Non-integrability of Hamiltonian systems through high order variational equations: Summary of results and examples

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Remembering Jürgen Moser 80 years after his birth

Abstract

This paper deals with non-integrability criteria, based on differential Galois theory and requiring the use of higher order variational equations. A general methodology is presented to deal with these problems. We display a family of Hamiltonian systems which require the use of order k variational equations, for arbitrary values of k, to prove non-integrability. Moreover, using third order variational equations we prove the non-integrability of a non-linear spring-pendulum problem for the values of the parameter that can not be decided using first order variational equations.

1 Introduction

Hamiltonian systems appear in multiple models of the sciences. They satisfy equations of the form

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q},$$

where H is assumed to be real analytic on some domain Ω of $\mathbb{R}^{2n}$. We consider the extension to a complex domain $\hat{\Omega}$ of $\mathbb{C}^{2n}$.

If $x = \{q, p\} \in \mathbb{C}^{2n}$ we consider solutions $x(t)$ with $t \in \hat{D} \subset \mathbb{C}$. The image of $\hat{D}$ by x is a Riemann surface Γ.

We shall consider integrability in the Liouville-Arnol'd sense:

There exist n first integrals $f_1, f_2, \ldots, f_n$ independent almost everywhere and in involution. Usually it is taken $f_1 = H$. In general the functions $f_1, f_2, \ldots, f_n$ will be considered *meromorphic* in a neighbourhood of a given solution $x(t)$.

The standing problem is to find *necessary conditions for integrability*, or, equivalently, *sufficient conditions for non-integrability.*

Integrable Hamiltonian systems have, in some sense, well ordered dynamics, while non-integrable ones are associated to some amount of *chaos*. Eventually the chaotic dynamics can be confined to the complex phase space without showing up in the real one. A chaotic behaviour implies lack of predictability, i.e., a sensitive dependence to initial conditions.

Typical Hamiltonian systems are non-integrable. To check non-integrability for a concrete Hamiltonian one can appeal to numerical techniques, which can be made rigorous by using Computed Assisted Proofs (CAP). A classical example is the Hénon-Heiles family, HHF, a family of 1–1 resonant systems at the origin

$$H = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2) + \frac{1}{3}q_1^3 + bq_1q_2^2. \tag{1}$$

The pioneer example [5] appears for $b = -1$. Figure 1 shows a Poincaré section of this problem on the level $h = 0.1$ with $q_2 = 0, p_2 > 0$. One can easily guess the presence of three hyperbolic periodic orbits. The heteroclinic connections between them split and create chaotic zones. There are also four elliptic periodic orbits. Furthermore, in these variables, the boundary is also an elliptic periodic orbit. Identifying all the points at the boundary we have $\mathbb{S}^2$ as Poincaré section, with 5 fixed points of index $+1$ and 3 of index -1.

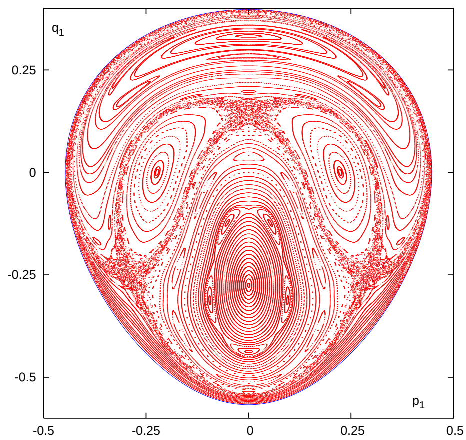

Figure 1: Poincaré section of (1) for $b = -1$ on $h = 0.1$.

Using the first order variational equations (see Section 2) Ito proved that for all except *four* of the values of b, HHF is non-integrable, see [6]. *Three* of the remaining cases are *trivially integrable*. The fourth case $b = 1/2$, that we denote as *degenerate Hénon-Heiles*, DHH, has been proved recently to be non-integrable, see [18].

In Section 2 we remind several non-integrability criteria, both in the perturbative case for $H(q, p) = H_0(p) + \varepsilon H_1(q, p)$ and in the general case. Some of these criteria are related to exponentially small phenomena. Then we present criteria based on differential Galois theory, including those which require the use of higher order variational equations. In Section 3 a general methodology to apply these criteria is introduced and justified. In problems in which some "reference solution" of the Hamilton equations is known explicitly, as in the cases presented in that paper, the methods described allow to decide, in a pure analytic way, about the lack of integrability.

After sketching, for completeness and as a preliminary example, the proof of the non-integrability of the DHH in Section 4, as presented in [18], we provide in Section 5 examples which require the use of order k variational equations, for arbitrary values of k, along simple solutions $x(t)$. We denote these examples as *generalized* DHH or GDHH.

It is relevant to stress here that the fact that for a given system, around a given simple enough solution, requires higher order variational to prove non-integrability is, in general, independent of the fact that the system could display large chaotic regions in numerical simulations. The "simple enough" solutions can be more degenerate than generic solutions. As an example, the GDHH systems show large chaotic regions in suitable domains.

The integrability of an interesting example of non-linear spring-pendulum, SP, model was studied in [10]. The model depends on two parameters k and a and when $a = -k$ the analysis could not lead to any conclusion. Using a suitable complex path and third order variational equations, non-integrability is proved in Section 6. Besides standard computations concerning the solutions of the involved equations, our proof requires a kind of blow up for one of the singularities in the first variational equations. A suitable scaling of the system, depending on k leads to a limit system which is parameter free. This limit system is also shown to be non-integrable.

Another similar system concerns the Swinging Atwood's Machine, SAM. The system depends, essentially, on one parameter, a mass ratio, and has been proved to be non-integrable except for a discrete set of values. Again using a suitable path and third order variational equations, non-integrability has been proved. We refer to [13] for the details.

A first difficulty to apply the techniques presented in Section 3 is the need to use a solution $x(t)$ which is explicitly known. A second difficulty is the choice of a suitable path for the complex time. A final difficulty is the possibility to integrate first and higher order variational equations in a simple and efficient way.

This leads, in a natural way, to the question of the numerical check of the necessary conditions for integrability along arbitrary paths γ of $t \in \mathbb{C}$. This approach has been taken in [14], where a general method, based on Taylor expansions both in time and in nearby initial conditions, is presented. The integration of arbitrary higher order variational equations or, equivalently, the transport of jets to any order along arbitrary paths is easily automatized.

The criteria and methodology presented in sections 2 and 3 can be applied to systems with an arbitrary (finite) number of degrees of freedom. However, for simplicity and to be able to have simple reference solutions, we confine our interest here to *systems with two degrees of freedom and having an invariant plane.*

Readers familiarised with integrability criteria and differential Galois theory can skip next Section.

Professor Jürgen Moser largely contributed, with key and beautiful papers, to the study of integrability and non-integrability. He was mainly using analytical and geometric tools to this end. Present paper deals with tools which, initially, have a more algebraic flavor. Hopefully different approaches will merge in the near future and will help understanding the dynamics.

2 Integrability criteria

The interest in deciding about the integrability of a Hamiltonian system has lead to the search of different kind of criteria.

2.1 Some criteria

Different methods are used for parametric families of systems, mainly based on the use of *first order variational equations*, VE, along $x(t)$, like Ziglin's method [24] and the so-called Melnikov methods. The basic idea of the last approach is to try to detect the *splitting of separatrices* of unstable periodic orbits, i.e., of hyperbolic fixed points on a suitable Poincaré map, as presented, e.g., in [20], either in the homoclinic or the heteroclinic case.

Some difficulties can appear when justifying Melnikov approach for perturbations of integrable systems $H(q,p) = H_0(p) + \varepsilon H_1(q,p)$. Indeed, if the unstable periodic orbit is created by the perturbation, then the dominant eigenvalues are of the form $1 + \mathcal{O}(\varepsilon)$. The related integrals are of the form $A\varepsilon^r \exp(-c/\varepsilon^s)$ for some positive values of c and s, but, in principle, the remainder can be $\mathcal{O}(\varepsilon^{r+1})$. One has to justify that the remainder contains also similar exponentially small factors.

This has been achieved in different cases. In particular the problem is related to the splitting of separatrices for area preserving maps APM. It is worth to mention here the seminal paper of Lazutkin in 1984 (see [7]) for the Chirikov's standard map, completed to a full proof in [3]. In [4] a variety of cases is examined numerically with high-precision computations. The presence of exponentially small splitting of separatrices is generic in problems in which hyperbolic periodic points appear as a perturbation of the identity, see [2]. For the splitting of separatrices which appear in resonant zones around an elliptic fixed point we refer to [23].

A different situation can appear in the *two-parameter case*. Assume the system depends on two small parameters ε, α. Then it can happen that the splitting is of the form $A\alpha\varepsilon^r \exp(-c/\varepsilon^s) + \mathcal{O}(\alpha^2\varepsilon^{r+1})$ and the first term dominates if α is sufficiently small ($\alpha << \exp(-c/\varepsilon^s)$). This approach was used for some problems in Celestial Mechanics [8, 9, 11, 12], where the first order variational equations were used directly. The role of ε was played by a mass ratio μ and the one of α by the Jacobi constant or the inverse of the semi-major axis. But it can lead to unrealistic bounds on the required size of α.

Ziglin's method uses monodromy matrices, solutions of first order VE along closed paths in Γ based on a regular point. Non-degeneracy conditions are required for the monodromy matrices (e.g. non-resonant eigenvalues). The results using Ziglin's method can be recovered using the approach of Section 2.2. This alternative approach is based on differential Galois theory and it is presented in next subsections. It is widely used and can be applied both to perturbative and *non-perturbative problems*, like in [19]. However, it provides no information if the necessary integrability conditions are satisfied.

2.2 The Morales-Ramis theory

The results summarized here are contained in [16, 17]. See also [15] for all the necessary background and technical details.

Consider the m-dimensional ODE $\dot{x} = f(x(t))$ and let $x(t)$ be a solution. The first VE along $x(t)$ is $\frac{d}{dt}A = Df(x(t))A$ and we consider the initial condition $A(t_0) = Id$, where $x_0 = x(t_0)$ is a regular point of f. If we consider closed paths on the Riemann surface Γ with base point x_0, one can associate to each path the corresponding monodromy matrix. The set of all these matrices form the monodromy group.

More generally, we can consider any linear ODE

$$\frac{d}{dt}A(t) = B(t)A(t). \tag{2}$$

We assume that the entries of B belong to some field of functions K. Let $\xi_{i,j}$ be the elements of a fundamental matrix of (2). Let L be the *extension* $K(\xi_{1,1}, \xi_{1,2}, \ldots, \xi_{m,m})$, trivially a differential field. Consider the *Galois group* G =Gal$(L \mid K)$, which is an algebraic group. Then the following result is obtained.

Theorem 1. *(Morales-Ramis) Under the assumptions above, if a Hamiltonian is integrable in a neighbourhood of* Γ *then the identity component* G^0 *of the Galois group of the first order VE along* Γ *is commutative.*

The identity component is taken using Zariski's topology. We also recall that the Galois group coincides with the Zariski closure of the monodromy group.

A delicate example of application of Theorem 1 can be seen in [19]. See also [18] for a long, but not exhaustive, list of examples where this Theorem has been used to detect non-integrability.

If G^0 is commutative there is nothing against integrability. This suggests to try to detect non-integrability at *higher order*.

2.3 Using higher order variational equations

There are Hamiltonian systems in which none of the previous methods gives a proof of non-integrability, even if there is a strong numerical evidence (e.g. by computing Poincaré sections, Lyapunov exponents, by frequency analysis, splitting of separatrices, etc). Methods based on *higher order variational equations* have been introduced recently.

Let $x(t) = \varphi(t, x_0)$ be the solution of $\dot{x} = f(x)$ with $\varphi(0, x_0) = x_0$. We consider as *fundamental* solutions of the k-th order VE, VE_k, based on x_0, the string of maps $(\varphi^{(1)}(t), \varphi^{(2)}(t), \dots, \varphi^{(k)}(t))$ such that

$$\varphi(t, y_0) = \varphi(t, x_0) + \varphi^{(1)}(t)(y_0 - x_0) + \ldots + \varphi^{(k)}(t)(y_0 - x_0)^k + \ldots,$$

i.e., the coefficients of the k-jet. Obviously $\varphi^{(1)}(t)$ is a solution of the first order VE=VE_1.

The $\varphi^{(k)}(t)$ satisfy linear non-homogeneous ODE, e.g.

$$\frac{d}{dt}\varphi^{(2)}(t) = Df(x(t))\varphi^{(2)}(t) + D^2 f(x(t))(\varphi^{(1)}(t))^2,$$

$$\frac{d}{dt}\varphi^{(3)}(t) = Df(x(t))\varphi^{(3)}(t) + 2D^2 f(x(t))(\varphi^{(2)}(t), \varphi^{(1)}(t)) + D^3 f(x(t))(\varphi^{(1)}(t))^3$$

with initial conditions $\varphi^{(2)}(t_0) = 0$, $\varphi^{(3)}(t_0) = 0$. See [18] for more explicit versions in terms of components. For further use we introduce the notation x_i, $x_{i;k}$, $x_{i;k_1,k_2}$, $x_{i;k_1,k_2,k_3}, \ldots$ for the components of x and the first, second, third, ... derivatives with respect to the initial conditions, that is, the components of $\varphi(t), \varphi^{(1)}(t), \varphi^{(2)}(t), \varphi^{(3)}(t), \ldots$, except by the presence of factorials.

Note that when $\varphi^{(1)}$ is available, all $\varphi^{(k)}$ are obtained by quadratures.

