \boldsymbol{A}_0 + \sum_{k=1}^{N_f}\Big(\boldsymbol{A}_k\cos(k\xi) + \boldsymbol{B}_k\sin(k\xi)\Big).$$
- We will discretize parameter space by looking for φ satisfying
$$\varphi(\xi+\rho) - \phi_{T_2}\big(\varphi(\xi_i)\big), \qquad i = 0 \div 2N_f,$$
for $\xi_i = i2\pi/(1+2N_f)$.

Indeterminations

We have two indeterminations to cope with.

- *(Invariant curve indetermination)* Assuming there exists a parametrization of a 2D torus,

$$\varphi(\theta_1 + t\omega_1, \theta_2 + t\omega_2) = \phi_t\big(\psi(\theta_1, \theta_2)\big),$$

not only $\varphi(\xi) = \psi(\xi, 0)$ satisfies

$$\varphi(\xi + \rho) = \phi_{T_2}\big(\varphi(\xi)\big),$$

but any $\varphi(\xi) := \psi(\xi, \eta_0)$ for $\eta_0 \in [0, 2\pi)$ also does.

  - This indetermination can be avoided by fixing a curve on the torus.
  - This can be done by prescribing a value for a coordinate of A_0.
  - It must be chosen by geometrical considerations.

Indeterminations

We have two indeterminations to cope with.

- *(Phase shift indetermination)* If $\varphi(\xi)$ satisifes $\varphi(\xi+\rho) = \phi_{T_2}\big(\varphi(\xi)\big)$, then, for any $\xi_0 \in \mathbb{R}$, $\varphi_{\xi_0}(\xi) = \varphi(\xi - \xi_0)$ also does.
  - This indetermination can be avoided by prescribing a coordinate of $\boldsymbol{A}_1$ to be zero.
  - Assume that $\boldsymbol{A}_1 = (A_1^1, \dots, A_1^6)$, $\boldsymbol{B}_1 = (B_1^1, \dots, B_1^6)$. If $(A_1^k, B_1^k) \neq (0,0)$, since

$$
\begin{aligned}
A_1^k \cos\big(k(\xi-\xi_0)\big) + B_1^k \sin\big(k(\xi-\xi_0)\big)& \\
= \quad & (A_1^k \cos k\xi_0 - B_1^k \sin k\xi_0) \cos k\xi \\
& \quad + (A_1^k \sin k\xi_0 + B_1^k \cos k\xi_0) \sin k\xi \\
=: \quad & \widetilde{A}_1^k \cos k\xi + \widetilde{B}_1^k \sin k\xi .
\end{aligned}
$$

we can always choose ξ_0 such that $\widetilde{A}_1^k = 0$.

The system of equations

We want to design a system of equations such that

- We are to prescribe values for the energy.
  For that, we add an additional equation.
- We want to overcome high instability.
  For that, we implement multiple shooting.

The system of equations

We will, therefore, look for $\varphi_0, \ldots, \varphi_{m-1}$ satysfiying

$$\begin{cases} H\big(\varphi_0(0)\big) - h & = & 0 \\ \varphi_{j+1}(\xi_i) - \phi_{T_2/m}\big(\varphi_j(\xi_i)\big) & = & 0, \quad j = 0 \div m-2, \quad i = 0, \ldots, 2N_f, \\ \varphi_0(\xi_i + \rho) - \phi_{T_2/m}\big(\varphi_{m-1}(\xi_i)\big) & = & 0, \quad i = 0 \div 2N_f, \end{cases}$$

where $\xi_i = i(2\pi)/(1+2N_f)$, $i = 0 \div 2N_f$, and the unknowns are

$$h, T_2, \rho, \boldsymbol{A}_0^0, \boldsymbol{A}_1^0, \boldsymbol{B}_1^0, \ldots, \boldsymbol{A}_{N_f}^0, \boldsymbol{B}_{N_f}^0, \ \ldots\ , \boldsymbol{A}_0^{m-1}, \boldsymbol{A}_1^{m-1}, \boldsymbol{B}_1^{m-1}, \ \ldots\ , \boldsymbol{A}_{N_f}^{m-1}, \boldsymbol{B}_{N_f}^{m-1}$$

with $h, T_2, \rho \in \mathbb{R}$, $\quad \boldsymbol{A}_i^j, \boldsymbol{B}_i^j \in \mathbb{R}^6$ and

$$\varphi_j(\xi) = \boldsymbol{A}_0^j + \sum_{l=0}^{N_f} \Big(\boldsymbol{A}_l^j \cos(l\xi) + \boldsymbol{B}_l^j \sin(l\xi) \Big).$$

This system is $\big(1 + 6m(1+2N_f)\big) \times \big(3 + 6m(1+2N_f)\big)$.

Computation of a torus

The tori we are looking for are embedded in 2–parametric families, which can be paramtrized by 2 parameters among h, ρ, T_2. Therefore, in order to compute a torus, we

- eliminate one coordinate of $\boldsymbol{A}_0^0$, in order to fix a curve on the torus,
- set a coordinate of $\boldsymbol{A}_1^0$ equal to zero and eliminate it, in order to get rid of the phase shift indetermination.
- eliminate two unknonws among h, T_2, ρ, in order to fixate a particular torus.

When applying Newton's method, we end up with a

$$\left(1+6m(1+2N_f)\right) \times \left(3+6m(1+2N_f)-4\right)$$

system of linear equations, with unique solution but which has more equations than uknowns.
This is not a problem, as long as we use the general rutine we have described, specifying the kernel dimension to be zero.

Error estimation

In order to estimate the error of the computed torus, we can consider a refinement of the discretization of the parameter space, that is,

$$\widetilde{\xi}_j = j\frac{2\pi}{M},$$

for $M \gg 1 + 2N_f$, and use as error estimate

$$\max_{j=0\div M} \left\| \begin{pmatrix} \left(\varphi_{l+1}(\widetilde{\xi}_j) - \phi_{T_2/m}(\varphi_l(\widetilde{\xi}_j))\right)_{l=0}^{m-2} \\ \varphi_0(\widetilde{\xi}_j + \rho) - \phi_{T_2/m}(\varphi_{m-1}(\widetilde{\xi}_j)) \end{pmatrix} \right\|$$

for some norm.

We can reduce this estimate by increasing N_f, but this increases the size of the linear system we need to solve, and this is the bottleneck of the procedure.

Globalization of a torus from an invariant curve

Assume we have φ satisfying $\varphi(\xi + \rho) = \phi_{T_2}\big(\varphi(\xi)\big)$.

Then a calculation shows that

$$\psi(\theta_1, \theta_2) := \phi_{\frac{\theta_2}{2\pi}T_2}\Big(\varphi\big(\theta_1 - \frac{\theta_2}{2\pi}\rho\big)\Big)$$

describes an invariant torus with frequency vector $(\rho/T_2, 2\pi/T_2)$, that is,

$$\phi_t\big(\psi(\theta_1, \theta_2)\big) = \psi\Big(\theta_1 + t\omega_1, \theta_2 + t\omega_2\Big).$$

with $\omega_1 = \rho/T_2, \omega_2 = 2\pi/T_2$.

If we need to integrate a trajectory on the computed torus for a long time, we just have to numerically integrate it from invariant curve to invariant curve.

Starting from a periodic orbit

- Let $\boldsymbol{x}_0$ be s.t. $\phi_T(\boldsymbol{x}_0) = \boldsymbol{x}_0$.
- Let $s_1, s_2, s_i = \lambda_i + \lambda_i^{-1}$ be the corresponding stability parameters.
- Assume $|s_1| < 1$, $s_1 = 2\cos\nu$ (i.e., $\lambda_1 = \cos\nu + i\sin\nu$).

