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STATION KEEPING CLOSE TO UNSTABLE EQUILIBRIUM POINTS WITH A SOLAR SAIL

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We have considered the movement of a solar sail in the Sun - Earth system. Using the Circular RTBP adding the solar radiation pressure as a model we have a 2D family of equilibrium points parametrised by the two angles defining the sail orientation. Most of these points are unstable and require a control strategy to maintain a sail on a small neighbourhood of the fixed point the for a long time. The knowledge of the variation of the invariant manifolds with respect to the sail orientation has permitted us design a control strategy to keep its trajectory close to one of these unstable points. This strategy has been tested for two known missions, the Polar Observer and the Geostorm Warning Mission. Simulations up to 30 years have been done taking into account errors in the position determination and on the sail orientation. In this paper we present details on the implementation of the control strategy and the results obtained for both missions.

INTRODUCTION

Solar Sailing is a proposed form of spacecraft propulsion using large membrane mirrors. The impact of the photons emitted by the Sun on the surface of the sail and their further reflection produce momentum on it. Although the acceleration produced by this reflection is smaller than the one achieved by a 'traditional' spacecraft it is continuous and unlimited. These makes long term missions more accessible.

It is well know that a solar sail is an orientable surface, the orientation of the sail is defined by two angles, the pitch (α) and yaw (δ) angle. Another important parameter is the sail lightness number (β) used to define the sail's effectiveness. In this paper we have considered that the sail is a perfectly reflecting surface so the force due to the solar radiation pressure is normal to the surface of the sail.

To model the dynamics of the sail we have taken the Sun - Earth Restricted Three Body Problem (RTBP) and added the solar radiation pressure effect. This model is a perturbation on the RTBP and depends on three parameters,, the sail lightness number β and the two angles defining the sail orientation α, δ. It is well know that the RTBP has five equilibrium points ($L_{1,\dots,5}$). For a small β these five equilibrium points are replaced by five continuous families of equilibria parametrised by the sail orientation (α, δ). As β

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increases 4 of these families connect and then the fixed points form 2 connected surfaces S_1 and S_2. In[4] it can be seen that S_1 is diffeomorfic to a sphere and S_2 is diffeomorfic to a torus.

We can classify these fixed points by their stability. In this paper we will focus on those fixed points that are unstable and that the differential of the flow on the fixed point has one pair of real eigenvalues and two pair of complex eigenvalues. Although the complex eigenvalues can have real positive part this one will be very small compared with the instability produced by the real eigenvalues. Hence, as a first approximation we will suppose that the linear dynamic around these fixed point is saddle $\times$ centre $\times$ centre. The real effect will be considered later.

Let p_0 be a fixed point for $\alpha = \alpha_0$ and $\delta = \delta_0$. Take $\vec{v}_1, \vec{v}_2$ the stable and unstable eigenvectors, $\vec{v}_3, \vec{v}_4$ and $\vec{v}_5, \vec{v}_6$ the real and complex part of the two complex eigenvectors. It is well know that $V = \{\vec{v}_i\}_{i=1,\ldots 6}$ are a basis and the couple $\{p_0; V\}$ is a reference system. From now on, the trajectory followed by the probe can be written in terms of this reference system, which will help us understand the linear dynamics around the equilibrium point.

When the probe is close to the fixed point p_0, the trajectory escapes along the unstable direction. We want to change the sail orientation $\alpha = \alpha_1, \delta = \delta_1$ so that the unstable direction of the new equilibrium point brings the trajectory close to the stable direction of p_0. Then we will restore the original orientation and so on. The projection of the trajectory on the central behaviour are rotations around the different equilibrium points. This process can produce an unbounded trajectory, so we have to take into account the central behaviour when choosing the new sail orientation.

In the literature we can find two different missions that need to keep a solar sail close to an unstable fixed point. These two missions are the Geostorm Enhanced Warning[1–4,7] and the Polar Observer[3,4] missions. We will give a small overview of the main objectives of these two missions and we will apply our control strategy to them. A 1000 simulations with random initial conditions have been done and the results are successful. As we will see for all of these initial conditions the control strategies manages to keep the sail close to the equilibrium point.

Finally, we have tested our control strategy with different fonts of errors. We have introduced errors on the sail orientation and errors in the position and velocity determination. The errors on the sail orientation will affect to the probe's trajectory and the second type of errors will affect to the decisions taken by the control strategy. We have made 1000 simulations with the same initial conditions as before including these two effects. We will discus the results.

EQUATIONS OF MOTION

To describe the dynamics of a solar sail we have supposed that Earth and Sun are point masses moving in a circular orbit around their common centre of mass. The sail is under the gravitational attraction of these two bodies and the effect of the solar radiation pressure. The units of mass, distance and time have been normalised so that the total mass of the system is 1, the Sun - Earth distance is 1 and the period of the orbit is 2π. With these units, the gravitational constant is also 1. We use a rotating reference system so that Earth and Sun are fixed on the x axis, z is perpendicular to the ecliptic plane and

y defines a orthogonal positive oriented reference system (Figure 1, left).

The force of the solar radiation pressure depends on the position of the sail, the orientation and the characteristic acceleration of the sail. The orientation of the sail is defined by two angles, say α and δ: α is the angle between the Sun - line and the projection on the ecliptic plane of the normal vector to the sail $\vec{n}$; δ is the angle between the ecliptic plane and $\vec{n}$ (see Figure 1, right). As the vector $\vec{n}$ cannot point towards the Sun then $\alpha \in [-\pi/2, \pi/2]$ and $\delta \in [-\pi/2, \pi/2]$. There are other possibilities to define these angles, see[1,4,5].

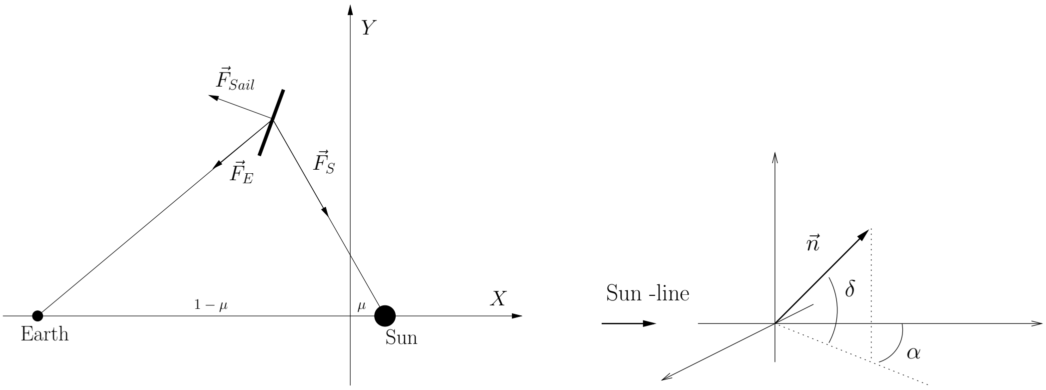

Figure 1 Left: Relation between the forces due to the sail and the gravitational attraction of the Earth and Sun. Right: Relation between the sail's angles (α and δ) and the the Sun - line