The equation for $\varphi^{(k)}$, $k > 1$ depends in a non-linear way on $\varphi^{(j)}$ for $j < k$, but, for any k, the equations for the entries of the $\varphi^{(j)}$ can be made linear by introducing additional variables (products of entries) which also satisfy linear ODE (see again [18] for details).

Hence, one can introduce the *k-th order Galois group* G_k as the Galois group associated to the linearized version of the variational equations up to order k. We can also introduce the k-th order monodromy as the monodromy obtained with the linearized version of the VE. The information it gives is equivalent to the information obtained by transporting the jet up to order k. That is, starting at the point $x_0 + \xi$ at time t_0 one has

$$\varphi(t; t_0, x_0 + \xi) = \sum_{0 \le |n| \le k} a_n(t)\xi^n + \mathcal{O}(|\xi|^{k+1}), \tag{3}$$

where n is a multiindex. The jet $\sum_{0\le|n|\le k} a_n(t_0)\xi^n$ when we return to t_0 moving along a path γ can be seen as the k-th order monodromy along γ, to be denoted as M_k^γ. The composition of elements in M_k^γ as a group is equivalent to the composition of jets.

Then, for any $k \ge 1$ the following extension of Theorem 1 holds:

Theorem 2. *([18]) Under the assumptions above, if the Hamiltonian is integrable then for any $k \ge 1$ the identity component $(G_k)^0$ of G_k is commutative.*

This result gives rise to *non-integrability criteria to all orders*. Note that these criteria can depend strongly on the reference solution $x(t)$ and on the paths taken on it. See [18] for details.

An obvious standing question (see [18] for other open problems) is the following. Assume $(G_k)^0$ is commutative for all k. Under which *additional conditions* one has that the Hamiltonian is integrable? It is known, see [18], that there exist systems and special solutions $x(t)$ for which all the $(G_k)^0$ are commutative despite there is evidence that the system is non-integrable.

The main purpose of present paper is to set up a methodology to check the necessary conditions and then to apply Theorem 2 to different examples.

3 A general methodology to test the non-integrability criteria

To decide that a system is non-integrable applying Theorem 2 it is enough to find a couple of closed paths ψ_1 and ψ_2, in the Riemann surface Γ, such that there exists a $k \ge 1$ satisfying

- The k-th order monodromies along them, $M_k^{\psi_1}$ and $M_k^{\psi_2}$, are in $(G_k)^0$.
- The commutator
$$[M_k^{\psi_1}, M_k^{\psi_2}] := M_k^{\psi_2^{-1}} \circ M_k^{\psi_1^{-1}} \circ M_k^{\psi_2} \circ M_k^{\psi_1}, \tag{4}$$
is not trivial, that is, different from the identity.

Typically one should resort to $G_k, k > 1$ when Theorem 1 gives no information. This means that it has been possible to find two closed paths ψ_1, ψ_2 in Γ such that $M_1^{\psi_1}, M_1^{\psi_2} \in (G_1)^0$ and they commute. For completeness we give some simple cases in which $M_1^{\psi_1}, M_1^{\psi_2} \in (G_1)^0$.

Assume that the Hamiltonian system has two degrees of freedom and an invariant plane Π. For concreteness we consider Π to be the (q_1, p_1)-plane. As the system in Π is integrable no obstruction can be found related only to the (q_1, p_1) variables. In particular the VE_1 uncouple into the "tangential" part TVE_1 in the (q_1, p_1) variables and the "normal" part NVE_1 in the (q_2, p_2) variables.

Lemma 1. *Under the above conditions let ψ_1, ψ_2 be two non-trivial paths in Γ, the Riemann surface corresponding to a solution $x(t)$ in Π. Hence the normal parts of the monodromy matrices along these paths are in $SL(2, \mathbb{C})$. Then, sufficient conditions so that $M_1^{\psi_1}$ and $M_1^{\psi_2}$ belong to $(G_1)^0$ are:*

i) The normal parts of the matrices $M_1^{\psi_1}, M_1^{\psi_2}$ are unipotent and they commute. The simplest case appears when both parts of these matrices are the identity.

ii) At least one of the two normal parts of the matrices $M_1^{\psi_1}$ and $M_1^{\psi_2}$ is non-resonant, that is, the eigenvalues are not roots of the unity, and the matrices commute.

The proof follows immediately from the classification of the algebraic subgroups of $SL(2,\mathbb{C})$. See Proposition 2.2 in [15]. An extension of this result to more degrees of freedom requires to know the algebraic subgroups of $SP(2n,\mathbb{C})$ and the corresponding identity components. This is not available in general, as far as the authors know. But in some simple cases, like $M_1^{\psi_j}, j=1,2$ being the identity or simple enough, they are in $(G_1)^0$. Furthermore, if one can not decide the non-integrability from Theorem 1 using a couple of paths ψ_1, ψ_2, then one can go the $\mathrm{VE}_k, k > 1$ using the same paths and Theorem 2.

Assume we have $M_1^{\psi_1}, M_1^{\psi_2} \in (G_1)^0$ and they commute. Next Lemma shows that to get obstructions to the integrability it is sufficient to check that for some $k > 1$ the monodromies $M_k^{\psi_1}$ and $M_k^{\psi_2}$ do not commute.

Lemma 2. *Assume $M_1^{\psi_1}, M_1^{\psi_2} \in (G_1)^0$ for two closed paths ψ_1, ψ_2 in Γ and they commute. Let us assume that there exists $k > 1$ such that $M_k^{\psi_1}$ and $M_k^{\psi_2}$ do not commute. Then the Hamiltonian is non-integrable in a neighbourhood of Γ.*

Proof. Let us consider a Riemann surface $\Gamma' \subset \Gamma$ obtained by deleting from Γ a closed set K containing all the singularities and leaving only in Γ' a small vicinity of $\psi_1 \cup \psi_2$.

By construction, the monodromy on Γ', $\mathcal{M}^{\Gamma'}$, is generated by $M_1^{\psi_1}$ and $M_1^{\psi_2}$ which are in the connected subgroup $(G_1)^0$. Then the Galois group $G_1^{\Gamma'}$ is connected. Also $G_k^{\Gamma'}$ is connected, see Corollary 8.1 in [15]. Then the component $(G_k^{\Gamma'})^0$ coincides with $G_k^{\Gamma'}$.

Applying Theorem 2 it follows that the Hamiltonian is non-integrable in a neighbourhood of Γ'. Hence, it is non-integrable in a neighbourhood of Γ. $\square$

In order to apply the above results one has to select, first, a couple of closed paths in a suitable Riemann surface Γ. Good candidates to be used as paths ψ_1, ψ_2 appear when the system has a real invariant plane and a separatrix $\gamma(t)$ on it, homoclinic to a hyperbolic fixed point. Close to the separatrix, in nearby levels of the energy there are periodic orbits which give rise to a closed path ψ_1. Beyond the real period it can happen that the solutions near $\gamma(t)$ have also a complex period giving rise to a closed path ψ_2. This is the case in classical systems, the Hamiltonian being of the form $H(q,p) = \frac{1}{2}(p,p) + U(q)$, if $\gamma(t)$ is homoclinic to a hyperbolic fixed point P. Then the changes $q = v, p = \mathrm{i}w, t = -\mathrm{i}s$ lead to a Hamiltonian $K(v,w) = \frac{1}{2}(w,w) - U(v)$ with respect to the new time s. The point P becomes elliptic for the new Hamiltonian, and it is surrounded by periodic orbits which correspond to periodic orbits of imaginary period of the Hamiltonian H.

In many typical examples the path $\psi_2^{-1} \circ \psi_1^{-1} \circ \psi_2 \circ \psi_1$ can be deformed in Γ, without passing through any singularity, until we obtain a path which has arcs close to several singularities. Then we are faced to a part of the analysis which can be done locally around these singularities and another, more global part, which involves the passage from the vicinity of a singularity to the vicinity of another one.

We should also remark that there is freedom in the choice of the level of energy h of the solution $x(t)$ that we consider. If the commutator at some order k is different from the identity for some level of energy h_0 then, due to the analytical dependence on h, it is also different from the identity for nearby, suitably chosen, values of h. We recall that, when the solutions of the first variational equations are available then the solutions of the $\mathrm{VE}_k, k > 1$ are obtained by quadratures along regular paths. In particular, in the case of classical Hamiltonians as before, the computations can be done along the separatrix and be used to prove non-integrability in a neighbourhood of a nearby periodic orbit.

It is also important to stress the following fact. Assume the Hamiltonian has an invariant plane Π and $x(t)$ is contained in Π. The possible lack of integrability should show up when we consider effects in the directions "normal" to Π. However, the part of VE_k in the normal directions depends also on the behaviour in the "tangent" directions. This can be checked in the examples studied in Section 6 and in [13], where one can see that the differential equations for variables like $x_{i;k_1,k_2}, x_{i;k_1,k_2,k_3}$, even if all the indices $i, k_1, k_2, k_3, \ldots$ correspond to normal variables, involve variables in Π.

Note that singularities can appear in the Riemann surface Γ for some values of t for the solution $x(t)$ itself but additional singularities can appear for the VE along $x(t)$. They must be taken also into account. An example of this situation will be given in Section 6.

But it can also happen that despite no singularities appear for the reference orbit $x(t)$, some singularities show up for the VE. Assume there are two of them, t^* and t^{**}, and let ψ_1, ψ_2 be two closed paths starting at a regular point and encircling t^* and t^{**}, respectively. If the monodromy matrices $M_1^{\psi_1}, M_1^{\psi_2}$ satisfy some of the conditions in Lemma 1 then to prove non-integrability one should still check that the commutator in (4) is not trivial. An illustration of this case is shown in [13].

In general no explicit solution is known for an arbitrary Hamiltonian with n degrees of freedom. But assume we are able to find, numerically, two paths ψ_1, ψ_2, such that $M_1^{\psi_j}, j = 1, 2$ are in $(G_1)^0$ and we can also compute $M_k^{\psi_j}, j = 1, 2$ along them for some $k > 1$. If the numerical computations give strong evidence that the commutator (4) to order k is not trivial, we can try to prove that this still holds when we account for the numerical errors (see [14] for several examples). In fact it possible to do a Computer Assisted Proof of non-integrability following these steps.

In the next sections we apply the methodology presented here to prove non-integrability of some systems in cases which can not be decided using only VE_1. All of them are classical Hamiltonians with two degrees of freedom and a separatrix, and it is checked that $M_1^{\psi_j}, j = 1, 2$ satisfy some of the conditions in Lemma 1, for some suitable paths, before proceeding to the computation of $M_k^{\psi_j}, j = 1, 2$ for $k > 1$. A different example, non involving any separatrix, can be found in [13].

4 A degenerate Hénon-Heiles system

We return to the DHH Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2) + \frac{1}{3}q_1^3 + \frac{1}{2}q_1 q_2^2. \tag{5}$$

This was the seminal example for the theory in [18] where the non-integrability of (5) was proved. We sketch here basic ideas of the proof because they will be useful in what follows.

As usual we denote also the variables (q_1, q_2, p_1, p_2) as (x_1, x_2, x_3, x_4). The system has two fixed points. One of them P_{ee} located at the origin is totally elliptic and in 1–1 resonance. The second one $P_{hp} = (-1, 0, 0, 0)$ is hyperbolic-parabolic (H-P). DHH is the only member of Hénon-Heiles Family with an H-P point.

The Hamiltonian (5) has an invariant plane $\Pi = \{x_2 = x_4 = 0\}$ On that plane the equations of motion are

$$\dot{x_1} = x_3, \qquad \dot{x_3} = -(1 + x_1)x_1 \tag{6}$$

and the solutions are given by elliptic integrals. In particular there is a separatrix Γ_0 on the energy level $H = h_0 = 1/6$, through P_{hp} given by

$$x_1(t) = \frac{3/2}{\cosh^2(t/2)} - 1, \qquad x_3(t) = \frac{-(3/2)\sinh(t/2)}{\cosh^3(t/2)}. \tag{7}$$

Hence, we are facing a classical Hamiltonian systems with a separatrix as described in Section 3. The Riemann surface Γ_{h_0} corresponding to the separatrix is a cylinder, while the Riemann surfaces Γ_h for nearby $h < h_0$ are tori. In both cases the solutions $x(t)$ have a double pole that be denote as t_*.

There exists numerical evidence that (5) is non-integrable (see [18] for a Poincaré section on the level $h = 1/5$) but for $h < 1/6$ the chaotic zones are hardly visible.

The equations for $\varphi^{(k)}$, $k \geq 1$, have a singularity only at t_*. All the $x_{i;k}$ (solution of VE_1) are known in terms of hyperbolic functions. In particular the normal variational equations NVE_1, are of the form

$$\dot{\xi_2} = \eta_2, \qquad \dot{\eta_2} = -(1 + x_1)\xi_2. \tag{8}$$

This implies that there are solutions of (8) which are proportional to the solutions x_1, x_3 of (6). In particular, if we start the solution in Γ at a point of the form $(x_1, 0)$ we have that the first column of the normal part of the monodromy at the end of the period, either the real or the imaginary one, returns to $\xi_2 = 1, \eta_2 = 0$. Hence, taking as ψ_1 and ψ_2 the paths along these periods, both matrices are unipotent and, hence, in $(G_1)^0$ according to Lemma 1.

Consider a small loop γ around t_*. Integration along γ cancels for all $x_{i;k}, x_{i;k_1,k_2}$, but it is *different from zero* for some components of $\varphi^{(3)}$ (e.g., it gives $\frac{72}{5} 2\pi\,\mathrm{i}$ for $x_{2;2,2,2}$).