A calculation shows that, for

$$\begin{aligned} \boldsymbol{\varphi}(\xi) &= \boldsymbol{x}_0 + \gamma\Big(\cos(\xi-\xi_0)\boldsymbol{v}_1 - \sin(\xi-\xi_0)\boldsymbol{v}_2\Big) \\ &= \boldsymbol{x}_0 + \gamma\Big((\boldsymbol{v}_1\cos\xi_0 - \boldsymbol{v}_2\sin\xi_0)\cos\xi + (\boldsymbol{v}_1\sin\xi_0 + \boldsymbol{v}_2\cos\xi_0)\sin\xi\Big) \end{aligned}$$

we have

$$L_{\phi_T}^{\boldsymbol{x}_0}\big(\boldsymbol{\varphi}(\xi)\big) = \boldsymbol{\varphi}(\xi+\nu).$$

where $L_{\phi_T}^{\boldsymbol{x}_0}$ is the linear approximation of ϕ_T around $\boldsymbol{x}_0$.

Starting from a periodic orbit

- Let $\boldsymbol{x}_0$ be s.t. $\phi_T(\boldsymbol{x}_0) = \boldsymbol{x}_0$.
- Let s_1, s_2, $s_i = \lambda_i + \lambda_i^{-1}$ be the corresponding stability parameters.
- Assume $|s_1| < 1$, $s_1 = 2\cos\nu$ (i.e., $\lambda_1 = \cos\nu + i\sin\nu$).

Recall that we have shown that,

$$L_{\phi_T}^{\boldsymbol{x}_0}\left(\varphi(\xi)\right) = \varphi(\xi + \nu).$$

Therefore, as initial seed to get a torus around the o.p., we can take

$$\begin{array}{rclcrcl} h & = & H(\boldsymbol{x}_0), & & \boldsymbol{A}_0 & = & \boldsymbol{x}_0, \\ T_2 & = & T, & & \boldsymbol{A}_1 & = & (\boldsymbol{v}_1\cos\xi_0 - \boldsymbol{v}_2\sin\xi_0), \\ \rho & = & \nu, & & \boldsymbol{B}_1 & = & (\boldsymbol{v}_1\sin\xi_0 + \boldsymbol{v}_2\cos\xi_0), \end{array}$$

$$\boldsymbol{A}_j = \boldsymbol{B}_j = 0, \quad j \geq 2.$$

Starting from a periodic orbit

$$
\begin{array}{rclcrcl}
h & = & H(\boldsymbol{x}_0), & & \boldsymbol{A}_0 & = & \boldsymbol{x}_0, \\
T_2 & = & T, & & \boldsymbol{A}_1 & = & (\boldsymbol{v}_1 \cos \xi_0 - \boldsymbol{v}_2 \sin \xi_0), \\
\rho & = & \nu, & & \boldsymbol{B}_1 & = & (\boldsymbol{v}_1 \sin \xi_0 + \boldsymbol{v}_2 \cos \xi_0), \\
\end{array}
$$

$$
\boldsymbol{A}_j = \boldsymbol{B}_j = 0, \quad j \geq 2.
$$

Note that:

- We can use ξ_0 to get one coordinate of A_1 equal to zero and, in this way, avoid the phase shift indetermination.
- For the nonlinear system $\rho \neq \nu$, but we don't know if either $\rho > \nu$ or $\rho < \nu$. The same happens with T_2 and h.
- The o.p. itselfs satisfies the equations of an invariant torus, and has a large basin of attraction as a zero of these equations.

We can avoid the problems of the last two points above at once by keeping constant one coordinate of $\boldsymbol{A}_1$ or $\boldsymbol{B}_1$ which is different from zero in the initial seed.

Starting from a p.o.: second method

We have seen that

$$L_{\varphi}(\xi) = \boldsymbol{x}_0 + \gamma\Big((\boldsymbol{v}_1 \cos\xi_0 - \boldsymbol{v}_2 \sin\xi_0)\cos\xi + (\boldsymbol{v}_1 \sin\xi_0 + \boldsymbol{v}_2 \cos\xi_0)\cos\xi\Big)$$

parametrizes a closed curve invariant by the linarized time–T flow. We can globalize this invariant curve to a 2D torus invariant by the linearized flow as

$$L_{\psi}(\theta_1,\theta_2) = L^{\boldsymbol{x}_0}_{\boldsymbol{\phi}_{(\theta_2/2\pi)T_2}}\big(L_{\varphi}(\theta_1 - \frac{\theta_2}{2\pi}\nu)\big)$$

where we denote, for an arbitrary function $\boldsymbol{G}$ and an arbitrary point $\boldsymbol{y}_0$, the linearization of $\boldsymbol{G}$ around $\boldsymbol{y}_0$ as

$$L^{\boldsymbol{y}_0}_{\boldsymbol{G}}(\boldsymbol{y}) = \boldsymbol{y}_0 + D\boldsymbol{G}(\boldsymbol{y}_0)(\boldsymbol{y} - \boldsymbol{y}_0).$$

Then, a calculation shows that

$$L^{\boldsymbol{\phi}_{(\theta_2/2\pi)T}(\boldsymbol{x}_0)}_{\boldsymbol{\phi}_t}\big(L_{\psi}(\theta_1,\theta_2)\big) = L_{\psi}\Big(\theta_1 + t\frac{\nu}{T}, \theta_2 + t\frac{2\pi}{T}\Big).$$

Starting from a p.o.: second method

$$L_{\phi_t}^{\phi_{\frac{\theta_2}{2\pi}T}(\boldsymbol{x}_0)}\left(L_{\boldsymbol{\psi}}(\theta_1,\theta_2)\right) = L_\psi\left(\theta_1 + t\frac{\nu}{T}, \theta_2 + t\frac{2\pi}{T}\right).$$

- The previous way to obtain an initial seed to compute a torus corresponds to take T_2 close to the period of the backbone periodic orbit (second period of thelinear torus).
- In some situations, we will want to get an initial seed for a torus with T_2 close to a normal period of the backbone p.o. (first period of the linear torus).
- For that, we can take as initial seed

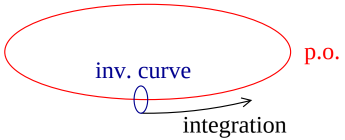
$$h = H(\boldsymbol{x}_0), \quad T_2 = \frac{2\pi}{\nu}T, \quad \rho = T_2\frac{2\pi}{T},$$

and $\boldsymbol{A}_i$, $\boldsymbol{B}_j$ the Fourier coefficients (easily obtained by a DFT) of $\{L_\psi(0, j\frac{2\pi}{N})\}_{j=0}^{N-1}$.

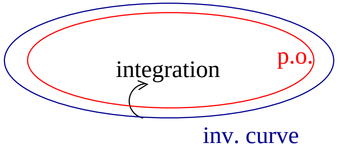
Starting longitudinally and transversally

In a somewhat sloopy/informal/unfortunate fashion,

- The first method of starting from a p.o. will be referred to as "starting longitudinally from a p.o.".



- The second method of starting from a p.o. will be referred to as "starting transversally from a p.o.".



Outline

Computation of a family of tori in the RTBP

Remember that the corresponding equations are

$$\begin{cases} H\big(\varphi_0(0)\big) - h &= 0 \\ \varphi_{j+1}(\xi_i) - \phi_{\delta/m}\big(\varphi_j(\xi_i)\big) &= 0, \quad j = 0 \div m-2, \quad i = 0, \dots, 2N_f, \\ \varphi_0(\xi_i + \rho) - \phi_{\delta/m}\big(\varphi_{m-1}(\xi_i)\big) &= 0, \quad i = 0 \div 2N_f, \end{cases}$$

with unknowns

$$h, T_2, \rho, \boldsymbol{A}_0^0, \boldsymbol{A}_1^0, \boldsymbol{B}_1^0, \dots, \boldsymbol{A}_{N_f}^0, \boldsymbol{B}_{N_f}^0, \ \dots \ , \boldsymbol{A}_0^{m-1}, \boldsymbol{A}_1^{m-1}, \boldsymbol{B}_1^{m-1}, \ \dots \ , \boldsymbol{A}_{N_f}^{m-1}, \boldsymbol{B}_{N_f}^{m-1}$$

Assume $\boldsymbol{A}_0^{k_1}, \boldsymbol{A}_1^{k_2} = 0$ are fixed in order to eliminate indeterminations, so that each value of the remaining coordinates corresponds at most to a torus.