In the rotating frame, the equations of motion for the probe are

$$
\begin{aligned}
\ddot{x} &= 2\dot{y} + x - (1-\mu)\frac{x-\mu}{r_{PS}^3} - \mu\frac{x+1-\mu}{r_{PT}^3} + \kappa\cos(\phi+\alpha)\cos(\psi+\delta), \\
\ddot{y} &= -2\dot{x} + y - \left(\frac{1-\mu}{r_{PS}^3} + \frac{\mu}{r_{PT}^3}\right) y + \kappa\sin(\phi+\alpha)\cos(\psi+\delta), \qquad (1) \\
\ddot{z} &= -\left(\frac{1-\mu}{r_{PS}^3} + \frac{\mu}{r_{PT}^3}\right) z + \kappa\sin(\psi+\delta),
\end{aligned}
$$

where $\kappa = \beta\dfrac{1-\mu}{r_{PS}^2}\cos^2\alpha\cos^2\delta$ is the force exerted by the sail with β as the sail lightness number. The angles ϕ, ψ refer to the position of the probe w.r.t. the Sun. They are given by

$$
\phi = \arctan\left(\frac{y}{x-\mu}\right), \qquad \psi = \arctan\left(\frac{z}{\sqrt{(x-\mu)^2+y^2}}\right), \qquad (2)
$$

with $\phi \in [-\pi, \pi]$ and $\psi \in [-\pi/2, \pi/2]$. Note that the equations of motion depend on three parameters: β, α and δ. It is clear that if $\beta = 0$ or $\alpha = \pm\pi/2$ or $\delta = \pm\pi/2$,i.e. sail aligned with the Sun - line, the equations are the same as in the RTBP.

EQUILIBRIUM POINTS

It is well known that the RTBP has five equilibrium points. It is easy to see that for small β, these points are replaced by five continuous families of equilibria parametrised by α, δ. The equations for the fixed points are obtained setting $\dot{x} = \ddot{x} = \dot{y} = \ddot{y} = \dot{z} = \ddot{z} = 0$ in Eq 1. As we have already mentioned, for $\alpha = \pm\pi/2$ or $\delta = \pm\pi/2$ this model coincides with the RTBP and hence, it has five well known equilibrium points $L_{1,\ldots,5}$ (Szebehely[6]).

We have computed these families numerically by means of a continuation method. For β fixed and small we have five disconnected families, as β increases four of these families merge and the fixed points form two disconnected surfaces, S_1 and S_2, parametrised by the two angles α and δ. It can be seen that S_1 is diffeomorphic to a sphere and contains L_2 and that S_2 is diffeomorphic to a torus and is located around the Sun and contains $L_{1,3,4,5}$. In Figures 2 and 3 we can see different slices of this surface for different values of β.

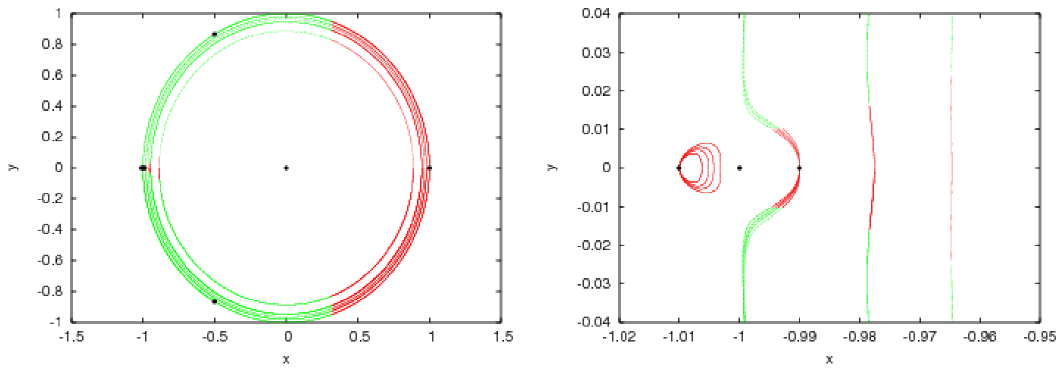

Figure 2 Equilibrium points in the $\{x, y\}$- plane for $\beta_1 = 0.06, \beta_2 = 0.1, \beta_3 = 0.15, \beta_4 = 0.3$. The red points have a pair of real eigenvalues and two pair of complex eigenvalues and the green points have three complex eigenvalues.

McInnes[4] has shown that the equilibrium points of this system are in general unstable but controllable. We can classify the fixed points in three classes depending on the eigenvalues of the differential matrix of the flow on the fixed point. The first class includes the equilibrium points that have three pair of complex eigenvalues. The second one includes those with one pair of complex eigenvalues and two pair of complex eigenvalues and the third class includes the ones with two pair of real eigenvalues an a pair of complex eigenvalues. In this work we will describe a control strategy around the equilibrium points of the second class.

Linearisation Around the Equilibrium Points

From now on the value of sail lightness number (β) will be considered fixed. Hence, the fixed points are parametrised by α and δ. We are interested in knowing the variation of the fixed points when we change α and δ.

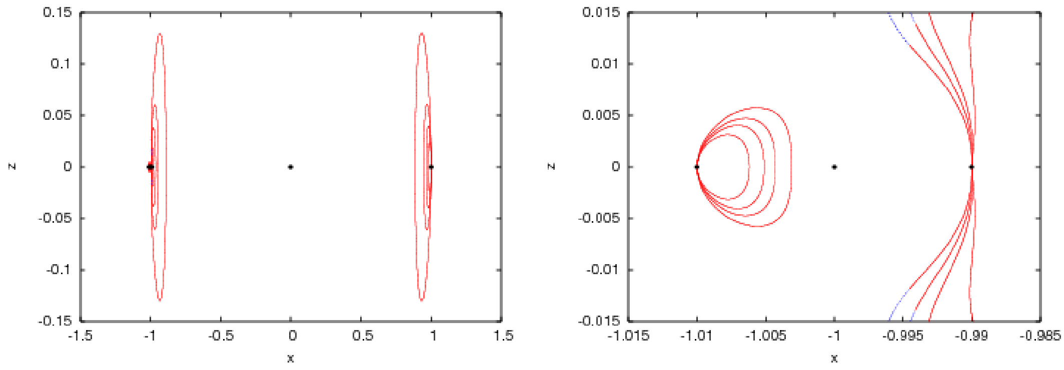

Figure 3 Equilibrium points in the $\{x, z\}$ plane for $\beta_1 = 0.06, \beta_2 = 0.1, \beta_3 = 0.15, \beta_4 = 0.3$. The red points have a pair of real eigenvalues and two pair of complex eigenvalues and the blue points have two pair of real eigenvalues and one pair of complex eigenvalues.

Let $p_0 = p(\alpha_0, \delta_0)$ be the coordinates of a fixed point of $\dot{X} = f(X, \alpha, \delta)$, that is, $f(p_0, \alpha_0, \delta_0) = 0$. The Implicit Function Theorem implies that $\frac{\partial p}{\partial \alpha}(\alpha_0, \delta_0)$ and $\frac{\partial p}{\partial \delta}(\alpha_0, \delta_0)$ are found by solving,

$$\begin{aligned} D_X f(p_0, \alpha_0, \delta_0) \frac{\partial p}{\partial \alpha}(\alpha_0, \delta_0) &= -\frac{\partial f}{\partial \alpha}(p_0, \alpha_0, \delta_0), \\ D_X f(p_0, \alpha_0, \delta_0) \frac{\partial p}{\partial \delta}(\alpha_0, \delta_0) &= -\frac{\partial f}{\partial \delta}(p_0, \alpha_0, \delta_0). \end{aligned}$$

As we do not know explicitly $p(\alpha, \delta)$ and the variation on the sail orientation will be small we will deal with its linear approximation. If $p(\alpha_0, \delta_0)$ is a given fixed point then,

$$p(\alpha, \delta) = p(\alpha_0, \delta_0) + Dp \cdot h, \tag{3}$$

where $h = (\alpha - \alpha_0, \delta - \delta_0)^T$ and $Dp = \left(\frac{\partial p}{\partial \alpha}(\alpha_0, \delta_0), \frac{\partial p}{\partial \delta}(\alpha_0, \delta_0)\right)$.