By continuity with respect to parameters along a regular γ this is also true for nearby energy levels $h < h_0$. The path can be deformed to a period parallelogram $\hat{\gamma}$. Hence the third order monodromy $M_3^{\hat{\gamma}}$ is different from the identity and, therefore, $(G_3)^0$ is not commutative, proving *non-integrability of DHH.*

We can interpret that result in terms of jet transport. After transporting along γ the initial variations ξ we recover ξ at first order, zero at second order and something different from zero at third order.

4.1 Dynamical information on the DHH system

System (5), beyond requiring third order variational equations along the separatrix to detect non-integrability, has some interesting dynamical properties. We list here some of them. They are obtained using standard normal form (NF) techniques supplemented by numerical analysis when dealing with global properties.

1) In contrast with the classical HH model, see Section 1, only two families of simple periodic orbits emanate from P_{ee}, that is, families whose limit period when the energy $h \to 0$ is 2π. For the classical HH eight families show up (see Section 1). In particular the two families for the DHH are elliptic for small h.

   Using $\Sigma = \{q_2 = 0\}$ as Poincaré section and (q_1, p_1) as coordinates on it, for small h, the periodic orbit living on Σ is the boundary of the admissible domain. The other periodic orbit, that we denote as *vertical p.o.* is seen on Σ as an elliptic fixed point located on $F_e = (q_1^v, 0)$, where $q_1^v = \mathcal{O}(h)$. The Poincaré map PM is very close to integrable. The rotation number around F_e is $\mathcal{O}(h)$. This can be expected because of the 1–1 resonance.

   It can also be derived from a NF study that the first Birkhoff coefficient of the Poincaré map around F_e is negative. Hence, points going away from F_e rotate around it with decreasing angular velocity under PM. This also guarantees the applicability of Moser's twist Theorem [20].

2) When changing h it has been checked that the first Birkhoff coefficient changes sign at $h \approx 0.11$. As this is far from zero, a classical analytic proof seems to be unfeasible. But the techniques which are presented in [14] allow to compute easily higher order Taylor representations of PM around F_e.

   As a consequence, the rotation number (of an integrable approximation of PM) around F_e passes through a maximum and the existence of *meandering curves* (see [22]) follows.

   All these variations of the rotation number are also associated to the creation of subharmonic periodic solutions.

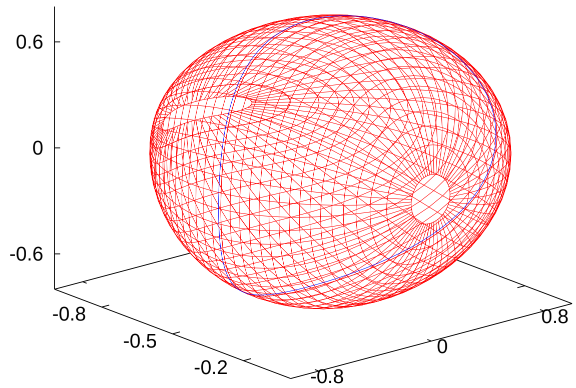

Figure 2: The global W^c_{hp} manifold, which coincides with a family of periodic orbits. See text for additional details.

3) The fact that system (5) has a hyperbolic-parabolic point shows that P_{hp} has a 2D centre manifold W^c_{hp}. A NF computation shows that, on W^c_{hp}, the Hamiltonian has a dominant part of the form $\frac{1}{2}y^2 + \frac{1}{4}x^4$. This implies the existence of a family of periodic orbits tending to P_{hp} when $h \to 1/6$. Let $\Delta = h - 1/6$. Then the period of these orbits behaves as $\mathcal{O}(\Delta^{-1/4})$ and the dominant eigenvalues of the monodromy matrix along it are extremely large due to the hyperbolic character in the normal direction and the very long period.

An interesting feature appears when trying to globalize W^c_{hp}. Locally, on each level of energy, should consist of a periodic orbit. A numerical continuation shows that this family of periodic orbits coincides with the family of vertical p.o. emanating from P_{ee} as described in 1). Figure 2 shows a representation of the family (and therefore of the global behaviour of W^c_{hp}) in $(q_1, q_2, \dot{q}_2)$ coordinates.

The approximated circular hole seen in the front part corresponds to the location of P_{ee} , while the elongated hole on the back corresponds to the location of P_{hp}. The blue curve is the periodic orbit which is found for the maximal value of h.

5 Systems requiring order k variational equations

Concerning the applicability of Theorem 2 one can ask if there exists examples such that, along some simple solution, it is necessary to go to an arbitrary high order to decide that they are non-integrable. In this section we show examples of this kind.

As an extension of DHH let us consider, for $n \geq 2$, the *generalized degenerate HH* problem, GDHH, with Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}q_1^2 + \frac{1}{3}q_1^3 + (1 + q_1)\frac{1}{n!}q_2^n \tag{9}$$

which for $n = 2$ gives DHH. We shall consider $n \geq 3$.

The only fixed points are the origin and the point P_{hp}, as before. Note that now the origin becomes elliptic-parabolic while P_{hp} is still hyperbolic-parabolic. This difference implies that the results for $n = 2$ and $n \geq 3$ are slightly different. The Hamiltonian (9) has the same invariant plane Π and the same separatrix Γ_0 as DHH.

Our goal in this section is to prove the following result

Theorem 3. *For any $n \geq 3$ system (9) is non-integrable in a neighbourhood of the separatrix Γ_0 sitting on the $q_2 = p_2 = 0$ plane. To decide the non-integrability one should use the order $n-1$ monodromy M^γ_{n-1} along a suitable path, all the lower order monodromies $M^\gamma_k, k < n-1$ being trivial.*

Proof. The separatrix has been given in (7) and the first order VE uncouple. The "tangential" part of the solution, containing $x_{1;1}, x_{3;1}, x_{1;3}, x_{3;3}$, is given in formula (42) in [18], but it is not necessary now. On the other hand one has $x_{2;2} = x_{4;4} = 1, x_{2,4} = t$, all the other elements $x_{i;j}$ being zero. This shows, in particular, that for the NVE_1 (i.e., the "normal" part) the monodromy matrices along both real and imaginary periods for levels of energy close to the level of the separatrix are unipotent and so Lemma 1 applies.

Let f_i be the components of the vector field and $f_{i;k}, f_{i;k_1,k_2}, f_{i;k_1,k_2,k_3}, \ldots$ the higher order derivatives. From the expressions of f_i it follows that all the derivatives of order higher than 1 are zero along the separatrix except

$$f_{3;1,1} = -2, \quad f_{3;k_1,\ldots,k_n} = -1, \quad f_{4;k_1,\ldots,k_{n-1}} = -(1 + x_1), \quad f_{4;1,k_1,\ldots,k_{n-1}} = -1,$$

where all the k_j indices are equal to 2.

Hence, for any k with $1 < k < n-1$ all the entries of VE_k of the form $x_{i;j_1,j_2,\ldots,j_k}$ with $i \in \{2, 4\}$, $j_m \in \{2, 4\}$, $m = 1, \ldots, k$ are identically zero.

The lowest order VE_k for which some of the components of the form $x_{i;j_1,j_2,\dots,j_k}$, as above, is different from zero is $k=n-1$. They are

$$\frac{d}{dt}\begin{pmatrix} x_{2;k_1\dots,k_{n-1}} \\ x_{4;k_1\dots,k_{n-1}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_{2;k_1\dots,k_{n-1}} \\ x_{4;k_1\dots,k_{n-1}} \end{pmatrix} + \begin{pmatrix} 0 \\ -(1+x_1) \end{pmatrix}, \tag{10}$$

where all the k_j indices are equal to 2, as before.

We can take a path Γ_0 starting at $t_0=0$ and do a loop around $t^*=\pi\,\mathrm{i}$. After the loop the solution is

$$\begin{pmatrix} x_{2;k_1\dots,k_{n-1}} \\ x_{4;k_1\dots,k_{n-1}} \end{pmatrix} = \int_{\Gamma_0}\begin{pmatrix} t(1+x_1) \\ -(1+x_1) \end{pmatrix} dt = \begin{pmatrix} -12\pi\,\mathrm{i} \\ 0 \end{pmatrix}.$$

For completeness, we can consider an equation like (10) where, among the derivation indices appears 2 a total of n_1 times and 4 a total of n_2 times, with $n_1+n_2=n-1$. The related integrals are

$$\int_{\Gamma_0}\begin{pmatrix} t^{n_2+1}(1+x_1) \\ -t^{n_2}(1+x_1) \end{pmatrix} dt = \begin{pmatrix} -12(n_2+1)(\pi\,\mathrm{i})^{n_2+1} \\ 12n_2(\pi\,\mathrm{i})^{n_2} \end{pmatrix} \tag{11}$$

which proves that all the elements $x_{i;k_1\dots,k_{n-1}}$, $i\in\{2,4\}$, $k_j\in\{2,4\}$ for all j are different from zero with the exception of the element $x_{4;2,\dots,2}$.

At that point we apply the same arguments for $h<h_0$, close to it, as before, and use the result of Lemma 2. Hence, in a neighbourhood of Γ, the system is detected to be non-integrable using $(G_{n-1})^0$ but not using $(G_k)^0$, $1<k<n-1$. □

6 A non-linear spring-pendulum problem

Consider a spring-pendulum problem with non-linear spring, having the Hamiltonian

$$H=\frac{1}{2}\left(p_r^2+\frac{p_\theta^2}{r^2}\right)-r\cos(\theta)+\frac{k}{2}(r-1)^2-\frac{a}{3}(r-1)^3, \tag{12}$$

where the real constant k is assumed to be positive. Simple solutions are obtained, for instance, for $p_\theta=0$ and $\theta=0$ or π. Letting aside the trivial case $k=a=0$ non-integrability has been proved if $k+a\neq 0$ in the nice paper [10].

These authors claim that for $a=-k$, along the chosen solution, the analysis around the imaginary singularity of $r(t)$ (see (13)) of the m-th order variational equations up to $m=7$, shows that no obstructions to integrability are found. Furthermore numerical computation of Poincaré maps gives strong evidence of non-integrability. This asks for a clarification. In this section we shall be concerned with the non-integrability.

When $a=-k$ we have a separatrix given by

$$r(t)=\rho+\frac{\alpha}{\cosh^2(\beta t)}, \qquad \theta=p_\theta=0, \tag{13}$$

where

$$\rho=(1-\gamma)/2,\quad \alpha=3\gamma/2,\quad \beta^2=k\gamma/4,\quad \gamma=(1+4/k)^{1/2},$$

similar to the DHH case. It is on the level of energy $h_0=\left(\frac{k}{12}+\frac{1}{3}\right)\gamma+\frac{k}{12}-\frac{1}{2}$.

Let us look first to VE_1 along (13). $\varphi^{(1)}$ is the solution of $\frac{d}{dt}\varphi^{(1)}(t) = B(t)\varphi^{(1)}(t)$, $\varphi^{(1)}(0) = Id$. It uncouples in two linear systems for r, p_r and θ, p_θ with matrices

$$B_1 = \begin{pmatrix} 0 & 1 \\ k(1-2r) & 0 \end{pmatrix}, \; B_2 = \begin{pmatrix} 0 & r^{-2} \\ -r & 0 \end{pmatrix}. \tag{14}$$

We note that, as $k > 0$, the parameters in (13) satisfy $\alpha > 0$, $\rho < 0$, $\alpha + \rho > 0$. But the separatrix (13) tends, for $t \to \pm\infty$, to the (non-physical) value $\rho < 0$. In particular it takes the value 0 for $t = t_\pm = \pm \cosh^{-1}(-\alpha/\rho)/\beta$. In other words, the radius $r(t)$ has period $\frac{\pi \mathrm{i}}{\beta}$ and a double pole at $t_* = \frac{\pi \mathrm{i}}{2\beta}$. Hence, B_2 has singularities at t_* and also at $t_\pm \in \mathbb{R}$, where $r(t_\pm) = 0$. A similar thing, existence of zeros of r and of singularities, happens for the (doubly) periodic nearby solutions on energy levels $h < h_0$, $h \approx h_0$. A suitable path can be chosen to prove

Theorem 4. *System (12) is non-integrable when $a = -k$.*

Proof. We shall give here the main steps of the proof. The computational details will be given in the next sections.

Using circles of small radius ε let $\gamma_{+,*,-}$ be paths emanating from the origin and encircling the respective singularities $t_{+,*,-}$ as shown in Figure 3, traveled counterclockwise. It is clear that the singularities depend analytically on the level of energy h. Then, as explained before, one can do the computations on the level h_0, when required.

We will show, first, that along each of $\gamma_{+,*,-}$ the variation of the entries of $\varphi^{(1)}$ cancels. Then one has to compute the variations of the entries of $\varphi^{(2)}$ and $\varphi^{(3)}$ along $\gamma_{+,*,-}$. After proving that the variations of $\varphi^{(2)}$ also cancel, it will be enough to show that some of the components of $\varphi^{(3)}$ has a contribution different from zero.

Furthermore by deforming the path $\gamma_- \circ \gamma_* \circ \gamma_+$ for a level of energy $h < h_0$, $h \approx h_0$ we obtain a circuit which is equivalent to the commutator $\psi_2^{-1} \circ \psi_1^{-1} \circ \psi_2 \circ \psi_1$ where ψ_1 and ψ_2 are the paths giving rise to real and imaginary periods, see Figure 3. In Lemma 3 (see subsection 6.2) it will be proved that we are in case ii) of Lemma 1. Hence the monodromy matrices along ψ_1, ψ_2 are in $(G_1)^0$ and one can proceed with the higher order monodromy according to Lemma 2.

