A renormalization operator for 1D maps under quasi-periodic perturbations

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Abstract. This paper concerns with the reducibility loss of (periodic) invariant curves of quasi-periodically forced one dimensional maps and its relationship with the renormalization operator. Let g_α be a one-parametric family of one dimensional maps with a cascade of period doubling bifurcations. Between each of these bifurcations, there exists a parameter value α_n such that g_{α_n} has a superstable periodic orbit of period 2^n. Consider a quasi-periodic perturbation (with only one frequency) of the one dimensional family of maps, and let us call ε the perturbing parameter. For ε small enough, the superstable periodic orbits of the unperturbed map become attracting invariant curves (depending on α and ε) of the perturbed system. Under suitable hypothesis, it is known that there exist two reducibility loss bifurcation curves around each parameter value $(\alpha_n, 0)$, which can be locally expressed as $(\alpha_n^+(\varepsilon), \varepsilon)$ and $(\alpha_n^-(\varepsilon), \varepsilon)$. We propose an extension of the classic one-dimensional (doubling) renormalization operator to the quasi-periodic case. We show that this extension is well defined and the operator is differentiable. Moreover, we show that the slopes of reducibility loss bifurcation $\frac{d}{d\varepsilon}\alpha_n^\pm(0)$ can be written in terms of the tangent map of the new quasi-periodic renormalization operator. In particular, our result applies to the families of quasi-periodic forced perturbations of the Logistic Map typically encountered in the literature. We also present a numerical study that demonstrates that the asymptotic behaviour of $\{\frac{d}{d\varepsilon}\alpha_n^\pm(0)\}_{n\geq 0}$ is governed by the dynamics of the proposed quasi-periodic renormalization operator.

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1. Introduction

This paper can be regarded as a continuation of the results presented in [6], both motivated by the study of period doubling bifurcation cascades of invariant curves in a family of autonomous one-dimensional maps under a quasi-periodic perturbation. The previous paper was focused on the existence of reducibility loss bifurcations created by the quasi-periodic perturbation, whereas the present one is concerned about the relationship of these bifurcations with the renormalization operator.

1.1. Preliminary definitions and concepts

Let us start with some basic definitions and concepts on quasi-periodically forced one dimensional maps. A *quasi-periodically forced one dimensional map* is a map of the form

$$\begin{array}{rccc} F: & \mathbb{T}\times\mathbb{R} & \rightarrow & \mathbb{T}\times\mathbb{R} \\ & \begin{pmatrix} \theta \\ x \end{pmatrix} & \mapsto & \begin{pmatrix} \theta+\omega \\ f(\theta,x) \end{pmatrix} \end{array} \tag{1}$$

where $\mathbb{T}=\mathbb{R}/\mathbb{Z}$, $f\in C^r(\mathbb{T}\times\mathbb{R},\mathbb{R})$ with $r\geq 1$ and $\omega\in\mathbb{T}\setminus\mathbb{Q}$. A quasi-periodically forced map determines a dynamical system on the cylinder, explicitly defined as

$$\left.\begin{array}{l} \bar{\theta}=\theta+\omega, \\ \bar{x}=f(\theta,x). \end{array}\right\} \tag{2}$$

A continuous function $u:\mathbb{T}\rightarrow\mathbb{R}$ is an *invariant curve* of (2) if and only if $u(\theta+\omega)=f(\theta,u(\theta))$, for all $\theta\in\mathbb{T}$. The value ω is known as the *rotation number* of the curve. The rotation number is said to be *Diophantine* if there exist $\gamma>0$ and $\tau\geq 1$ such that

$$|q\omega-p|\geq\frac{\gamma}{|q|^{\tau}},\qquad\text{for all }(p,q)\in\mathbb{Z}\times\mathbb{Z}\setminus\{0\}.$$

An equivalent way to define an invariant curve is to require the set $\{(\theta,x)\in\mathbb{T}\times\mathbb{R}\mid x=u(\theta)\}$ to be invariant by F, where F is the function defined by (1). On the other hand, note that F^n is also a quasi-periodically forced map. Given a function $u:\mathbb{T}\rightarrow\mathbb{R}$, we say that u is a *n-periodic invariant curve* of F if u is invariant by F^n (and there is no smaller n satisfying such condition).

Given $x=u_0(\theta)$ an invariant curve of (2) of class C^r $(r\geq 1)$, its linearized normal behaviour is described by the following linear skew product:

$$\left.\begin{array}{rcl} \bar{\theta} & = & \theta+\omega, \\ \bar{x} & = & a(\theta)x, \end{array}\right\} \tag{3}$$

where $a(\theta)=D_x f(\theta,u_0(\theta))$ is of class C^{r-1}, $x\in\mathbb{R}$ and $\theta\in\mathbb{T}$.

A linear skew product like (3) is called *reducible* if, and only if, there exists a change of variable $x=c(\theta)y$, continuous with respect to θ, such that (3) becomes

$$\left.\begin{array}{rcl} \bar{\theta} & = & \theta+\omega, \\ \bar{y} & = & by, \end{array}\right\}$$

where b does not depend on θ. An invariant curve is called *reducible* if its linearized normal behaviour (3) is reducible. A n-periodic invariant curve is reducible if it is reducible for F^n.

In the case that $a(\cdot)$ is a C^∞ function and ω is Diophantine, the skew product (3) is reducible if, and only if, $a(\cdot)$ has no zeros [7]. Due to this property, the reducibility loss can be characterized as a codimension one bifurcation as follows:

Consider a one-parametric family of linear skew-products

$$\left.\begin{array}{rcl} \bar{\theta} & = & \theta + \omega, \\ \bar{x} & = & a(\theta, \mu) x, \end{array}\right\} \tag{4}$$

where ω is Diophantine, μ belongs to an open set of $\mathbb{R}$ and a is a C^∞ function of θ and μ. We say that the system (4) undergoes a *reducibility loss bifurcation* at μ_0 if

(i) $a(\cdot, \mu)$ has no zeros for $\mu < \mu_0$,

(ii) $a(\cdot, \mu)$ has a double zero at θ_0 for $\mu = \mu_0$,

(iii) $\frac{d}{d\mu} a(\theta_0, \mu_0) \neq 0$.

On the other hand, consider a system like (2) with f a C^∞ function, which depends (smoothly) on a one dimensional parameter μ (we denote this dependence as $f = f_\mu$), having an invariant curve $u = u_\mu$. We will say that *the invariant curve undergoes a reducibility loss bifurcation* if the family of linear skew-products (4), where $a(\theta, \mu) = D_x f_\mu(\theta, u_\mu(\theta))$, undergoes a reducibility loss bifurcation. Note that the reducibility loss takes place when the number of zeros of $\theta \mapsto a(\theta, \mu)$ goes from 0 to 2 as μ crosses μ_0. The number of zeros of a is invariant under linear changes of variables (for more details see Section 3 in [7]). For a given value of μ, if the number of zeros (counting their multiplicity) of $\theta \mapsto a(\theta, \mu)$ is finite, this number will be called the *degree of non-reducibility* of the skew-product. In what follows, we will refer to degree of non-reducibility simply as degree. Note that, when a is C^∞ and ω is Diophantine, degree zero is equivalent to reducibility.

We are interested on maps of the form

$$\begin{array}{rccc} F_{\alpha,\varepsilon}: & \mathbb{T} \times \mathbb{R} & \rightarrow & \mathbb{T} \times \mathbb{R} \\ & \begin{pmatrix} \theta \\ x \end{pmatrix} & \mapsto & \begin{pmatrix} \theta + \omega \\ f(\theta, x, \alpha, \varepsilon) \end{pmatrix}, \end{array} \tag{5}$$

where ω is Diophantine, α and ε are real parameters and f is of the form

$$f(\theta, x, \alpha, \varepsilon) = g(x, \alpha) + \varepsilon h(\theta, x, \alpha, \varepsilon), \tag{6}$$

where g and h are C^∞ functions. The function g is assumed to have a cascade of period doubling bifurcations. This means that the system $\bar{x} = g(x, \alpha)$ also has a sequence of superstable periodic orbits. We recall that a superstable periodic orbit is a periodic orbit with a critical point, i.e., a point with zero derivative.

If x_0 is an attracting fixed point of $x \mapsto g(x, \alpha_0)$ then for $|\alpha - \alpha_0|$ and $|\varepsilon|$ small enough, there exists a unique invariant curve $x = x(\theta, \alpha, \varepsilon)$ of (5) such that it is smooth with respect to all its arguments and $x(\theta, \alpha_0, 0) = x_0$ for all $\theta \in \mathbb{T}$ (see Section 2 in [7]). Moreover, the function

$$a(\theta, \alpha, \varepsilon) = \frac{\partial f}{\partial x}(\theta, x(\theta, \alpha, \varepsilon), \alpha, \varepsilon),$$

describing the linearized normal behaviour around the curve satisfies, for $\varepsilon = 0$, $a(\theta, \alpha_0, 0) = \frac{\partial g}{\partial x}(x_0, \alpha_0)$. In general, if $\left|\frac{\partial g}{\partial x}(x_0, \alpha_0)\right| \neq 1$ (i.e., the point is hyperbolic), the implicit function theorem can be applied to prove the existence of an invariant curve close to x_0 for α close to α_0 and ε close to 0. Moreover, if $\dfrac{\partial g}{\partial x}(x_0, \alpha_0) \neq 0$, there exists a neighbourhood of $(\alpha, \varepsilon) = (\alpha_0, 0)$ small enough for which $a(\theta, \alpha, \varepsilon) \neq 0$ for any $\theta \in \mathbb{T}$. Hence the invariant curve is reducible.

In the case where $\dfrac{\partial g}{\partial x}(x_0, \alpha_0) = 0$, x_0 is a superattracting fixed point of g, it is shown in [6] that the degree of the invariant curve depends on the critical points of the function

$$H(\theta) = h(\theta - \omega, x_0, \alpha_0, 0) \frac{\partial^2 g}{\partial x^2}(x_0, \alpha_0) + \frac{\partial h}{\partial x}(\theta, x_0, \alpha_0, 0).$$

Concretely, for each simple zero θ_0 of $H'(\theta)$ and for $|\varepsilon|$ small enough, there exists a function $\alpha = \alpha(\varepsilon)$ such that $\alpha(0) = \alpha_0$ and the curve $(\alpha(\varepsilon), \varepsilon)$ is a curve of change of degree for the unique invariant curve $x = z(\theta, \alpha, \varepsilon)$ obtained as the continuation of the invariant curve $z(\theta, \alpha_0, 0) \equiv x_0$.

1.2. Summary of results

Consider a two parametric family of maps like (5) such that f can be written as (6). We are interested in the case when the one-parameter family $g(\cdot, \alpha)$ has a cascade of period doubling bifurcations. Between each of these bifurcations, a superstable periodic orbit is known to exist. An example of such a family is the well-known Logistic Map $g(x, \alpha) = \alpha x(1 - x)$. Let α_n denote the parameter value for which g has a superattracting 2^n-periodic orbit. If functions g and h satisfy suitable conditions, the results summarized above (see also [6]) guarantee the existence of two bifurcation curves (in the parameter plane) of change of degree. These can be locally written as $(\alpha_n^+(\varepsilon), \varepsilon)$ and $(\alpha_n^-(\varepsilon), \varepsilon)$, with $\alpha_n^+(0) = \alpha_n^-(0) = \alpha_n$. When the family of maps f is C^∞ these bifurcation curves correspond to reducibility loss bifurcation. Concretely the results apply to typical quasi-periodic perturbations of the Logistic Map encountered in the literature [1, 3, 5, 10, 11]. Nonetheless, the results summarized above do not provide any information about the asymptotic behaviour of the bifurcation curves when n goes to infinity. The slopes of this bifurcation curves $\frac{d}{d\varepsilon}\alpha_n^\pm(0)$ have been shown to have very interesting asymptotic behaviour [12], demonstrating the existence of some universality and self-renormalization properties. The phenomena of universality and self-renormalization are a bit different with respect to the one-dimensional case. Concretely, the rotation number is shown to have a crucial role.

The sequence of parameter values $\{\alpha_n\}_{n \geq 0}$ for which the family $g(\cdot, \alpha)$ has a superstable periodic orbit are known to accumulate following a universal geometric ratio known as the Feigenbaum constant. This phenomenon is well understood and explained by the one-dimensional renormalization operator [2]. The main goal of this paper is to explore the effect of a quasi-periodic perturbation on the one-dimensional (doubling) renormalization operator.

With this aim, we introduce in Section 2 a slightly modified renormalization operator for one-dimensional maps which can be extended to the quasi-periodically forced case. We show that this extension of the operator is well defined and differentiable.

In Section 3 we show that, for a suitable class of two-parameter quasi-periodically forced maps, the slopes of reducibility loss bifurcation $\frac{d}{d\varepsilon}\alpha_n^{\pm}(0)$ can be written in terms of the tangent map of the (quasi-periodic) renormalization operator.

In Section 4 we perform a numerical study of the asymptotic behaviour of the slopes $\frac{d}{d\varepsilon}\alpha_n^{\pm}(0)$ based on the results developed in the previous sections. Concretely we show that the asymptotic behaviour is governed by the dynamics of the newly introduced renormalization operator. Section 5 contains the proofs of our results, which have been moved there for the sake of clarity.

2. Renormalization of 1D maps under quasi-periodic forcing

An essential idea behind the renormalization operator is its definition as the self composition of a map followed by an affine transformation which defines a change of scale. This way, the periodic orbits with high period are turned into periodic orbits of lower period by the renormalization operation, leading to the well-known self-similarity relationship between the cascades of period doubling bifurcations as a consequence of the existence of a fixed point of the renormalization operator. The extension of the renormalization operator to the quasi-periodically forced case proposed here is based on this same idea: to define the operator as the operation obtained from the self composition plus an affine transformation.

In Section 2.1 we introduce a setup for the 1D renormalization that is suitable for introducing a quasi-periodic forcing of the one-dimensional map later on. In Section 2.2 we define the renormalization operator for a quasi-periodically forced map. In Section 2.3 we show that the operator is differentiable and introduce the tangent maps associated to it. To simplify the reading, proofs have been moved to Section 5.

2.1. A setup for the 1D renormalization operator

Given a small value $\delta \geq 0$, let $\mathcal{M}_\delta$ denote the space of C^r $(r \geq 1)$ even maps ψ of the interval $I_\delta = [-1-\delta, 1+\delta]$ into itself such that

M1. $\psi(0) = 1$,

M2. $x\psi'(x) < 0$ for $x \neq 0$.