STATION KEEPING

In this section we will focus on a linearly unstable equilibrium point and we will use the information of its local dynamics to design a control strategy. The idea is to change the sail orientation (i.e. the phase space) to make the system act as we wish. When the spacecraft is in a neighbourhood of a fixed point the linear approximation gives an accurate description of the dynamics. As we have already said we are interested in controlling unstable points that have two real eigenvalues ($\pm\lambda$) and two pairs of complex eigenvalues. One of the pairs of complex eigenvalues can have non-zero real part ($\nu_1 \pm i\omega_1$) and the other pair of eigenvalues is purely imaginary ($\pm\omega_2$).

From now on we will make the assumption that $\nu_1 = 0$, supposing that the dynamic given in this central direction is a rotation around the origin instead of spiralling inwards or outwards, depending on the sign of ν_1. If $\nu_1 < 0$ this supposition just adds more difficulties on the central behaviour as this is naturally stable. Instead, if $\nu_1 > 0$ our assumption might make the control strategy fail in the real case. But we will see that the control strategy designed reduce the amplitude of the central behaviour. If this reduction is bigger than the expansion that the spiral experiences we will be able to control the sail. Besides, this spiralling is really small and it is not very relevant for short times. Hence, here we assume that the linear dynamics around the fixed points is saddle $\times$ centre $\times$ centre.

Let p_0 be the fixed point for $\alpha = \alpha_0$ and $\delta = \delta_0$. If the sail is close to p_0 its trajectory will escape through the unstable direction. We want to change the orientation of the sail ($\alpha = \alpha_1, \delta = \delta_1$) so that the unstable direction of this new fixed point sends the probe back to the neighbourhood of p_0. Then, we restore the initial orientation of the sail, $\alpha = \alpha_0$ and $\delta = \delta_0$, and so on. It is important to note that, during this process, the projection of the dynamics into the central part of the equilibria can grow: as the central behaviour are rotations around each of the fixed points, the composition of central motions with different centre of rotation can result in an unbounded growth of the central component of the motion. For this reason we have to be careful when we chose the sail orientation. We have to be able to control the instability given by the unstable direction and to make sure that the central behaviour does not grow.

As we have already said we are supposing that the linear behaviour of the fixed points is saddle $\times$ centre $\times$ centre. Hence, let $p_0 \in \mathbb{R}^6$ be the fixed point and $\pm\lambda$, $\pm i\omega_1$ and $\pm i\omega_2$ the eigenvalues. If the sail orientation is slightly changed then p_0 is also slightly changed as well as the eigenvalues and eigenvectors.

From now on we will describe the trajectory of the probe by its projection on the three different planes centred on p_0. The first one is generated by the two eigenvector with real eigenvalues ($\vec{v}_1$, $\vec{v}_2$), where the saddle behaviour is described. The other two are generated by the real and imaginary part of the two pairs of complex eigenvectors ($\vec{v}_i$, $\vec{v}_{i+1}$ for $i = 3, 5$). The projection of the orbit on these two planes describes the central behaviour of the motion. In this reference system the trajectory of the probe is given by $(x_1(t), y_1(t), x_2(t), y_2(t), x_3(t), y_3(t))$.

We want to obtain a sail orientation so that the unstable direction of the new fixed point brings the probe back to a neighbourhood close to the initial fixed point p_0. As we know the fixed points live in a 2D surface and we have a 6D phase space, so we have some limitations in the positions of the new fixed point.

For the moment let us suppose that there are no limitations in the position of a new fixed point and we will try to understand the behaviour of the probe when the sail changes in the projection on the saddle plane and on a centre plane. Understanding these behaviour we will be able to design a control strategy.

Behaviour on the Saddle Plane

Suppose that for $\alpha = \alpha_0$ and $\delta = \delta_0$ the fixed point is at the origin so the motion of the saddle part is,

$$\left.\begin{array}{rcl} x_1(t) & = & x_{10}e^{\lambda(t-t_0)} \\ y_1(t) & = & y_{10}e^{-\lambda(t-t_0)} \end{array}\right\}, \tag{4}$$

where (x_{10}, y_{10}) is the initial condition.

When the sail orientation is changed $\alpha = \alpha_0 + \epsilon_\alpha$ and $\delta = \delta_0 + \epsilon_\delta$ the fixed point and

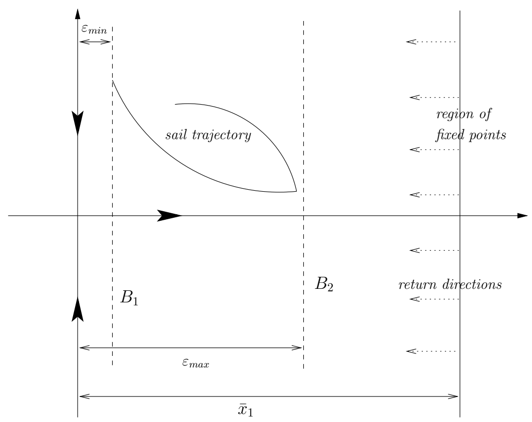

Figure 4 Representation of the important parameters in the control of the saddle part. The bounce region is the location of the future fixed points, we will chose one of them, and the bounce direction are the eigendirections for those fixed points.

the eigenvalues and eigenvectors change slightly. From now on we will just consider that the eigenvectors are the same as the ones at the origin. If $(\bar{x}_1, \bar{y}_1)$ is the new fixed point and $\pm\bar{\lambda}$ are the real eigenvalues for $(\bar{x}_1, \bar{y}_1)$, then the movement of the probe is,

$$\left.\begin{array}{rcl} \bar{x}_1(t) &=& \bar{x}_1 + (\bar{x}_{10} - \bar{x}_1)e^{\bar{\lambda}(t-t_0)} \\ \bar{y}_1(t) &=& \bar{y}_1 + (\bar{y}_{10} - \bar{y}_1)e^{-\bar{\lambda}(t-t_0)} \end{array}\right\}, \tag{5}$$

where $(\bar{x}_{10}, \bar{y}_{10})$ is the initial condition.

To control the saddle behaviour we will define two bounds $B_1 = \{x_1 = \varepsilon_{min}\}$ (the minimal distance to the stable direction) and $B_2 = \{x_1 = \varepsilon_{max}\}$ (the maximal distance to the stable direction), that define the region of movement (between B_1 and B_2). When the trajectory reaches one of these two bounds the sail orientation is changed.

If the sail orientation is fixed to $\alpha = \alpha_0$ and $\delta = \delta_0$ the trajectory followed is given by Eq 4 and goes from B_1 to B_2. When the sail orientation is changed to $\alpha = \alpha_1$ and $\delta = \delta_1$ the trajectory is given by Eq 5 and goes from B_2 to B_1.

