ON STRANGE ATTRACTORS IN A CLASS OF PINCHED SKEW PRODUCTS

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Abstract. In this note we construct strange attractors in a class of skew product dynamical systems.

A dynamical system of the class is a bundle map of a trivial bundle whose base is a compact metric space and the fiber is the non-negative half real line. The map on the base is a homeomorphism preserving an ergodic measure. The fiber maps either are strictly monotone and strictly concave or collapse at zero (pinching condition). The points on the base space whose fibers collapse are the pinched points of the skew product. We also assume that the set of pinched points has zero measure, and that there is a pinched point whose orbit is dense in the base space. Moreover, we assume that the zero-section is a super-repeller, in the sense that it is invariant and its Lyapunov exponent is $+\infty$.

For such a skew product dynamical system, we prove the existence of a measurable but non-continuous invariant graph, whose Lyapunov exponent is negative. We will refer to such an object as a strange attractor.

Since the dynamics on the strange attractor is the one given by the base homeomorphism, we will say that it is a strange chaotic attractor or a strange non-chaotic attractor depending on the fact that the dynamics on the base is chaotic or non-chaotic.

1. Introduction

The existence of attractive non-continuous invariant graphs in non-autonomous systems is a topic that has generated great interest, specially for the case of quasiperiodically forced dynamical systems, that is bundle maps over irrational rotations. In this context, such objects are known as strange non-chaotic attractors or SNA, since the dynamics is the one given by the external irrational rotation. Their existence was first conjectured in the seminal works [GOPY84] and [Kan84], and motivated an explosion of numerical studies (see the review [PNR01]). This is a fascinating problem for which there are also rigorous results, although for rather specific systems. So, Herman [Her83], even before the term SNA was invented, proved the existence of such objects for quasi-periodically Moebious transformations (see e.g. also the more recent [HP06]). The proofs of the existence of SNA for the so-called pinched skew-products introduced in [GOPY84] go back to the works of Keller [Kel96], and Bezhaeva and Oseledets [BO96] (see e.g. the further studies [Gle02, Jäg07]). We emphasize that, besides these rigorous results for specific models, there are other examples in the literature for which the existence of SNA remains unclear [BSV05, JT05, HS05].

In this note we consider such a problem of existence of strange attractors in a class of skew product dynamical systems, which are bundle maps of a trivial bundle whose base is a compact metric space and the fiber is the non-negative half real line. The bundle maps of this class have fiber maps that are either strictly monotone and strictly concave or collapse at zero. This is the so-called *pinching condition* mentioned above. We will refer to the points on the base space whose fibers collapse as the *pinched points* of the skew product. We also assume that the set of pinched points has zero measure, ant that there is a pinched point whose orbit is dense in the base space. Moreover, we also assume that the zero-section is a super-repeller, meaning that it is invariant and its Lyapunov exponent in $+\infty$. Since the dynamics on the attractor is that of the motion on the base of the bundle, its dynamical properties are inherited from the base motion. As a result, the non-continuous invariant graphs we construct are strange chaotic attractors or strange non-chaotic attractors depending on the fact that the base motion is chaotic or non-chaotic.

Formulation. We start now by giving a precise formulation of the problem we want to study and, in particular, the class of skew product dynamical systems we will consider along this paper.

Let $X = \Theta \times [0, +\infty[= \{(\theta, x) \ / \ \theta \in \Theta, x \geq 0\}$ be a trivial bundle over the compact metric space Θ. Let $\omega : \Theta \to \Theta$ be a homeomorphism, and let μ be a Borel probability ergodic measure. In this note we study the dynamics of the bundle map (or skew product)

$$
\begin{array}{rccc}
T: & \Theta \times [0, +\infty[& \longrightarrow & \Theta \times [0, +\infty[\\
 & (\theta, x) & \longrightarrow & (\omega(\theta), g(\theta) f(x))
\end{array}
\tag{1.1}
$$

covering ω, where:

(g0) $g : \Theta \to [0, +\infty[$ is a continuous function;

(g1) g is log-integrable with respect to μ, that is

$$\int_\Theta \log g(\theta) \ d\mu > -\infty;$$

(f0) $f : [0, +\infty[\to [0, +\infty[$ is a continuous function, C^1 in $]0, +\infty[$;

(f1) $f(0) = 0$;

(f2) $f' :]0, +\infty[\to]0, +\infty[$ is strictly decreasing (so f is strictly increasing and strictly concave);

(f3) $\displaystyle\lim_{x \to 0^+} \frac{x f'(x)}{f(x)} = q_0$, with $0 < q_0 < 1$.