The tori we are looking for are embedded in two–parametric families, so we have to fix one more parameter in order to use the pseudo–arclength method.

Computation of a family of tori in the RTBP

Remember that the corresponding equations are

$$\left\{ \begin{array}{rcll} H\big(\varphi_0(0)\big) - h & = & 0 & \\ \varphi_{j+1}(\xi_i) - \phi_{\delta/m}\big(\varphi_j(\xi_i)\big) & = & 0, & j = 0 \div m-2, \quad i = 0, \ldots, 2N_f, \\ \varphi_0(\xi_i + \rho) - \phi_{\delta/m}\big(\varphi_{m-1}(\xi_i)\big) & = & 0, & i = 0 \div 2N_f, \end{array} \right.$$

with unknowns

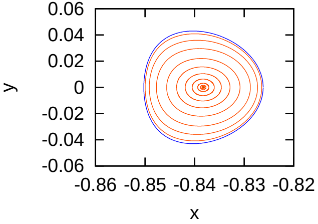
$$h, T_2, \rho, \boldsymbol{A}_0^0, \boldsymbol{A}_1^0, \boldsymbol{B}_1^0, \ldots, \boldsymbol{A}_{N_f}^0, \boldsymbol{B}_{N_f}^0, \ \ldots \ , \boldsymbol{A}_0^{m-1}, \boldsymbol{A}_1^{m-1}, \boldsymbol{B}_1^{m-1}, \ \ldots \ , \boldsymbol{A}_{N_f}^{m-1}, \boldsymbol{B}_{N_f}^{m-1}$$

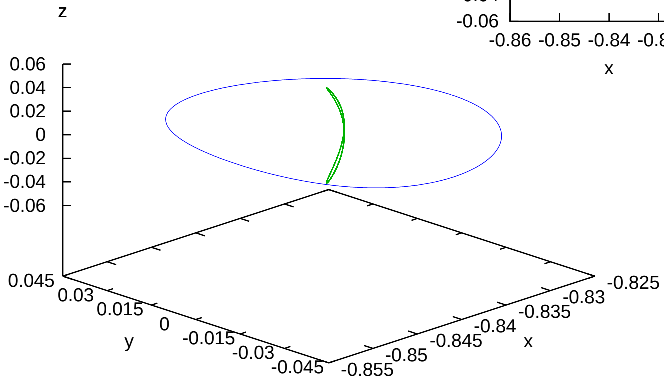
Assume $\boldsymbol{A}_0^{k_1}, \boldsymbol{A}_1^{k_2} = 0$ are fixed in order to eliminate indeterminations, so that each value of the remaining coordinates corresponds at most to a torus.
Interesting cases are

- to fix ρ to a number with good Diophantine properties,
- to fix h, in order to follow an iso–energetic family.

Example: 2D tori containing Lissajous orbits around L_1

From vertical p.o. (longitudinally) to planar (transversally)





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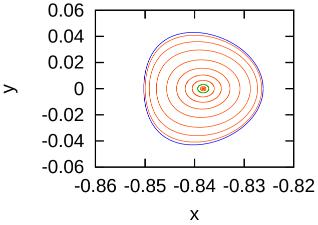

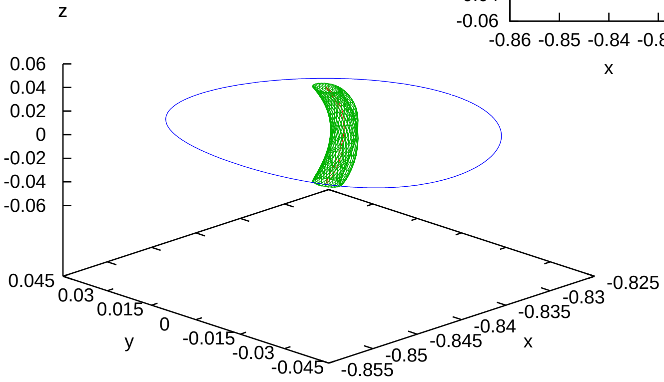

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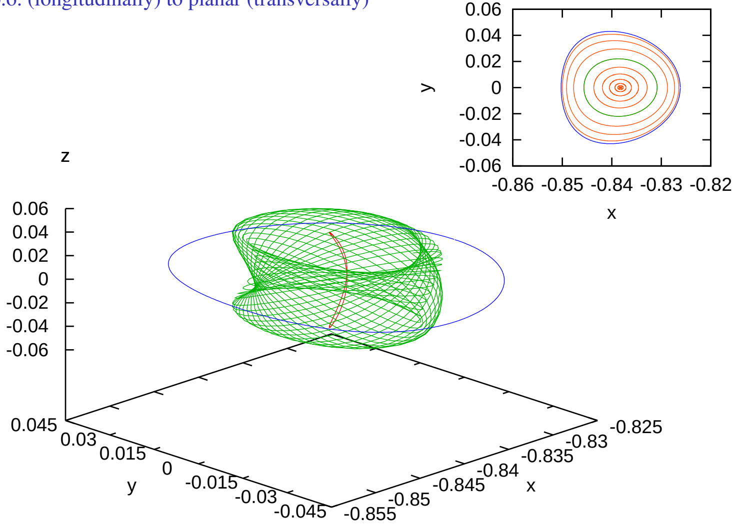

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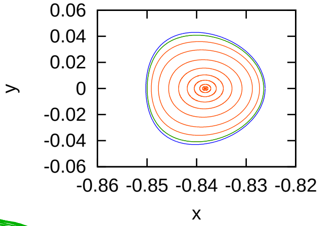
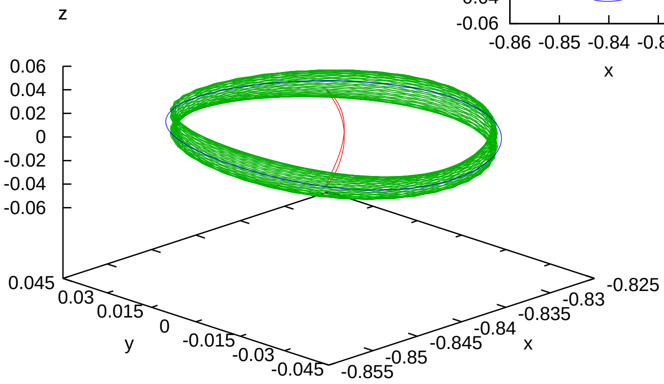
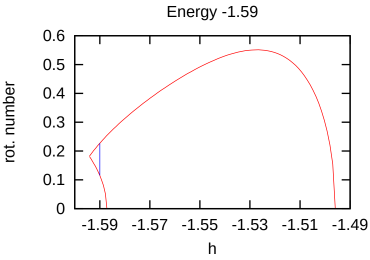
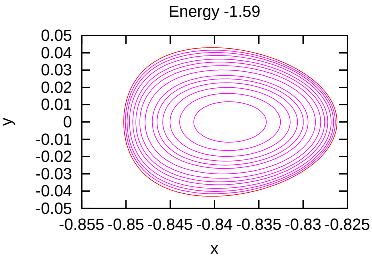




Example: 2D tori containing Lissajous orbits around L_1

Full family of 2D tori[8]





Variables:

- h : energy (Hamiltonian)
- ρ : (horizontal) **rotation number**:

$$\rho = 2\pi\left(\frac{\omega_{\text{horiz}}}{\omega_{\text{vert}}} - 1\right) = 2\pi\left(\begin{array}{c}\text{num. of horiz. turns}\\ \text{per vertical period}\end{array}\right) - 2\pi$$