In order to control the instability the new fixed point $(\bar{x}_1, \bar{y}_1)$ must satisfy $\bar{x}_1 > \varepsilon_{max}$. As we are supposing that all the eigenvectors are the same then the new fixed points unstable direction will bring the probe back to B_1 (see Figure 4).

Behaviour on a Centre Plane

Suppose that for $\alpha = \alpha_0$, $\delta = \delta_0$ the fixed point is at the origin and let (x_{20}, y_{20}) be the initial condition. Then if $r_0 = \sqrt{x_{20}^2 + y_{20}^2}$ and $\tau_0 = \arctan\left(\frac{y_{20}}{x_{20}}\right)$ the motion on the central plane is,

$$\left.\begin{array}{rcl} x_2(t) &=& r_0 \cos(\omega_1(t-t_0) + \tau_0) \\ y_2(t) &=& r_0 \sin(\omega_1(t-t_0) + \tau_0) \end{array}\right\}. \tag{6}$$

When the sail orientation is changed to $\alpha = \alpha_0 + \epsilon_\alpha$ and $\delta = \delta_0 + \epsilon_\delta$, the fixed point changes as well as the eigenvalues and eigenvectors. As before we will just consider that the eigenvectors are the same as the ones at the origin. If $(\bar{x}_2, \bar{y}_2)$ is the new fixed point and $\pm i\bar{\omega}_1$ are the pair of complex eigenvalues for $(\bar{x}_2, \bar{y}_2)$. Then,

$$\left.\begin{array}{rcl} \bar{x}_2(t) & = & \bar{x}_2 + \bar{r}_0 \cos(\bar{\omega}_1(t-t_0) + \bar{\tau}_0) \\ \bar{y}_2(t) & = & \bar{y}_2 + \bar{r}_0 \sin(\bar{\omega}_1(t-t_0) + \bar{\tau}_0) \end{array}\right\}, \tag{7}$$

where $\bar{r}_0 = \sqrt{(\bar{x}_2 - \bar{x}_{20})^2 + (\bar{y}_2 - \bar{y}_{20})^2}$, $\bar{\tau}_0 = \arctan\left(\dfrac{\bar{y}_2 - \bar{y}_{20}}{\bar{x}_2 - \bar{x}_{20}}\right)$ and $(\bar{x}_{20}, \bar{y}_{20})$ is the initial condition.

The movement in the centre plane will be a sequence of rotations around each of the fixed points. The composition of rotations around different fixed points does not need to be bounded and we would like to place the fixed points so that this movement does not grow. In fact we will find a sequence of fixed points so that the trajectory tends to the origin.

As we have already said, we are assuming that for $\alpha = \alpha_0$, $\delta = \delta_0$ the fixed point is at the origin and the trajectory is an arc starting at the initial condition (x_{20}, y_{20}) and radius $r_0 = \sqrt{x_{20}^2 + y_{20}^2}$. Let (ζ_2, η_2) be the point where the sail orientation is changed, we want to find a fixed point $(\bar{x}_2, \bar{y}_2)$ so that arc described around $(\bar{x}_2, \bar{y}_2)$ ends closer to the origin than (x_{20}, y_{20}).

Depending on the position of $(\bar{x}_2, \bar{y}_2)$ with respect to the (ζ_2, η_2) the arc will or will not be totally included in the disk centred at the origin and radius r_0 (D_0). We are interested in taking fixed point so that the arc described by the probe is totally included in D_0. It is easy to see that if the new fixed points must be placed on line between the origin and (ζ_2, η_2). More over if $(\bar{x}_2, \bar{y}_2) = (\zeta_2/2, \eta_2/2)$ the distance to the origin will be minimised. Figure 5 is a schematic representation of the sequence of fixed points that must be taken so that the centre trajectory tends to the origin.

Choosing the New Sail Orientation (α, δ)

We have found an ideal sequence of fixed points to control the instability of the fixed point. As it has already mentioned the fixed points live on a 2D surface parametrised by α and δ in a 6D phase space. So we might not be able to find a sail orientation α_1 and δ_1 so that the fixed point is one of the described before.

As we do not know explicitly the 2D surface of fixed points $(p(\alpha, \delta))$ and the variation on the sail orientation will be small we will deal with the linear approximation of this surface. So we would like to find $h = (\alpha - \alpha_0, \delta - \delta_0)^T$ such that,

$$\bar{p} - p_0 = Dp \cdot h, \tag{8}$$

where $\bar{p}$ is the desired new fixed point. The position of $\bar{p}$ is as described previously, the saddle part must be fixed on the appropriate side of the saddle and both centres have to be at the middle distance between the position of the probe at the change moment and p_0. Notice that Eq 8 is Eq 3 rewritten and that it has 6 equations and 2 unknowns. We would like to find α_1, δ_1 such that $\|\bar{p} - p(\alpha_1, \delta_1)\|$ is small enough.

Notice that although $\|\bar{p} - p(\alpha_1, \delta_1)\|$ can be small $p(\alpha_1, \delta_1)$ may not be able to control the instability due to the saddle part: If the projection of $p(\alpha_1, \delta_1)$ on the saddle plane is on the left hand side of B_2 then the unstable direction of $p(\alpha_1, \delta_1)$ will not bring the

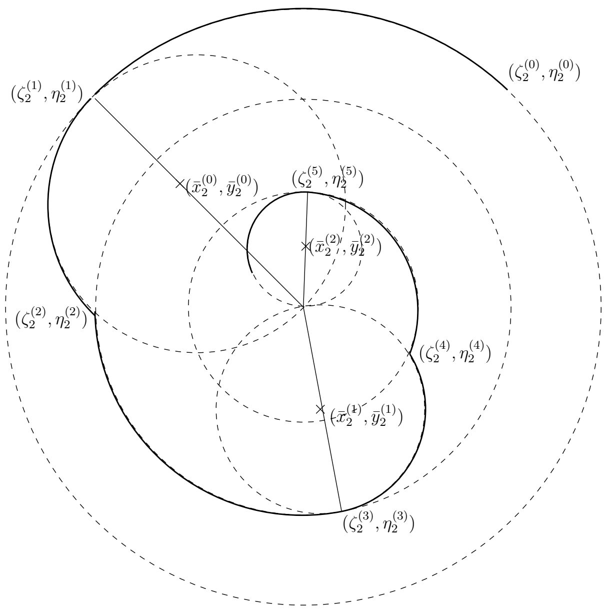

Figure 5 Sequence of fixed points and the projection of the probe's trajectory in the centre plane.

probe back (see Figure 6). So we will fix one of the components of $\bar{p}$, having to find the fixed points in a 1D surface.

We will now give more details of the process described above. As said before, $\bar{p}$ is the ideal position for the new fixed point. The coordinates of this position are in the $\{p_0; \vec{v}_1, \ldots, \vec{v}_6\}$ reference system and Eq 8 is in synodical coordinates, so we must change the base. Let M_v be the matrix that has $\vec{v}_i$ for $i = 1, \ldots, 6$ as columns and $s = (s_1, \ldots, s_6)$ are the coordinates of $\bar{p}$ in this reference system. If $A = M_v^{-1} Dp$ then Eq 8 becomes,

$$s^T = A \cdot h. \tag{9}$$

To avoid problems in the saddle behaviour we will fix s_1:

1. If $a_{11} = a_{12} = 0 \Longleftrightarrow \frac{\partial p}{\partial \alpha}, \frac{\partial p}{\partial \delta} \perp \vec{v}_1$:

   In this case there are no fixed points using the linear approximation for which its saddle behaviour brings the sail back.