We are interested in the asymptotic behavior of the (forward) orbits of T. in particular, in the existence of attractors that are graphs of measurable functions $\Phi : \Theta \to [0, \infty[$, that are $\Gamma_\Phi = \{(\theta, \Phi(\theta)) \ / \ \theta \in \Theta\}$. That is, for a point $(\theta, x) \in X$, we are interested in the behavior of the distance $|\Phi(\theta_n) - x_n|$ of the points $(\theta_n, x_n) = T^n(\theta, x)$ of the orbit to the graph Γ_Φ when n goes to $+\infty$.

As we will see in Theorem 1.1, there is an attractor Γ_Φ for the skew product T. What makes a crucial difference for the regularity of the attractor is if g has zeros (pinched case) or not (invertible case). Notice that if g has zeros, full fibers of the bundle collapse at zero. In such a case, the points on the invariant graph in pinched fibers go to the zero-section, and remain there under iteration of the map.

The main result. The main result of this paper is

Theorem 1.1. *Let T be a bundle map* (1.1) *satisfying the assumptions* (g0), (g1), (f0), (f1), (f2), (f3) *described above. Let us assume also that*

(gf) *$Mp_\infty < 1$, where $M = \max_{\theta \in \Theta} g(\theta)$, $p_\infty = \inf_{x>0} \dfrac{f(x)}{x}$.*

Then, there exists a function $\Phi : \Theta \to [0, +\infty[$ which is

(a) *bounded and upper-semicontinuous, so it is measurable;*
(b) *log-integrable, so its set of zeros is null;*

and its graph $\Gamma_\Phi = \{(\theta, \Phi(\theta)) \ / \ \theta \in \Theta\}$ is

(c) *invariant under T, i.e. for all $\theta \in \Theta$, $g(\theta) f(\Phi(\theta)) = \Phi(\omega(\theta))$;*
(d) *attractive, i.e. for a.e. $\theta \in \Theta$, for all $x > 0$, there exist positive constants $\alpha_x < 1$ and $C_{\theta,x}$ such that*

$$|\Phi(\theta_n) - x_n| \leq C_{\theta,x} \alpha_x^n |\Phi(\theta) - x| \overset{n\to+\infty}{\longrightarrow} 0. \tag{1.2}$$

Moreover,

(e) *for a.e. $\theta \in \Theta$, for all $x > 0$, the Lyapunov exponent of the orbit,*

$$\lambda(\theta, x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log(g(\theta_k) f'(\theta_k)),$$

does exist and equals the Lyapunov exponent of the graph,

$$\lambda_\Phi = \int_\Theta \log g(\theta) \ d\mu + \int_\Theta \log f'(\Phi(\theta)) \ d\mu.$$

The continuity or strangeness of Φ depends on the existence of zeros of g:

(z0) *If g has not zeros, Φ is continuous and has not zeros.*
(z1) *If g has zeros, Φ is not necessarily continuous and the set of zeros of Φ is a G_δ set of zero measure. If, moreover, one of the zeros of g has a dense orbit by ω, then the set of zeros of Φ is a residual set of zero measure, and Φ is not continuous.*

Remark 1.2. *In the case* (z0) *above, the convergence of orbits to the attractor is uniform, that is we can make the constants* $C_{\theta,x}$ *independing on* θ, x *in a compact neighborhood of* Γ_Φ *and, moreover, the convergence to the attractor is for all the points in such a neighborhood. Hence,* Γ_Φ *is a Uniform Attractor. Moreover, since* Φ *is continuous, we will say that* Γ_Φ *is a Regular Attractor.*

In the case (z1) *above, the convergence of orbits to the attractor is not uniform, and the constants* $C_{\theta,x}$ *explode for certain* θ*s. Hence,* Γ_Φ *is a Non-uniform Attractor. If, moreover, the orbit of a pinched point is dense in* Θ*, the graph can not be continuous, just log-integrable, and we will say that* Γ_Φ *is a Strange Attractor.*

Remark 1.3. *The dynamics on the attractor* Γ_Φ *is the one given by the homeomorphism* $\omega : \Theta \to \Theta$*. So, we will say that the attractor is Chaotic or Non-Chaotic relatively to the dynamical behavior of* ω*.*