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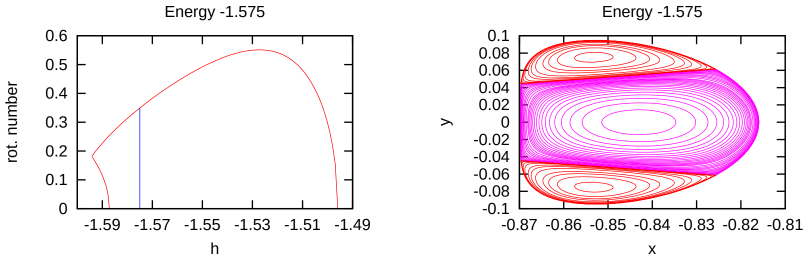

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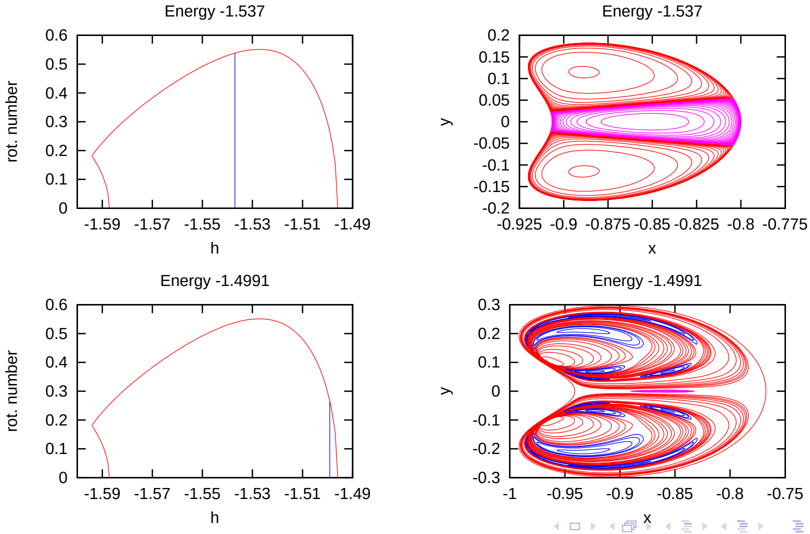
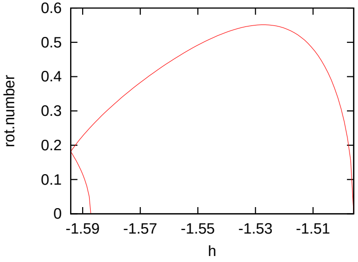


Global parametrization of families using interpolation

- In order to globally describe families or tori[10], we can
  - Compute a fine grid of tori covering the whole family we are interested in.
  - Interpolate between computed tori in order to obtain the ones not in the grid.
- For instance, for invariant tori containing Lissajous orbits around L_1:
  - The interpolation is done in the h–ρ representation (2D Lagrange iterated).
  - Final product: a routine that returns Fourier series for φ from h, ρ.



Global parametrization of families using interpolation

Some data

- Number of tori on the grid:

$$25433.$$

- Total processor time for tori:

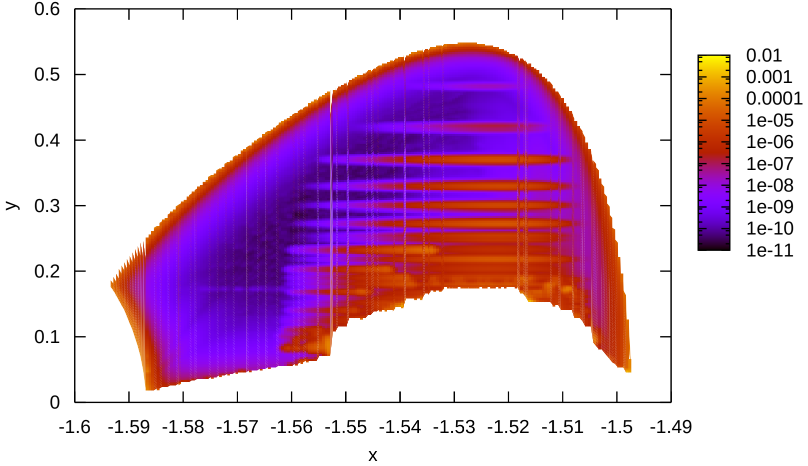
$$836.88 \text{ hours } (34.8 \text{ days}).$$

  (When using a cluster, divide by the number of processes).
- Sizes of binary files storing all the Fourier coefficients of the grid:

$$180 \text{ MB}.$$

Global parametrization of families using interpolation

Interpolation error for tori



Outline

Computation

- Let $\boldsymbol{x}_0$ be an i.c. of a T–periodic orbit: $\phi_T(\boldsymbol{x}_0) = \boldsymbol{x}_0$.
- $\boldsymbol{\varphi}(\theta) = \phi_{\frac{\theta}{2\pi}T}(x_0)$ parametrizes the p.o.
- Let $\Lambda \in \operatorname{Spec} D\phi_T(\boldsymbol{x}_0)$, $v \in V_\Lambda\left(D\phi_T(\boldsymbol{x}_0)\right)$,
  ($\Lambda > 1$ unst. mani, $\Lambda < 1$ stb. one).
  Then $\boldsymbol{v}(\theta) = \Lambda^{-\frac{\theta}{2\pi}} D\phi_{\frac{\theta}{2\pi}T}(\boldsymbol{x}_0)v$ parametrizes the tangent vectors to the manifold.
- $\overline{\boldsymbol{\psi}}(\theta,\xi) = \boldsymbol{\varphi}(\theta) + \xi\boldsymbol{v}(\theta)$ parametrizes the linear approximation of the manifold.
  (Can be evaluated **for small ξ only**).
  Satisfies
  $$\phi_t\left(\overline{\boldsymbol{\psi}}(\theta,\xi)\right) = \overline{\boldsymbol{\psi}}(\theta + t\omega, e^{t\lambda}\xi) + O(\xi^2)$$
  for $\omega = \frac{2\pi}{T}$, $\lambda = \frac{\omega \ln \Lambda}{2\pi}$.

Globalistaion

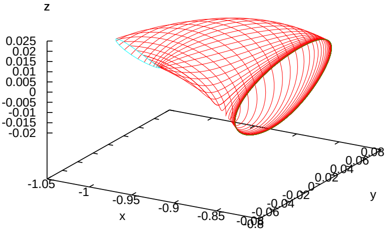
The manifold can be globalised by numerical integration:
for each ξ, take m such that $\Lambda^{-m}\xi$ small and compute

$$\mathbf{\Psi}(\theta,\xi) = \phi_{mT}\left(\theta, \Lambda^{-m}\xi\right).$$

$\mathbf{\Psi}$ satisfies

$$\phi_t\left(\mathbf{\Psi}(\theta,\xi)\right) = \mathbf{\Psi}(\theta + t\omega, e^{t\lambda}\xi) + O\left((\Lambda^{-m}\xi)^2\right).$$

Example: 2D unstable manifold associated to a Halo orbit (Moon branch).



Outline

Manifolds of tori[9]

- Assume $\theta \to \boldsymbol{\varphi}(\theta)$ parametrizes an invariant curve:

$$\phi_{T_2}\big(\boldsymbol{\varphi}(\theta)\big) = \boldsymbol{\varphi}(\theta + \rho)$$

We want to find $\Lambda \in \mathbb{C}$ and $\boldsymbol{u} : \mathbb{R} \to \mathbb{R}^6$, 2π–periodic, s.t.

$$D\phi_{T_2}\Big(\boldsymbol{\varphi}(\theta - \rho)\Big)\boldsymbol{u}(\theta - \rho) = \Lambda \boldsymbol{u}(\theta),$$

which can be compactly written as

$$\mathcal{C}\boldsymbol{u} = \Lambda \boldsymbol{u},$$

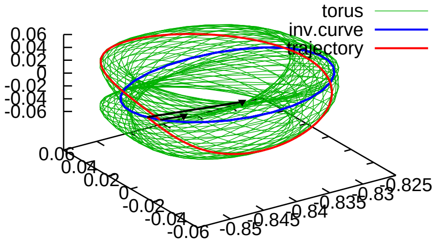
with

$$(\mathcal{C}\boldsymbol{u})(\theta) = D\phi_{T_2}\big(\boldsymbol{\varphi}(\theta - \rho)\big)\boldsymbol{u}(\theta - \rho),$$

and $\boldsymbol{u}$ is expanded as a (truncated) Fourier series.