2. If $|a_{11}| = \max(|a_{11}|, |a_{12}|)$:

$$s_1 = a_{11} h_1 + a_{12} h_2 \Rightarrow h_1 = \frac{s_1 - a_{12} h_2}{a_{11}}, \tag{10}$$

3. If $|a_{12}| = \max(|a_{11}|, |a_{12}|)$:

$$s_1 = a_{11} h_1 + a_{12} h_2 \Rightarrow h_2 = \frac{s_1 - a_{11} h_1}{a_{12}}, \tag{11}$$

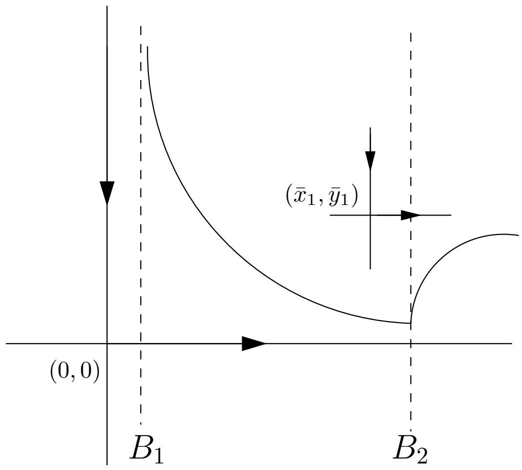

Figure 6 Possible position of the new fixed point $p(\alpha_1, \delta_1)$ in the saddle projection that will not control the unstable behaviour.

This reduces Eq 9 into $\hat{s} = \hat{A} \cdot \hat{h}$ (5 equations and 1 unknown).
Then $\hat{h}$ so that $\|\hat{s} - \hat{A} \cdot \hat{h}\|$ is minimal is,

$$\hat{h} = (\hat{A}^T \hat{A})^{-1} \hat{A}^T s. \tag{12}$$

Summary of the Control Algorithm

Suppose p_0 is a fixed points for $\alpha = \alpha_0, \delta = \delta_0$ that is linearly unstable. Let $\{\lambda_i, \vec{v}_i\}_{i=1,\dots,6}$ be the eigenvalues and eigenvectors for $D_X f(p_0)$. We will fix a reference system $\{p_0; \vec{v}_1, \dots, \vec{v}_6\}$ where,

- p_0 is the fixed point.
- $\vec{v}_1$ is the unstable eigenvector $(+\lambda)$.
- $\vec{v}_2$ is the stable eigenvector $(-\lambda)$.
- $\vec{v}_3, \vec{v}_4$ is the couple that defines one of the central movements $(\pm i\omega_1)$.
- $\vec{v}_5, \vec{v}_6$ defines the second central movement $(\pm i\omega_2)$.

From now on the trajectories will be seen in this reference system $(x(t^*) = \sum s_i \vec{v}_i)$, being $(s_1, \dots, s_6)$ the coordinates of the trajectory.

Let ε_{max} be the maximal distance we will allow to escape from the fixed point and ε_{min} the closest distance to the fixed point, needed when the probe is coming back. These constants depend on the mission objectives and the dynamical properties of the region around p_0.

We start with the probe close to the fixed point p_0 with $\alpha = \alpha_0$, $\delta = \delta_0$. When $|s_1| > \varepsilon_{max}$, the probe is far from p_0, we chose the appropriate α_1, δ_1 that takes the probe back to a neighbourhood of p_0 and change the sail orientation ($\alpha = \alpha_1$, $\delta = \delta_1$). When

$|s_1| < \varepsilon_{min}$, the sail is close to p_0 and we change the sail orientation back to $\alpha = \alpha_0$, $\delta = \delta_0$. This process is then restarted.

MISSION APPLICATION

We have seen a technique that uses dynamical system tools and permits a solar sail maintain its trajectory close to an unstable fixed point. We would like to illustrate how this control technique behaves with missions that are now being developed as the Geostorm and the Polar Observer missions.

First we will give a brief introduction of the aims of each mission. Second, we have tested our control strategy for both missions for 1000 simulations with random initial conditions. We will see that the probe manages to maintain its trajectory close to the fixed point for all these simulations. Finally, we will see the effect of errors during the control strategy. We have considered errors on the position and velocity determinations and errors on the sail orientation.

The Geostorm Mission

Its primary goal is to provide enhanced warning of geomagnetic storms to take preventive actions to protect vulnerable systems. Geomagnetic storms are principally the result of Coronal Mass Ejections (CME), the violent release of large volumes of plasma from the solar corona. The impact of CME on the Earth's magnetosphere can change its magnetic field and produce electromagnetic storms.

Currently predictions of future activity are made by the National Oceanic Atmospheric Administration (NOAA) Space Environment Centre in Colorado using terrestrial data and real-time solar wind data obtained from the Advanced Compositions Explorer (ACE) spacecraft. The ACE spacecraft is stationed on a halo orbit near L_1, at about 0.01 AU from the Earth.

The enhanced storm warning provided by ACE is limited by the need to orbit the L_1 point. However, since solar sails add an extra force to the dynamics of the orbit, the location of L_1 can be artificially displaced. The goal of Geostorm is to station a solar sail twice as far from the Earth than L_1 while remaining close to the Sun - Earth line as can be seen in Figure 7. As the CME will be detected earlier than by ACE, the warning times will be at least doubled.

In this paper we are just interested in testing our station keeping process. We want the sail to be at a double distance from the Sun - Earth L_1 point, so we need a characteristic acceleration of $a_0 = 0.3\text{mm/s}^2$. As we also want a constant communications with the Earth we must displace the probe approximately 10^o from the Sun - Earth line. In this region the fixed points are linearly unstable (see Figure 2).

We have taken a reference system ($\{p_0; \vec{v}_1, \dots, \vec{v}_6\}$) such that the p_0 is a fixed point satisfying the requirements explained above for a fixed sail orientation α_0, δ_0. We have done 1000 simulations with different initial conditions chosen in a random way. The control strategy has been applied up to 30 years and we have measured for each one the time between manoeuvres, the variation of the sail orientation (α, δ) and the variation of the trajectory w.r.t p_0.

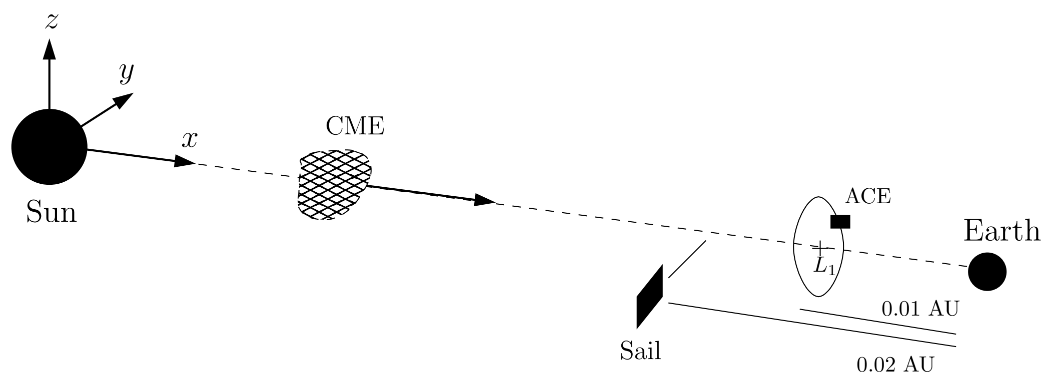

Figure 7 Schematic representation of the position of the Geostorm Mission (not to scale).