Set $a = \psi(1)$, $a' = (1+\delta)a$ and $b' = \psi(a')$. We define $\mathcal{D}(\mathcal{R}_\delta)$ as the set of $\psi \in \mathcal{M}_\delta$ such that

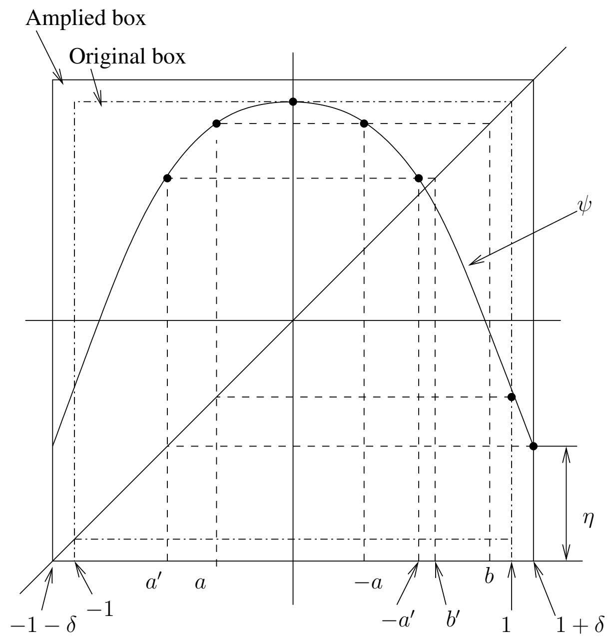
D1. $a < 0$,

D2. $1 > b' > -a'$,

D3. $\psi(b') < -a'$.

Condition M1 and M2 require the map to be unimodal with 0 as the critical point. Condition D1, D2 and D3 ensure that the intervals $[-a', a']$ and $[b', 1]$ do not overlap and each one is mapped into the other. In Figure 1 we include a schematic plot of a map in $\mathcal{M}_\delta$ where the geometric meaning of the values a, a' and b' is shown.

Figure 1. Schematic plot of a map in $\mathcal{M}_\delta$. The geometric meaning of the values a, a', b', δ and η are shown.



Definition 2.1. We define the (1D doubling) *renormalization operator* $\mathcal{R}_\delta : \mathcal{D}(\mathcal{R}_\delta) \to \mathcal{M}_\delta$ as

$$\mathcal{R}_\delta(\psi)(x) = \frac{1}{a}\psi \circ \psi(ax).$$

with $a = \psi(1)$. Let $\mathcal{D}^n(\mathcal{R}_\delta)$ denote the set of functions which are n times renormalizable:

$$\mathcal{D}^n(\mathcal{R}_\delta) = \left\{ f_0 \in \mathcal{M}_\delta \,\middle|\, f_i = \mathcal{R}^i_\delta(f_0) \in \mathcal{D}(\mathcal{R}_\delta), \text{ for } i = 0, \ldots, n-1, \right\}$$

Proposition 2.2. *The operator $\mathcal{R}_\delta$ is well defined. Moreover, for any fixed point $\Phi \in \mathcal{M}_0$ of $\mathcal{R}_0$, there exists δ_0 such that Φ extends to a fixed point of $\mathcal{R}_\delta$ for any $\delta \in (0, \delta_0)$.*

The setup for the renormalization operator given above is a small modification of the one introduced in [8], which is recovered for $\delta = 0$. The modification done here is to ensure that one dimensional maps can be quasi-periodically perturbed further on and remain well defined. Note that the definition given above has been done in the C^r topology. Let us introduce now the topology of real analytic function, in which a fixed point of $\mathcal{R}_0$ is known to exist.

Let W be an open set in $\mathbb{C}$ containing the real interval I_δ. Consider $\mathcal{A}(W)$ the Banach space of even functions that are bounded and real analytic in W, equipped with the supremum norm. Consider the subsets defined by

$$\begin{aligned} \mathcal{A}_0(W) &= \{\phi \in \mathcal{A}(W) \,|\, \phi(0) = 0,\ \phi'(0) = 0\}, \\ \mathcal{A}_1(W) &= \{\phi \in \mathcal{A}(W) \,|\, \phi(0) = 1,\ \phi'(0) = 0\}. \end{aligned} \tag{7}$$

Note that given a function $\phi \in \mathcal{A}(W)$, it is necessary to have $aW = \bigcup_{z\in W}\{az\} \subset W$ and $\phi(aW) \subset W$ in order to have $\mathcal{R}_0(\phi)$ well defined on its domain.

Theorem 2.3 (Lanford [8, 9]). *For $W = \{z \in \mathbb{C}$ such that $|z^2 - 1| < \frac{5}{2}\}$ there exist a function $\Phi \in \mathcal{A}_1(W)$ such that:*

(i) Φ *is a fixed point of* $\mathcal{R}_0$.

(ii) There exists an open neighbourhood V of Φ in $\mathcal{A}_1(W)$ where $\mathcal{R}_0$ is differentiable.

(iii) $D\mathcal{R}_0(\Phi)$ is hyperbolic on $\mathcal{A}_0(W)$ with a one dimensional expanding subspace, whose eigenvalue is positive.

(iv) $\mathrm{Cl}(aW) \subset W$ *and* $\Phi\left(\mathrm{Cl}(aW)\right) \subset W$ *where* $\mathrm{Cl}(\cdot)$ *denotes the closure of a set.*

2.2. Extension to the quasi-periodically forced case

Consider a quasi-periodically forced map, with its domain restricted to the compact cylinder $\mathbb{T} \times I_\delta$:

$$\begin{array}{rccc} F: & \mathbb{T} \times I_\delta & \rightarrow & \mathbb{T} \times I_\delta \\ & \begin{pmatrix} \theta \\ x \end{pmatrix} & \mapsto & \begin{pmatrix} \theta + \omega \\ f(\theta, x) \end{pmatrix}, \end{array} \tag{8}$$

with ω an irrational number, f a C^r function and $I_\delta = [-(1+\delta), 1+\delta]$.

Any map F as above can be identified with a couple (ω, f) in $\mathbb{T} \times C^r(\mathbb{T} \times I_\delta, I_\delta)$. Note that ω is not assumed to be Diophantine yet, this will be necessary when we are concerned with invariant curves of F, but not for the definition of the renormalization operator.

Given $F = (\omega, f)$ a quasi-periodically forced map, we want to define its renormalization $R(F)$ as an affine transformation applied to $F^2 = F \circ F$ (the self-composition of F). The map F^2 is of the form $(2\omega, f^2)$, with $f^2(\theta, x) = f(\theta+\omega, f(\theta, x))$. The extension of the renormalization operator that we propose is of the form $R(F) = (2\omega, \mathcal{T}_\omega(f))$, with $\mathcal{T}_\omega(f)$ an affine transformation of f^2. In other words, the affine transformation applied to F^2 is trivial in the ω-component. With this definition we obtain an operator that preserves the skew structure of these maps.

We give now the definition of $\mathcal{T}_\omega(f)$, the operator on its non-trivial component. Let us identify $C^r(I_\delta, I_\delta)$ with its natural inclusion in $C^r(\mathbb{T} \times I_\delta, I_\delta)$ defined as $[i(f)](\theta, x) = f(x)$ for any $f \in C^r(I_\delta, I_\delta)$. With this identification $C^r(I_\delta, I_\delta)$ is a subspace of $C^r(\mathbb{T} \times I_\delta, I_\delta)$ and the operator

$$\begin{array}{rccc} p_0: & C^r(\mathbb{T} \times I_\delta, I_\delta) & \rightarrow & C^r(I_\delta, I_\delta) \\ & f & \mapsto & \displaystyle\int_0^1 f(\theta, x) d\theta, \end{array} \tag{9}$$

defines a projection. Then, we can consider $\mathcal{M}_\delta$ and $\mathcal{D}(\mathcal{R}_\delta)$ as sets in both $C^r(I_\delta, I_\delta)$ and $C^r(\mathbb{T} \times I_\delta, I_\delta)$ depending on the context. Consider also $\mathcal{X}_\delta$, the set defined as

$$\mathcal{X}_\delta = \{f \in C^r(\mathbb{T} \times I_\delta, I_\delta) |\, p_0(f) \in \mathcal{M}_\delta\}.$$

Definition 2.4. Given a function $g \in \mathcal{X}_\delta$, we define the *(quasi-periodic doubling) renormalization* of g as

$$[\mathcal{T}_\omega(g)](\theta, x) := \frac{1}{\hat{a}} g(\theta + \omega, g(\theta, \hat{a}x)), \tag{10}$$

where $\hat{a} = \displaystyle\int_0^1 g(\theta, 1)d\theta$.

The operator $\mathcal{T}_\omega$ restricted to the set $\mathcal{D}(\mathcal{R}_\delta)$ coincides with the operator $\mathcal{R}_\delta$. Therefore, any fixed point of $\mathcal{R}_\delta$ extends to a fixed point of $\mathcal{T}_\omega$. Consider the set $\mathcal{D}(\mathcal{T}_\omega) = \{g \in \mathcal{X}_\delta \,|\, \mathcal{T}_\omega(g) \in \mathcal{X}_\delta\}$, then the renormalization operator $\mathcal{T}_\omega$ is defined from $\mathcal{D}(\mathcal{T}_\omega)$ to $\mathcal{X}_\delta$. The following result implies that $\mathcal{D}(\mathcal{T}_\omega)$ contains an open neighbourhood (in $\mathcal{X}_\delta$) of $\mathcal{D}(\mathcal{R}_\delta)$, where the operator is well defined.

Proposition 2.5. *For any function $\phi \in \mathcal{D}(\mathcal{R}_\delta)$ there exists an open neighbourhood $V \subset \mathcal{D}(\mathcal{T}_\omega)$ of ϕ such that $\mathcal{T}_\omega : V \to C^r(\mathbb{T} \times I_\delta, I_\delta)$ is a well defined continuous map.*

At this point we can go back to the definition of R, the renormalization operator for a quasi-periodically forced map $F = (\omega, f)$ of the form (8).

Definition 2.6. Consider

$$\begin{aligned} X &= \{(\omega, f) \in \mathbb{T} \times C^r(\mathbb{T} \times I_\delta, I_\delta) \,|\, f \in \mathcal{X}_\delta\}, \\ \mathcal{D}(R) &= \{(\omega, f) \in \mathbb{T} \times C^r(\mathbb{T} \times I_\delta, I_\delta) \,|\, f \in \mathcal{D}(\mathcal{T}_\omega)\}. \end{aligned}$$

We define the *quasi-periodically forced renormalization operator* as

$$\begin{array}{rccc} R: & \mathcal{D}(R) & \to & X \\ & (\omega, f) & \mapsto & (2\omega, \mathcal{T}_\omega(f)). \end{array}$$

Let $\mathcal{D}^n(R)$ denote the domain of maps (ω, f) which are n-times renormalizable, in other words

$$\mathcal{D}^n(R) = \left\{ (\omega_0, f_0) \in X \;\middle|\; \begin{array}{c} f_i \in \mathcal{D}(\mathcal{T}_{\omega_i}), \text{ for } i = 0, \dots, n-1, \\ \text{where } (\omega_i, f_i) := R(\omega_{i-1}, f_{i-1}) \end{array} \right\}.$$

Note that for any map $g \in \mathcal{D}(\mathcal{R}_\delta)$ we have $\mathcal{T}_\omega(g) = \mathcal{R}_\delta(g)$ and therefore the dynamics of (ω, f) by R is not coupled, in the sense that each component is independent of the other. For instance, consider Φ the fixed point of $\mathcal{R}_\delta$ given by Theorem 2.3. Then Φ is also fixed by $\mathcal{T}_\omega$ and the set $\mathbb{T} \times \{\Phi\}$ is invariant by R and the dynamics in ω is determined by the expansive map $\omega_{n+1} = 2\omega_n$.

2.3. Differentiability of the operator and its tangent map

As we will see in Section 3, one of the reasons to introduce the (quasi-periodically forced) renormalization operator is to study the image by R of functions of the form $(\omega_0, f_0) + \varepsilon(0, h_0)$ where ω_0 is a Diophantine number, $f_0 \in \mathcal{D}(\mathcal{R}_\delta)$, ε is a small parameter and $h_0 \in T_{\omega_0, f_0} X$. With this aim, we have the following result on the differentiability of the renormalization operator $\mathcal{T}_\omega$.

Theorem 2.7. *Given $\phi \in \mathcal{D}(\mathcal{T}_\omega)$, $\phi \in C^{r+s}$, there exists an open neighbourhood U of ϕ in $\mathcal{D}(\mathcal{T}_\omega)$ with the C^{r+s} topology, such that $\mathcal{T}_\omega : U \to \mathcal{X}_\delta$ is a C^s operator.*

There is a "loss of differentiability", in the sense that a C^{r+s} function gives place to a C^s operator in the C^r topology. Alternatively, one can consider a setup of the operator on the analytic functions: Let B_ρ be a complex band of width ρ around the real numbers ($B_\rho = \{z = x + iy \in \mathbb{C}$ such that $|y| < \rho\}$) and W be an open set in $\mathbb{C}$ containing the real interval I_δ. Consider $\mathcal{B}(B_\rho, W)$ the space of functions $f : B_\rho \times W \to \mathbb{C}$ such that:

(i) f is holomorphic in $B_\rho \times W$ and continuous in the closure of $B_\rho \times W$.

(ii) f is real analytic (it maps real numbers to real numbers).

(iii) f is 1-periodic in the first variable, i. e. $f(\theta+1, z) = f(\theta, z)$ for any $(\theta, z) \in B_\rho \times W$.

It is not difficult to check that the space $\mathcal{B}(B_\rho, W)$ endowed with the supremum norm is a Banach space.

We want to consider the quasi-periodic renormalization operator $\mathcal{T}_\omega$ extended to an open set of $\mathcal{B}(B_\rho, W)$. Given $f \in \mathcal{B}(B_\rho, W)$, a necessary condition to have $\mathcal{T}_\omega(f)$ well defined is that $f(B_\rho \times \hat{a}(f)W) \subset W$ (where $aW = \bigcup_{z \in W}\{az\}$). The following result ensures the existence of a neighbourhood of the Feigenbaum fixed point where the operator is well defined and differentiable.