Along γ_* the variational equations up to order 7 give a trivial contribution, according to [10]. Additional evidence (see [14]) seems to indicate that along γ_* they cancel to all orders. In the sequel it will be proved up to order 3. The inclusion of γ_+ and γ_- in the path is essential to prove Theorem 4.

It is immediate from the symmetries of the equations (locally, around $t = 0$, and starting with initial conditions of the form (13)) that the parity of an element x_i, $x_{i;k_1}$, $x_{i;k_1,k_2}$, $x_{i;k_1,k_2,k_3}$, if it is not identically zero, is the same as the parity of

$$\mathcal{P} = \#\{i, k_1, \ldots, k_s \in \{3, 4\}\}, \tag{15}$$

where s denotes the order of the variational equations. Furthermore we observe that the path γ_- is minus the path γ_+. We shall prove, in subsections 6.2 and 6.3, that all the non-zero values of the components of $\varphi^{(3)}$ at the end of γ_+ are purely imaginary.

Hence, it is easy to check that the non-zero elements of $\varphi^{(3)}$ along γ_+ are imaginary and keep the sign when we pass to γ_- if $\mathcal{P}$ in (15) is even and change sign if it is odd. As the monodromy up to third order along $\gamma_{+,-}$ is the identity plus third order terms, it is

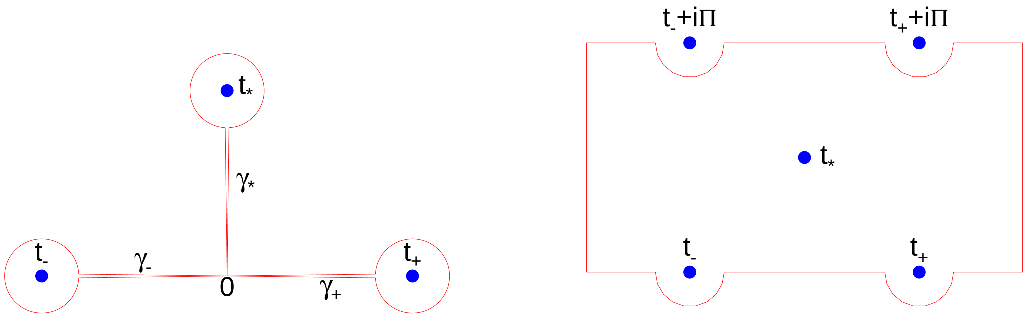

Figure 3: Left: Path for the proof of Theorem 4 around the three singularities of the variational equations. Right: the path deformed to a period parallelogram, avoiding the singularities. Here iΠ denotes the imaginary period on the energy level h.

enough to double the value of the elements at the end of γ_+ which keep sign when passing from γ_+ to γ_-.

Concretely, the non-zero elements in $\varphi^{(3)}$ at the end of γ_+ are of the form:

$$x_{2;2,2,2} = a\,\mathrm{i},\ x_{2;2,2,4} = -c\,\mathrm{i},\ x_{2;2,4,4} = d\,\mathrm{i},\ x_{2;4,4,4} = -e\,\mathrm{i},$$

$$x_{4;2,2,2} = b\,\mathrm{i},\ x_{4;2,2,4} = -a\,\mathrm{i},\ x_{4;2,4,4} = c\,\mathrm{i},\ x_{4;4,4,4} = -d\,\mathrm{i},$$

where a, b, c, d, e are real positive. Therefore, only

$$x_{2;2,2,2} = 2a\,\mathrm{i},\ x_{4;2,2,4} = -2a\,\mathrm{i},\ x_{2;2,4,4} = 2d\,\mathrm{i},\ x_{4;4,4,4} = -2d\,\mathrm{i}$$

remain at the end of the full loop. It is clear that the structure of the coefficients ($x_{2;2,2,2} + x_{4;2,2,4} = x_{2;2,2,4} + x_{4;2,4,4} = x_{2;2,4,4} + x_{4;4,4,4} = 0$) at the end of γ_+ follows immediately from symplectiness of the return time map. Therefore, the only thing to prove, beyond the fact that some elements are zero is $a > 0, d > 0$.

The computational details are provided in Subsection 6.2.

Using the continuity and deformation arguments with respect to the level of energy, as in previous cases, this proves that system (13) is non-integrable for $a = -k, k > 0$. □

6.1 A limit Hamiltonian when $k \to 0$

When $k \to 0$ in (12) the role of the spring disappears and the solution (13) goes to infinity. A suitable scaling of variables and time can be introduced as follows:

$$(R, \Theta, P_R, P_\Theta, s) = (k^{1/2}r, \theta, k^{1/4}p_r, k^{3/4}p_\theta, k^{1/4}t),$$

giving rise to the Hamiltonian

$$H = \frac{1}{2}\left(P_R^2 + \frac{P_\Theta^2}{R^2}\right) - R\cos(\Theta) + \frac{1}{3}R^3 - k^{1/2}\frac{1}{2}R^2.$$

This system is analytic with respect to the parameter $k^{1/2}$. This suggests to study the limit system

$$H = \frac{1}{2}\left(P_R^2 + \frac{P_\Theta^2}{R^2}\right) - R\cos(\Theta) + \frac{1}{3}R^3, \tag{16}$$

which is parameter-free, like the classical Hill's problem [19]. The basic solution for R is like (13) with $(\rho,\alpha,\beta)=(-1,2,1/\sqrt{2})$ and singularities $(s_*,s_\pm)=(\pi/\sqrt{2},\pm\sqrt{2}\log(\sqrt{3}+\sqrt{2})$, all of them now finite.

All the computations done for (12) are repeated for (16). The approximate values for a,d are $a_0\approx 18.10054308, d_0\approx 3.24984189$ (see [14]). Furthermore, undoing the scaling, we recover the values $a=a_0+\mathcal{O}(k^{1/2})$, $d=d_0k^{3/2}+\mathcal{O}(k^2)$. The scaling will be useful in the proof of Theorem 4, but the numerical information above is not used and given only as a complement.

6.2 Analytical details for the proof of Theorem 4

This section is devoted to perform the computations needed in the proof of Theorem 4. The first variational equations uncouple in two systems

$$\dot{\xi}_1=\xi_3,\qquad \dot{\xi}_3=-k(2r-1)\xi_1, \tag{17}$$

and

$$\dot{\xi}_2=r^{-2}\xi_4,\qquad \dot{\xi}_4=-r\xi_2, \tag{18}$$

where $r=\rho+\alpha/\cosh^2(\beta t)$ as defined in (13) Following the notation introduced in Subsection 2.3 we write

$$\Phi_1(t)=\begin{pmatrix} x_{1;1}(t) & x_{1;3}(t)\\ x_{3;1}(t) & x_{3;3}(t)\end{pmatrix},\qquad \Phi_2(t)=\begin{pmatrix} x_{2;2}(t) & x_{2;4}(t)\\ x_{4;2}(t) & x_{4;4}(t)\end{pmatrix}, \tag{19}$$

for the fundamental matrices of the systems (17) and (18), respectively, such that $\Phi_1(0)=I$, and $\Phi_2(0)=I$. We remark that trivially

$$x_{i;j}=0\quad \text{if the parity of } i \text{ and } j \text{ is different.} \tag{20}$$

The system (17) can be solved explicitly. We obtain

$$\begin{aligned} x_{1;1}(t)&=-\frac{5}{8}\left(1-\frac{3}{\cosh^2(\beta t)}+\frac{2}{5}\cosh^2(\beta t)+3\beta t\frac{\sinh(\beta t)}{\cosh^3(\beta t)}\right),\\ x_{1;3}(t)&=-\frac{1}{2\alpha\beta^2}\dot{r}(t),\\ x_{3;1}(t)&=\dot{x}_{1;1}(t),\qquad x_{3;3}(t)=\dot{x}_{1;3}(t).\end{aligned} \tag{21}$$

Lemma 3. *On energy levels $h<h_0$, $h\approx h_0$ the matrix Φ_2 along the imaginary period is non-resonant. Furthermore it commutes with the one along the real period.*

Proof. The commutativity follows immediately from the absence of logarithmic terms in the solution of (18) along the three paths $\gamma_{+,*,-}$. Let us proceed to show that the monodromy matrix along the imaginary period is non-resonant for values of $h<h_0$ arbitrarily close to h_0.

According to the discussion following Lemma 2 concerning classical Hamiltonians with a separatrix, the periodic solutions with imaginary period are equivalent to real periodic solutions of the Hamiltonian $K=\frac{1}{2}p_r^2-[-r+\frac{k}{2}(r-1)^2+\frac{k}{3}(r-1)^3]$ around the fixed point $r=r_f:=\frac{1}{2}(1-\gamma)<0, p_r=0$. Introducing a local variable $\sigma=r-r_f$ the equation for the periodic orbits is $\sigma'' = -A^2\sigma + B\sigma^2$, where $A^2 = k(1-2r_f)$, $B = k$ and $' = d/ds, s = \mathrm{i}t$. Small amplitude solutions, with a small parameter ε related to changes in the energy level, can be obtained using Lindstedt-Poincaré method, see [21]. They are of the form

$$\sigma(s) = \varepsilon(z+z^{-1}) + \varepsilon^2(a_{2,0} + a_{2,2}(z^2+z^{-2})) + \varepsilon^3(a_{3,3}(z^3+z^{-3})) + \mathcal{O}(\varepsilon^4),$$

where $z = \exp(\mathrm{i}\omega s)$, $\omega = \omega_0 + \omega_2\varepsilon^2 + \mathcal{O}(\varepsilon^4)$. The NVE_1 along these solutions are $\xi_2' = r^{-2}\xi_4$, $\xi_4' = r\xi_2$ which can be written as a second order linear differential equation. It is useful to introduce $u = r\xi_2$ which satisfies a Hill-like equation of the form

$$u'' + (C^2 + D\sigma)u = 0, \quad C^2 = k(1-r_f), \quad D = -k.$$

The solutions are found using Normal Form methods, see [1].

Summarizing, the monodromy matrices along the imaginary periodic solutions of the initial Hamiltonian H are symplectic matrices of the form $\begin{pmatrix} a & b \\ c & a \end{pmatrix}$, because of the symmetry of the periodic solutions Moreover $a = \cos(2\pi\nu(\varepsilon))$, $\nu(\varepsilon) = \nu_0 + \nu_2\varepsilon^2 + \mathcal{O}(\varepsilon^4)$ where

$$\nu_0 = \sqrt{(1-r_f)/(1-2r_f)}, \quad \nu_2 = \nu_0 \frac{3+14r_f}{3(1-2r_f)^2(3-2r_f)}.$$

Resonance is produced if $\nu(\varepsilon) \in \mathbb{Q}$. If $\nu_2 \neq 0$ then by changing ε there is a set of full measure in ε which avoids resonance. For $r_f = -3/14$, which corresponds to $k = 196/51$, either some of the coefficients $\nu_{2j}, j > 1$ is non-zero or all of them are zero. But in the last case one has $\nu(\varepsilon) = \nu_0 = \sqrt{17/20}$, which is a quadratic irrational and prevents again from resonance. $\square$

The solutions of (18) will be obtained as power series in a neighbourhood of the singularities $t_*, t_\pm = \pm\hat{t}$. We can write (18) as

$$\ddot{\xi}_2 + \frac{2\dot{r}}{r}\dot{\xi}_2 + \frac{1}{r}\xi_2 = 0. \tag{22}$$

Assume that x_{f1}, x_{f2} are two linearly independent solutions of (22). Then we shall write

$$x_{2;2}(t) = a\, x_{f1}(t) + b\, x_{f2}(t), \qquad x_{2;4}(t) = c\, x_{f1}(t) + d\, x_{f2}(t), \tag{23}$$

for some constants a, b, c, d. Using (18) we have

$$x_{4;2}(t) = r^2\dot{x}_{2;2}(t), \qquad x_{4;4}(t) = r^2\dot{x}_{2;4}(t). \tag{24}$$

The second order variational equations can be written in components as

$$\dot{x}_{i;j_1,j_2} = \sum_{k=1}^{4} f_{i;k}\, x_{k;j_1,j_2} + \sum_{k_1,k_2=1}^{4} f_{i;k_1,k_2}\, x_{k_1;j_1}\, x_{k_2;j_2}, \quad 1 \le i, j_1, j_2 \le 4. \tag{25}$$

For a fixed pair of indices (j_1, j_2), equations in (25) uncouple in two linear systems

$$\begin{aligned} \dot{x}_{1;j_1,j_2} &= x_{3;j_1,j_2}, & \dot{x}_{2;j_1,j_2} &= r^{-2}x_{4;j_1,j_2} + h_{2;j_1,j_2}, \\ \dot{x}_{3;j_1,j_2} &= k(1-2r)x_{1;j_1,j_2} + h_{3;j_1,j_2}, & \dot{x}_{4;j_1,j_2} &= -rx_{2;j_1,j_2} + h_{4;j_1,j_2}, \end{aligned} \tag{26}$$

where the functions $h_{2;j_1,j_2}, h_{3;j_1,j_2}, h_{4;j_1,j_2}$ depend on r and the corresponding solutions of the first order variational equations. They are summarized (besides a permutation of indices), together with the integrands $I_{i;j_1,j_2}, i = 1, \ldots, 4$ which appear in the solutions, in tables 1 and 2, where to simplify the expressions we have introduced

$$D_{2,2} := x_{2;2}^2 - 2r\dot{x}_{2;2}^2, \quad D_{2,4} := x_{2;4}^2 - 2r\dot{x}_{2;4}^2, \quad D_M := x_{2;2}x_{2;4} - 2r\dot{x}_{2;2}\dot{x}_{2;4}.$$

The indices j_1, j_2 such that they or their permutation do not appear in tables 1 and 2 give rise to variables identically zero. This is mainly due to (20).