Remark 1.4. *Theorem 1.1 works for the skew-product*

$$T(\theta, x) = (\omega(\theta), g(\theta)x^\alpha)$$

where $0 < \alpha < 1$*. In fact, in Section 5 we find rather explicit formulae for the invariant graph of this paradigmatic example.*

Remark 1.5. *Notice that if the ergodic measure* μ *is topological, that is the measure of any non-empty open set is positive, then the homeomorphism* ω *is topologically transitive, and there exist base points whose orbits are dense. The question is if some of these base points are also pinched.*

If, moreover, ω *is uniquely ergodic and the invariant measure* μ *is topological, then* ω *is minimal, and all the orbits are dense.*

Related results. The consequences of Theorem 1.1 are close to the results of Keller's paper [Kel96], in which the present note is highly inspired. There are, however, several differences.

Among the hypotheses, notice that in [Kel96], and most of the papers in the literature dealing with the so called Strange Non-Chaotic Attractors, the skew product map is defined over an irrational rotation on the torus $\Theta = \mathbb{R}/\mathbb{Z}$, while here this assumption is considerably generalized. As a result, the chaoticity or not of the strange attractor depends on the dynamics on the base manifold.

In Keller's paper, the function f is C^1 in the closed interval $[0, +\infty[$, implying that $\lim_{x\to 0^+} \dfrac{xf'(x)}{f(x)} = 1$, while in this note the function f is not differentiable at 0, in fact $f'(0) = +\infty$. So, even some of the arguments are similar to those in [Kel96], the ones that in [Kel96] rely on the hypothesis $f'(0) < +\infty$ have to be overcome. Hypotheses such as $f'(0) = +\infty$ appear in other contexts, for instance in Economics is known as an Inada condition.

Notice also that from the log-integrability assumption on the function g, which is not assumed in [Kel96], we are able of proving the log-integrability of the invariant graph. We emphasize that log-integrability is close to the property of temperedness, typical in Ergodic Theory.

Finally, we emphasize that the fact that the forcing introduced by θ is multiplicative is not essential for our results, and they could be adapted to more general pinched skew-products in the spirit of [Gle02, Jäg07].

Overview. In Section 2 we establish several elementary properties of the functions g and f that will be useful in the sequel. In Section 3 we start the study of the asymptotic behavior of the orbits of T, in particular we bound their (upper) Lyapunov exponent. In Section 4 we adapt the results of [Kel96] to prove the existence of an upper-semicontinuous invariant graph, and we prove also that it is log-integrable, finishing the proof of Theorem 1.1. Section 5 is devoted to the explicit computation of the invariant graph for the paradigmatic example of Remark 1.4, and we also obtain a lower bound for the invariant graph in the general case.

2. Preliminary results

In this section we state several very elementary properties of the functions g and f, that will be important for our purposes.

Properties of the function g. Since $g : \Theta \to [0, +\infty[$ is a continuous function, we can define

$$m = \min_{\theta\in\Theta} g(\theta) \geq 0 \ , \ M = \max_{\theta\in\Theta} g(\theta).$$

Notice that the cases $m > 0$ and $m = 0$ are rather different. If $m > 0$, the function g has not zeros, and the map T is invertible. If $m = 0$, the function g has zeros, and whole fibers of the bundle X, those that are supported on zeros of g, are mapped to the zero section.

The following proposition says that, even if $m = 0$, the map T is invertible from a measure theoretic point of view. The result has to do with the measure of the set of zeros of g.

Proposition 2.1. *The set of zeros of g, $Z_g = \{\theta \in \Theta \ / \ g(\theta) = 0\}$, is null. Moreover, the set of points that are eventually zero of g, $\tilde{Z}_g = \{\theta \in \Theta \ / \ \exists k \geq 0 : g(\omega^k(\theta)) = 0\}$, is null.*

Proof. Since $g : \Theta \to [0, +\infty[$ is a continuous and non-negative log-integrable function, $\mu(Z_g) = 0$, because otherwise its integral would be $-\infty$.

Moreover, since

$$\tilde{Z}_g = \bigcup_{k\geq 0} Z_{g\circ\omega^k} = \bigcup_{k\geq 0} \omega^{-k}(Z_g),$$

and ω preserves the measure μ, $\mu(\tilde{Z}_g) = 0$. □

Properties of the function f. From the hypothesis on the function f, it is important for us to consider the continuous functions $p, q :]0, +\infty[\to]0, +\infty[$ defined by

$$p(x) = \frac{f(x)}{x} \ , \ q(x) = \frac{xf'(x)}{f(x)}. \tag{2.1}$$

The following proposition states several properties of those functions.