Theorem 2.8. *Let Φ be the fixed point given by Theorem 2.3 and W its domain of definition. Let $\mathcal{B}(B_\rho, W)$ be the space of real analytic functions defined above, endowed with the supremum norm. For a sufficiently small ρ, there exists $U \subset \mathcal{B}(B_\rho, W)$, an open neighbourhood of Φ, such that $\mathcal{T}_\omega(\Psi)$ is well defined for any $\Psi \in U$. Moreover $\mathcal{T}_\omega$ is Fréchet differentiable for any $\Psi \in U$ and its derivative is equal to*

$$\begin{aligned}[D\mathcal{T}_\omega(\Psi, h)](\theta, x) = {} & \frac{1}{a}(\partial_x \Psi)(\theta + \omega, \Psi(\theta, ax))h(\theta, ax) + \frac{1}{a}h(\theta + \omega, \Psi(\theta, ax)) \\ & + \frac{b}{a}(\partial_x \Psi)(\theta + \omega, \Psi(\theta, ax))(\partial_x \Psi(\theta, ax))x - \frac{b}{a^2}\Psi(\theta + \omega, \Psi(\theta, ax)),\end{aligned} \tag{11}$$

with $a = \int_0^1 \Psi(\theta, 1)d\theta$ and $b = \int_0^1 h(\theta, 1)d\theta$.

Given a quasi-periodically forced map $F = (\omega, f) \in X$ we are interested only on (infinitesimal) perturbations $\delta F \in T_F X$ that preserve the skew product structure of maps of the form (8). For this reason we will only consider tangent vectors of the form $\delta F = (0, \delta f)$. We can incorporate this in the tangent bundle of X and (re)define it as:

$$TX = \left\{(\omega, f, h) \,|\, (\omega, f) \in X, \quad (0, h) \in T_{(\omega, f)}X\right\}.$$

We can define now S, the *tangent map associated to* R, in the C^r topology as

$$\begin{array}{rccc} S: & \mathcal{D}(S) & \to & TX \\ & (\omega, f, h) & \mapsto & (2\omega, \mathcal{T}_\omega(f), D\mathcal{T}_\omega(f)h), \end{array} \tag{12}$$

with $\mathcal{D}(S) := \{(\omega, f, h) \in TX \,|\, (\omega, f) \in \mathcal{D}(R)$ and $f \in C^{r+1}(\mathbb{T} \times I_\delta, I_\delta)\}$.

In the next section we focus on families of maps F_ε which can be written as $F_\varepsilon = (\omega_0, f_0) + \varepsilon(0, h_0)$. The interest on the tangent map S is that it describes $R^n(F_\varepsilon)$ up to the first order in ε. Concretely, if we have $(\omega_k, f_k, h_k) = S^k(\omega_0, f_0, h_0)$, then

$$R^n(F_\varepsilon) = R^n((\omega_0, f_0) + \varepsilon(0, h_0)) = (\omega_k, f_k) + \varepsilon(0, h_k) + O(\varepsilon^2).$$

For further discussion, the following subspaces will be also relevant. Consider

$$TX^{(0)} := \{(\omega, f, h) \in TX \,|\, f \in \mathcal{M}_\delta \text{ and } h(\theta, x) = h_0(x)\} \tag{13}$$

and

$$TX^{(k)} := \left\{ (\omega, f, h) \in TX \,\middle|\, f \in \mathcal{M}_\delta \text{ and } h(\theta, x) = \begin{array}{c} h_c(x)\cos(2\pi k\theta) \\ +h_s(x)\sin(2\pi k\theta) \end{array} \right\}, \tag{14}$$

for any $k > 0$.

Proposition 2.9. *The spaces $TX^{(k)}$ are invariant by S for any $k \geq 0$. In other words, for any $(\omega, f, h) \in TX^{(k)} \cap \mathcal{D}(S)$, $k \geq 0$, we have that $S(\omega, f, h) \in TX^{(k)}$.*

3. Reducibility loss and quasi-periodically forced renormalization

Consider a two parametric family of quasi-periodically forced maps,

$$\begin{array}{cccc} F_{\alpha,\varepsilon}: & \mathbb{T} \times I_\delta & \rightarrow & \mathbb{T} \times I_\delta \\ & \left(\begin{array}{c} \theta \\ x \end{array}\right) & \mapsto & \left(\begin{array}{c} \theta + \omega \\ f_0(\theta, x, \alpha, \varepsilon) \end{array}\right), \end{array} \tag{15}$$

with ω Diophantine, α and ε real parameters and f_0 a function of the form

$$f_0(\theta, x, \alpha, \varepsilon) = g_0(x, \alpha) + \varepsilon h_0(\theta, x, \alpha, \varepsilon), \tag{16}$$

where g_0 and h_0 are C^r functions ($r \geq 3$). Let f_n denote the second component of F^{2^n}, that is, $F^{2^n}(\theta, x) = (\theta + 2^n\omega, f_n(\theta, \omega))$. Using the differentiability of F it follows that f_n is also of the form

$$f_n(\theta, x, \alpha, \varepsilon) = g_n(x, \alpha) + \varepsilon h_n(\theta, x, \alpha, \varepsilon),$$

with $g_n(x) = g^{2^n}(x)$ and $h_n(\theta, x, \alpha, \varepsilon) = \frac{\partial f_n}{\partial \varepsilon}(\theta, x, \alpha, 0) + O(\varepsilon)$.

Let us introduce now a couple of definitions which characterize the family of maps $F_{\alpha,\varepsilon}$. In this section x_0 will denote the critical value of g, although in our framework we have $x_0 = 0$ due to the definition of M_δ.

Definition 3.1. Consider a one-parametric family of maps in the interval $g_\alpha : I_\delta \rightarrow I_\delta$, with $g_\alpha \in \mathcal{M}_\delta$, for any $\alpha \in J$, with J a closed interval of the real line. We will say that the family g_α has a *complete cascade of superattracting periodic orbits* if there exists a strictly increasing sequence of values of the parameter $\{\alpha_n\}_{n\geq 1}$ such that, for each n, $g(\,\cdot\,, \alpha_n) \in \mathcal{D}^{n-1}(\mathcal{R}_\delta)$ and $g(\,\cdot\,, \alpha_n)$ has a superattracting periodic orbit of period 2^n (and not 2^{n-1}). More concretely, we ask for the conditions

(i) $\dfrac{\partial g}{\partial x}(x_0, \alpha) = 0$, $\dfrac{\partial^2 g}{\partial x^2}(x_0, \alpha) \neq 0$, for all α.

(ii) $g_n(x_0, \alpha_n) = x_0$, $\dfrac{\partial g_n}{\partial \alpha}(x_0, \alpha_n) \neq 0$, for all $n \geq 0$, where g_n denotes g^{2^n}.

Definition 3.2. Consider $F_{\alpha,\varepsilon}$ a two parametric family of quasi-periodically forced maps of the form (15); with ω Diophantine, α and ε real parameters, and f_0 a function of the form (16) with g and h are C^r functions ($r \geq 3$). Let f_n denote the function such that $F^{2^n}(\theta, x) = (\theta + 2^n\omega, f_n(\theta, \omega))$. Consider $g_n(x) = g^{2^n}(x)$ and h_n given in (16) as before. Assume that $\{g_\alpha\}_{\alpha\in J}$ has a complete cascade of superattracting periodic orbits and let $\{\alpha_n\}_{n\geq 1}$ denote the corresponding sequence of parameter values. We will say that h is an *admissible quasi-periodic perturbation* if:

(i) for any $g_\alpha \in D^n(R)$ there exists ε_0 small enough, such that $g_\alpha + \varepsilon h$ belong to $D^n(R)$ for any $0 \leq \varepsilon \leq \varepsilon_0$.

(ii) for each $n \geq 1$ the function

$$H_n(\theta) = h_n(\theta - 2^n\omega, x_0, \alpha_n, 0)\frac{\partial^2 g_n}{\partial x^2}(x_0, \alpha_n) + \frac{\partial h_n}{\partial x}(\theta, x_0, \alpha_n, 0)$$

has exactly two non-degenerate extrema: a maximum and a minimum.

Definition 3.1 characterizes the one-dimensional part of the family before the quasi-periodic forcing, whereas Definition 3.2 characterizes the type of quasi-periodic perturbation to consider.

When the function H_n has (only) a non-degenerate maximum and minimum, Theorem 2.1 in [6] applies to $F^{2^n}_{\alpha,\varepsilon}$. This implies that there exist exactly two curves of change of degree between degrees 0 and 2 in the parameter space.

Remark 3.3. When the function h which defines $F_{\alpha,\varepsilon}$ is of the form

$$h(\theta, x, \alpha, \varepsilon) = h_0(x, \alpha) + h_c(x, \alpha)\cos 2\pi\theta + h_s(x, \alpha)\sin 2\pi\theta + O(\varepsilon).$$

with $|h_c(x_0, \alpha_n)| + |h_s(x_0, \alpha_n)| \neq 0$ for all $n \geq 0$, then there exists a set $\Omega \subset \mathbb{T} \setminus \mathbb{Q}$ such that h is admissible for any $\omega \in \Omega$ and Ω is of full Lebesgue measure (see Theorem 3.1 in [6]).

The following theorem complements the results cited above by giving a explicit relationship between the bifurcation curves and the renormalization operator introduced in this paper.

Theorem 3.4. *Consider $F_{\alpha,\varepsilon}$ a two parametric family of quasi-periodically forced maps of the form (15); with ω Diophantine, α and ε are real parameters, and f of the form (16) with g and h C^∞ functions. Assume that g_α has a complete cascade of superattracting periodic orbits and let $\{\alpha_n\}_{n\geq 1}$ be the corresponding sequence of parameter values. We also assume that h is an admissible quasi-periodic perturbation in the sense of Definition 3.2.*

Then, for each $n \geq 1$, there exist ε_0 and exactly two functions $\alpha = \alpha_n^+(\varepsilon)$ and $\alpha = \alpha_n^-(\varepsilon)$, such that $\alpha_n^\pm(0) = \alpha_n$ and $(\alpha_n^\pm(\varepsilon), \varepsilon)$ are reducibility loss bifurcation curves, for any $0 \leq \varepsilon < \varepsilon_0$. Moreover, the functions $\alpha_n^\pm(\varepsilon)$ satisfy

$$\frac{\mathrm{d}\alpha_n^\pm(0)}{\mathrm{d}\varepsilon} = \frac{G^\pm\left(S^{n-1}\left(\omega, g(\cdot, \alpha_n), h(\cdot, \cdot, \alpha_n, 0)\right)\right)}{G^\pm\left(S^{n-1}\left(\omega, g(\cdot, \alpha_n), \frac{\partial g}{\partial \alpha}(\cdot, \alpha_n)\right)\right)}, \tag{17}$$

where S is the tangent map associated to the renormalization operator given by (12) and $G^\pm$ are given by

$$G^+(\omega_0, g_0, h_0) = \max_{\theta\in\mathbb{T}} G(\omega_0, g_0, h_0), \quad G^-(\omega_0, g_0, h_0) = \min_{\theta\in\mathbb{T}} G(\omega_0, g_0, h_0),$$

with

$$\begin{aligned}\left[G(\omega_0, g_0, h_0)\right](\theta) := & \left[\frac{\partial g_0}{\partial x}\left(g_0(x_0)\right) h_0(\theta - 2\omega_0, x_0) + h_0(\theta - \omega_0, g_0(x_0))\right]\frac{\partial^2 g_0}{\partial x^2}(x_0) \\ & + \frac{\partial h_0}{\partial x}(\theta, x_0).\end{aligned} \tag{18}$$

4. A preliminary numerical study of the renormalization operator

The goal of this section is to show numerical evidences that the renormalization operator (for quasi-periodically forced 1-D maps) proposed in the previous sections can be used to understand the self-similarity and universality properties of certain Forced Logistic Maps.

Given $F_{\alpha,\varepsilon}$ a two parametric family as in the hypothesis of Theorem 3.4, consider $\alpha_n^+(\varepsilon)$ one of the functions that define a reducibility loss bifurcation curve. Note that this function also depends on the rotation number ω, therefore let us write $\alpha_n^+(\varepsilon) = \alpha_n^+(\varepsilon, \omega)$ and let us denote

$$\alpha_n'(\omega) := \frac{\mathrm{d}\alpha_n^+(0,\omega)}{\mathrm{d}\varepsilon},$$

in order to simplify the notation.

In this section we estimate the values $\alpha_n'(\omega)$ for certain families of Forced Logistic Maps using the newly defined renormalization operator. In [12] this values were obtained using a completely different procedure based on the direct computation of the bifurcation curve. We show that both estimates coincide, which is a further sanity test on the definition of the operator.

Moreover, [12] contains strong numerical evidence that $\alpha_n'(\omega)/\alpha_{n-1}'(2\omega)$ converges to a constant when $n \to \infty$ for some families of quasi-periodically Forced Logistic Maps. This convergence is relevant because it determines an affine relationship between the bifurcation diagram of the family of maps for different values of ω. In Figure 2 there is a schematic representation of the kind of self-similarity found in the parameter space of these quasi-periodically forced Logistic Maps (see also Figure 2 in [12] for the original plot). The affine relationship between the parameter spaces is determined by the Feigenbaum constant in the direction of the parameter α and by the limit of $\alpha_n'(\omega)/\alpha_{n-1}'(2\omega)$ in the direction of the parameter ε. We believe that the theory developed here can be used to reduce the problem of self-similarity to the dynamics of the renormalization operator as happens in the 1D case.

4.1. Discretization of the tangent map S

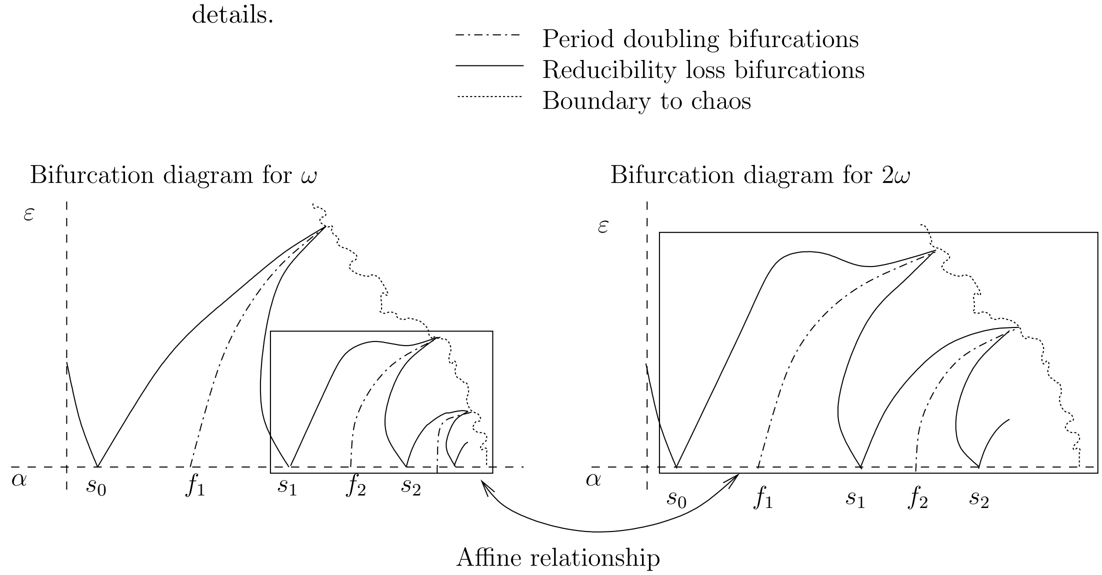
In this section we introduce a discretization of the tangent map S that will be studied numerically in the next subsection. The discretization proposed here is a slight modification of the one introduced by Lanford in [8] (see also [9]).