- We an discretize the previous equation using FFT.

Manifolds of tori



$$D\phi_{T_2}\Big(\varphi(\theta-\rho)\Big)\boldsymbol{u}(\theta-\rho) = \Lambda \boldsymbol{u}(\theta)$$

Manifolds of tori

Notations for the DFT

Given data $\{f_j\}_{j=0}^{N-1}$, we denote

$$
\begin{aligned}
F_{\{f_j\}_{j=0}^{N-1}}(k) &= \frac{1}{N}\sum_{j=0}^{N-1} f_j e^{-i2\pi\frac{k}{N}j}, && k = 0,\dots,N-1,\\
A_{\{f_j\}_{j=0}^{N-1}}(k) &= \frac{\delta_k}{N}\sum_{j=0}^{N-1} f_j \cos(2\pi\frac{k}{N}j), && k = 0 \div N/2,\\
B_{\{f_j\}_{j=0}^{N-1}}(k) &= \frac{2}{N}\sum_{j=0}^{N-1} f_j \sin(2\pi\frac{k}{N}j), && k = 1 \div N/2-1,
\end{aligned}
$$

($\delta_0 = \delta_{\frac{N}{2}} = 1$, $\delta_k = 2$ for $k = 0, \frac{N}{2}$). If $f_j = f(\theta_j)$ for $\theta_j = j2\pi/N$ and f is 2π–periodic,

$$
\begin{aligned}
f(\theta) \approx\ & A_{\{f_j\}_{j=0}^{N-1}}(0) + \sum_{k=0}^{N/2}\Big(A_{\{f_j\}_{j=0}^{N-1}}(k)\cos(k\theta) + B_{\{f_j\}_{j=0}^{N-1}}(k)\sin(k\theta)\Big)\\
& + A_{\{f_j\}_{j=0}^{N-1}}(N/2)\cos((N/2)\theta).
\end{aligned}
$$

Manifolds of tori

Discretization of the $\mathcal{C}$ operator

To discretize $\mathcal{C}\boldsymbol{u}$, we approximate the Fourier coeff. of $\mathcal{C}\boldsymbol{u}$ by the DFT

$$(\mathcal{C}\boldsymbol{u})(\theta) \approx \overline{\boldsymbol{A}}_0 + \sum_{k=1}^{N/2-1} \Big(\overline{\boldsymbol{A}}_k \cos(k\theta) + \overline{\boldsymbol{B}}_k \sin(k\theta)\Big) + \overline{\boldsymbol{A}}_{N/2} \cos((N/2)\theta)$$

If we take $\boldsymbol{u}$ of the form

$$\boldsymbol{u}(\theta) = \boldsymbol{A}_0 + \sum_{k=1}^{N/2-1} \Big(\boldsymbol{A}_k \cos(k\theta) + \boldsymbol{B}_k \sin(k\theta)\Big) + \boldsymbol{A}_{N/2} \cos((N/2)\theta),$$

and denote

$$\begin{aligned} \boldsymbol{X} &= (\boldsymbol{A}_0, \boldsymbol{A}_1, \boldsymbol{B}_1, \ldots, \boldsymbol{A}_{N/2-1}, \boldsymbol{B}_{N/2-1}, \boldsymbol{A}_{N/2}), \\ \overline{\boldsymbol{X}} &= (\overline{\boldsymbol{A}}_0, \overline{\boldsymbol{A}}_1, \boldsymbol{B}_1, \ldots, \overline{\boldsymbol{A}}_{N/2-1}, \overline{\boldsymbol{B}}_{N/2-1}, \overline{\boldsymbol{A}}_{N/2}), \end{aligned}$$

then, for a suitable (finite–dim.) matrix C,

$$\overline{\boldsymbol{X}} = C\boldsymbol{X}.$$

Manifolds of tori

Discretization of the $\mathcal{C}$ operator

The discrete version of $\mathcal{C}\boldsymbol{u} = \Lambda \boldsymbol{u}$ is

$$C\boldsymbol{X} = \Lambda \boldsymbol{X}$$

Recall:

$$(\mathcal{C}\boldsymbol{u})(\theta) = D\phi_{T_2}\big(\varphi(\theta-\rho)\big)\boldsymbol{u}(\theta-\rho).$$

Denote: $\boldsymbol{w}_k = (0,\dots,\overset{(k)}{1},\dots,0)$, $k = 1 \div 6$.
The coefficients of the C matrix can be computed from

$$\begin{aligned} & F_{\{D\phi_{T_2}(\varphi(\theta_l-\rho))\boldsymbol{w}_j e^{ik(\theta_l-\rho)}\}_{l=0}^{N-1}}(m) \\ &= \quad e^{-ik\rho} F_{\{D\phi_{T_2}(\varphi(\theta_l-\rho))\boldsymbol{w}_j\}_{l=0}^{N-1}}(m-k), \qquad \begin{array}{c} j=1\div 6 \\ k=0\div N/2 \\ m=0\div N/2 \end{array}, \end{aligned}$$

where the latter term can be computed in a *single* FFT.

Manifolds of tori

Structure of the spectrum

- The eigenvalues of the discretized equation appear grouped in circles.
- If the torus is reducible, there are as many circles as eigenvalues of the reduced matrix ("monodromy matrix"), and each circle contains one of them.
- Apart from unit circles, there will be 2 cicles containing Λ, Λ^{-1}, for some $\Lambda > 0$. We are interested in the latter.
  The corresponding eigenvectors, $\boldsymbol{u}_\Lambda(\theta)$, $\boldsymbol{u}_{\Lambda^{-1}}(\theta)$, give the vectors tangent to the manifolds we look for.
- There are some additional issues on the accuracy of the computed eigenvalues. See (Jorba, 2001) for details.

Manifolds of tori

Globalisation of an inv. curve to the whole torus

Once we have the (linear approximation of the) inv. manifolds corresponding to an inv. curve inside the torus, we globalize them to the whole torus as usual:

$$\boldsymbol{v}(\theta_1, \theta_2) = \Lambda^{-\theta_2/2\pi} D\phi_{\frac{\theta_2 T_2}{2\pi}}\left(\boldsymbol{\varphi}(\theta_1 - \frac{\theta_2}{2\pi}\rho)\right)\boldsymbol{u}(\theta_1 - \frac{\theta_2}{2\pi}\rho)$$

satisfies

$$D\phi_t(\boldsymbol{\varphi}(\theta_1, \theta_2))\boldsymbol{v}(\theta_1, \theta_2) = e^{\lambda t}\boldsymbol{v}(\theta_1 + t\omega_1, \theta_2 + t\omega_2),$$

being

$$\begin{aligned} \lambda &= \ln(\Lambda)/T_2, \\ \omega_1 &= \rho/T_2, \\ \omega_2 &= 2\pi/T_2. \end{aligned}$$