Proposition 2.2. *The functions p, q defined in* (2.1) *satisfy the following properties:*

(p0) *p is strictly decreasing, and*

$$f'(0) = \lim_{x\to 0^+} p(x) = +\infty \ , \ \lim_{x\to+\infty} p(x) = p_\infty \geq 0;$$

(q0) *for all $x > 0$, $q(x) < 1$.*

Proof. Since the function f satisfies the hypothesis (f0)-(f2), and in particular f is strictly concave, then for all $x > 0$

$$f'(x) < \frac{f(x)}{x},$$

which proves (q0). Notice that

$$p'(x) = \frac{f'(x)x - f(x)}{x^2} = \frac{f(x)}{x^2}(q(x) - 1) < 0,$$

so p is strictly decreasing. As a result

$$\lim_{x\to+\infty} p(x) = \inf_{x>0} p(x) = p_\infty.$$

Since both the functions f' and p are strictly decreasing, the limits $f'_0 = \lim_{x\to 0^+} f'(x)$ and $p_0 = \lim_{x\to 0^+} p(x)$ do exist, although they could by $+\infty$. In fact, by the Mean Value Theorem, both limits are equal. Finally, the limit can not be finite, because in such a case $\lim_{x\to 0^+} q(x) = 1$. This proves (p0). □

Remark 2.3. *The function f satisfies what in Economics are known as Inada conditions:*

$$\lim_{x\to 0^+} f'(x) = +\infty \ , \ \lim_{x\to+\infty} f'(x) = p_\infty.$$

Even in many cases one assumes that $p_\infty = 0$. Notice that in such a case the assumption (gf) *of Theorem 1.1 is automatic.*

Remark 2.4. *The function q is what in Economics is known as the elasticity of the function f. Since $f(0) = 0$ and f is strictly concave, the elasticity is bounded from above by* 1*. In fact, under assumption* (f3)*, the elasticity never reaches* 1*. We define $q(0) = q_0$, making the function q continuous in $[0, +\infty[$.*

Remark 2.5. *If we write* $q_f(x) = xf'(x)/f(x)$*, in order to emphasize the* f*-dependence of* q*, notice that*

$$q_{f_1+f_2}(x) \le \max(q_{f_1}(x), q_{f_2}(x)) \ , q_{af}(x) = q_f(x),$$

where f_1,f_2, f *are functions satisfying* (f0)-(f3) *and* a *is a positive constant. Notice also that* $q_{f_1 f_2}(x) = q_{f_1}(x) + q_{f_2}(x)$.

As a result, the linear combination $f(x) = a_1 f_1(x) \dots a_n f_n(x)$ *of functions* $f_1, \dots f_n$ *satisfying* (f0)-(f3) *with positive coefficients* $a_1, \dots a_n$ *satisfies also* (f0)-(f3).

Finally, we give some examples of functions satisfying the assumptions (f0)-(f3).

Example 2.6. *Our paradigmatic example is* $f(x) = x^\alpha$*, where* $\alpha \in]0, 1[$*. This function has constant elasticity* $q(x) = \alpha$*. Moreover,* $p(x) = x^{\alpha-1}$*, so* $p_\infty = 0$.

Example 2.7. *The functions*

$$f(x) = x^\alpha (1 + cx)^{-\beta},$$

where $0 < \beta \le \alpha < 1$ *and* $c > 0$*, have elasticity* $q(x) = \alpha - c\beta x(1+cx)^{-1} \le \alpha$*, so* $q_0 = \alpha$ *and* q *is strictly decreasing. Moreover,* $p(x) = x^{\alpha-1}(1 + cx)^{-\beta}$*, so* $p_\infty = 0$.

Example 2.8. *The third example is the family of functions*

$$f(x) = px + x^\alpha$$

where $\alpha \in]0, 1[$ *and* $p > 0$*. The elasticity is* $q(x) = 1 - (1 - \alpha)x^\alpha(px + x^\alpha)^{-1} < 1$*, so* $q_0 = \alpha$ *and* q *is strictly increasing (with supremum* 1*). In this case,* $p(x) = p + x^{1-\alpha}$*, so* $p_\infty = p > 0$.