On the left hand side of Figure 8 we can see for each simulation the maximum and minimum time between manoeuvres. As we can see the minimum time between manoeuvres is around 40 days and the maximum time is around 146 days. On the right hand side of Figure 8 we have the maximum angular variation between the fixed point (p_0) and the probe's trajectory seen from the Earth for each simulation. Notice that the maximum variation experienced is less than 0.45 degrees. The variation of the sail orientation is reflected in the variation of two angles α and δ. For these simulations we have seen that α varies around 0.0695 degrees every time the sail orientation is changed and δ varies around 0.005 degrees.

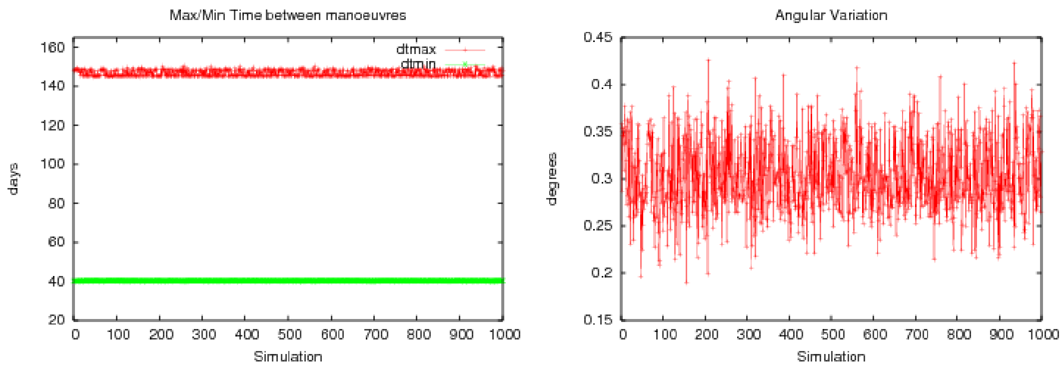

Figure 8 Left: maximum and minimum time between manoeuvres vs number of simulation. Right: maximum angular variation between p_0 and the probe trajectory vs number of simulation.

Finally in Figure 9 we can see one particular orbit after applying the control strategy on the $\{x, y\}$- plane (left), the $\{x, z\}$- plane (middle) and the 3D trajectory (left). Figure 10 shows the projection of this orbit on the saddle plane generated by the eigenvector $\vec{v}_1, \vec{v}_2$ (left) and the projection on the other two central planes $\vec{v}_3, \vec{v}_4$(middle) and $\vec{v}_5, \vec{v}_6$ (right).

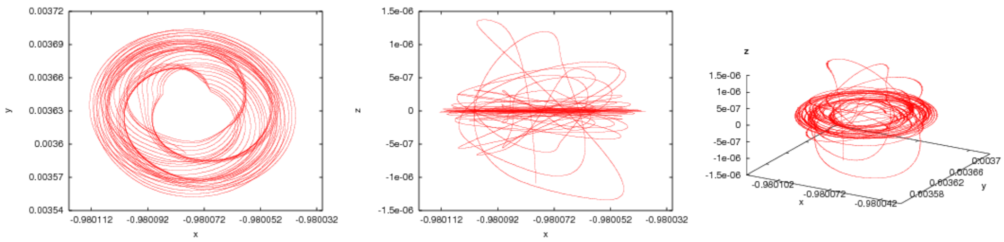

Figure 9 **Trajectory followed by the probe for 30 years. Left: $\{x, y\}$- projection, Middle: $\{x, z\}$- projection, Right: $\{x, y, z\}$- projection**

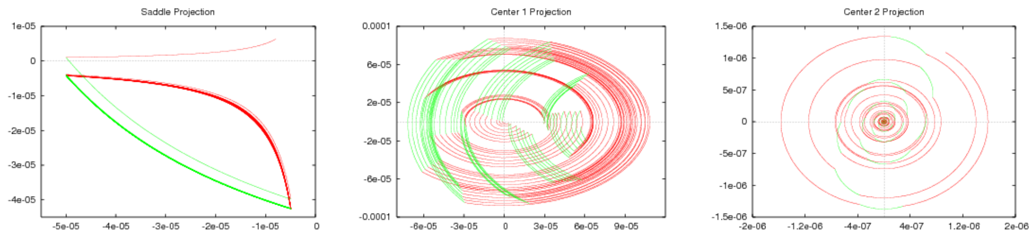

Figure 10 **Trajectory followed by the probe for 30 years. Left: Projection of the trajectory on the saddle plane, Middle: Projection of the trajectory in one of the centre planes $(\vec{v}_3, \vec{v}_4)$, Right: Projection of the trajectory in the other centre planes $(\vec{v}_5, \vec{v}_6)$. In red the trajectory when the sail is set to α_0, δ_0 and in green other orientations.**

The Polar Observer Mission

High latitude regions are of importance for a number of military, commercial and environmental interests. During the cold war the Arctic was a strategically important region, also the growing interest for the oil and mineral extraction of these regions may lead to a growing demand for communication services. The Arctic and Antarctic are also of great environmental importance and there is a requirement for relaying data from remote weather stations and automated monitoring platforms. Additional environmental requirements for polar services include continuous imaging of polar weather systems and monitoring of polar ice coverage for climate studies between others.

As solar sails provide a wide range of new artificial equilibrium points, some of these equilibrium points can be used to place a sail to have constant viewing of the Polar regions of the Earth. As the Earth's inclination is of about 23.4^o we must place the solar sail at 66.6^o from the ecliptic plane. Notice that as the Earth orbits around the Sun, the sail will maintain its fixed position with respect to the Earth but it will not always have the same view at the pole due to the Earth's inclination, see Figure 11. Having the sail perfectly situated on the north pole during the summer solstice, and during the winter solstice the sail will appear displaced over the horizon, having still some imaging of the north pole.

As the sail performance increases the equilibrium points come closer to the Earth. But as a first mission we are dealing with small resolution sails. We will take a characteristic

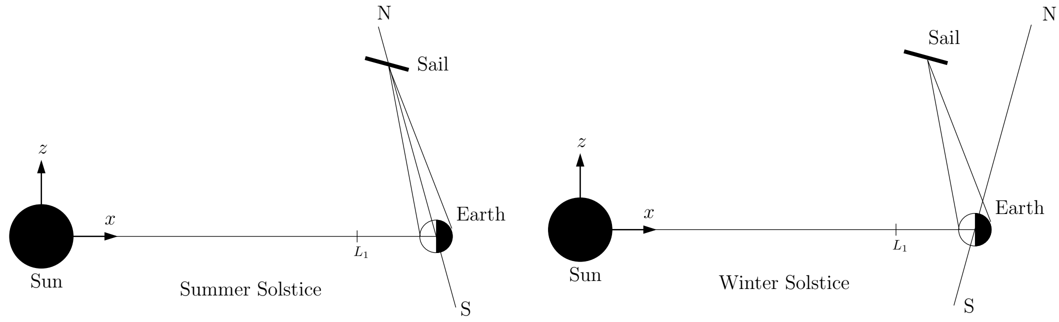

Figure 11 Schematic representation of the Polar Observer Mission (not to scale).

acceleration $a_0 = 0.46\text{mm/s}^2$ as it is the minimum sail's characteristic acceleration that makes the solar sail be placed over the north pole. In these region the points are also linearly unstable (see Figure 3).