Globalisation of the manifold

- φ param. inv. curve ($\phi_{T_2}\big(\varphi(\xi)\big) = \varphi(\xi+\rho)$)
  $\Longrightarrow \psi(\theta_1,\theta_2) = \phi_{\frac{\theta_2}{2\pi}T_2}\Big(\varphi\big(\theta_1 - \frac{\theta_2}{2\pi}\rho\big)\Big)$ param whole 2D torus.
  Satisfies: $\phi_t\big(\psi(\theta_1,\theta_2)\big) = \psi(\theta_1 + t\omega_1, \theta_2 + t\omega_2)$,
  with $\omega_1 = \frac{\rho}{T_2}, \omega_2 = \frac{2\pi}{T_2}$.
- u param. vec. tg. mani. inv. curve ($D\phi_{T_2}\big(\varphi(\xi)\big)u(\xi) = \Lambda u(\xi+\rho)$)
  $\Longrightarrow v(\theta_1,\theta_2) = \Lambda^{-\frac{\theta_2}{2\pi}} D\phi_{\frac{\theta_2}{2\pi}T_2}\Big(\varphi\big(\theta_1 - \frac{\theta_2}{2\pi}\rho\big)\Big)u\big(\theta_1 - \frac{\theta_2}{2\pi}\rho\big)$.
  Satisfies: $D\phi_t\big(\psi(\theta_1,\theta_2)\big)v(\theta_1,\theta_2) = \Lambda^{-\frac{t\omega_2}{2\pi}} v(\theta_1 + t\omega_1, \theta_2 + t\omega_2)$.
- **Linear approximation to the manifold**:

$$\overline{\psi}(\theta_1,\theta_2,\xi) = \psi(\theta_1,\theta_2) + \xi v(\theta_1,\theta_2)$$

  Satisfies: $\phi_t\big(\overline{\psi}(\theta_1,\theta_2,\xi)\big) = \overline{\psi}(\theta_1 + t\omega_1, \theta_2 + t\omega_2, e^{t\lambda}\xi) + O(\xi^2)$,
  with $\lambda = \frac{\omega_2 \ln\Lambda}{2\pi}$.
  Can be evaluated **for small ξ only**.

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- $\boldsymbol{u}$ param. vec. tg. mani. inv. curve ($D\phi_{T_2}\big(\varphi(\xi)\big)\boldsymbol{u}(\xi) = \Lambda\boldsymbol{u}(\xi+\rho)$)
  $\Longrightarrow \boldsymbol{v}(\theta_1,\theta_2) = \Lambda^{-\frac{\theta_2}{2\pi}} D\phi_{\frac{\theta_2}{2\pi}T_2}\Big(\varphi\big(\theta_1 - \frac{\theta_2}{2\pi}\rho\big)\Big)\boldsymbol{u}\big(\theta_1 - \frac{\theta_2}{2\pi}\rho\big)$.
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Globalisation of the manifold

- **Linear approximation to the manifold**:

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- Globalization of the manifold:
  for each ξ, take m such that $\Lambda^{-m}\xi$ small and compute

$$\Psi(\theta_1, \theta_2, \xi) = \phi_{mT_2}\Big(\overline{\psi}(\theta_1 - m\rho, \theta_2, \Lambda^{-m}\xi)\Big)$$

  ($m > 0$ for unst. manifold, $m < 0$ for stb. manifold)
  Ψ satisfies

$$\phi\big(\Psi(\theta_1, \theta_2, \xi)\big) = \Psi(\theta_1 + t\omega_1, \theta_2 + t\omega_2, e^{t\lambda}\xi) + O\big((\Lambda^{-m}\xi)^2\big)$$

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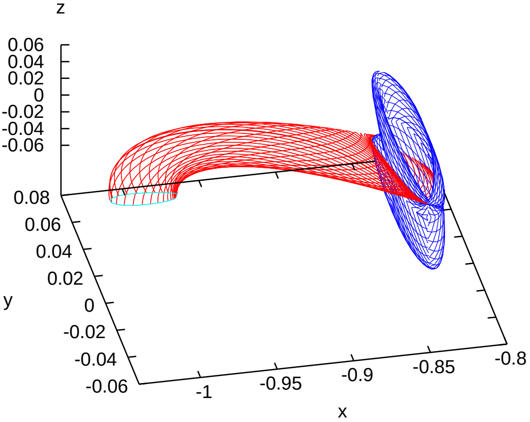
An example

3D unstable manifold associated to a Lissajous orbit

$$\xi \in [0, 2\pi], \qquad \eta = 0$$



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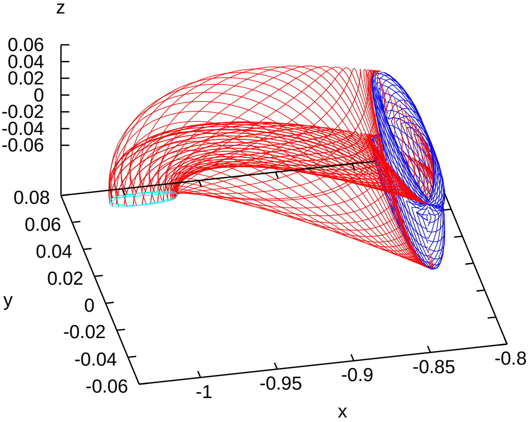

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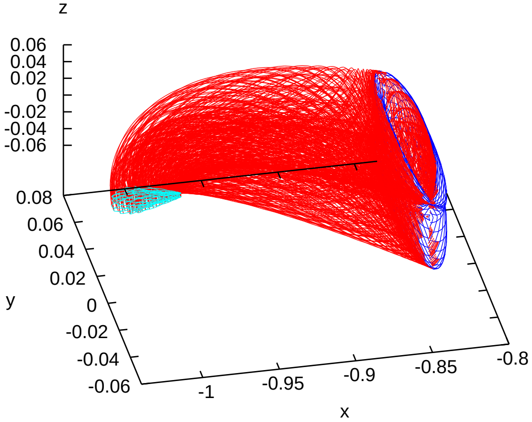

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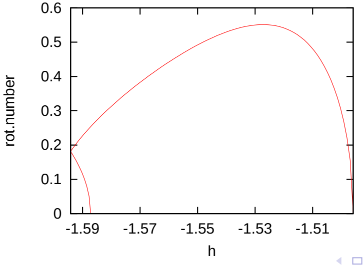

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  $$25433.$$

- Total processor time for tori:

  $$836.88 \text{ hours } (34.8 \text{ days}).$$

  (When using a cluster, divide by the number of processes).
- Total processor time for manifolds (unstable+stable):

  $$79.19 \text{ hours}.$$

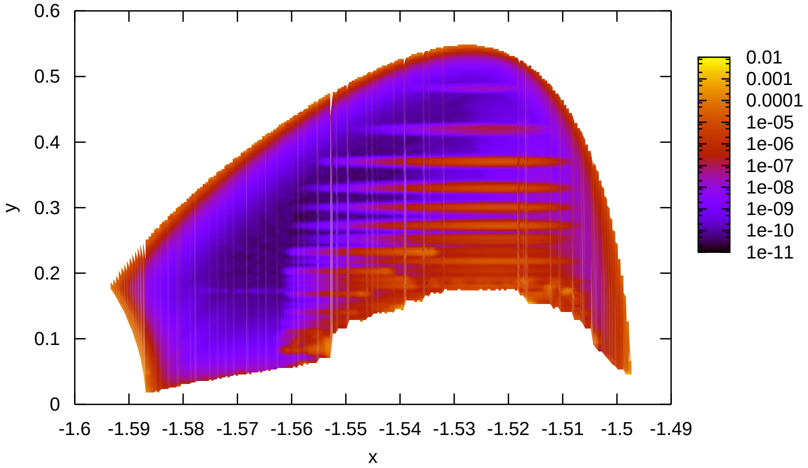
- Sizes of binary files storing all the Fourier coefficients of the grid:

  $$180 \text{ MB}.$$

  - File for tori: 180 MB.
  - Files for unst. and stb. manifolds: 128 MB each.