j_1	j_2	$h_{3;j_1,j_2}$	$I_{1;j_1,j_2}$	$I_{3;j_1,j_2}$
1	1	$-2k\,x_{1;1}^2$	$2kx_{1;1}^2\,x_{1;3}$	$-2kx_{1;1}^3$
1	3	$-2k\,x_{1;1}\,x_{1;3}$	$-2kx_{1;1}\,x_{1;3}^2$	$-2kx_{1;1}^2\,x_{1;3}$
2	2	$-D_{2;2}$	$x_{1;3}D_{2,2}$	$-x_{1;1}D_{2,2}$
2	4	$-D_M$	$x_{1;3}D_M$	$-x_{1;1}D_M$
3	3	$-2k\,x_{1;3}^2$	$2kx_{1;3}^3$	$-2kx_{1;1}\,x_{1;3}^2$
4	4	$-D_{2;4}$	$x_{1;3}D_{2,4}$	$-x_{1;1}D_{2,4}$

Table 1: Independent terms $h_{3;j_1,j_2}$ in (26) and integrands $I_{1;j_1,j_2}, I_{3;j_1,j_2}$ in (27).

j_1	j_2	$h_{2;j_1,j_2}$	$h_{4;j_1,j_2}$	$I_{2;j_1,j_2}$	$I_{4;j_1,j_2}$
1	2	$-2r^{-1}\,x_{1;1}\,\dot{x}_{2;2}$	$-x_{1;1}x_{2;2}$	$x_{1;1}D_M$	$-x_{1;1}D_{2,2}$
1	4	$-2r^{-1}\,x_{1;1}\,\dot{x}_{2;4}$	$-x_{1;1}x_{2;4}$	$x_{1;1}D_{2,4}$	$-x_{1;1}D_M$
2	3	$-2r^{-1}\,x_{1;3}\,\dot{x}_{2;2}$	$-x_{1;3}x_{2;2}$	$x_{1;3}D_M$	$-x_{1;3}D_{2,2}$
3	4	$-2r^{-1}\,x_{1;3}\,\dot{x}_{2;4}$	$-x_{1;3}x_{2;4}$	$x_{1;3}D_{2,4}$	$-x_{1;3}D_M$

Table 2: Independent terms $h_{2;j_1,j_2}, h_{4;j_1,j_2}$ in (26) and integrands $I_{2;j_1,j_2}, I_{4;j_1,j_2}$ in (27).

j_1	j_2	j_3	$I_{2;j_1,j_2,j_3}$
2	2	2	$3x_{1;2,2}D_M - rx_{2;2}^3x_{2;4}$
2	2	4	$x_{1;2,2}D_{2,4} + 2x_{1;2,4}D_M - rx_{2;2}^2x_{2;4}^2$
2	4	4	$2x_{1;2,4}D_{2,4} + x_{1;4,4}D_M - rx_{2;2}x_{2;4}^3$
4	4	4	$3x_{1;4,4}D_{2,4} - rx_{2;4}^4$

Table 3: $I_{2;j_1,j_2,j_3}$.

j_1	j_2	j_3	$I_{4;j_1,j_2,j_3}$
2	2	2	$-3x_{1;2,2}D_{2,2} + rx_{2;2}^4$
2	2	4	$-x_{1;2,2}D_M - 2x_{1;2,4}D_{2,2} + rx_{2;2}^3x_{2;4}$
2	4	4	$-2x_{1;2,4}D_M - x_{1;4,4}D_{2,2} + rx_{2;2}^2x_{2;4}^2$
4	4	4	$-3x_{1;4,4}D_M + rx_{2;2}x_{2;4}^3$

Table 4: $I_{4;j_1,j_2,j_3}$.

The solutions of the linear systems (26), with initial conditions equal to zero, can be obtained in terms of some quadratures involving the solutions of the first order variational equations as

$$\begin{pmatrix} x_{1;j_1,j_2} \\ x_{3;j_1,j_2} \end{pmatrix} = \Phi_1(t) \begin{pmatrix} \int_0^t I_{1;j_1,j_2} \\ \int_0^t I_{3;j_1,j_2} \end{pmatrix}, \qquad \begin{pmatrix} x_{2;j_1,j_2} \\ x_{4;j_1,j_2} \end{pmatrix} = \Phi_2(t) \begin{pmatrix} \int_0^t I_{2;j_1,j_2} \\ \int_0^t I_{4;j_1,j_2} \end{pmatrix}, \qquad (27)$$

where $I_{i;j_1,j_2}, i = 1, \dots 4$ are given in tables 1 and 2.

In a similar way, we can obtain the third order variational equations. For our purpose we only need to consider the variables $x_{2;j_1,j_2,j_3}$ and $x_{4;j_1,j_2,j_3}$, for $(j_1, j_2, j_3) \in \mathcal{I}$, where $\mathcal{I} = \{(2, 2, 2), (2, 2, 4), (2, 4, 4), (4, 4, 4)\}$. We get

$$\begin{pmatrix} x_{2;j_1,j_2,j_3} \\ x_{4;j_1,j_2,j_3} \end{pmatrix} = \Phi_2(t) \begin{pmatrix} \int_0^t I_{2;j_1,j_2,j_3} \\ \int_0^t I_{4;j_1,j_2,j_3} \end{pmatrix}, \tag{28}$$

where $I_{i;j_1,j_2,j_3}$ are given in tables 3 and 4

Remark 1. *The integrals $\int_0^t x_{1;j} D_{2,2}$, $\int_0^t x_{1;j} D_{2,4}$, $\int_0^t x_{1;j} D_M$, for $j = 1, 3$ are linear combinations of I_i, J_i, $i = 1, 2, 3$ where*

$$I_i = \int_0^t x_{1;1} D_i, \qquad J_i = \int_0^t x_{1;3} D_i, \qquad i = 1, 2, 3,$$

being

$$D_1 = x_{f1}^2 - 2r\dot{x}_{f1}^2, \quad D_2 = x_{f2}^2 - 2r\dot{x}_{f2}^2, \quad D_3 = x_{f1}x_{f2} - 2r\dot{x}_{f1}\dot{x}_{f2}.$$

Remark 2. *Using the expressions of $x_{1;2,2}, x_{1;2,4}$, and $x_{1;4,4}$, derived from (27), $\int_0^t I_{i;j_1,j_2,j_3}$, for $i = 2, 4$, $(j_1, j_2, j_3) \in \mathcal{I}$ can be written as linear combinations of $\int_0^t A_{i,j}$ and $\int_0^t B_{i,j}$ where*

$$A_{i,j} := \dot{I}_i J_j - \dot{J}_i I_j = x_{1;1} D_i \int_0^t x_{1;3} D_j - x_{1;3} D_i \int_0^t x_{1;1} D_j, \quad i, j = 1, 2, 3, \tag{29}$$

$$B_{i,j} := r x_{f1}^i x_{f2}^j, \qquad 0 \le i, j \le 4, \quad i + j = 4, \tag{30}$$

the dot denoting the derivative with respect to t.

The next proposition shows that the variations of the entries of $\varphi^{(2)}$ and $\varphi^{(3)}$ cancel along γ_*.

Proposition 1. *For any $k > 0$, $x_{i;j_1,j_2}$, $i = 1, \dots, 4$, $1 \le j_1, j_2 \le 4$ and $x_{i;j_1,j_2,j_3}$, for $i = 2, 4$, $(j_1, j_2, j_3) \in \mathcal{I}$ cancel along the path γ_* defined at the beginning of Section 6.*

Proof The proof goes as follows. First, we shall develop the solutions of the first order variational equations around the singularity t_*. This will show that $\Phi_1(t)$ and $\Phi_2(t)$ do not change after moving along a small circle around t_* (see Figure 3). Then we shall prove that the residues at t_* of $I_{i;j_1,j_2}$ and $I_{i;j_1,j_2,j_3}$ given in the tables 1 to 4 are zero. This implies that going through the small circle around t_* the variables do not change. Then the proof finishes by taking into account that the variation going up and down from a neighbourhood of zero to a neighbourhood of t_* trivially cancel. In the rest of the proof we shall made explicit the required computations.

Let us consider a neighbourhood of $t_* = \pi\,\mathrm{i}/(2\beta)$. We introduce $\tau = t - t_*$. Then, the following expansions hold

$$r(\tau) = \sum_{i \ge -2} r_i \tau^i, \qquad x_{1;1}(\tau) = \sum_{i \ge -3} e_i \tau^i, \qquad x_{1;3}(\tau) = \sum_{i \ge -3} d_i \tau^i, \tag{31}$$

where the main coefficients r_i, e_i, d_i are given in the appendix 1.

Furthermore, using Frobenius method we easily see that

$$x_{f1}(\tau) = \alpha_0 + \alpha_4\tau^4 + \alpha_6\tau^6 + \ldots, \qquad x_{f2}(\tau) = \tau^5(\hat{\alpha}_0 + \hat{\alpha}_2\tau^2 + \hat{\alpha}_4\tau^4 + \ldots), \tag{32}$$
$$\alpha_4 = -\frac{k}{24}\alpha_0, \qquad \hat{\alpha}_2 = \frac{5k}{42}\hat{\alpha}_0, \qquad \hat{\alpha}_4 = \frac{k(k-2)}{144}\hat{\alpha}_0,$$

and $\alpha_0, \hat{\alpha}_0$ are arbitrary values different from zero. Then

$$D_1 = \alpha_0^2 + O(\tau^4), \quad D_2 = O(\tau^6), \quad D_3 = O(\tau^5). \tag{33}$$

This implies that the residues at t_* of $x_{1;1}D_i$, $x_{1;3}D_i$, for $i = 1, 2, 3$ are zero, and the same is true for $I_{i;j_1,j_2}$, for $i = 1, 3$ and $(j_1, j_2) \in \{(2, 2), (2, 4), (4, 4)\}$, and for $I_{2;j_1,j_2}, I_{4;j_1,j_2}$.

The residues of $x_{1;1}^2 x_{1;3}$, $x_{1;1}^3$, $x_{1;3}^3$, $x_{1;1}x_{1;3}^2$ at t_* are the following ones

$$\begin{aligned}
\mathcal{R}_*(x_{1;1}^2 x_{1;3}) &= 2e_{-3}(e_1 d_1 + e_5 d_{-3}) + e_1^2 d_{-3} + e_{-3}^2 d_5 = 0,\\
\mathcal{R}_*(x_{1;1} x_{1;3}^2) &= 2d_{-3}(e_1 d_1 + e_{-3} d_5) + e_5 d_{-3}^2 + e_{-3} d_1^2 = 0,\\
\mathcal{R}_*(x_{1;1}^3) &= 3e_{-3}(e_1^2 + e_{-3}e_5) = 0,\\
\mathcal{R}_*(x_{1;3}^3) &= 3d_{-3}(d_1^2 + d_{-3}d_5) = 0.
\end{aligned}$$

Let us consider now $I_{2;j_1,j_2,j_3}, I_{4;j_1,j_2,j_3}$ given in tables 3 and 4. Using the Remark 2 we only need to compute the residues of $B_{i,j}$ and $A_{i,j}$. It is immediate to see that $B_{i,j}$ and $A_{i,j}$ have residue zero for any (i, j) except for $A_{1,1}$. However a simple computation shows that

$$\mathcal{R}_*(A_{1,1}) = \alpha_0^4(e_{-3}d_1 - e_1 d_{-3}) = 0,$$

where the expansions (31) have been used. □

For the remaining part of the proof of Theorem 4 we begin with the analysis of the behaviour along γ_+.

Proposition 2. *For any $k > 0$, $x_{i;j_1,j_2}$, $i = 1, \ldots, 4$, $\leq j_1, j_2 \leq 4$ cancel along the path γ_+ defined at the beginning of Section 6. Furthermore, $x_{i;j_1,j_2,j_3}$, for $i = 2, 4$, $(j_1, j_2, j_3) \in \mathcal{I}$ do not cancel simultaneously along γ_+.*

Proof The proof follows the same steps as the proof of Proposition 1.

Let us consider a neighbourhood of t_+. We shall keep for the coefficients the notation introduced in the proof of the Proposition 1. Let be $\tau = t - t_+$. We obtain

$$r(\tau) \;=\; \sum_{i\geq 1} r_i\tau^i, \quad x_{1;1}(\tau) = \sum_{i\geq 0} e_i\tau^i, \quad x_{1;3}(\tau) = -\frac{1}{2\alpha\beta^2}\sum_{i\geq 1} i r_i \tau^{i-1}, \tag{34}$$

where the main coefficients r_i, e_i are given in the Appendix.

Furthermore we obtain the following linearly independent solutions of (22)

$$\begin{aligned}
&x_{f1}(\tau) = \alpha_0 + \alpha_1\tau + \alpha_2\tau^2 + \ldots,\\
&x_{f2}(\tau) = \frac{\hat{\alpha}_{-1}}{\tau} + \hat{\alpha}_1\tau + \hat{\alpha}_2\tau^2 + \ldots,\\
&\alpha_1 = -\frac{1}{2r_1}\alpha_0, \quad \alpha_2 = \frac{1}{4r_1^2}\alpha_0, \qquad \hat{\alpha}_1 = \frac{k}{3}\hat{\alpha}_{-1}, \quad \hat{\alpha}_2 = -\frac{k}{72r_1}(3 - k + \gamma(4 + k))\hat{\alpha}_{-1},
\end{aligned} \tag{35}$$

where $\alpha_0, \hat{\alpha}_{-1}$ are nonzero arbitrary values. Then trivially, $I_{i;1,1}, I_{i;1,3}, I_{i;3,3}$, for $i = 1, 3$ have residues equal to zero at t_+.