Let W be an open set in $\mathbb{C}$. Consider $\mathcal{A}(W)$ the Banach space of functions that are holomorphic on W, continuous on its closure, real analytic, and equipped with the supremum norm on W.

Let $\mathbb{D}(z_0, \rho)$ be the complex disc centered on z_0 with radius ρ. Given a function $\xi \in \mathcal{A}(\mathbb{D}(z_0, \rho))$, we can consider the following Taylor expansion of ξ around z_0,

$$\xi(z) = \sum_{k=0}^{\infty} \xi_k \left(\frac{z - z_0}{\rho} \right)^k .$$

Figure 2. Schematic representation of the bifurcations diagram of the Forced Logistic Map, for rotation number equal to ω (left) and 2ω (right). See the text for more details.



The truncation of this Taylor series at order N induces a projection defined as

$$\begin{array}{rccc} p_{(N)}: & \mathcal{A}(\mathbb{D}(z_0,\rho)) & \to & \mathbb{R}^{N+1} \\ & \xi & \mapsto & (\xi_0, \xi_1, \dots, \xi_N). \end{array}$$

On the other hand we have its pseudo-inverse by the left

$$\begin{array}{rccc} i_{(N)}: & \mathbb{R}^{N+1} & \to & \mathcal{A}(\mathbb{D}(z_0,\rho)) \\ & (\xi_0, \xi_1, \dots, \xi_N) & \mapsto & \displaystyle\sum_{k=0}^{N} \xi_k \left(\frac{z-z_0}{\rho}\right)^k ; \end{array}$$

in other words $i_{(N)} \circ p_{(N)}$ is the identity on $\mathbb{R}^{N+1}$. Note also that both maps are linear.

Let W be an open set in $\mathbb{C}$ containing the disc $\mathbb{D}(z_0,\rho)$. Given a map $T : \mathcal{A}(W) \to \mathcal{A}(W)$, we can approximate its restriction to $\mathcal{A}(\mathbb{D}(z_0,\rho))$ by the discretization $T^{(N)} : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ defined as $T^{(N)} := p_{(N)} \circ T \circ i_{(N)}$. If the disc $\mathbb{D}(z_0,\rho)$ is strictly contained in W, then it is not difficult to see that $i_{(N)} \circ T^{(N)}(\xi)$ converges to $T(\xi)$ (in the supremum norm) as $N \to \infty$.

Consider the tangent map S given by (12) in the analytic topology restricted to the subspaces $TX^{(k)}$ given by (13) and (14) for $k \geq 0$. In the case $(\omega, f, h) \in TX^{(0)}$ we have that (ω, f, h) can be identified with an element in $\mathbb{T} \times A(\mathbb{D}(z_0,\rho)) \times A(\mathbb{D}(z_0,\rho))$. In the case $(\omega, f, h) \in TX^{(k)}$ for $k > 0$, (ω, f, h) can be rewritten as $(\omega, f, h_c, h_s) \in \mathbb{T} \times A(\mathbb{D}(z_0,\rho))^3$. In both cases, we can approximate the tangent map using the discretization described above, obtaining approximations in $\mathbb{T} \times \mathbb{R}^{2(N+1)}$ and $\mathbb{T} \times \mathbb{R}^{3(N+1)}$ respectively.

4.2. Applicability to some families of Forced Logistic Maps

The most prototypical example of quasi-periodically forced 1D maps encountered in the literature is the Quasi-Periodically Forced Logistic Map:

$$\left.\begin{array}{rcl} \bar{\theta} & = & \theta + \omega, \\ \bar{x} & = & \alpha x(1-x) + \varepsilon h(\theta, x), \end{array}\right\} \tag{19}$$

This is the Logistic Map $\bar{x} = \alpha x(1-x)$ plus a quasi periodic forcing $h(\theta, x)$ times a perturbation parameter ε. This forcing has been typically taken as either multiplicative $h(\theta, x) = \alpha x(1-x)\cos(2\pi\theta)$ or additive $h(\theta, x) = \cos(2\pi\theta)$. The former case sometimes has been refereed as the Driven Logistic Map to differentiate it from the later, but essentially both maps have very similar behaviour (see [5] and references therein). Note that these two maps do not completely fit into the theory developed in the previous sections because the family of maps does not belong to $\mathcal{B}(B_\rho, W)$. This problem can be easily solved applying a suitable change of variables. For $\alpha > 2$ we can consider the affine change of variables given by $y = ax + b$, with $a = \frac{4}{\alpha-2}$ and $b = -\frac{2}{\alpha-2}$. If we apply this change of variables to the family (19) when $h_{\alpha,\varepsilon}(\theta, x) = \cos(2\pi\theta)$ we obtain the family:

$$\left.\begin{array}{rcl} \bar{\theta} & = & \theta + \omega, \\ \bar{y} & = & 1 - \dfrac{\alpha(\alpha-2)}{4} y^2 + \dfrac{4\varepsilon}{\alpha-2}\cos(2\pi\theta). \end{array}\right\} \tag{20}$$

If we apply the same change of variables when $h_{\alpha,\varepsilon}(\theta, x) = \alpha x(1-x)\cos(2\pi\theta)$ we obtain this other family

$$\left.\begin{array}{rcl} \bar{\theta} & = & \theta + \omega, \\ \bar{y} & = & \alpha\left(\dfrac{\alpha}{\alpha-2} - \dfrac{\alpha(\alpha-2)}{4} y^2\right)(1 + \varepsilon\cos(2\pi\theta)) - \dfrac{2}{\alpha-2}. \end{array}\right\} \tag{21}$$

With this new setup, both families of maps belong to $\mathcal{B}(B_\rho, W)$ for any $\alpha \in (2, 4)$ and ε small enough. The Logistic Map is well known to have a complete cascade of period doubling bifurcations [2]. On the other hand, the quasi-periodic perturbation of both (20) and (21) is admissible for a full measure set of values of ω (see Remark 3.3). Therefore, Theorem 3.4 is applicable to both maps. Moreover, functions g, h and $\frac{\partial g}{\partial \alpha}$ are such that $(\omega, g, \frac{\partial g}{\partial \alpha}) \in TX^{(0)}$ and $(\omega, g, h) \in TX^{(1)}$.

4.3. Results and conclusions

Let us notice that Theorem 3.4 not only guarantees the existence of reducibility loss bifurcations, but equation (17) also gives an explicit expression of the slopes of the bifurcations in terms of the renormalization operator. Actually, this formula has been used to compute a numerical approximation of the values $\alpha_n'(\omega) := \frac{\mathrm{d}\alpha_n^+(0,\omega)}{\mathrm{d}\varepsilon}$ as follows. The parameter values α_n for which the 1D map has a super-attracting periodic orbit of period 2^n have been computed numerically by means of a Newton method applied to their invariance equation. From equations (20) and (21) it is easy to derive expressions for $g(\cdot, \alpha)$, $h(\cdot, \cdot, \alpha, \varepsilon)$ and $\frac{\partial g}{\partial \alpha}(\cdot, \alpha)$ for each respective map. Note that, for the maps

Table 1. Numerical estimation of $\alpha'_n(\omega)$ for the map (20) on the left and the map (21) on the right, both for $\omega = \frac{\sqrt{5}-1}{2}$. The values $\bar{\epsilon}_a$ correspond to the discrepancy between the current values and the estimates in Table 5 and Table 1 of [12] in absolute terms.

n	$\alpha'_n(\omega)$	$\bar{\epsilon}_a$	n	$\alpha'_n(\omega)$	$\bar{\epsilon}_a$
1	-8.1607837043e+00	2.1e-11	1	-5.8329149229e+00	2.5e-11
2	-1.1166652707e+01	3.4e-10	2	-8.4942599432e+00	8.2e-12
3	-2.1221554117e+01	5.9e-11	3	-1.6351279467e+01	8.2e-11
4	-1.4564213015e+01	5.4e-11	4	-1.1252460775e+01	4.5e-10
5	-1.5837452605e+01	3.2e-10	5	-1.2243326651e+01	5.8e-11
6	-2.3384207858e+01	2.1e-10	6	-1.8079693906e+01	4.8e-10
7	-4.4925217655e+01	1.5e-11	7	-3.4735234068e+01	5.0e-10
8	-3.8261700375e+01	4.1e-10	8	-2.9583312211e+01	3.6e-10
9	-5.3763965692e+01	8.1e-10	9	-4.1569457725e+01	4.8e-10
10	-1.0213020110e+02	3.7e-08	10	-7.8965495553e+01	3.1e-08
11	-9.6355685476e+01	1.0e-07	11	-7.4500733376e+01	7.9e-08

Table 2. Numerical estimation of the values $\delta_n^{(0)}$ and $\delta_n^{(1)}$ given by formulas (22) and (23). The values on the left correspond to the map (20) and the values on the right correspond to (21).

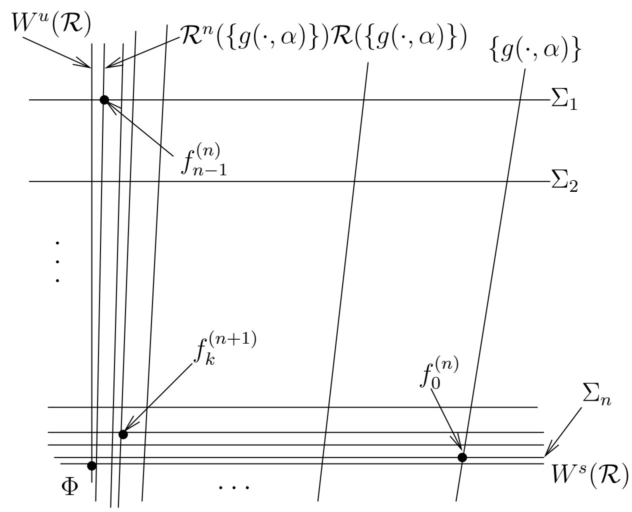
n	$\delta_n^{(0)}$	$\delta_n^{(1)}$	n	$\delta_n^{(0)}$	$\delta_n^{(1)}$
2	4.7580060014	8.5122626469	2	4.7580060014	8.4562747297
3	4.6657579609	8.2259755572	3	4.6657579609	7.6923936058
4	4.6691632933	8.2037877410	4	4.6691632933	7.5777607120
5	4.6689496835	8.1989359357	5	4.6689496835	7.5534974650
6	4.6691592857	8.1981051338	6	4.6691592857	7.5485106372
7	4.6691882755	8.1979405877	7	4.6691882755	7.5474557033
8	4.6691988457	8.1979132949	8	4.6691988457	7.5472371211
9	4.6692009362	8.1979076070	9	4.6692009362	7.5471904539
10	4.6692014624	8.1979064625	10	4.6692014624	7.5471805271
11	4.6692015842	8.1979062216	11	4.6692015842	7.5471784050

(20) and (21), $(\omega, g(\cdot, \alpha_n), h(\cdot, \cdot, \alpha_n, 0))$ and $\left(\omega, g(\cdot, \alpha_n), \frac{\partial g}{\partial \alpha}(\cdot, \alpha_n)\right)$ belong, respectively, to the spaces $TX^{(1)}$ and $TX^{(0)}$ given by (14) and (13). Then using the discretization of the operator seen in Section 4.1 we have computed a numerical approximation of $S^{n-1}(\omega, g(\cdot, \alpha_n), h(\cdot, \cdot, \alpha_n, 0))$ and $S^{n-1}\left(\omega, g(\cdot, \alpha_n), \frac{\partial g}{\partial \alpha}(\cdot, \alpha_n)\right)$ (S is defined in (12)). Then, we have evaluated these functions to compute the values of $\alpha'_n(\omega)$ given by (17).

The results are shown in Table 1 for the maps (20) and (21). The values $\alpha'_n(\omega)$ were also computed in [12] via a completely different procedure based on a continuation method with extended precision. In Table 1 we display the discrepancies between both computations in absolute value, namely $\bar{\epsilon}_a$.

One of the aims to develop the renormalization operator R and its tangent map S is to understand why $\alpha'_n(\omega)/\alpha'_{n-1}(2\omega)$ converges to a constant. Applying equation (17)

Figure 3. Schematic representation of the dynamics of $\mathcal{R}_\delta$ around its fixed point Φ. See the text for the definition of each of the elements displayed.



to $\alpha_n'(\omega)$ and $\alpha_{n-1}'(2\omega)$ and rearranging the terms we have

$$\frac{\alpha_n'(\omega)}{\alpha_{n-1}'(2\omega)} = \left(\delta_n^{(0)}\right)^{-1} \cdot \delta_n^{(1)},$$

with

$$\delta_n^{(0)} = \frac{G^+\left(S^{n-1}\left(\omega, g(\cdot,\alpha_n), \frac{\partial g}{\partial \alpha}(\cdot,\alpha_n)\right)\right)}{G^+\left(S^{n-2}\left(2\omega, g(\cdot,\alpha_{n-1}), \frac{\partial g}{\partial \alpha}(\cdot,\alpha_{n-1})\right)\right)}, \tag{22}$$

$$\delta_n^{(1)} = \frac{G^+\left(S^{n-1}\left(\omega, g(\cdot,\alpha_n), h(\cdot,\cdot,\alpha_n,0)\right)\right)}{G^+\left(S^{n-2}\left(2\omega, g(\cdot,\alpha_{n-1}), h(\cdot,\cdot,\alpha_{n-1},0)\right)\right)}. \tag{23}$$

Using the same numerical procedure as before we have estimated $\delta_n^{(0)}$ and $\delta_n^{(1)}$ for the maps (20) and (21). The results are shown in Table 2.