As before we have also taken a reference system $\{p_0; \vec{v}_1, \ldots, \vec{v}_6\}$ so that the fixed point p_0 satisfies the required conditions for a fixed sail orientation α_0, δ_0. We have also done a 1000 simulations applying the control strategy up to 30 years with random initial conditions and measured the time between manoeuvres, the variation of the sail orientation and the angular variation w.r.t p_0.

In the left hand side of Figure 12 we can see for each simulation the maximum and minimum time between manoeuvres. Now the minimum time between manoeuvres is always around 59 days and the maximum time is around 187 days. In the right hand side of Figure 12 we have the maximum angular variation between the fixed point (p_0) and the probe's trajectory for each simulation, where the maximum variation experienced is around 0.3 degrees. In these simulations we have seen that α varies around 0.03 degrees and δ around 0.02 degrees every time the sail orientation is changed.

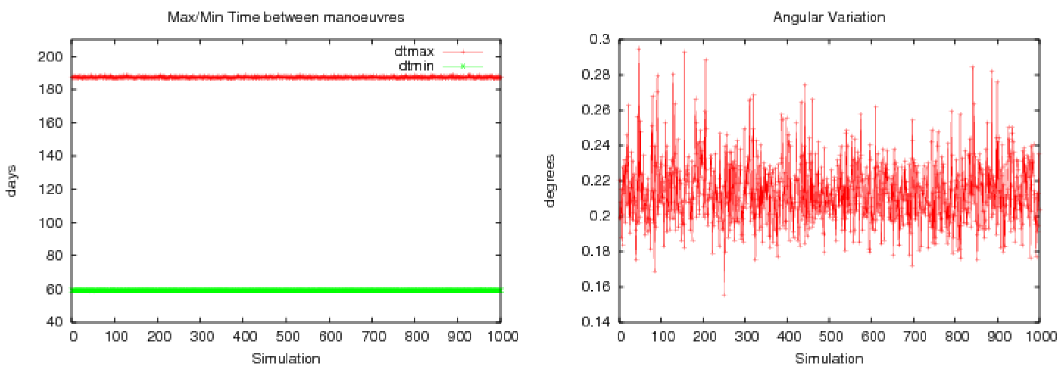

Figure 12 Left: maximum and minimum time between manoeuvres vs number of simulation. Right: maximum angular variation between p_0 and the probe trajectory vs number of simulation.

Finally Figure 14 shows the projection of this orbit on the saddle plane generated by the eigenvector $\vec{v}_1, \vec{v}_2$ (left) and the projection on the other two central planes $\vec{v}_3, \vec{v}_4$(middle) and $\vec{v}_5, \vec{v}_6$ (right).

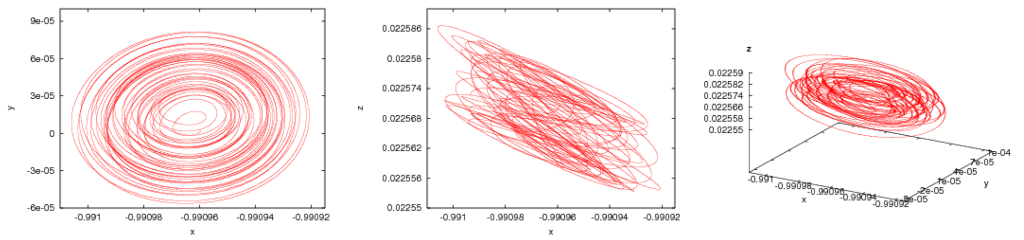

Figure 13 Trajectory followed by the probe for 30 years. Left: $\{x, y\}$- projection, Middle: $\{x, z\}$- projection, Right: $\{x, y, z\}$- projection

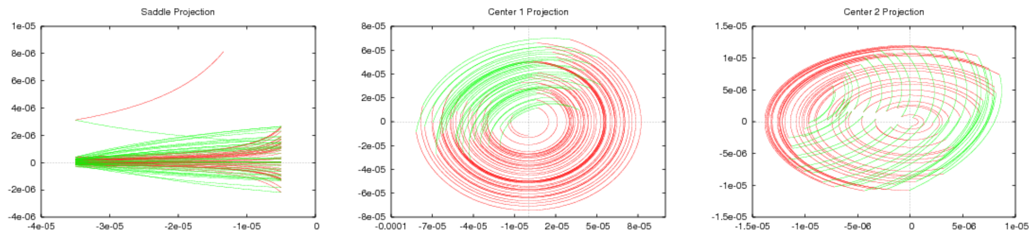

Figure 14 Trajectory followed by the probe for 30 years. Left: Projection of the trajectory on the saddle plane, Middle: Projection of the trajectory in one of the centre planes ($\vec{v}_3, \vec{v}_4$), Right: Projection of the trajectory in the other centre planes ($\vec{v}_5, \vec{v}_6$). In red the trajectory when the sail is set to α_0, δ_0 and in green other orientations.

Sensitivity to Errors

It is a known fact that during a mission the position and velocity of the probe will not be determined exactly, this has an effect on the decisions taken by the control algorithm. Errors on the sail orientation will also be made and have an important effect in the probe's trajectory. We will see the effect of these errors in our control strategy.

Let us first consider the errors on the determination of the position and velocity of the probe. As we have seen in previous sections the sail orientation will be changed when the probe is at a certain distance of the fixed point in the saddle plane projection. Each time the algorithm asks itself if the sail orientation has to be changed, the probe's position in the phase space has some small error. If the sail orientation is changed the new fixed point will be found using the wrong position of the probe. If this errors are not very big the difference between changing the sail orientation a little before or after in time will not affect on the control of the probe.

We have supposed that all the errors follow a normal distribution with mean value 0. We have taken a precision on the space slant of 1m and $2-3$milli-arc-seconds in the angle determination. The precision in speed is around $20-30$micro/seconds. These errors magnitudes reflect as errors of order 10^{-8} in the saddle plane projection, these effects are almost neglected.

We have done 1000 simulation taking the same initial conditions as before adding the uncertainty in the position and velocity measurement. We will see that the results obtained similar for all the 1000 simulations in both missions the probe's trajectory does not escape after 30 years. The average time between manoeuvres is slightly changed and so are the angular variation on the trajectories position w.r.t the initial fixed point seen from the Earth (see Table 1 and 2).

Let us now consider the errors due to the sail orientation, these errors have a more important effect on the sail trajectory and the controllability of the probe. Each time the sail orientation is changed an error in its orientation is made ($\alpha = \alpha_1 + \epsilon_\alpha$, $\delta = \delta_1 + \epsilon_\delta$). Then the new fixed point p_1 is shifted $p(\alpha, \delta) = p(\alpha_1, \delta_1) + \epsilon_p$ and so do the stable and unstable directions $\vec{v}_{1,2}(\alpha, \delta) = \vec{v}_{1,2}(\alpha_1, \delta_1) + \epsilon_v$. These variations can make the probe's trajectory not come close to p_0, as $p(\alpha, \delta)$ can be placed on the incorrect side of the saddle or the the central behaviour can blow up.