Moreover we easily compute the residues of $x_{1;i}D_j$, $i = 1, 3$, $j = 1, 2, 3$ as

$$
\begin{array}{rcl}
\mathcal{R}_+(x_{1;1}D_1) &=& 0, \\
\mathcal{R}_+(x_{1;1}D_2) &=& \hat{\alpha}_{-1}^2 r_1(ke_0 - 2e_2) = 0, \\
\mathcal{R}_+(x_{1;1}D_3) &=& e_0\hat{\alpha}_{-1}(\alpha_0 + 2\alpha_1 r_1) = 0, \\
\mathcal{R}_+(x_{1;3}D_1) &=& 0, \\
\mathcal{R}_+(x_{1;3}D_2) &=& -\dfrac{2}{\alpha\beta^2}\hat{\alpha}_{-1}r_1(-2r_3\hat{\alpha}_{-1} + \hat{\alpha}_1 r_1) = 0, \\
\mathcal{R}_+(x_{1;3}D_3) &=& -\dfrac{1}{2\alpha\beta^2}\hat{\alpha}_{-1}r_1(\alpha_0 + 2\alpha_1 r_1) = 0.
\end{array}
$$

Using (34) and (35) we have that the coefficient of τ^{-1} in $A_{i,j}$ is equal to 0 except for $A_{1,2}, A_{2,1}$ and $A_{2,2}$. In these cases we have the following values

$$
\begin{array}{rcl}
\mathcal{R}_+(A_{1,2}) &=& \mathcal{R}_+(A_{2,1}) = \dfrac{1}{2\alpha\beta^2}\alpha_0^2\hat{\alpha}_{-1}^2 r_1(e_1 r_1 - e_0), \qquad (36) \\
\mathcal{R}_+(A_{2,2}) &=& \dfrac{8}{\alpha\beta^2}\hat{\alpha}_{-1}^3\hat{\alpha}_2 r_1^2(e_1 r_1 - e_0) + \dfrac{2}{\alpha\beta^2}\hat{\alpha}_{-1}^4\left(e_0(9r_1r_3^2 - 4r_3 + r_1r_4 - 5r_1^2r_5) + \right. \\
&& \left. + e_1r_1(-9r_1r_4 + 4r_3) - 3e_2r_1^2r_3 + 2e_3r_1^2 + e_4r_1^3\right). \qquad (37)
\end{array}
$$

Furthermore

$$
\begin{array}{l}
\mathcal{R}_+(B_{0,4}) = \dfrac{3}{2}\hat{\alpha}_{-1}^4 kr_1, \qquad \mathcal{R}_+(B_{2,2}) = \alpha_0^2\hat{\alpha}_{-1}^2 r_1, \qquad (38) \\
\mathcal{R}_+(B_{1,3}) = \hat{\alpha}_{-1}^3(\dfrac{\alpha_0}{2} + r_1\alpha_1) = 0, \quad \mathcal{R}_+(B_{4,0}) = \mathcal{R}_+(B_{3,1}) = 0\,.
\end{array}
$$

Therefore

$$
\begin{array}{l}
\mathcal{R}_+(rx_{2;2}^4) = 6a^2b^2\mathcal{R}_+(B_{2,2}) + b^4\mathcal{R}_+(B_{0,4}), \\
\mathcal{R}_+(rx_{2;2}^3x_{2;4}) = 3ab(ad + bc)\mathcal{R}_+(B_{2,2}) + b^3d\mathcal{R}_+(B_{0,4}), \\
\mathcal{R}_+(rx_{2;2}^2x_{2;4}^2) = (a^2d^2 + b^2c^2 + 4abcd)\mathcal{R}_+(B_{2,2}) + b^2d^2\mathcal{R}_+(B_{0,4}), \\
\mathcal{R}_+(rx_{2;2}x_{2;4}^3) = 3cd(ad + bc)\mathcal{R}_+(B_{2,2}) + bd^3\mathcal{R}_+(B_{0,4}), \\
\mathcal{R}_+(rx_{2;4}^4) = 6c^2d^2\mathcal{R}_+(B_{2,2}) + d^4\mathcal{R}_+(B_{0,4}).
\end{array}
$$

The residues of $I_{i;j_1,j_2,j_3}$ at t_+ can be written in terms of

$$
X_1 = \mathcal{R}_+(A_{1,2}) - \mathcal{R}_+(B_{2,2}), \qquad X_2 = 3\mathcal{R}_+(A_{2,2}) - \mathcal{R}_+(B_{0,4}). \qquad (39)
$$

However $X_1 = 0$ and we simply obtain

$$
\begin{array}{ll}
\mathcal{R}_+(I_{2;2,2,2}) = b^3dX_2, & \mathcal{R}_+(I_{4;2,2,2}) = -b^4X_2, \\
\mathcal{R}_+(I_{2;2,2,4}) = b^2d^2X_2, & \mathcal{R}_+(I_{4;2,2,4}) = -b^3dX_2, \\
\mathcal{R}_+(I_{2;2,4,4}) = bd^3X_2, & \mathcal{R}_+(I_{4;2,4,4}) = -b^2d^2X_2, \\
\mathcal{R}_+(I_{2;4,4,4}) = d^4X_2, & \mathcal{R}_+(I_{4;4,4,4}) = -bd^3X_2,
\end{array} \qquad (40)
$$

where

$$X_2 = 2\hat{\alpha}_{-1}^4 \sqrt{\frac{k(1+2\gamma)}{3}} \, \frac{2\gamma^2-\gamma-1}{(\gamma+1)^2} \neq 0.$$

We note that the following relations hold

$$\mathcal{R}_+(I_{2;2,2,2}) = -\mathcal{R}_+(I_{4;2,2,4}), \quad \mathcal{R}_+(I_{2;2,2,4}) = -\mathcal{R}_+(I_{4;2,4,4}), \quad \mathcal{R}_+(I_{2;2,4,4}) = -\mathcal{R}_+(I_{4;4,4,4}).$$

To finish the proof of the lemma we must prove that $b \neq 0$ and $d \neq 0$. Let us introduce

$$C = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad \Phi_f(t) = \begin{pmatrix} x_{f1} & x_{f2} \\ r^2\dot{x}_{f1} & r^2\dot{x}_{f2} \end{pmatrix}.$$

Using (23) we have that $C = (\Phi_f(0))^{-1}$. Then

$$b = -\frac{1}{D} r^2(0)\dot{x}_{f1}(0), \qquad d = \frac{1}{D} x_{f1}(0)$$

where $D = \det(\Phi_f(0))^{-1} \neq 0$. The next lemma will finish the proof of Proposition 2.

Lemma 4. *Let be r and x_{f1} as defined in (34) and (35) respectively. Then x_{f1} and $r^2\dot{x}_{f1}$ do not cancel at $t = 0$.*

We shall prove this lemma in the next subsection. □

6.3 Proof of Lemma 4

Let us consider the first order variational system in (18) written as

$$\dot{x} = r^{-2}y, \qquad \dot{y} = -rx, \tag{41}$$

where we remind the expression of $r(t)$ from (13)

$$r(t) = \frac{1}{2}(1-\gamma) + \frac{3\gamma}{2\cosh^2(\beta t)}, \quad \beta = \beta(\gamma) = \sqrt{\frac{\gamma}{\gamma^2-1}}. \tag{42}$$

We recall that $x_{f1}(t), y_{f1}(t) = r^2(t)\dot{x}_{f1}(t)$ is a solution of (41) in a neighbourhood of the singularity $t = t_+$.

To prove Lemma 4 we shall perform a blow up of the singularity t_+ in order to see $x_{f1}(t), y_{f1}(t)$ as a branch of the stable manifold of a saddle point for some regularized system. In this way the proof of Lemma 4 is reduced to study the behaviour of such a stable manifold. To do this we shall use qualitative techniques.

For the proof it will be necessary to consider separately the cases $\gamma \leq 7/5$ and $\gamma \geq 7/5$ and introduce different formulations for the equation (41).

We start with a technical lemma concerning t_+ as a function of γ

$$t_+ = t_+(\gamma) = \frac{1}{2\beta} \ln\left(\frac{1+\sqrt{(1+2\gamma)/(3\gamma)}}{1-\sqrt{(1+2\gamma)/(3\gamma)}}\right).$$

Lemma 5. *Let γ be such that $1 < \gamma \leq 7/5$. Then $t_+(\gamma)$ is an increasing function such that $\lim_{\gamma\to1^+} t_+(\gamma) = 0$. Moreover $t_+(\gamma)$ goes to infinity as $\gamma \to \infty$.*

Proof We compute

$$\frac{dt_+}{d\gamma}=\frac{1}{2\beta(\gamma-1)}\left(\frac{(\gamma^2+1)\,t_+}{\beta(\gamma+1)^2(\gamma-1)}-\sqrt{\frac{3}{\gamma(1+2\gamma)}}\right)>\frac{1}{2\beta(\gamma-1)}\left(\frac{(\gamma^2+1)}{2\gamma(\gamma+1)}\ln Y(\gamma)-1\right),$$

where

$$Y(\gamma):=\frac{1+\sqrt{(1+2\gamma)/(3\gamma)}}{1-\sqrt{(1+2\gamma)/(3\gamma)}} \tag{43}$$

is a decreasing function of γ. Then if $1<\gamma\leq 7/5$, we get $Y(\gamma)\geq Y(7/5)=\left(\frac{\sqrt{21}+\sqrt{19}}{\sqrt{2}}\right)^2$ and

$$\frac{dt_+}{d\gamma}>\frac{1}{2\beta(\gamma-1)}\left(\frac{(\gamma^2+1)}{\gamma(\gamma+1)}\ln\left(\frac{\sqrt{21}+\sqrt{19}}{\sqrt{2}}\right)-1\right)>0.$$

□

Figure 5 shows a plot of $t_+(\gamma)$.

We note that $r(t)$ is an even function of t, decreasing for $t>0$, such that $r(t)>0$ for $0\leq t<t_+$ and $r(0)=(1+2\gamma)/2>3/2$ for $\gamma>1$. Let us denote by t_1 the value of t such that $r(t_1)=1$, that is,

$$t_1=t_1(\gamma)=\frac{1}{2\beta}\ln\left(\frac{1+\sqrt{(2\gamma-1)/(3\gamma)}}{1-\sqrt{(2\gamma-1)/(3\gamma)}}\right). \tag{44}$$

We introduce polar coordinates (R,θ) in the following way

$$rx=R\cos\theta,\qquad y=R\sin\theta$$

and we change the independent variable through $ds=r^{-1}dt$. It turns out that the equation for θ is independent of R. However it will be convenient to consider the following planar autonomous differential system for t,θ

$$\frac{dt}{ds}=r(t),\qquad \frac{d\theta}{ds}=-r(t)\cos^2\theta-\dot{r}(t)\sin\theta\cos\theta-\sin^2\theta, \tag{45}$$

where

$$\dot{r}(t)=\frac{dr}{dt}=-3\gamma\beta\frac{\tanh(\beta t)}{\cosh^2(\beta t)}.$$

The system (45) can be extended analytically to $t=t_+$. In fact the vertical line $t=t_+$ is invariant under the flow defined by (45).

Using the periodicity of (45) with respect to θ we only need to consider the domain

$$\mathcal{R}=\{(t,\theta)\,|\,0\leq t\leq t_+,-\pi/2\leq\theta\leq\pi/2\}.$$

There are two equilibrium points of (45) in $\mathcal{R}$, $P_0=(t_+,0)$ and $P_1=(t_+,\theta_+)$ where $\theta_+=\arctan(-\dot{r}(t_+))\in(0,\pi/2)$. We remark that θ_+ depends on γ. It is easy to check that for any $\gamma>1$, P_1 is an attractor and P_0 is a saddle point being the stable direction given by the vector $(-2,1)$ (see Figure 4). Let us denote as W^s the branch of the stable manifold of P_0 with $t<t_+$. We note that for any $t\in[0,t_+)$ with $\theta=0$ or $\theta=\pi/2$ we have

$$\frac{dt}{ds}>0,\quad \frac{d\theta}{ds}<0.$$

Then, following W^s backwards in time it intersects $t=0$ at some point with $\theta=\theta_s(\gamma)$.

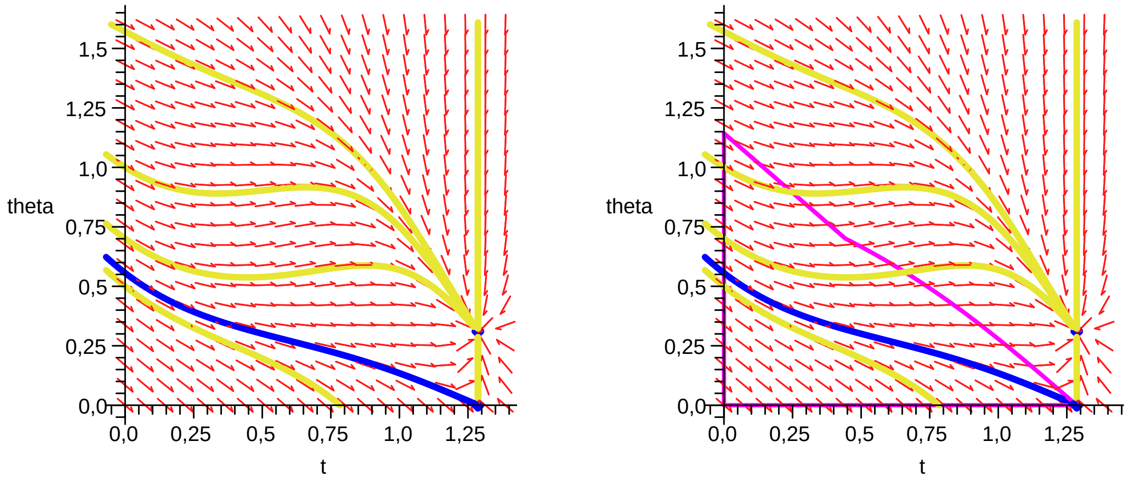

Figure 4: Left: Phase portrait of system (45) for $\gamma = 1.2$ in the plane (t, θ). The dark curve corresponds to W^s. Right: The domain $\mathcal{R}_0$ for $\gamma = 1.2$.