We can observe that $\delta_n^{(0)}$ converges to the Feigenbaum constant for both maps, whereas $\delta_n^{(1)}$ converges to a constant that depends on the map. In other words, $\delta_n^{(1)}$ converges to a constant that is not universal. Same computations can be repeated for different values of ω for both maps, in all cases one obtains that $\delta_n^{(0)}$ converges to the universal Feigenbaum constant whereas $\delta_n^{(1)}$ converges to a constant that also depends on ω.

It is relatively easy to understand why $\delta_n^{(0)}$ converges to the Feigenbaum constant. Consider the following functions:

$$f_0^{(n)} = g(\cdot,\alpha_n); \qquad f_k^{(n)} = \mathcal{R}_\delta\left(f_{k-1}^{(n)}\right), \qquad k = 1,\ldots,n-1.$$

$$u_0^{(n)} = \frac{\partial g}{\partial \alpha}(\cdot,\alpha_n); \quad u_k^{(n)} = D\mathcal{R}_\delta\left(f_{k-1}^{(n)}\right) u_{k-1}^{(n)}, \quad k = 1,\ldots,n-1.$$

Since $f_0^{(n)}$ is a one-dimensional map and u_k is a perturbation without quasi-periodic component, we have that $\mathcal{T}_\omega$ and $\mathcal{R}_\delta$ coincide. Hence

$$S^{n-1}\left(w, g(\cdot,\alpha_n), \frac{\partial g}{\partial\alpha}(\cdot,\alpha_n)\right) = \left(2^{n-1}\omega, f_{n-1}^{(n)}, u_{n-1}^{(n)}\right).$$

On the other hand, note that for any constant $\kappa > 0$ we have

$$\kappa G^{\pm}(\omega, g, h) = G^{\pm}(\omega, g, \kappa h).$$

Using this on (22) it is easy to check that

$$\delta_n^{(0)} = \frac{G^+\left(2^{n-1}\omega, f_{n-1}^{(n)}, \frac{u_{n-1}^{(n)}}{\|u_{n-1}^{(n)}\|}\right)}{G^+\left(2^{n-1}\omega, f_{n-2}^{(n-1)}, \frac{u_{n-2}^{(n-1)}}{\|u_{n-2}^{(n-1)}\|}\right)} \cdot \frac{\|u_{n-1}^{(n)}\|}{\|u_{n-2}^{(n-1)}\|}.$$

In Figure 3 we have an schematic representation of the functions $f_{k-1}^{(n)}$ in terms of the dynamics of the classic 1D renormalization operator $\mathcal{R}_\delta$, see also [2] for technical details on the one dimensional version of the operator. In this case, the operator has a unique eigenvalue bigger than one while all the rest are contained in the unit disc. Its unstable invariant manifold $W^u(\mathcal{R}_\delta)$ is one dimensional and its stable invariant manifold $W^s(\mathcal{R}_\delta)$ has codimension one. Let Σ_n denote the set of maps in $\mathcal{M}_\delta$ (with zero topological entropy) for which 0 is a 2^n-periodic orbit. Since $\mathcal{R}_\delta(\Sigma_n) \subset \Sigma_{n-1}$, the manifolds Σ_n accumulate to $W^s(\mathcal{R}_\delta)$.

Note that $f_0^{(n)} = \Sigma_n \cap \{g(\cdot,\alpha)\}$, therefore $f_0^{(n)}$ tends to $W^s(\mathcal{R}_\delta) \cap \{g(\cdot,\alpha)\}$ when $n \to \infty$ (see Figure 3). On the other hand, one has that $f_{n-1}^{(n)} = \mathcal{R}_\delta^{n-1}(\{g(\cdot,\alpha)\}) \cap \Sigma_1$, therefore $f_{n-1}^{(n)}$ tends to $W^u(\mathcal{R}_\delta) \cap \Sigma_1$ when $n \to \infty$ (see also Figure 3). Finally, the vectors $\frac{u_{n-1}^{(n)}}{\|u_{n-1}^{(n)}\|} \to v^*$ as $n \to \infty$, with v^* the unit tangent vector to $W^u(\mathcal{R}_\delta)$ at its intersection with Σ_1. Hence, $G^+\left(2^{n-1}\omega, f_{n-1}^{(n)}, \frac{u_{n-1}^{(n)}}{\|u_{n-1}^{(n)}\|}\right)$ and $G^+\left(2^{n-1}\omega, f_{n-2}^{(n-1)}, \frac{u_{n-2}^{(n-1)}}{\|u_{n-2}^{(n-1)}\|}\right)$ converge to the same value and the asymptotic behaviour of $\delta_n^{(0)}$ is the same as the ratio $\frac{\|u_{n-1}^{(n)}\|}{\|u_{n-2}^{(n-1)}\|}$.

The iteration for $\mathcal{R}_\delta$ from $f_0^{(n)}$ to $f_{n-1}^{(n)}$ corresponds to a passage close to a saddle point. When n increases, more and more iterates lay arbitrarily close to the the fixed point Φ. This makes the ratio $\frac{\|u_{n-1}^{(n)}\|}{\|u_{n-2}^{(n-1)}\|}$ to converge to the dominant eigenvalue of $\mathcal{R}_\delta$, which is the Feigenbaum constant.

In the case of $\delta_n^{(1)}$ the situation must be somehow similar, but there is a real dependency of $D\mathcal{T}_\omega$ on ω, which gives place to a much more complicated dynamics. We conjecture that $\delta_n^{(1)}$ converges to a value, but this limit depends on ω and on the initial quasi-periodic perturbation.

We believe that the asymptotic behaviour of $\delta_n^{(1)}$ (and consequently $\frac{\alpha_n'(\omega)}{\alpha_{n-1}'(2\omega)}$) is completely determined by the dynamics of S. Therefore, a deeper understanding of the dynamics of S is crucial to understand the self-renormalization properties of 1D maps under quasi-periodic forcing.

5. Proofs

5.1. Proofs of results in Section 2

Proof of Proposition 2.2. To check that the operator is well defined we need to prove that $\mathcal{R}_\delta(\psi)$ belongs to $\mathcal{M}_\delta$ for any $\psi \in \mathcal{D}(\mathcal{R}_\delta)$.

Let $\Psi = \mathcal{R}_\delta(\psi)$, then

$$\Psi(0) = \frac{1}{a}\psi \circ \psi(0) = \frac{1}{a}\psi(1) = 1.$$

Note that

$$x\Psi'(x) = x\psi'(ax)\psi'(\psi(ax)).$$

Using that $x\psi'(x) < 0$ for any $x \in I_\delta \setminus \{x = 0\}$ it follows that $x\psi'(-ax) > 0$, for any $x \in I_\delta \setminus \{x = 0\}$. On the other hand, for any $x \in I_\delta$ we have that $\psi(-ax) \in [b', 1]$. Using again $x\psi'(x) < 0$ and $0 < a < a' < b'$ it follows that $\psi'(\psi(ax)) < 0$ for any $x \in I_\delta$. Then we can conclude that $x\Psi'(x) < 0$ for any $x \in I_\delta \setminus \{x = 0\}$.

To prove that $\mathcal{R}_\delta(\psi)$ belongs to $\mathcal{M}_\delta$ we have to check that Ψ maps x inside the set I_δ for any $x \in I_\delta$. Using $\Psi(0) = 1$ and the monotonicity consequences of $x\Psi'(x) < 0$ we have that $\Psi(1+\delta) \leq \Psi(x) \leq \Psi(0)$ for any $x \in I_\delta$. Since $\Psi(0) = 1$ we only have to check that $\Psi(1 + \delta) > -(1 + \delta)$. Using $\Psi(1 + \delta) = \frac{1}{a}\psi(b')$, $a' = (1 + \delta)a < 0$ and $\psi(b') < -a'$ the inequality follows.

A fixed point ϕ of $\mathcal{R}_0$ can be extended to the real line using recursively the invariance equation $\psi(x) = \frac{1}{a}\psi \circ \psi(ax)$ because $|a| < 1$. We also have to check that $\psi \in \mathcal{D}(\mathcal{R}_\delta)$ for a sufficiently small δ. Using again that $|a| < 1$, we have that $a = f(1) > -1$. Therefore, there exist δ_0, such that we have $f(1 + \delta) > -1 - \delta$ for any $\delta \in (0, \delta_0)$. □

We will need the following lemma to prove Proposition 2.5.

Lemma 5.1. *Given f a function in $\mathcal{M}_\delta$, define $\eta := f(1 + \delta) + 1 + \delta$ (see Figure 1). For any $h(x, \theta) \in C^r(\mathbb{T} \times I_\delta, I_\delta)$ with $p_0(h) = 0$, $\|h\|_{C^r} < \delta$ and $\|h\|_{C^r} < \eta$, we have that $g = f + h$ belongs to $\mathcal{X}_\delta$.*

Proof. Note that $p_0(g) = p_0(f) + p_0(h) = f \in \mathcal{M}_\delta$, then it is only necessary to check that $g \in C^r(\mathbb{T} \times I_\delta, I_\delta)$ to prove that $g \in \mathcal{X}_\delta$. The map g is C^r for being the linear combination of C^r maps, hence g belongs to $C^r(\mathbb{T} \times I_\delta, I_\delta)$ if $g(\theta, x) \in I_\delta$ for any $\theta \in \mathbb{T}$ and $x \in I_\delta$.

To check the upper bound note that $f \in \mathcal{M}_\delta$, then $f(x) \leq f(0) = 1$ for any $x \in I_\delta$ and

$$g(\theta, x) = f(x) + h(\theta, x) \leq f(x) + \|h\|_{C^r} \leq f(0) + \varepsilon = 1 + \varepsilon,$$

for any $(\theta, x) \in \mathbb{T} \times I_\delta$.

To check the lower bound, we first note that η only depends on f and it is always greater than or equal to 0. On the other hand, $f(x) \geq f(1 + \delta)$ for any $x \in I_\delta$. Hence,

$$g(\theta, x) = f(x) + h(\theta, x) \geq f(x) - \|h\|_{C^r} \geq f(1 + \delta) - \|h\|_{C^r} = -1 - \delta + \eta - \|h\|_{C^r},$$

which implies that $g(\theta, x) \geq -1-\delta$ if $\|h\|_{C^r} < \eta$. $\square$

Proof of Proposition 2.5. We construct the neighbourhood V in the statement of Proposition 2.5 as the intersection of different open sets.

Consider the value $\hat{a} = \int_0^1 g(\theta, 1) d\theta$ in the Definition 2.4 as a functional operator $\hat{a} : \mathbb{C}^r(\mathbb{T} \times I_\delta, I_\delta) \to I_\delta$. Note that $\hat{a}$, as operator, is equal to the evaluation map at $x = 1$ composed with the projection p_0 given by (9), therefore it is a continuous map. For any $\phi \in \mathcal{D}(\mathcal{R}_\delta)$ we have $-1 < \hat{a}(\phi) < 0$. Consider J_0 an open interval around $\hat{a}(\phi)$ such that $J_0 \subset (-1, 0)$. Then the set $U_1 = \hat{a}^{-1}(J_0)$ defines an open neighbourhood of ϕ.

For any function $g \in U_1$ (using $-1 < \hat{a}(g) < 0$) we have

$$\sup_{(\theta,x)\in\mathbb{T}\times I_\delta} |g(\theta, \hat{a}x)| \leq \sup_{(\theta,x)\in\mathbb{T}\times I_\delta} |g(\theta, x)| \leq 1+\delta.$$

Hence $g(\theta, \hat{a}x)$ is well defined for any $g \in U_1$ and $(\theta, x) \in \mathbb{T} \times I_\delta$. Using the smoothness of the composition map (see [4]), the operator $F_2 : U_1 \subset C^r(\mathbb{T} \times I_\delta, I_\delta) \to C^r(\mathbb{T} \times I_\delta, I_\delta)$, defined as $[F_2(g)](\theta, x) = g(\theta + \omega, g(\theta, \hat{a}x))$ is a well defined continuous map for any $g \in U_1$. Finally note that $\mathcal{T}_\omega$ is obtained as $\frac{1}{\hat{a}(g)} F_2(g)$, therefore $\mathcal{T}_\omega$ is continuous and it is well defined in U_1.

Consider

$$U_2 := \{g \in U_1 \text{ such that } |\mathcal{T}_\omega(g)(\theta, x)| < 1+\delta,\ \forall (\theta, x) \in \mathbb{T} \times I_\delta\}.$$

As ϕ does not depend on θ, it is easy to check that U_2 is an open subset of U_1 with $\phi \in U_2$. Now let us construct a smaller neighbourhood of ϕ such that it is contained in $\mathcal{D}(\mathcal{T}_\omega)$. Consider $F_4 : U_2 \to [0, +\infty)$ given by $F_4(g) = \|\mathcal{T}_\omega(g) - p_0(\mathcal{T}_\omega(g))\|_{C^r}$ and $F_5 : U_2 \to \mathbb{R}$ given by $F_5(g) = 1 + \delta + [\mathcal{T}_\omega(g)](1+\delta) - \|\mathcal{T}_\omega(g) - p_0(\mathcal{T}_\omega(g))\|_{C^r}$. Both maps are well defined and continuous. We can define the set V in the statement of Proposition 2.5 as

$$V := U_2 \cap F_4^{-1}([0, \delta)) \cap F_5^{-1}((-\delta, \delta)).$$

Using $\phi \in \mathcal{D}(\mathcal{R}_\delta)$ it follows that $F_4(\phi) = 0$ and $0 < F_5(\phi) < \delta$, therefore $\phi \in V$.