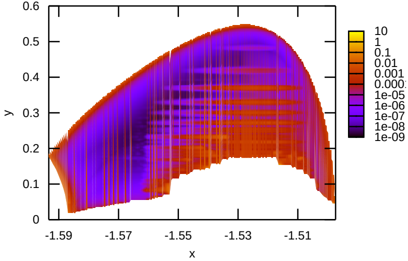
Global parametrization of families using interpolation

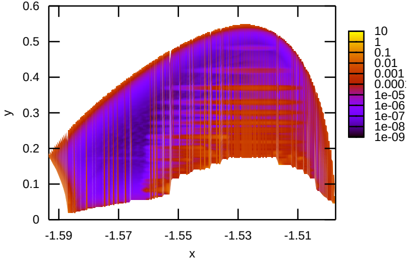
Interpolation error for tori



Global parametrization of families using interpolation

Interpolation error for unstable and stable manifolds





Outline

Finding connections

General setting

Some references: [7, 6, 4, 3]

Finding connections

General setting

- Let
  - $\overline{\psi}^u(\theta, \xi)$ param. of lin. app. of unst. mani. of departure object. ($\theta \in \mathbb{T}^1$ for p.o., $\theta \in \mathbb{T}^2$ for tori).
  - $\overline{\psi}^s(\theta, \xi)$ param. of lin. app. of stb. mani. of arrival object.
- Let $\Sigma = \{g(x) = 0\}$ a hypersurface of section that the manifolds are known to intersect.
- Consider two associated Poincaré maps:
  - P_Σ^+: integrating forward,
  - P_Σ^-: integrating backwards.
- Choose ξ_{small} small enough so that the linear approximation is valid.
- Look for a zero of

$$F(\theta^u, \theta^s) = P_\Sigma^+(\overline{\psi}^u(\theta^u, \xi_{\text{small}})) - P_\Sigma^-(\overline{\psi}^u(\theta^s, \xi_{\text{small}}))$$

where
  - $\theta^u, \theta^s \in \mathbb{T}^1$ for p.o.,
  - $\theta^u, \theta^s \in \mathbb{T}^2$ for tori.

Finding connections

General setting

- Let
  - $\overline{\psi}^u(\theta, \xi)$ param. of lin. app. of unst. mani. of departure object. ($\theta \in \mathbb{T}^1$ for p.o., $\theta \in \mathbb{T}^2$ for tori).
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  - $\theta^u, \theta^s \in \mathbb{T}^2$ for tori.

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  - $\theta^u, \theta^s \in \mathbb{T}^1$ for p.o.,
  - $\theta^u, \theta^s \in \mathbb{T}^2$ for tori.

Finding connections

General setting

- Let
  - $\overline{\psi}^u(\theta,\xi)$ param. of lin. app. of unst. mani. of departure object. ($\theta \in \mathbb{T}^1$ for p.o., $\theta \in \mathbb{T}^2$ for tori).
  - $\overline{\psi}^s(\theta,\xi)$ param. of lin. app. of stb. mani. of arrival object.
- Let $\Sigma = \{g(\boldsymbol{x}) = 0\}$ a hypersurface of section that the manifolds are known to intersect.
- Consider two associated Poincaré maps:
  - P_Σ^+: integrating forward,
  - P_Σ^-: integrating backwards.
- Choose ξ_{small} small enough so that the linear approximation is valid.
- Look for a zero of

$$F(\theta^u,\theta^s) = P_\Sigma^+(\overline{\psi}^u(\theta^u,\xi_{\text{small}})) - P_\Sigma^-(\overline{\psi}^u(\theta^s,\xi_{\text{small}}))$$

where
  - $\theta^u,\theta^s \in \mathbb{T}^1$ for p.o.,
  - $\theta^u,\theta^s \in \mathbb{T}^2$ for tori.

Finding connections

General setting

- Let
  - $\overline{\psi}^u(\theta, \xi)$ param. of lin. app. of unst. mani. of departure object. ($\theta \in \mathbb{T}^1$ for p.o., $\theta \in \mathbb{T}^2$ for tori).
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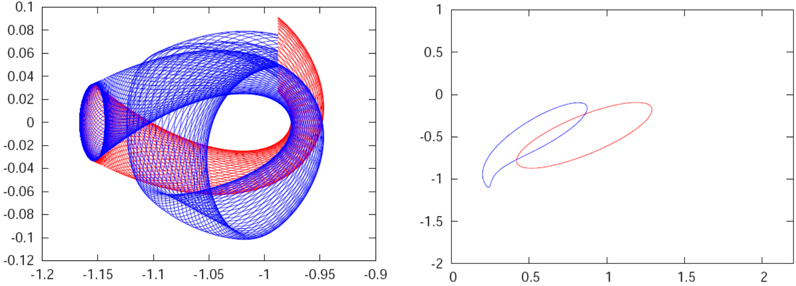
  where
  - $\theta^u, \theta^s \in \mathbb{T}^1$ for p.o.,
  - $\theta^u, \theta^s \in \mathbb{T}^2$ for tori.

Finding connections

The case of p.o.

In the case of p.o., the unstable and stable manifolds are 2D tubes, so $\{P_\Sigma^+(\overline{\psi}^u(\theta,\xi_{\text{small}}))\}_{\theta\in\mathbb{T}^1}$ and $\{P_\Sigma^-(\overline{\psi}^s(\theta,\xi_{\text{small}}))\}_{\theta\in\mathbb{T}^1}$ are S^1–like closed curves.

The number of cuts may be an issue.



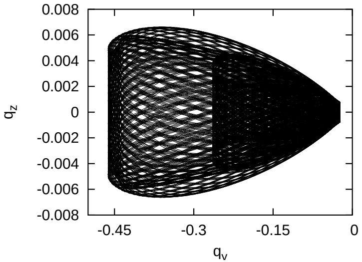
x, y plane $\quad$ p_x, p_y plane

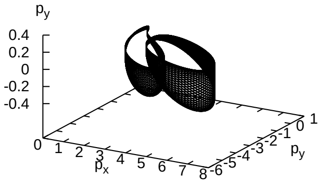
Invariant tori

For 2D tori, $\{P^+_\Sigma(\overline{\psi}^u(\theta,\xi_{\rm small}))\}_{\theta\in\mathbb{T}^2}$ and $\{P^-_\Sigma(\overline{\psi}^s(\theta,\xi_{\rm small}))\}_{\theta\in\mathbb{T}^2}$ are again 2D tori, so their intersections are not easy to visualize.
The representation of "clouds of points"

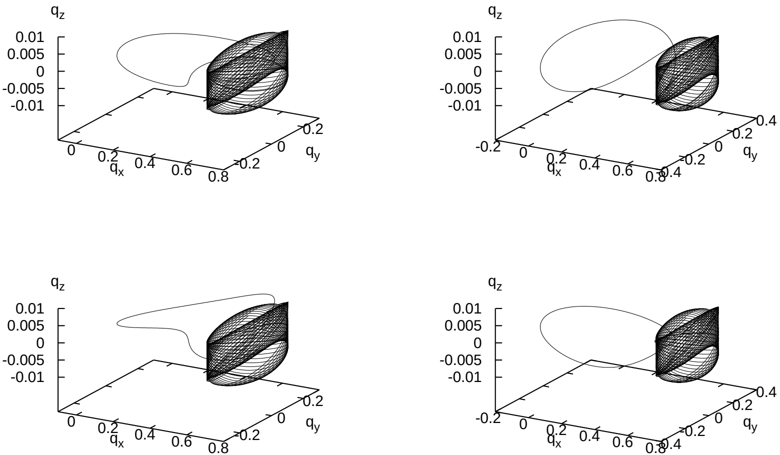
$$P^+_\Sigma(\overline{\psi}^u(\theta_1,\theta_2,\xi_{\rm small}))), \qquad P^-_\Sigma(\overline{\psi}^s(\theta_1,\theta_2,\xi_{\rm small})),$$

for a grid of values of (θ_1,θ_2), may help to locate possible connections, which then can be refined by looking for a zero of $F(\theta^u,\theta^s)$.