3. Orbits

Given $(\theta, x) \in X$, we define the (forward) orbit $\{(\theta_n, x_n) = T^n(\theta, x) \ / \ n \ge 0\}$. Its (forward) upper Lyapunov exponent is

$$\overline{\lambda}(\theta, x) = \limsup_{n\to\infty} \frac{1}{n} \log L_n(\theta, x),$$

where

$$L_n(\theta, x) = \frac{\partial x_n}{\partial x_0} = \prod_{k=0}^{n-1} g(\theta_k) f'(x_k).$$

If the lim sup is a lim we will write $\lambda(\theta, x)$, the Lyapunov exponent.

In the definitions above, if $\theta \notin \tilde{Z}_g$, we take $\overline{\lambda}(\theta, 0) = \lambda(\theta, 0) = +\infty$. In a sense, the zero-section Γ_0 is a super-repeller.

In the following proposition, we show that the orbits of T are bounded.

Proposition 3.1. *For all* $(\theta, x) \in X$ *there exist* $K_x > 0$ *such that for all* $n \ge 0, x_n \le K_x$.

Proof. Since the function p is strictly decreasing, with $\lim_{x\to 0^+} p(x) = +\infty$ (see Proposition 2.2) and $\lim_{x\to+\infty} p(x) = p_\infty \geq 0$ with $p_\infty < M^{-1}$ (see assumption (gf) in Theorem 1.1), there exist a unique $R > 0$ such that $p(R) = M^{-1}$. As a result:

- for all $\theta \in \Theta, x \leq R$, $\pi_x \circ T(\theta, x) = g(\theta)f(x) \leq Mf(R) = R$;
- for all $\theta \in \Theta, x \geq R$, $\pi_x \circ T(\theta, x) = g(\theta)f(x) \leq MxM^{-1} = x$.

The proof of the proposition follows by taking $K_x = \max(R, x)$. □

Since f is strictly increasing, it is obvious that the orbits are ordered.

Proposition 3.2. *If $\theta \in \Theta$ and $0 \leq x \leq y$, then for all $n \geq 0$, $0 \leq x_n \leq y_n$. Moreover, if $\theta \notin \tilde{Z}_g$ and $0 < x < y$, then for all $n \geq 0$, $0 < x_n < y_n$.*

In the following proposition, we prove that the orbits come closer exponentially fast.

Proposition 3.3. *For all $\theta \in \Theta$ and $0 < x < y$, there exist positive constants $\alpha_x < 1$ and C_x such that for all $n \geq 0$, $y_n - x_n \leq C_x \alpha_x^n (y - x)$. As a result,*

$$\lim_{n\to\infty} (y_n - x_n) = 0 \ , \ \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} (y_k - x_k) = 0.$$

Proof. We will assume that $\theta \notin \tilde{Z}_g$, otherwise the result is trivial. Notice that, from Proposition 3.2, for all $n \geq 0$, $0 < x_n < y_n$. Since

$$\frac{y_n - x_n}{y_{n-1} - x_{n-1}} = g(\theta_{n-1}) \frac{f(y_{n-1}) - f(x_{n-1})}{y_{n-1} - x_{n-1}} \leq g(\theta_{n-1}) f'(x_{n-1}) = q(x_{n-1}) \frac{x_n}{x_{n-1}},$$

where in the inequality we use that f is concave, then

$$y_n - x_n \leq q(x_0) \dots q(x_{n-1}) \frac{x_n}{x_0} (y_0 - x_0) \leq \frac{K_x}{x} \alpha_x^n (y - x),$$

where

$$\alpha_x = \sup_{0<t\leq K_x} q(t).$$

As a result, $C_x = K_x/x = \max(R/x, 1)$ and $\alpha_x < 1$. □

Using similar arguments, it is easy to see that the upper Lyapunov exponent of a positive orbit is negative.

Proposition 3.4. *For all $\theta \notin \tilde{Z}_g$ and $x > 0$, $\bar{\lambda}(\theta, x) \leq \log \alpha_x < 0$.*

Proof. Since

$$L_n(\theta, x) = \prod_{k=0}^{n-1} g(\theta_k) f'(x_k) = \frac{x_n}{x_0} \prod_{k=0}^{n-1} q(x_k) \tag{3.1}$$

then

$$L_n(\theta, x) \leq C_x \alpha_x^n$$

and the proof follows immediately. □

4. Invariant graphs

Let $\psi : \Theta \to [0, +\infty[$ be a measurable function, and $\Gamma_\psi = \{(\theta, \psi(\theta)) \ / \ \theta \in \Theta\}$ be its graph. Notice that T transfers the graph of ψ to the graph of the measurable function $\mathcal{T}\psi$ defined as

$$\mathcal{T}\psi(\theta) = g(\omega^{-1}(\theta)) f(\psi(\omega^{-1}(\theta))). \tag{4.1}$$