We only have to check $V \subset \mathcal{D}(\mathcal{T}_\omega)$ to finish the proof. This is equivalent to show that $\mathcal{T}_\omega(g) \in \mathcal{X}_\delta$ for any $g \in V$. Given $g \in W$, we have $\mathcal{T}_\omega(g) = p_0(\mathcal{T}_\omega(g)) + \mathcal{T}_\omega(g) - p_0(\mathcal{T}_\omega(g))$. From $g \in U_2$ it follows that $p_0(\mathcal{T}_\omega(g)) \in \mathcal{M}_\delta$. Moreover, $g \in F_4^{-1}([0,\delta))$ implies $\|\mathcal{T}_\omega(g) - p_0(\mathcal{T}_\omega(g))\|_{C^r} < \delta$ and $g \in F_5^{-1}([0,\delta))$ implies $\|\mathcal{T}_\omega(g) - p_0(\mathcal{T}_\omega(g))\|_{C^r} < 1 + \delta + [p_0(\mathcal{T}_\omega)](1+\delta)$. We can apply Lemma 5.1 (using $p_0(\mathcal{T}_\omega(g) - p_0(\mathcal{T}_\omega(g))) = 0$) to conclude that $\mathcal{T}_\omega(g) \in \mathcal{X}_\delta$. $\square$

Proof of Theorem 2.7. Given a function $g \in \mathcal{D}(\mathcal{T}_\omega) \subset C^{r+s}(\mathbb{T} \times I_\delta, I_\delta)$ we have $[\mathcal{T}_\omega(g)](\theta, x) := \dfrac{1}{\hat{a}} g(\theta + \omega, g(\theta, \hat{a}x))$ with $\hat{a} = \displaystyle\int_0^1 g(\theta, 1) d\theta$. Note that $\hat{a}$ (as a functional operator) can be written as $\hat{a} = [p_0(g)](1)$, where p_0 is given by (9). The function $p_0 : C^{r+s}(\mathbb{T} \times I_\delta, I_\delta) \to C^{r+s}(I_\delta, I_\delta)$ is actually a linear bounded operator, therefore it is C^∞. On the other hand the evaluation of a C^{r+s} function at a given value is also a C^{r+s} operator (it is linear and continuous). Therefore $\hat{a}(\cdot)$ as operator is C^{r+s} as well.

The map $\mathcal{T}_\omega(g)$ can be written as the composition of different functions, which are the self-composition of g, a translation in the θ variable and a scalar multiplication by a (and its inverse) in the x variable. Each one of these operations are C^{r+s} functions with respect to g except the composition of g with itself which is only a C^s map in the C^r topology (for details see [4]). Therefore we can only conclude that $\mathcal{T}_\omega$ is a C^s operator. □

Proof of Theorem 2.8. Note that the function Φ can be considered both as a function in $\mathcal{B}(B_\rho, W)$ and as a function in $\mathcal{A}_1(W)$ (see (7)). Let us denote by Φ the former case and by Φ_1 the later case. By Theorem 2.3 we have that $\mathrm{Cl}(a(\Phi_1)W) \subset W$ and $\Phi_1(\mathrm{Cl}(a(\Phi_1)W)) \subset W$, where $\mathrm{Cl}(\cdot)$ denotes the closure of a set. For any ρ we have $\Phi(B_\rho \times W) = \Phi_1(W)$, therefore we have $\mathrm{Cl}(a(\Phi)W) \subset W$ and $\Phi(\mathrm{Cl}(B_\rho \times a(\Phi)W)) \subset W$. Then, there exists U_1 a neighbourhood of Φ, such that $\mathrm{Cl}(a(\Psi)W) \subset W$ and $\Psi(\mathrm{Cl}(B_\rho \times a(\Psi)W)) \subset W$ for any $\Psi \in U_1$. Then we have that $\mathcal{T}_\omega$ is well defined in U_1.

To prove the differentiability of $\mathcal{T}_\omega$ we will check directly that its Fréchet derivative is given by (11). From $\mathrm{Cl}(\Phi(B_\rho \times aW)) \subset W$, and the fact of W being bounded it follows that $\mathrm{Cl}(\Phi(B_\rho \times aW))$ is compact. Consider the following filtration of sets in the complex plane

$$\mathrm{Cl}(\Phi(B_\rho \times aW)) = K_0 \subset V_0 \subset K_1 \subset V_1 \subset K_2 \subset V_2 = W,$$

with each K_i compact and each V_i open, for $i = 0, 1, 2$.

Consider now $U_2 \subset U_1$ the open neighbourhood of Φ in $\mathcal{B}(B_\rho, W)$ formed by the $\Psi \in \mathcal{B}(B_\rho, W)$ such that

$$\Psi(B_\rho \times aW) \subset V_0.$$

For any map $\Psi \in U_2$ we have that $\mathrm{Cl}(\Psi(B_\rho \times \hat{a}W)) \subset K_1$.

On the other hand, from $K_2 \subset W$ and the fact that K_2 is compact and W open, it follows that there exists a value $r > 0$ such that for any $x \in K_2$ the ball centered on x with radius r is contained in W. Then for any map $f \in \mathcal{B}(B_\rho, W)$ we have

$$\partial_x f(\theta, x) = \frac{1}{2\pi i} \int_{|z-x|=r} \frac{f(\theta, z)}{(z-x)^2} dz.$$

Then it follows easily that, for any $f \in \mathcal{B}(B_\rho, W)$ and $x \in K_2$ we have

$$|\partial_x f(\theta, x)| \leq \frac{1}{r} \|f\|_\infty. \tag{24}$$

Modifying the same argument, we can check that

$$|\partial^2_{x^2} f(\theta, x)| \leq \frac{2}{r^2} \|f\|_\infty. \tag{25}$$

Consider $\Psi \in U_2$, and $h \in \mathcal{B}(B_\rho, W)$ with $\|h\|_\infty$ small. We want to compute $\mathcal{T}_\omega(\Psi + h)$ up to $O(\|h\|_\infty^2)$. First of all we have,

$$\begin{aligned}\mathcal{T}_\omega(\Psi + h) = \frac{1}{\hat{a}(\Psi + h)} &\left[\Psi(\theta + \omega, \Psi(\theta, \hat{a}(\Psi + h)x) + h(\theta, \hat{a}(\Psi + h)x))\right. \\ &\left.+ h(\theta + \omega, \Psi(\theta, \hat{a}(\Psi + h)x) + h(\theta, \hat{a}(\Psi + h)x))\right].\end{aligned} \tag{26}$$

To simplify the notation consider

$$a = \int_0^1 \Psi(\theta, 1) d\theta, \quad b = \int_0^1 h(\theta, 1) d\theta.$$

Then we have $\hat{a}(\Psi + h) = a + b$, and

$$|b| \leq \int_0^1 |h(\theta, 1)| d\theta \leq \|h\|_\infty.$$

Since $\Psi \in U_2$ we have that for any h with $\|h\|_\infty$ sufficiently small, $\Psi + h \in U_2$, therefore we have that $\Psi(\theta, (a+b)x) + h(\theta, (a+b)x) \in V_1$. Using the Taylor expansion with respect to x up to second order is not difficult to check that

$$\begin{aligned} \Psi(\theta + \omega, \Psi(\theta, (a+b)x) + h(\theta, (a+b)x)) &= \Psi(\theta + \omega, \Psi(\theta, (a+b)x)) \\ &+ (\partial_x \Psi)(\theta + \omega, \Psi(\theta, (a+b)x)) h(\theta, (a+b)x) + R_2(\theta, x), \end{aligned} \tag{27}$$

and, using (25),

$$|R_2(\theta, x)| \leq \frac{2}{r^2} \|\Psi\|_\infty \|h\|_\infty^2 = O(\|h\|_\infty^2). \tag{28}$$

Analogously it is easy to check

$$h(\theta + \omega, \Psi(\theta, (a+b)x) + h(\theta, (a+b)x)) = h(\theta + \omega, \Psi(\theta, (a+b)x)) + R_1(\theta, x), \tag{29}$$

and, using (24),

$$|R_1(\theta, x)| \leq \frac{1}{r} \|h\|_\infty \|h\|_\infty = O(\|h\|_\infty^2). \tag{30}$$

As $|b| = O(\|h\|_\infty)$, applying Taylor expansion and the bound (24) on K_2 it follows easily that

$$\Psi(\theta, (a+b)x) = \Psi(\theta, ax) + (\partial_x \Psi)(\theta, ax) bx + O(\|h\|_\infty^2), \tag{31}$$

$$h(\theta, (a+b)x) = h(\theta, ax) + O(\|h\|_\infty^2). \tag{32}$$

Using that $\Psi \in U_2$ we have that $\Psi(\theta, ax) + (\partial_x \Psi)(\theta, ax) bx$ belongs to $V_1 \subset K_2$ for $\|h\|_\infty$ sufficiently small. Now we can combine this fact with the bounds (24) and (25) with equations (31) and (32) to prove that

$$h(\theta + \omega, \Psi(\theta, (a+b)x)) = h(\theta + \omega, \Psi(\theta, ax)) + O(\|h\|_\infty^2).$$

Using now equations (29) and (30) we obtain

$$h(\theta + \omega, \Psi(\theta, (a+b)x) + h(\theta, (a+b)x)) = h(\theta + \omega, \Psi(\theta, ax)) + O(\|h\|_\infty^2). \tag{33}$$

With a similar argument it follows that

$$(\partial_x \Psi)(\theta + \omega, \Psi(\theta, (a+b)x)) = (\partial_x \Psi)(\theta + \omega, \Psi(\theta, ax)) + O(\|h\|_\infty),$$

and

$$\begin{aligned} \Psi(\theta + \omega, \Psi(\theta, (a+b)x)) &= \Psi(\theta + \omega, \Psi(\theta, ax)) \\ &+ (\partial_x \Psi)(\theta + \omega, \Psi(\theta, ax))(\partial_x \Psi)(\theta, ax) bx + O(\|h\|_\infty^2). \end{aligned}$$

Replacing the last two equations in (27) and using the bound given by (28) yields

$$\begin{aligned}\Psi(\theta+\omega, \Psi(\theta,(a+b)x) + h(\theta,(a+b)x)) &= \Psi(\theta+\omega, \Psi(\theta, ax)) \\ &+(\partial_x\Psi)(\theta+\omega, \Psi(\theta,ax)) + (\partial_x\Psi)(\theta, ax)bx \\ &+(\partial_x\Psi)(\theta+\omega, \Psi(\theta,ax))h(\theta,(a+b)x)) + O(\|h\|_\infty^2). \end{aligned} \tag{34}$$

Finally, recall that $|b| = O(\|h\|_\infty)$, therefore

$$\frac{1}{a+b} = \frac{1}{a} - \frac{b}{a^2} + O(\|h\|_\infty^2).$$

When replacing this value and the ones of (33) and (34) in (26) it follows that

$$\|\mathcal{T}_\omega(\Psi+h) - \mathcal{T}_\omega(\Psi) - D\mathcal{T}_\omega(\Psi)h\|_\infty = O\left(\|h\|_\infty^2\right),$$

which proves the differentiability of the operator in the analytic topology. □

Proof of Proposition 2.9. Consider $(\omega, f, h) \in TX^{(k)} \cap \mathcal{D}(S)$, for any $k \geq 0$ we have $f \in \mathcal{D}(\mathcal{R}_\delta) \subset \mathcal{X}_\delta$, therefore $\mathcal{T}_\omega(f) = \mathcal{R}_\delta(f) \in \mathcal{X}_\delta$.

In the case $(\omega, f, h) \in TX^{(0)} \cap \mathcal{D}(S)$, we have that h has no periodic component, therefore it belongs to the tangent space of $\mathcal{M}_\delta$. Since f belongs to $\mathcal{D}(\mathcal{R}_\delta) \subset \mathcal{X}_\delta$ we have

$$D\mathcal{T}_\omega(f)h = D\mathcal{R}_\delta(f)h.$$

Hence $D\mathcal{T}_\omega(f)h$ will be a function with no periodic component, implying that $S(\omega, f, h) \in TX^{(0)}$.

In the case $(\omega, f, h) \in TX^{(k)} \cap \mathcal{D}(S)$, for any $k > 0$, we have that

$$h(\theta, x) = h_c(x)\cos(2\pi k\theta) + h_s(x)\sin(2\pi k\theta),$$

for suitable functions h_c and h_c. On the other hand, we have $f \in \mathcal{D}(\mathcal{R}_\delta)$, then let us write $f(\theta, x) = f(x)$. Using (11) is not difficult to check that

$$\begin{aligned}[D\mathcal{T}_\omega(f,h)](\theta,x) = & \frac{1}{a}(\partial_x f)(f(ax))\left(h_c(ax)\cos(2\pi k\theta) + h_s(ax)\sin(2\pi k\theta)\right) \\ & + \frac{1}{a}h_c(f(ax))\left(\cos(2\pi k\omega)\cos(2\pi k\theta) - \sin(2\pi k\omega)\sin(2\pi k\theta)\right) \\ & + \frac{1}{a}h_s(f(ax))\left(\sin(2\pi k\omega)\cos(2\pi k\theta) + \cos(2\pi k\omega)\sin(2\pi k\theta)\right)\end{aligned}$$

with $a = f(1)$. Grouping terms it follows easily that $S(\omega, f, h) \in TX^{(k)}$. □

5.2. Proof of Theorem 3.4

The main technical issue in this proof is the loss of differentiability of the renormalization operator (see Theorem 2.7). To get around this problem we have assumed that the family of maps is C^∞, taking a suitable topology for every case.

Let us introduce first some preliminary lemmas on the existence of periodic invariant curves and their reducibility.

Lemma 5.2. *Consider F_0 a map like (8) represented by a pair (ω, f_0) with ω irrational and $f \in C^r(\mathbb{T}\times I_\delta, I_\delta)$. Assume that $F_0 = (\omega, f_0)$ has an invariant curve $z_0 : \mathbb{T} \to I_\delta$ with negative Lyapunov exponent. Then there exists a neighbourhood U of f_0 in $C^r(\mathbb{T}\times I_\delta, I_\delta)$ and a function $z(\omega,\cdot) : U \to C^r(\mathbb{T}, I_\delta)$ such that $z(\omega, f)$ is an invariant curve of $F = (\omega, f)$, for any $f \in U$. This invariant curve is the continuation of z_0 and $z(\omega,\cdot) : U \to C^r(\mathbb{T}, I_\delta)$ is a C^r operator.*

Proof. The function $z(\omega,\cdot) : U \to C^r(\mathbb{T}, I_\delta)$ is obtained applying the Implicit Function Theorem (IFT) to

$$\begin{array}{llcl} J_\omega : & C^r(\mathbb{T} \times I_\delta, I_\delta) \times C^0(\mathbb{T}, I_\delta) & \to & C^0(\mathbb{T}, I_\delta) \\ & (f, z) & \mapsto & [J_\omega(f, z)](\theta) := f(\theta, z(\theta))) - z(\theta + \omega). \end{array}$$

If the IFT is applicable at the point (f_0, z_0), then we will have that there exist a neighbourhood $\tilde{U}$ of f_0 and a function $z(\omega,\cdot) : \tilde{U} \to C^0(\mathbb{T}, I_\delta)$, such that $z(\omega,\cdot)$ is the continuation of z_0. The differentiability (C^r) of J_ω with respect to z follows from the result of Irwing [4] on the smoothness of the composition map and the fact that $z(\theta) \mapsto z(\theta+\omega)$ is a linear bounded operator with respect to u. For any $f \in C^r(\mathbb{T}\times I_\delta, I_\delta)$ and $z, u \in C^0(\mathbb{T}, I_\delta)$ we have that the function $D_z J_\omega(f, z)u \in C^0(\mathbb{T}, I_\delta)$ is given by

$$[D_z J_\omega(f, z)u](\theta) = D_x f(\theta, u(\theta))v(\theta) - v(\theta + \omega).$$

It is immediate to verify that $D_u J_\omega(f, u)$ is a bounded operator. Therefore, the only remaining hypothesis of the Implicit Function Theorem is the existence of the bounded inverse of $D_z J_\omega(f, z)$. In [7], it was shown that this is the case if, and only if, the Lyapunov exponent of the invariant curve is negative. Since this is true by hypothesis, the IFT is applicable at (f_0, z_0).