Invariant tori



Outline

Homoclinics of periodic orbits

Consider:

- $h \in \mathbb{R}$ an energy level,
- $g_1 : \mathbb{R}^n \longrightarrow \mathbb{R}$ a function defining a section for a periodic orbit,
- $\boldsymbol{x} \in \mathbb{R}^n$ an i.c. of a p.o. with period T.
- $\Lambda^u \in \operatorname{Spec} D\phi_T(\boldsymbol{x})$, $\Lambda^u > 1$, $\boldsymbol{v}^u \in V_{\Lambda^u}\big(D\phi_T(\boldsymbol{x})\big)$,
- $\Lambda^s \in \operatorname{Spec} D\phi_T(\boldsymbol{x})$, $0 < \Lambda^s < 1$, $\boldsymbol{v}^s \in V_{\Lambda^s}\big(D\phi_T(\boldsymbol{x})\big)$,
- $g_2 : \mathbb{R}^n \longrightarrow \mathbb{R}$ a function defining a section to match the manifolds,
- $\theta^u, \theta^s \in \mathbb{T}$ starting phases on the linear appr. of the unstable and stable manifolds, respectively,
- $T^u, T^s \in \mathbb{R}$ time to intersect the g_2 section from the unstable and stable manifolds, respectively

Homoclinics of periodic orbits

To find or to continuate[11] an homoclinic connection of a p.o., we can solve

$$
\left\{
\begin{array}{l}
\begin{array}{rcl}
H(\boldsymbol{x}) - h & = & 0 \\
g_1(\boldsymbol{x}) & = & 0 \\
\phi_T(\boldsymbol{x}) - \boldsymbol{x} & = & 0
\end{array}
\\[2ex]
\begin{array}{rcl|rcl}
\|\boldsymbol{v}^u\|^2 - 1 & = & 0 & \|\boldsymbol{v}^s\|^2 - 1 & = & 0 \\
D\phi_T(\boldsymbol{x})\boldsymbol{v}^u - \Lambda^u \boldsymbol{v}^u & = & 0 & D\phi_T(\boldsymbol{x})\boldsymbol{v}^s - \Lambda^s \boldsymbol{v}^s & = & 0
\end{array}
\\[2ex]
\begin{array}{rcl}
g_2\left(\phi_{T^u}\left(\overline{\psi}^u(\theta^u, \xi_{\text{small}})\right)\right) & = & 0 \\
g_2\left(\phi_{-T^s}\left(\overline{\psi}^s(\theta^s, \xi_{\text{small}})\right)\right) & = & 0
\end{array}
\\[2ex]
\phi_{T^u}\left(\overline{\psi}^u(\theta^u, \xi_{\text{small}})\right) - \phi_{-T^s}\left(\overline{\psi}^s(\theta^s, \xi_{\text{small}})\right) = 0
\end{array}
\right.
$$

If ambient space is 6–dimensional, we have

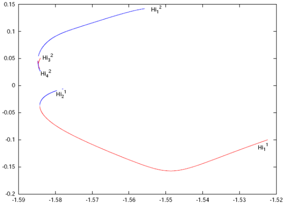
- 30 equations,
- 26 unknowns: $h, \boldsymbol{x}, T, \Lambda^u, \boldsymbol{v}^u, \Lambda^s, \boldsymbol{v}^s, \theta^u, T^u, \theta^s, T^s$.

Homoclinics of periodic orbits

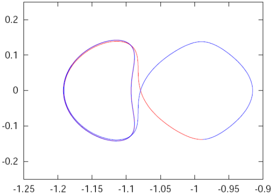
$$
\left\{
\begin{array}{c}
\begin{array}{rcl}
H(\boldsymbol{x}) - h & = & 0 \\
g_1(\boldsymbol{x}) & = & 0 \\
\phi_T(\boldsymbol{x}) - \boldsymbol{x} & = & 0
\end{array}
\\[2em]
\begin{array}{rcl|rcl}
\|\boldsymbol{v}^u\|^2 - 1 & = & 0 & \|\boldsymbol{v}^s\|^2 - 1 & = & 0 \\
D\phi_T(\boldsymbol{x})\boldsymbol{v}^u - \Lambda^u \boldsymbol{v}^u & = & 0 & D\phi_T(\boldsymbol{x})\boldsymbol{v}^s - \Lambda^s \boldsymbol{v}^s & = & 0
\end{array}
\\[2em]
\begin{array}{rcl}
g_2\left(\phi_{T^u}\left(\overline{\psi}^u(\theta^u, \xi_{\text{small}})\right)\right) & = & 0 \\
g_2\left(\phi_{-T^s}\left(\overline{\psi}^s(\theta^s, \xi_{\text{small}})\right)\right) & = & 0
\end{array}
\\[2em]
\phi_{T^u}\left(\overline{\psi}^u(\theta^u, \xi_{\text{small}})\right) - \phi_{-T^s}\left(\overline{\psi}^s(\theta^s, \xi_{\text{small}})\right) = 0
\end{array}
\right.
$$

- To find an homoclinic connection: fix $h, \boldsymbol{x}, T, \Lambda^u, \boldsymbol{v}^u, \Lambda^s, \boldsymbol{v}^s$,
- To continuate it: let everything free.

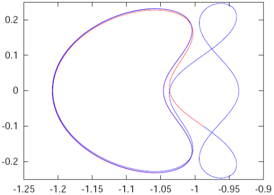
Examples



h, y plane



x, y plane, $h = -1.560$



x, y plane, $h = -1.522$

[1] E. L. Allgower and K. Georg.
Numerical continuation methods, volume 13 of *Springer Series in Computational Mathematics*.
Springer-Verlag, Berlin, 1990.
An introduction.

[2] E. Anderson, Z. Bai, C. Bischof, J. Demmel, J. Dongarra, J. D. Croz, A. Greenbaum, S. Hammarling, A. McKenney, S. Ostrouchov, and D. Sorensen.
LAPACK Users' Guide.
SIAM, second edition, 1995.

[3] E. Canalias.
Contribution to Libration Orbit Mission Design using Hyperbolic Invariant Manifolds.
PhD thesis, Universitat Politécnica de Catalunya, 2007.

[4] E. Canalias and J. J. Masdemont.
Homoclinic and heteroclinic transfer trajectories between planar Lyapunov orbits in the sun-earth and earth-moon systems.
Discrete Contin. Dyn. Syst., 14(2):261–279, 2006.

[5] G. H. Golub and C. F. van Loan.
Matrix Computations.
The Johns Hopkins University Press, Baltimore and London, 3rd edition, 1996.

[6] G. Gómez, M. Marcote, and J. M. Mondelo.
The invariant manifold structure of the spatial Hill's problem.
Dynamical Systems. An International Journal, 20(1):115–147, 2005.

[7] G. Gómez and J. J. Masdemont.
Some zero cost transfers between halo orbits.
Advances in the Astronautical Sciences, 105:1199–1216, 2000.

[8] G. Gómez and J. M. Mondelo.
The dynamics around the collinear equilibrium points of the RTBP.
Phys. D, 157(4):283–321, 2001.

[9] À. Jorba.
Numerical computation of the normal behaviour of invariant curves of n-dimensional maps.
Nonlinearity, 14(5):943–976, 2001.

[10] J. Mondelo, E. Barrabés, G. Gómez, and M. Ollé.
Numerical parametrisations of libration point trajectories and their invariant manifolds.
2007 AAS/AIAA Astrodynamics Specialist Conference, Mackinac Island, Michigan, August 19-23.

[11] J. Mondelo, E. Barrabés, and M. Ollé.
Numerical continuation of homoclinic connections of periodic orbits.
In preparation.

[12] E. Olmedo.
On the parallel computation of invariant tori.
PhD thesis, Universitat de Barcelona, 2007.

[13] C. Simó.
On the analytical and numerical approximation of invariant manifolds.
In D. Benest and C. Froeshlé, editors, *Modern methods in Celestial Mechanics*, pages 285–330. Editions Frontières, 1990.