Remark 3. *Using (34) and (35) we have*

$$\frac{y_{f1}(t)}{r(t)x_{f1}(t)} \to 0, \quad \textit{as} \quad t \to t_+.$$

Then in the blown up variables, the solution $x_{f1}(t), y_{f1}(t)$ of (45) projects on W^s in the (t, θ)-plane. We note that any other solution of (41) defined in a small neighbourhood of t_+, with $t - t_+ < 0$ will project in an orbit of (45) which goes to an attractor as t tends to t_+. Our purpose is to prove that for any $\gamma > 1, 0 < \theta_s(\gamma) < \pi/2$. This would imply that $0 < \dfrac{y_{f1}(0)}{r(0)x_{f1}(0)} < \infty$ for any $0 \leq t < t_+$ and then, $x_{f1}(0) \neq 0$ and $y_{f1}(0) \neq 0$.

Lemma 6. *Let be $1 < \gamma \leq 7/5$. Then $0 < \theta_s(\gamma) < \pi/2$.*

Proof Using Lemma 5, $t_+(\gamma) < t_+(7/5) < \pi/2$ for any $1 < \gamma \leq 7/5$.

Let us introduce

$$f(t) = f(t; \gamma) := \arctan(t_+ - t), \quad 0 \leq t \leq t_+. \tag{46}$$

Assume γ is fixed. Let A be the point $(t, \theta) = (t_1, f(t_1))$ where t_1 is defined in (44). We define

$$\begin{aligned} L_1 &= \{(t, \theta) \,|\, 0 \leq t \leq t_+, \ \theta = 0\}, \\ L_2 &= \{(t, \theta) \,|\, t_1 \leq t \leq t_+, \ \theta = \arctan(t_+ - t)\}, \\ L_3 &= \{(t, \theta) \,|\, 0 \leq t \leq t_1, \ \theta = -t + t_1 + f(t_1)\}, \\ L_4 &= \{(t, \theta) \,|\, t = 0, \ 0 \leq \theta \leq t_1 + f(t_1)\}. \end{aligned}$$

We shall denote by $\mathcal{R}_0$ the closed domain bounded by L_i, $i = 1, 2, 3, 4$ (see Figure 4).

We claim that for any $1 < \gamma \leq 7/5$

$$t_1 + f(t_1; \gamma) < \pi/2. \tag{47}$$

Then, $\mathcal{R}_0 \subset \mathcal{R}$. To prove (47) we note that the graph of $f(t; \gamma)$ is below the straight line $\theta = -t + \pi/2$ for values of $\gamma > 1$ but near 1. A tangency occurs at $t = t_+(\gamma)$ for $\gamma = \gamma_t$ such that $t_+(\gamma_t) = \pi/2$. Then $\gamma_t > 7/5$.

A local analysis of (45) at P_0 shows that in a neighbourhood of P_0, W^s is contained in $\mathcal{R}_0$. We shall prove that if $0 \leq t < t_+$, then W^s is contained in $\mathcal{R}_0$ and then $0 < \theta_s(\gamma) < \pi/2$.

Let (t_0, θ_0) be a point in the boundary of $\mathcal{R}_0$, and $(t(s), \theta(s))$ the solution of (45) such that $t(0) = t_0, \theta(0) = \theta_0$. We say that (t_0, θ_0) is an exit point of $\mathcal{R}_0$ if and only if there exists $s_0 > 0$ such that

$$(t(s), \theta(s)) \in \mathcal{R}_0 \qquad \text{for} \qquad -s_0 \leq s \leq 0$$

and

$$(t(s), \theta(s)) \notin \mathcal{R}_0 \qquad \text{for} \qquad 0 < s \leq s_0.$$

We shall prove that the points in L_1, L_2 and L_3, except the equilibrium P_0, are exit points of $\mathcal{R}_0$. Then following backwards in time, W^s intersect $t = 0$ at some point of L_4. This will finish the proof of the lemma.

It is clear that the points of L_1 with $0 < t < t_+$ are exit points of $\mathcal{R}_0$. Let us consider a point of L_2. The scalar product of the vector field and the gradient of the function $g(t, \theta) := \theta - \arctan(t_+ - t)$ on a point of L_4 is given by

$$\frac{1}{1 + (t_+ - t)^2} r(t) - r(t) \cos^2 \theta - \dot{r}(t) \sin \theta \cos \theta - \sin^2 \theta = - \sin \theta \cos \theta (\dot{r}(t) + t_+ - t).$$

Then, $(t, \theta) \in L_2$ with $t_1 < t < t_+$ is an exit point of $\mathcal{R}_0$, if and only if

$$h_2(t; \gamma) := \dot{r}(t) + t_+ - t < 0.$$

For a fixed value of γ, one has $h_2(0; \gamma) = t_+ > 0$, $h_2(t_+; \gamma) = \dot{r}(t_+) < 0$. Moreover, using that $\ddot{r}(t) = 1 + kr(1 - r)$, we get $\dfrac{dh_2}{dt} = \ddot{r}(t) - 1 = kr(1 - r)$. Therefore, at $t = t_1$, h_2 has a negative minimum and $h_2(t; \gamma) < 0$ for any $t_1 < t < t_+$.

In a similar way, $(t, \theta) \in L_3$ is an exit point of $\mathcal{R}_0$, if and only if

$$h_3(t; \gamma) := (r(t) - 1) \sin^2 \theta - \dot{r}(t) \sin \theta \cos \theta > 0, \quad 0 \leq t \leq t_1.$$

The inequality above holds because $r(t) \geq 1$ for $0 \leq t \leq t_1$, with $r(t_1) = 1$, and $\dot{r}(t) < 0$. This finishes the proof. $\square$

We recall that as γ goes to ∞, $t_+(\gamma)$ becomes unbounded and the arguments used in above lemma do not apply in this case. To deal with big values of γ we perform an scaling of variables and time in (41). The scaling is the one used in Subsection 6.1 adapted to the present variables. We introduce

$$X = k^{-1} x, \quad Y = k^{-1/4} y, \quad \sigma = k^{1/4} t$$

and recalling $k = 4/(\gamma^2 - 1)$ we obtain

$$\frac{dX}{d\sigma} = \hat{r}^{-2} Y, \qquad \frac{dY}{d\sigma} = -\hat{r} X, \tag{48}$$

where

$$\hat{r}(\sigma) = \sqrt{k} \left(\frac{1}{2}(1 - \gamma) + \frac{3\gamma}{2 \cosh^2(\hat{\beta}\sigma)} \right), \quad \hat{\beta} = \hat{\beta}(\gamma) = \frac{1}{\sqrt{2}} \left(\frac{\gamma^2}{\gamma^2 - 1} \right)^{1/4}. \tag{49}$$

In fact, $\hat{r}(\sigma) = \sqrt{k} r(t)$. The system (48) has a singularity when $\hat{r}(\sigma) = 0$ that is, at $\sigma = \sigma_+$

$$\sigma_+ = \sigma_+(\gamma) = \left(\frac{4}{\gamma^2 - 1}\right)^{1/4} t_+.$$

A plot of $\sigma_+(\gamma)$ shows that it is bounded (see Figure 5).

Lemma 7. *Let us assume $\gamma > 7/5$. Then $\sigma_+(\gamma)$ is a decreasing function such that* $\lim_{\gamma\to\infty} \sigma_+(\gamma) = \frac{\sqrt{2}}{2} \ln\left(\frac{1+\sqrt{2/3}}{1-\sqrt{2/3}}\right) = 1.62099\ldots$.

Proof We compute

$$\frac{d\sigma_+}{d\gamma} = \frac{1}{2\sqrt{2}\gamma^{3/2}(\gamma^2-1)^{3/4}}\left(\ln Y(\gamma) - 2\sqrt{3}(\gamma+1)\sqrt{\frac{\gamma}{1+2\gamma}}\right),$$

where $Y(\gamma)$ is defined in (43). Then $Y(\gamma) \leq Y(7/5)$. The polynomial

$$p(\gamma) = 12\gamma(\gamma+1)^2 - (1+2\gamma)(\ln Y(7/5))^2$$

has a unique positive zero for some $\gamma < 1$. Then

$$\ln Y(\gamma) - 2\sqrt{3}(\gamma+1)\sqrt{\frac{\gamma}{1+2\gamma}} < \ln Y(7/5) - 2\sqrt{3}(\gamma+1)\sqrt{\frac{\gamma}{1+2\gamma}} < 0.$$

$\square$

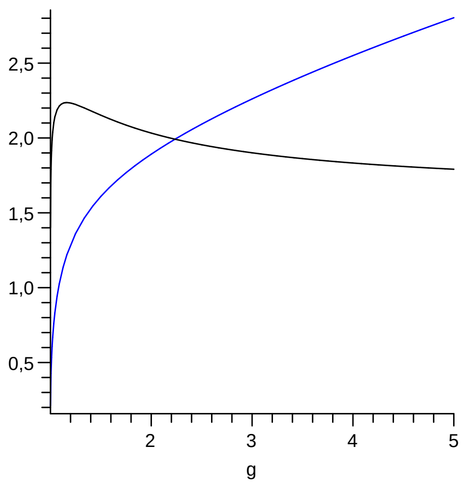

Figure 5: Plot of $t_+(\gamma)$, the increasing function, and $\sigma_+(\gamma)$.

As before we introduce polar coordinates and a new independent variable, $\hat{s}$,

$$\hat{r}X = \hat{R}\cos\varphi, \quad Y = \hat{R}\sin\varphi, \quad d\hat{s} = \hat{r}^{-1}dt.$$

Following the same steps as before we get an autonomous planar system for σ, φ which can be extended analytically to the singularity $\sigma = \sigma_+$

$$\begin{aligned} \frac{d\sigma}{d\hat{s}} &= \hat{r}(\sigma), \\ \frac{d\varphi}{d\hat{s}} &= -\hat{r}(\sigma)\cos^2\varphi - \hat{r}'(\sigma)\cos\varphi\sin\varphi - \sin^2\varphi, \end{aligned} \tag{50}$$

where $\hat{r}'(\sigma) = \frac{d\hat{r}}{d\sigma}$.

The system (50) has two equilibrium points in

$$\hat{\mathcal{R}} = \{(\sigma, \varphi) \,|\, 0 \le \sigma \le \sigma_+, -\pi/2 \le \varphi \le \pi/2\}.$$

We are interested in the behaviour of the stable manifold of the saddle point $\hat{P}_0$, $(\sigma, \varphi) = (\sigma_+, 0)$. Let us denote by $\varphi_s(\gamma)$ the value of φ at the intersection point of the left branch, W^s, with $\sigma = 0$ backwards in time.

Lemma 8. *Let be* $\gamma > 7/5$. *Then* $0 < \varphi_s(\gamma) < \pi/2$.

Proof We introduce

$$f_1(\sigma) = f_1(\sigma;\gamma) := \arctan(\sigma_+(\gamma) - \sigma), \quad 0 < \sigma < \sigma_+(\gamma), \quad f_2(\sigma) := -\frac{\pi}{5}\sigma + \frac{\pi}{2}.$$

Using Lemma 7 we have $\sigma_+(\gamma) < \sigma_+(7/5) < 2.2$ for $\gamma > 7/5$. So we can define the following sets (see Figure 6)

$$L_1 = \{(\sigma, \varphi) \,|\, 0 \le \sigma \le \sigma_+, \ \varphi = 0\},$$

$$L_2 = \{(\sigma, \varphi) \,|\, 1 \le \sigma \le \sigma_+, \ \varphi = f_1(\sigma)\},$$
$$L_3 = \{(\sigma, \varphi) \,|\, 0 \le \sigma \le 1, \ \varphi = f_2(\sigma)\},$$
$$L_4 = \{(\sigma, \varphi) \,|\, \sigma = 0, \ 0 \le \varphi \le \pi/2\},$$
$$L_5 = \{(\sigma, \varphi) \,|\, \sigma = 1, \ f_1(1) \le \varphi \le f_2(1)\}.$$

We note that L_5 is well defined because $f_1(1) \le f_2(1)$. To prove this inequality we use that $f_1(\sigma;\gamma)$ is a decreasing function of γ. In particular by taking $\sigma = 1$, we get

$$f_1(1;\gamma) < f_1(1;7/5) < \frac{3\pi}{10} = f_2(1).$$

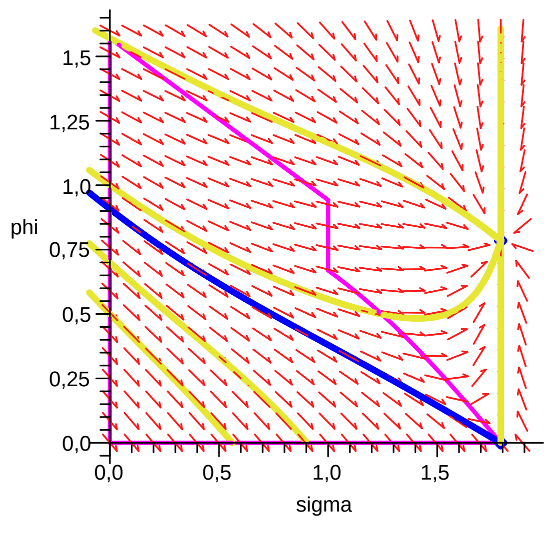

Figure 6: Phase portrait and domain $\hat{\mathcal{R}}_0$ for the system (50) in the plane (σ, φ) .