Also in [7], it is shown that the Lyapunov exponent varies continuously with respect to a map $f \in C^r(\mathbb{T} \times I_\delta, I_\delta)$. Therefore there exist $U \subset \tilde{U}$ an open neighbourhood of f_0, such that the Lyapunov exponent of $z(\omega, f)$ is negative for any $f \in U$. Now, applying Theorem 3.1 in [13] we have that any invariant curve $z(\omega, f)$ with negative Lyapunov exponent is as smooth as the function f. In other words, we have $z(\omega, f) \in C^r(\mathbb{T}, I_\delta)$ for any $f \in U$. □

In order to consider the 2^k-periodic invariant curves of a map F like (8) we need to consider the invariant curves of F^{2^k}. To this aim, let us introduce the operator Q such that $Q^k(F) = F^{2^k}$. Given $f \in C^r(\mathbb{T} \times I_\delta, I_\delta)$ and $\omega \in \mathbb{T}$, consider $q_\omega : C^r(\mathbb{T} \times I_\delta, I_\delta) \to C^r(\mathbb{T} \times I_\delta, I_\delta)$ the operator defined as

$$[q_\omega(f)]\,(\theta, x) := f(\theta + \omega, f(\theta, x)).$$

Then, $Q : \mathbb{T} \times C^r(\mathbb{T} \times I_\delta, I_\delta) \to \mathbb{T} \times C^r(\mathbb{T} \times I_\delta, I_\delta)$ is defined as

$$Q(F) = Q(\omega, f) = (2\omega, q_\omega(f)). \tag{35}$$

Given $F_0 = (\omega_0, f_0)$ a map like (8) with $f_0 \in C^r(\mathbb{T} \times I_\delta, I_\delta)$ and ω irrational. We have that F_0 has a 2^k-periodic invariant curve if, and only if, $Q^k(F_0)$ has an invariant curve.

As it happens with the renormalization operator $\mathcal{T}_\omega$, we have that $q_\omega(f) : C^r(\mathbb{T} \times I_\delta, I_\delta) \to C^r(\mathbb{T} \times I_\delta, I_\delta)$ can only be shown to be continuous. To show that q_ω is a C^s operator, it is necessary to consider q_ω as an operator from $C^{r+s}(\mathbb{T} \times I_\delta, I_\delta)$ to $C^r(\mathbb{T} \times I_\delta, I_\delta)$. The same applies to the operator Q. To solve this problem we take a C^s topology, where s depends on the period 2^k of the periodic invariant curves. This is not a problem because the original family $F_{\alpha,\varepsilon}$ is assumed to be C^∞ and the limit $k \to \infty$ is not taken at any point.

Lemma 5.3. *Let $F_{\alpha,\varepsilon}$ be a two parametric family as in the hypothesis of Theorem 3.4. Let α_k be the parameter value for which g has superattracting periodic orbit of period 2^k. Then, there exist U_k an open neighbourhood of $f(\cdot,\cdot,\alpha_k,0)$ in the $C^{(k+1)r}$ topology and a function $z_k(\omega_0,\cdot) : U_k \to C^r(\mathbb{T}, I_\delta)$ such that $z_k(\omega_0, f)$ is a 2^k periodic invariant curve of $F = (\omega_0, f)$ for any $f \in U_k$. This invariant curve is the continuation of superattracting periodic orbit of period 2^k and $z_k(\omega_0,\cdot)$ is a C^r operator.*

Proof. By definition, a curve z_k is 2^k-periodic invariant by a map $F = (\omega, f)$ if, and only if, z_k is an invariant curve by $F^{2^k} = Q^k(\omega, f)$. For $(\omega_0, f(\cdot,\cdot,\alpha_k,0))$ we have that $z_0 \equiv 0$ is a 2^k-periodic invariant curve. Therefore, $z_0 \equiv 0$ is an invariant curve of $(\omega_k, f_k) := Q^k(\omega_0, f(\cdot,\cdot,\alpha_k,0))$. Moreover, $D_x f(\theta, z_0(\theta), \alpha_k, 0) = 0$ for any $\theta \in \mathbb{T}$, then the Lyapunov exponent of z_0 is $-\infty$. Applying Lemma 5.2, we have that there exist a neighbourhood U_1 of f_k (in the C^r topology) and a function $z_1(\omega_k,\cdot) : U_1 \to C^r(\mathbb{T}, I_\delta)$ which is the continuation of the invariant curve $z_0 \equiv 0$.

Let q^k_ω be the second component of the map $Q^k(\omega,\cdot)$. Using the results on the differentiability of the composition map from [4], it is not difficult to check that $q^k_\omega : C^{(k+1)r}(\mathbb{T} \times I_\delta, I_\delta) \to C^r(\mathbb{T} \times I_\delta, I_\delta)$ is a C^r operator. Then, there exists U_k a sufficiently small neighbourhood of $f(\cdot,\cdot,\alpha_k,0)$ such that $q^k_\omega(U_k) \subset U_1$ and the operator $z_k(\omega_0,\cdot) : U_k \to C^r(\mathbb{T}, I_\delta)$ defined as $z_k(\omega_0, f) = z_1(Q^k(\omega_0, f))$ is a C^r operator. □

At this point let us introduce some lemmas on the function that defines the degree of a periodic invariant curve. Consider $K : \mathbb{T} \times C^{2r}(\mathbb{T} \times I_\delta, I_\delta) \times C^r(\mathbb{T}, I_\delta) \to C^r(\mathbb{T}, I_\delta)$ the operator defined as:

$$[K(\omega, f, z)](\theta) := D_x f(\theta + \omega, f(\theta, z(\theta))) D_x f(\theta, z(\theta)). \tag{36}$$

Lemma 5.4. *Consider U_1 an open subset $C^{2r}(\mathbb{T} \times I_\delta, I_\delta)$ and $z_1(\omega,\cdot) : U_1 \to C^r(\mathbb{T}, I_\delta)$ a C^r operator such that $z_1(\omega, f)$ is a 2-periodic invariant curve with negative Lyapunov exponent for any $f \in U_1$. Let us define*

$$K_1(\omega, f) := K(\omega, f, z_1(\omega, f)). \tag{37}$$

Then, the function $K_1(\omega,\cdot) : U_1 \to C^r(\mathbb{T} \times I_\delta)$ defined by (37) is C^{r-1}.

Moreover, if f_0 is a function such that $f_0(\theta, x) = g(x)$ for some $g \in C^{2r}(I_\delta, I_\delta)$ and this function g has a 2-periodic orbit x_0 such that $g'(x_0) = 0$, then we have that

$$[D_f K_1(\omega, f_0)h](\theta) = \tag{38}$$
$$\frac{\partial g}{\partial x}(g(x_0)) \left(\frac{\partial^2 g}{\partial x^2}(x_0) \left[\frac{\partial g}{\partial x}(g(x_0)) h(\theta - 2\omega, x_0) + h(\theta - \omega, g(x_0)) \right] + \frac{\partial h}{\partial x}(\theta, x_0) \right).$$

Note that the degree of $z_1(\omega, f)$ is determined by the number of zeros of $K_1(\omega, f)$.

Proof. The differentiability of the function K_1 follows using the same arguments as in Lemma 5.3. The function $z_1(\omega, \cdot) : U_1 \to C^r(\mathbb{T}, I_\delta)$ can alternatively be obtained applying the IFT to $\tilde{J}_\omega : C^{2r}(\mathbb{T} \times I_\delta, I_\delta) \times C^0(\mathbb{T}, I_\delta) \to C^0(\mathbb{T}, I_\delta)$ defined by $[\tilde{J}_\omega(f, z_1)](\theta) := f(\theta+\omega, f(\theta, z_1(\theta))) - z_1(\theta + 2\omega)$. As in Lemma 5.2, the IFT applies because z_1 has negative Lyapunov exponent. Then, also from the IFT we have that

$$D_f z_1(\omega, f)h = -(D_z \tilde{J}_\omega(f, z_1(\omega, f))^{-1} \circ (D_f \tilde{J}_\omega(f, z_1(\omega, f)))h. \tag{39}$$

In general, $(D_z \tilde{J}_\omega(f, z_1(\omega, f)))^{-1}$ does not have an explicit form. In the particular case where $f_0(\theta, x) = g(x)$ with $z_1(\theta, f_0) = x_0$ and $g'(x_0) = 0$, it is not difficult to check that $\left(\left[D_z \tilde{J}_\omega(f_0, z_1(\omega, f_0))\right]^{-1} \ell\right)(\theta) = -\ell(\theta - 2\omega)$. On the other hand, it is easy to check that $\left[D_f \tilde{J}_\omega(f_0, z_1(\omega, f_0))h\right](\theta) = \frac{\partial g}{\partial x}(z_1(\omega, f_0)h(\theta, x_0) + h(\theta+\omega, g(z_1(\omega, f_0)))$, then using (39) it follows

$$\left[D_f z_1(\omega, f_0)h\right](\theta) = \frac{\partial g}{\partial x}(g(x_0))h(\theta - 2\omega, x_0) + h(\theta - \omega, g(x_0)). \tag{40}$$

To compute $[D_f K_1(\omega, f)h]$ we can use the chain rule on (37). Then, using again that $f_0(\theta, x) = g(x)$ with $D_x g(z_1(\omega, f_0)) = 0$, it is not hard to see that

$$\left[D_f K_1(\omega, f)h\right](\theta) = \frac{\partial g}{\partial x}(g(x_0)) \left(\frac{\partial^2 g}{\partial x^2}(x_0)\left[D_f z_1(\omega, f_0)h\right](\theta) + \frac{\partial h}{\partial x}(\theta, x_0)\right).$$

Replacing (40) into the equation above follows (38). □

Consider R the (quasi-periodic) renormalization operator as in Definition 2.6 and let $\mathcal{D}(R)$ be its domain of definition. Given a scalar $a \neq 0$ and $f \in C^r(\mathbb{T} \times I_\delta, I_\delta)$ consider the operators

$$\left[l_a(f)\right](\theta, x) := \frac{1}{a} f(\theta, ax), \text{ and } L_a(\omega, f) := (\omega, l_a(f)).$$

From the definition of R it is easy to check that

$$R(\omega, f) = (L_{a(\omega,f)} \circ Q)(\omega, f), \text{ with } a(\omega, f) := \int_0^1 f(\theta, 1)d\theta.$$

Lemma 5.5. *Let $\mathcal{D}^k(R)$ denote the set of maps k-times renormalizable as in Definition 2.6. Given $(\omega, f) \in \mathcal{D}^k(R)$ we have that*

$$R^k(\omega, f) = L_{a\left(R^{k-1}(\omega,f)\right)\cdots a(R(\omega,f))a(\omega,f)} \circ Q^k(\omega, f)$$

where $a\left(R^i(\omega, f)\right) \neq 0$ for any $i = 0, \cdots, k-1$.

Proof. For any $(\omega, f) \in \mathcal{D}^i(R)$ we have that $a(R^{i-1}(\omega, f)) \neq 0$, otherwise $R^i(\omega, f)$ would not be well defined. On the other hand, it is not difficult to check that L_a and Q commute, in other words $L_{a(\omega,f)} \circ Q = Q \circ L_{a(\omega,f)}$. Note also that $L_a \circ L_b = L_{ab}$. Using this properties it is straightforward to check that

$$R^n(\omega, f) = L_{a(R^{n-1}(\omega,f))\cdots a(R(\omega,f))a(\omega,f)} \circ Q^n(\omega, f).$$

□

Lemma 5.6. *Assume that there exist U_1 an open subset of $C^{2r}(\mathbb{T}\times I_\delta, I_\delta)$ and $z_1(\omega,\cdot): U_1 \to C^r(\mathbb{T}, I_\delta)$ such that $z_1(\omega,f)$ is a 2-periodic invariant curve of (ω,f) for any $f\in U_1$. Assume also that there exist a scalar $a\neq 0$ (which can depend on (ω,f)) such that $l_a(f)\in C^{2r}(\mathbb{T}\times I_\delta, I_\delta)$ for any $f\in U_1$.*

Then, the function $\tilde{z}_1(\omega,\cdot): U_1\to C^r(\mathbb{T}, I_\delta)$ defined as

$$[\tilde{z}_1(\omega,f)](\theta) := \frac{1}{a}[z_1(\omega,f)](\theta),$$

is a 2-periodic invariant curve by $L_a(\omega,f)$ for any $f\in U_1$. Moreover, for any $f\in U_1$ we have

$$K(L_a(\omega,f),\tilde{z}_1(\omega,f)) = K_1(\omega,f).$$

Proof. Since $z_1(\omega,f)$ is a 2-periodic invariant curve we have

$$\begin{aligned}[z_1(\omega,f)](\theta+2\omega) &= f\left(\theta+\omega, f(\theta,[z_1(\omega,f)](\theta))\right)\\ &= a\,\frac{1}{a}f\left(\theta+\omega, a\,\frac{1}{a}f\left(\theta, a\,\frac{1}{a}[z_1(\omega,f)](\theta)\right)\right),\end{aligned}$$

for any $\theta\in\mathbb{T}$. Dividing both sides by a it follows that $\tilde{z}_1(\omega,f)$ is a 2-periodic invariant curve of $L_a(\omega,f)$.