Depending on the nature of the region where the fixed point is placed the control strategy will be able to deal with bigger errors in the sail orientation. It will all depend on the variation of the fixed point and the eigenvectors with respect to the sail orientation.

We have also done 1000 simulations taking the same initial conditions introducing the uncertainties on sail orientation and the probe's position and velocity. We will see that for different magnitude of errors on the sail orientation the control strategy is also able to maintain all the simulations close of the equilibrium point up to 30 years.

Tables 1 and 2 sumarises the results of all these simulations for the Geostorm and Polar Observer missions respectively. On the first line we have the results for the simulations when no errors are taken into account. On the second line there are the results when only error on the position and velocity determination are made. Finally the third line contains the results when all the errors are taken into account (sail orientation and position + velocity determination). We must mention that the errors on the sail orientation for each mission are different. For the Geostorm the errors on the sail orientation are of order 0.01^o while for the Polar Observer the errors made are of order 0.001^o. As we will see latter on this is due to the dynamics around the different equilibrium points. Column 2 shows the % of simulations that succeed in controlling the probe, column 3 and 4 have the average maximum and minimum time between manoeuvres respectively and column 5 has the average angular variation of the trajectory w.r.t the fixed points p_0 seen from the Earth.

Table 1
STATISTICS FOR THE GEOSTORM WARNING MISSION

	Succ. Sim.	Max. Time	Min. Time	Ang. Vari.
No Error	100%	146.77 days	40.20 days	0.3^o
Error Pos.	100%	146.82 days	40.19 days	0.3^0
Error Orient.	100%	376.78 days	32.60 days	1.2^o

In Table 1 we can see that for the Geostorm mission all of the 1000 simulations succeed even if errors on the sail orientation or on the determination of the probe's position and velocity are made. As we can see there is practically no change between including or not the error in the position determination, but it does change if we introduce errors on the sail orientation. This is due to the big variation in the fixed points position when these last errors are taken into account. As we can see the angular variation between the probe's trajectory and the fixed point is almost doubled when all the errors are taken into account, this is because now the probe's trajectory moves on both sides of the saddle as can be seen in Figure 15. In this figure we can see the difference between the trajectory followed by the probe when errors in the sail orientation are added and the trajectory when no errors are taken into account for the Geostorm.

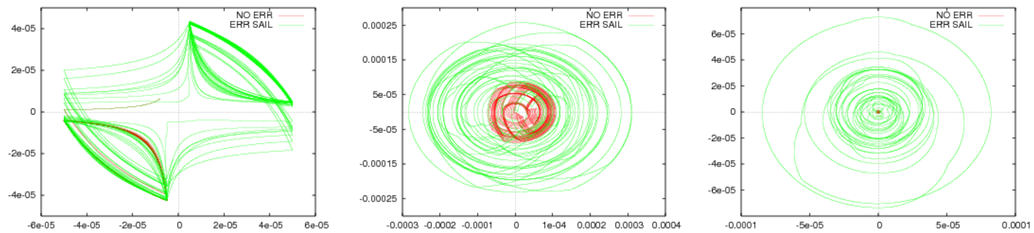

Figure 15 Geostorm: In red the trajectory followed by the probe when no error during the control algorithm are made. In green the trajectory followed by the probe when errors on the position and sail orientation are made. From left to right the projection of the sail's trajectory on the saddle, centre 1 and centre 2 planes.

In Table 2 we have the results for 1000 simulations for the Polar Observer. Our control strategy, in this particular mission, is no able to deal with errors on the sail orientation of order 0.01^o. As we have seen this magnitudes of errors were acceptable in the Geostorm, this is due to the nature of the region: we recall that for the Polar Observer the variation of the sail orientation was $\alpha \approx 0.03^o$ and $\delta \approx 0.01^o$. This is why we need more precision on the sail orientation to be able to control the probe.

Table 2
STATISTICS FOR THE POLAR OBSERVER MISSION

	Succ. Sim.	Max. Time	Min. Time	Ang. Vari.
No Error	100%	187.36 days	59.16 days	0.21^o
Error Pos.	100%	187.86 days	58.89 days	0.21^0
Error Orient.	100%	246.85 days	55.08 days	0.36^o

As it happened on the Geostorm mission we can see that the time between manoeuvres practically does not change when the errors on the position determination are made, the same happens for the angular variation, but it does change when errors on the sail orientation are introduced.

In Figure 16 we can see two different trajectories for the same initial condition, one

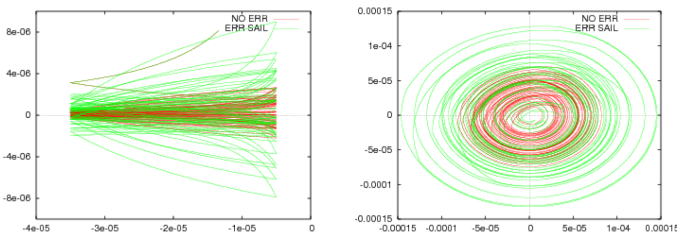

Figure 16 Polar Observer: In red the trajectory followed by the probe when no error during the control algorithm are made. In green the trajectory followed by the probe when errors on the position and sail orientation are made. From left to right the projection of the sail's trajectory on the saddle, centre 1 and centre 2 planes.

made with no errors on the control strategy and the other with errors during the control strategy for the Polar Observer.

We must also mention that there is a big relation between the variation on the sail orientation $(\Delta\alpha, \Delta\delta)$ and the two bound that define the region of movement on the saddle projection ($B_1 = \{x_1 = \varepsilon_{min}\}$ and $B_2 = \{x_1 = \varepsilon_{max}\}$). Let $\Delta\varepsilon = \varepsilon_{max} - \varepsilon_{min}$, then as $\Delta\varepsilon$ gets bigger the variation in the sail orientation is bigger. Although $\Delta\varepsilon$ cannot be too big because we would have problems in the station keeping algorithm. Notice that ε_{max} must be inside the limits where the linear behaviour is valid to define the behaviour of the probe, which is strictly related with the nature of the fixed point and its vicinity.

CONCLUSIONS

In this paper we present a new way of controlling a solar sail close to an unstable fixed point by using dynamical system tools. We have studied the natural dynamic of the system close to a fixed point and the variation of this one when the sail orientation (α, δ) is change. This knowledge has permitted us design a control strategy that maintains a probe's trajectory close to an unstable fixed point.

We have tested this strategy with to different missions, the 'Geostorm Warning Mission' and the 'Polar Observer'. In both cases the probe managed to stay close to the fixed point for 30 years. We have also tested the controllability of the algorithm including errors in the prediction of the probe's position and velocity and errors in the sail orientation angles (α, δ). We have seen that the errors on the position and velocity do not produce important changes in the sail's trajectory and its controllability. The errors on the sail's orientation are more relevant and give more variations on the trajectory followed by the probe.

As we have seen, the controllability of the sail is strictly related with the nature of the neighbourhood of the fixed point where we want to maintain the sail close. If the variations of the fixed points and the eigendirections is understood, we can be able to understand the difficulties of the control strategy. As we have seen in the Polar Observer Mission, more precision on the sail orientation was required to be able to control the probe.

Finally let us mention that the strategy proposed here does not require previous planing, the decisions taken by the probe depend only on its position on the phase space, that is known at each moment. These is really helpful as you do not to have to plan the control strategy in advance, and errors made during the manoeuvres can be rectified easily.

ACKNOWLEDGEMENTS

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