Let us denote by $\hat{\mathcal{R}}_0$ the domain in $\hat{\mathcal{R}}$ bounded by L_i for $i = 1, \ldots, 5$. We define exit points of $\hat{\mathcal{R}}_0$ as in the proof of Lemma 6.

We claim that for $\gamma > 7/5$, the points of L_1, L_2, L_3 and L_5, except the equilibrium point, are exit points of $\hat{\mathcal{R}}_0$.

A local analysis of the equilibrium point $\hat{P}_0$ shows that in a neighbourhood of $\hat{P}_0$, W^s is contained in $\hat{\mathcal{R}}_0$. We shall prove that this is true for any $0 \leq \sigma < \sigma_+$. Therefore going backwards in time, W^s leaves $\hat{\mathcal{R}}_0$ through some point in L_4 and then $0 < \varphi_s(\gamma) < \pi/2$.

It is clear that the orbits leave $\hat{\mathcal{R}}_0$ through L_1 and L_5. Let us consider a point $(\sigma, f_1(\sigma)) \in L_2$. Following the same steps as in the proof of Lemma 6, the condition for an exit point is

$$\hat{h}_2(\sigma;\gamma) := \hat{r}'(\sigma) + \sigma_+ - \sigma < 0 \qquad \text{for} \quad 1 \leq \sigma < \sigma_+. \tag{51}$$

However, $\hat{h}_2(0;\gamma) > 0$, $\hat{h}_2(\sigma_+;\gamma) < 0$ and $\hat{h}_2$ has a unique zero in the range $0 < \sigma < \sigma_+$. Moreover

$$\hat{h}_2(1;\gamma) < -\frac{6}{\sqrt{2}} \frac{\tanh(\hat{\beta}(7/5))}{\cosh^2(\hat{\beta}(7/5))} + \sigma_+(7/5) - 1 < 0.$$

Then (51) follows.

For a point of L_3, the condition to be an exit point of $\hat{\mathcal{R}}_0$ is the following

$$\hat{h}_3(\sigma;\gamma) := \left(\frac{\pi}{5} - \sin^2(\pi\sigma/5)\right) \hat{r}(\sigma) - \hat{r}'(\sigma)\sin(\pi\sigma/5)\cos(\pi\sigma/5) - \cos^2(\pi\sigma/5) > 0$$

for $0 \leq \sigma \leq 1$. However $\hat{\beta}(\gamma) < \hat{\beta}(7/5) < 0.9$ and then $\hat{h}_3(\sigma;\gamma) > H(\sigma)$ where

$$\begin{aligned} H(\sigma) \;&:=\; \left(\frac{\pi}{5} - \sin^2(\pi\sigma/5)\right)\left(-1 + \frac{3}{\cosh^2(0.9\sigma)}\right) + \\ &\frac{6}{\sqrt{2}} \frac{\tanh(\sigma/\sqrt{2})}{\cosh^2(0.9\sigma)} \sin(\pi\sigma/5)\cos(\pi\sigma/5) - \cos^2(\pi\sigma/5). \end{aligned}$$

Now, it is a simple exercise to prove that $H(\sigma) > 0$ for $0 \leq \sigma \leq 1$. However to make easier the reading we sketch a proof. We write

$$\cosh^2(0.9\sigma) H(\sigma) = p(\sigma) + q(\sigma),$$

where

$$p(\sigma) = \frac{3\pi}{5} - \left(\frac{\pi}{5} + \cos\left(\frac{2\pi\sigma}{5}\right)\right)\cosh^2(0.9\sigma), \quad q(\sigma) = \frac{3}{\sqrt{2}}\tanh\left(\frac{\sigma}{\sqrt{2}}\right)\sin\left(\frac{2\pi\sigma}{5}\right) - 3\sin^2\left(\frac{\pi\sigma}{5}\right).$$

Figure 7 shows a plot of the functions p and q.

We have that p is decreasing in $[0,1]$ and $p(\sigma) \geq 0$ if $\sigma \in [0, 0.8]$. Moreover, if $\sigma \in [0,1]$, q is positive and has a unique maximum at some point greater than 0.8. Hence, for $\sigma \in [0, 0.8]$ one has $p(\sigma) + q(\sigma) \geq p(0.8) = 0.00555... > 0$. For $\sigma \in [0.8, 1]$ we have $p(\sigma) + q(\sigma) > p(1) + \min\{q(0.8), q(1)\} = 0.151... > 0$.

To prove that $p(\sigma)$ is a decreasing function, we note that $p'(\sigma) = 0$ implies $\tanh(0.9\sigma) = (2\pi/5)\sin(2\pi\sigma/5)/\left(1.8\left(\pi/5 + \cos(2\pi\sigma/5)\right)\right)$. The function on the left side is increasing and concave, and the function on the right side, is increasing and convex. After evaluating these two functions at $\sigma = 1$ one can see that the equation above do not have any solution for $\sigma \in (0,1]$. So, $p'(\sigma)$ does not change sign. Indeed it is negative.

The claim for $q(\sigma)$ follows easily from the fact that $q'(\sigma)/\sin(2\pi\sigma/5)$ is a decreasing function for $\sigma \in (0,1]$ and $q'(0.8) > 0$, and $q'(1) < 0$.

This ends the proof. $\square$

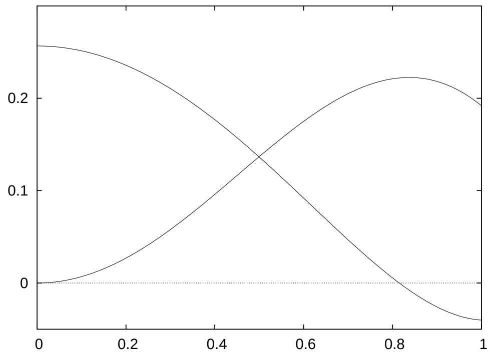

Figure 7: The function p (decreasing) and q (with a maximum) used to prove $H > 0$.

7 Appendix

Coefficients involved in (31):

$$r_{-2} = -\frac{\alpha}{\beta^2}, \quad r_{-1} = 0, \quad r_0 = \frac{\alpha}{3} + \rho, \quad r_1 = 0, \quad r_2 = -\frac{\alpha\beta^2}{15}, \quad r_3 = 0,$$
$$r_4 = \frac{2}{189}\alpha\beta^4, \quad r_5 = 0, \quad r_6 = -\frac{1}{675}\alpha\beta^6,$$
$$e_{-3} = \frac{15t_*}{8\beta^2}, \quad e_{-2} = e_{-1} = e_0 = 0, \quad e_1 = -\frac{\beta^2 t_*}{8}, \quad e_2 = 0,$$
$$e_3 = \frac{5}{126}\beta^4 t_*, \quad e_4 = \frac{\beta^4}{7}, \quad e_5 = -\frac{1}{120}\beta^6 t_*, \quad e_6 = 0, \quad e_7 = \frac{1}{693}\beta^8 t_*,$$
$$d_i = -\frac{1}{2\alpha\beta^2}(i+1)r_{i+1}, \quad i \geq 0, \quad d_{-3} = \frac{1}{\alpha\beta^2}r_{-2}, \quad d_{-2} = d_{-1} = 0,$$

where $\alpha = 3\gamma/2, \beta = \sqrt{\gamma/(\gamma^2-1)}, \gamma = \sqrt{1+4/k}$.

Coefficients involved in (34):

$$r_1 = 2\beta\rho\tanh(\beta t_+), \quad r_2 = \frac{1}{2}, \quad r_3 = \frac{k}{6}r_1,$$
$$r_4 = -\frac{k^2}{288}(\gamma-1)(4\gamma^2-5\gamma-5), \quad r_5 = \frac{k(k-6)}{120}r_1,$$
$$e_0 = -\frac{(6\gamma^2+5\gamma-5)}{8\gamma(\gamma-1)} + \frac{5r_1}{8\gamma}t_+, \quad e_1 = \frac{3}{8}\frac{(5-10\gamma+9\gamma^2)}{\gamma(\gamma-1)^2}r_1 + \frac{5}{8\gamma}t_+,$$
$$e_2 = -\frac{1}{16}\frac{k(6\gamma^2+5\gamma-5)}{\gamma(\gamma-1)} + \frac{5k}{16\gamma}r_1 t_+,$$
$$e_3 = \frac{kr_1(25-50\gamma+25\gamma^2+12\gamma^3)}{48\gamma(\gamma-1)^2} - t_+\frac{5k^2(\gamma-1)(4\gamma^2-5\gamma-5)}{576\gamma},$$
$$e_4 = -\frac{k^2(-25+25\gamma+45\gamma^2-63\gamma^3+30\gamma^4)}{384\gamma(\gamma-1)} - t_+\frac{5k^2 r_1(3\gamma^2-5)}{384\gamma},$$
$$e_5 = -\frac{k^2 r_1(-175+350\gamma-70\gamma^2-210\gamma^3+81\gamma^4)}{1920\gamma(\gamma-1)^2} - t_+\frac{k^3(\gamma-1)(9\gamma^3+49\gamma^2-35\gamma-35)}{4608\gamma},$$

where $\rho = (1-\gamma)/2$.

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References

[1] H. Broer, C. Simó, Resonance tongues in Hill's equations: a geometric approach, *J. Diff. Equations* **166** (2000) 290–327.

[2] E. Fontich, C. Simó, The Splitting of Separatrices for Analytic Diffeomorphisms, *Erg. Th. & Dyn. Systems* **10** (1990) 295–318.

[3] V.G. Gelfreich, A proof of the exponentially small transversality of the separatrices for the standard map, *Comm. Math. Phys.* **201** (1999) 155–216.

[4] V.G. Gelfreich, C. Simó, High-precision computations of divergent asymptotic series and homoclinic phenomena, *Discrete and Continuous Dyn. Syst. B* **10** (2008) 511–536.

[5] M. Hénon, C. Heiles, The applicability of the third integral of motion: Some numerical experiments, *Astronom. J.* **69** (1964) 73–79.

[6] H. Ito, Non-integrability of the Hénon-Heiles system and a theorem of Ziglin, *Koday Math. J.* **8** (1985) 129–138.

[7] V.F. Lazutkin, Splitting of separatrices for the Chirikov standard map, *J. Math. Sci.* **128** (2005), 2687–2705 (translated from the Russian manuscript originally published in VINITI 1984).

[8] J. Llibre, C. Simó, Homoclinic phenomena in the three-body problem, *J. Diff. Eq.* **37** (1980) 444–465.

[9] J. Llibre, C. Simó, Oscillatory solutions in the planar restricted three-body problem, *Math. Ann.* **248** (1980) 153–184.

[10] A.J. Maciejewski, M. Przybylska, J.-A. Weil, Non-integrability of the generalized spring-pendulum problem, *J. Phys. A: Math. Gen.* **37** (2004) 2579–2597.

[11] R. Martínez, C. Pinyol, Parabolic orbits in the elliptic restricted three body problem, *J. of Differential Equations* **111** (1994), 299–339.

[12] R. Martínez, C. Simó, A note on the existence of heteroclinic orbits in the planar three–body problem, in *Seminar on Dynamical Systems*, S. Kuksin, V. Lazutkin, J. Pöschel, editors, Birkhäuser 1993, 129–139.

[13] R. Martínez, C. Simó, Non-integrability of the degenerate cases of the Swinging Atwood's Machine using higher order variational equations, 2009, in preparation.

[14] R. Martínez, C. Simó, Efficient numerical implementation of integrability criteria based on high order variational equations, 2009, in preparation.

[15] J.J. Morales-Ruiz, *Differential Galois Theory and Non-Integrability of Hamiltonian systems*, Progress in Mathematics vol. 179, Birkhäuser 1999.

[16] J.J. Morales-Ruiz, J-P. Ramis, Galoisian obstructions to integrability of Hamiltonian systems, *Methods and Applications of Analysis* **8** (2001) 33–96.

[17] J.J. Morales-Ruiz, J-P. Ramis, Galoisian Obstructions to integrability of Hamiltonian Systems II, *Methods and Applications of Analysis* **8** (2001) 97–112.

[18] J.J. Morales-Ruiz, J-P. Ramis, C. Simó, Integrability of Hamiltonian Systems and Differential Galois Groups of Higher Variational Equations, *Ann. Scient. Éc. Norm. Sup.* 4^e série **40** (2007) 845–884.

[19] J. J. Morales, C. Simó, S. Simón, Algebraic proof of the non-integrability of Hill's Problem, *Ergodic Theory and Dynamical Systems* **25** (2005), 1237–1256.

[20] J. Moser, *Stable and Random Motions in Dynamical Systems*, Annals of Mathematics Studies, Princeton Univ. Press 1973.

[21] H. Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, Gauthier–Villars, 1892, 1893, 1899.

[22] C. Simó, Curves of Perturbations of Non-Twist Integrable Area Preserving Maps, *Regular and Chaotic Dynamics* **3** (1998) 180–195.

[23] C. Simó, A. Vieiro, Resonant zones, inner and outer splittings in generic and low order resonances of APM, to appear in *Nonlinearity* (2009).

[24] S.L. Ziglin, Branching of solutions and non-existence of first integrals in Hamiltonian mechanics I, *Funct. Anal. Appl.* **16** (1982) 181–189.