Let us define $\tilde{f}(\theta,x)=\frac{1}{a}f(\theta,ax)$. Then it follows easily that

$$D_x\tilde{f}(\theta,x) = D_x f(\theta,ax).$$

Using this, we have

$$\begin{aligned}[K(\omega,\tilde{f},\tilde{z}_1(\omega,f))](\theta) &= D_x\tilde{f}\left(\theta+\omega,\tilde{f}(\theta,[\tilde{z}_1(\omega,f)](\theta))\right) D_x\tilde{f}\left(\theta,[\tilde{z}_1(\omega,f)](\theta)\right)\\ &= D_x f\left(\theta+\omega, a\,\frac{1}{a}f\left(\theta, a\,\frac{1}{a}[z_1(\omega,f)](\theta)\right)\right) D_x\tilde{f}\left(\theta, a\,\frac{1}{a}[z_1(\omega,f)](\theta)\right)\\ &= [K(\omega,f,z_1(\omega,f))](\theta),\end{aligned}$$

for any $\theta\in\mathbb{T}$.

□

Proof of Theorem 3.4. Let us show first that the existence of the functions $\alpha_n^{\pm}(\varepsilon)$ follows from [6] (Theorem 2.1). Let f_n denote the function such that $F^{2^n}(\theta,x) = (\theta+2^n\omega, f_n(\theta,\omega))$. Using the differentiability of F it follows that f_n is also of the form $f_n(\theta,x,\alpha,\varepsilon) = g_n(x,\alpha)+\varepsilon h_n(\theta,x,\alpha,\varepsilon)$, with $g_n(x)=g^{2^n}(x)$ and $h_n(\theta,x,\alpha,\varepsilon) = \frac{\partial f_n}{\partial\varepsilon}(\theta,x,\alpha,0)+O(\varepsilon)$. For each parameter value α_n the hypothesis of h being an admissible quasi-periodic perturbation, ensures that the function

$$H_n(\theta) = h_n(\theta-2^n\omega,x_0,\alpha_n,0)\frac{\partial^2 g_n}{\partial x^2}(x_0,\alpha_n)+\frac{\partial h_n}{\partial x}(\theta,x_0,\alpha_n,0)$$

has a unique non-degenerate maximum and minimum. Concretely $H_n'(\theta)$ has at least two simple zeros (one for the maximum and one for the minimum), therefore Theorem 2.1 in [6] is applicable to F^{2^n} at the parameter value $(\alpha,\varepsilon)=(\alpha_n,0)$. Then, for each of these two zeros there exist a curve of change of degree 0 to degree 2. Denote by $(\alpha_n^+(\varepsilon),\varepsilon)$ the one associated to the maximum and $(\alpha_n^-(\varepsilon),\varepsilon)$ the one associated to the minimum. Since the family $F_{\alpha,\varepsilon}$ is C^∞ and ω is Diophantine, we have that this curves of change of degree correspond to reducibility loss bifurcations.

Now it is left to prove that the slope of the curve is given by (17). When $n = 1$, the result could be proved applying the second part of Theorem 2.1 in [6] to the map F^2. Instead, we use an alternative argument which is a bit more complicated for the case $n = 1$, but that can be generalized to $n > 1$. We consider these two cases ($n = 1$ and $n > 1$) separately.

Case $n = 1$. Consider z_1 the 2-periodic invariant curve given by Lemma 5.3 and U_1 the open neighbourhood of $f_0(\cdot,\cdot,\alpha_1,0)$ where it is defined. Given $f \in U_1$, the degree of $z_1(\omega, f)$ is determined by the number of zeros of the function $K_1(\omega, f)$ given by (37). Concretely we have a change of degree when the number of zeros of $K_1(\omega, f)$ changes. Any 2-periodic invariant curve of (ω, f) with $\max_{\theta\in\mathbb{T}} [K_1(\omega, f)](\theta) < 0$ or $\min_{\theta\in\mathbb{T}} [K_1(\omega, f)](\theta) > 0$ will have degree 0. When $\max_{\theta\in\mathbb{T}} [K_1(\omega, f)](\theta) = 0$ (resp. $\min_{\theta\in\mathbb{T}} [K_1(\omega, f)](\theta) = 0$) we have that, except for degenerate cases, the degree of the 2-periodic invariant curve changes from 0 to 2. Then, the sets

$$\Sigma_1^+(\omega) = \left\{ f \in U_1 \;\middle|\; K^+(\omega, f) := \max_{\theta\in\mathbb{T}} [K_1(\omega, f)](\theta) = 0 \right\},$$

and

$$\Sigma_1^-(\omega) = \left\{ f \in U_1 \;\middle|\; K^-(\omega, f) := \min_{\theta\in\mathbb{T}} [K_1(\omega, f)](\theta) = 0 \right\}.$$

determine the boundary between curves with reducibility degree 0 and 2.

Given a two parameter family $\{F_{\alpha,\varepsilon}\}$ like (15) we have that $\{F_{\alpha_1^+(\varepsilon),\varepsilon}\} = \Sigma_1^+(\omega) \cap \{F_{\alpha,\varepsilon}\}$ and $\{F_{\alpha_1^-(\varepsilon),\varepsilon}\} = \Sigma_1^-(\omega) \cap \{F_{\alpha,\varepsilon}\}$. The degenerate cases are avoided due to the hypothesis of h being an admissible quasi-periodic perturbation.

At this point let us expand $K_1(\omega, f_0(\cdot,\cdot,\alpha,\varepsilon))$ in terms of (α,ε) around $(\alpha_1, 0)$. Consider the Taylor expansion of $f_0(\cdot,\cdot,\alpha,\varepsilon)$ at $\varepsilon = 0$:

$$f_0(\cdot,\cdot,\alpha,\varepsilon) = f_0(\cdot,\cdot,\alpha,0) + \frac{\partial f_0}{\partial\varepsilon}(\cdot,\cdot,\alpha,0)\varepsilon + O(\varepsilon^2)$$

Recall that $f_0(\cdot,\cdot,\alpha,0)$ does not depend on the periodic variable θ, in other words $f_0(\theta,x,\alpha,0) = g_0(x,\alpha)$. Then, its two periodic invariant curve $z(\cdot,\alpha,0)$ does not depend on the periodic variable θ either. This means that $K_1(\omega, f_0(\cdot,\cdot,\alpha,0))$ is a scalar value completely independent of θ. Hence,

$$\begin{aligned}[K_1(\omega, f_0(\cdot,\cdot,\alpha,\varepsilon))](\theta) = K_1(\omega, f_0(\cdot,\cdot,\alpha,0)) + \left[D_f K_1(\omega, f_0(\cdot,\cdot,\alpha,0)) \frac{\partial f_0}{\partial\varepsilon}(\cdot,\cdot,\alpha,0) \right](\theta)\,\varepsilon \\ + O(\varepsilon^2),\end{aligned}$$

for any $\varepsilon \geq 0$.

Using the differentiability of K_1 (Lemma 5.4) and that $\alpha(\varepsilon) = \alpha_1 + \beta_1\varepsilon + O(\varepsilon^2)$, we can expand $K_1(\omega, f_0(\cdot,\cdot,\alpha,\varepsilon))$ at $\varepsilon = 0$,

$$[K_1(\omega, f_0(\cdot,\cdot,\alpha,\varepsilon))](\theta) = K_1(\omega, f_0(\cdot,\cdot,\alpha_1,0)) + D_f K_1(\omega, f_0(\cdot,\cdot,\alpha_1,0)) \frac{\partial f_0}{\partial\alpha}(\cdot,\cdot,\alpha_1,0)\beta_1\varepsilon$$

$$+\left[D_f K_1(\omega, f_0(\cdot,\cdot,\alpha_1,0))\frac{\partial f_0}{\partial \varepsilon}(\cdot,\cdot,\alpha_1,0)\right](\theta)\,\varepsilon + O(\varepsilon^2). \tag{41}$$

Recall that $\{F_{\alpha_1^\pm(\varepsilon),\varepsilon}\} = \Sigma_1^\pm(\omega)\cap\{F_{\alpha,\varepsilon}\}$, then

$$\max_{\theta\in\mathbb{T}} K_1(\omega, f_0(\cdot,\cdot,\alpha_1^+(\varepsilon),\varepsilon))(\theta) = 0, \quad \min_{\theta\in\mathbb{T}} K_1(\omega, f_0(\cdot,\cdot,\alpha_1^-(\varepsilon),\varepsilon))(\theta) = 0,$$

for any $0 \le \varepsilon \le \varepsilon_0$, with ε_0 sufficiently small. Assume that $\alpha_1^\pm(\varepsilon) = \alpha_1 + \beta_1^\pm\varepsilon + O(\varepsilon^2)$. Using the expansion of $K_1(\omega, f_0(\cdot,\cdot,\alpha,\varepsilon))$ given by (41) it follows

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\alpha_1^+(0^+) = \beta_1^+ = -\frac{\max_{\theta\in\mathbb{T}}\left(\left[D_f K_1(\omega, f_0(\cdot,\cdot,\alpha_1,0))\frac{\partial f_0}{\partial\varepsilon}(\cdot,\cdot,\alpha_1,0)\right](\theta)\right)}{D_f K_1(\omega, f_0(\cdot,\cdot,\alpha_1,0))\frac{\partial f_0}{\partial\alpha}(\cdot,\cdot,\alpha_1,0)}. \tag{42}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\alpha_1^-(0^+) = \beta_1^- = -\frac{\min_{\theta\in\mathbb{T}}\left(\left[D_f K_1(\omega, f_0(\cdot,\cdot,\alpha_1,0))\frac{\partial f_0}{\partial\varepsilon}(\cdot,\cdot,\alpha_1,0)\right](\theta)\right)}{D_f K_1(\omega, f_0(\cdot,\cdot,\alpha_1,0))\frac{\partial f_0}{\partial\alpha}(\cdot,\cdot,\alpha_1,0)}. \tag{43}$$

If we replace the value of $D_f K_1(\omega, f_0(\cdot,\cdot,\alpha_1,0))$ given by (38) and compare with the function G given by (18) then we obtain that $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\alpha_1^\pm(0^+)$ is given by (17).

Case $n>1$. Consider z_n the 2^n-periodic invariant curve given by Lemma 5.3 and U_n the open neighbourhood of $f(\cdot,\cdot,\alpha_n,0)$ where it is defined. This open neighborhood is considered in the $C^{r(n+1)}$ topology, with $r\ge 3$. Given $F=(\omega_0,f_0)$ with $f_0\in U_n$, let (ω_n,f_n) be the couple such that $(\omega_n,f_n)=F^{2^n}$. The degree of z_n is determined by the number of zeros of the function $D_x f_n(\theta, z_n(\theta))$.

Let Q be the self composition operator defined in (35), then $(\omega_n,f_n) = F^{2^n} = Q^n(F)$. It is not difficult to check that

$$\begin{aligned} D_x f_n(\theta, z_n(\theta)) &= D_x f_{n-1}(\theta+\omega_{n-1}, f_{n-1}(\theta), z_n(\theta)) D_x f_{n-1}(\theta, z_n(\theta)) \\ &= \left[K(Q^{n-1}(\omega_0,f_0), z_n(\omega_0,f_0))\right](\theta), \end{aligned}$$

where K is the function given by (36). Note that, for any $f_0\in U_n$, $z_n(\omega_0,f_0)$ is a 2^k-periodic invariant curve of (ω_0,f_0) if, an only if, $z_n(\omega_0,f_0)$ is a 2-periodic invariant curve of $Q^{n-1}(\omega_0,f_0)$. In other words, $z_n(\omega_0,f_0)$ can be understood as $z_1(Q^{n-1}(\omega_0,f_0))$, with $z_1(\omega_{n-1},\cdot)$ defined in a neighbourhood U_1 of $Q^{n-1}(\omega_0,f_0)$. Consider K_1 as in Lemma 5.4, then

$$\begin{aligned} D_x f_n(\theta, z_n(\theta)) &= \left[K(Q^{n-1}(\omega_0,f_0), z_n(\omega_0,f_0))\right](\theta) \\ &= \left[K(Q^{n-1}(\omega_0,f_0), z_1(Q^{n-1}(\omega_0,f_0)))\right](\theta) \\ &= \left[K_1(Q^{n-1}(\omega_0,f_0))\right](\theta). \end{aligned} \tag{44}$$

By hypothesis we have that g_α (the one dimensional family that defines $F_{\alpha,0}$) has a complete cascade of superattracting periodic orbits. This implies that $F_{\alpha_n,0}\in\mathcal{D}^{n-1}(R)$ for any $n\ge 1$. Using that h is an admissible quasi-periodic perturbation (see Definition 3.2) we have that $F_{\alpha,\varepsilon}\subset\mathcal{D}^{n-1}(R)$ for any (α,ε) close to $(\alpha_n,0)$. Using Lemma 5.5 we have that

$$R^k(\omega_0,f_0) = L_{a\left(R^{k-1}(\omega_0,f_0)\right)\cdots a(R(\omega_0,f_0))a(\omega_0,f_0)}\circ Q^k(\omega_0,f_0),$$

for any $(\omega_0, f_0) \in \mathcal{D}^{n-1}(R)$ with $f_0 \in U_n$. Using Lemma 5.6 we have that $K_1(R^{n-1}(\omega, f)) = K_1(Q^{n-1}(\omega, f))$, therefore

$$D_x f_n(\theta, z_n(\theta)) = \left[K_1(R^{n-1}(\omega_0, f_0))\right](\theta).$$

Concretely, we have a change of degree when the number of zeros of $K_1(R^{n-1}(w_0, f_0))$ changes. Then the sets

$$\Sigma_n^+(\omega) = \left\{ f \in U_n \,\middle|\, \max_{\theta\in\mathbb{T}} \left[K_1(R^{n-1}(\omega, f))\right](\theta) = 0 \right\},$$

and

$$\Sigma_n^-(\omega) = \left\{ f \in U_n \,\middle|\, \min_{\theta\in\mathbb{T}} \left[K_1(R^{n-1}(\omega, f))\right](\theta) = 0 \right\}$$

determine the boundary between curves with reducibility degree 0 and degree 2.

Given a two parametric $F_{\alpha,\varepsilon}$ like (15) we have that $\{F_{\alpha_n^+(\varepsilon),\varepsilon}\} = \Sigma_n^+(\omega) \cap \{F_{\alpha,\varepsilon}\}$ and $\{F_{\alpha_n^-(\varepsilon),\varepsilon}\} = \Sigma_n^-(\omega) \cap \{F_{\alpha,\varepsilon}\}$. The problem can be reduced now to the case $n = 1$ by taking $\tilde{F}_{\alpha,\varepsilon} = R^{n-1}(F_{\alpha,\varepsilon})$ at the parameter value $(\alpha_n, 0)$. The differential of $\tilde{F}_{\alpha,\varepsilon}$ with respect to α and ε can be obtained through the tangent map S, given by (12). When this differential is replaced into (42) and (43) the directions given by (17) are obtained. □

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