In particular, we say that ψ is invariant under T, or that Γ_ψ is an invariant graph, if for all $x \in \Theta$,

$$g(\theta) f(\psi(\theta)) = \psi(\omega(\theta)). \tag{4.2}$$

In other words, ψ is invariant iff $\mathcal{T}\psi = \psi$. We then define its Lyapunov exponent as

$$\lambda_\psi = \int_\Theta \log g(\theta)\ d\mu + \int_\Theta \log f'(\psi(\theta))\ d\mu.$$

For example, the zero-section Γ_0 is an invariant graph and its Lyapunov exponent is $\lambda_0 = +\infty$.

In the following two propositions we review the results of [Kel96] to construct an attractive invariant graph, which is unique up to sets of zero measure.

Proposition 4.1. *Let $\psi : \Theta \to [0, +\infty[$ be an invariant measurable function and let $Z_\psi = \{\theta \in \Theta \ / \ \psi(\theta) = 0\}$ be its set of zeros. Then:*

- *Either $\mu(Z_\psi) = 0$ or $\mu(Z_\psi) = 1$;*
- *If $\mu(Z_\psi) = 0$ and if $\tilde{\psi} : \Theta \to [0, +\infty[$ is an invariant measurable function with $\mu(Z_{\tilde{\psi}}) = 0$, then $\psi(\theta) = \tilde{\psi}(\theta)$ for a.e. θ.*

Proof. Since $f(0) = 0$, the set of zeros Z_ψ is invariant under ω (i.e. $\psi(\theta) = 0$ implies $\psi(\omega(\theta)) = 0$). Since μ is an ergodic invariant measure for ω, either $\mu(Z_\psi) = 0$ or $\mu(Z_\psi) = 1$.

Let us now consider the function $\Delta(\theta) = |\psi(\theta) - \tilde{\psi}(\theta)|$, and for $N > 0$ we define $\Delta_N(\theta) = \min(\Delta(\theta), N)$. That is, Δ is the limit point of the increasing sequence of non-negative and integrable functions Δ_N. For each N, using Birkhoff Ergodic Theorem, for a.e. $\theta \in \Theta$

$$\int_\Theta \Delta_N(\theta)\, d\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Delta_N(\omega^k(\theta)) \leq \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Delta(\omega^k(\theta)) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\psi(\theta)_k - \tilde{\psi}(\theta)_k|,$$

and the last limit is zero provided that, moreover, $\theta \notin Z_\psi$ and $\theta \notin Z_{\tilde{\psi}}$. See Proposition 3.3. As a result, for each N we have $\Delta_N(\theta) = 0$ for a.e. θ, and so then $\Delta(\theta) = 0$ for a.e. θ. □

Now we construct an invariant graph of T, which will be positive a.e. by Proposition 4.4.

Proposition 4.2. *Let $\Phi : \Theta \to [0, +\infty[$ be the limit point function defined as*

$$\Phi(\theta) = \lim_{n\to\infty} \pi_x \circ T^n(\omega^{-n}(\theta), R),$$

where R is given in Proposition 3.1. Then:

- *Φ is bounded by R from above, and upper-semicontinuous;*
- *Φ is invariant under T;*
- *Z_Φ is a (possibly empty) G_δ set.*

Proof. The key point in [Kel96] is that the transfer operator of graphs defined in (4.1), $\mathcal{T}$, preserves the order of the graphs. That is, for two functions $\psi_1, \psi_2 : \Theta \to [0, +\infty]$,

If $\forall \theta \in \Theta, \psi_1(\theta) \le \psi_2(\theta)$, then $\forall \theta \in \Theta, \mathcal{T}\psi_1(\theta) \le \mathcal{T}\psi_2(\theta)$.

For $n \ge 0$, let $\Phi_n : \Theta \to [0, +\infty[$ be the function defined by $\Phi_n(\theta) = \pi_x \circ T^n(\omega^{-n}(\theta), R)$. The sequence of functions Φ_n satisfies the recurrence relation

$$\Phi_0(\theta) = R,\ \Phi_n(\theta) = \mathcal{T}\Phi_{n-1}(\theta). \tag{4.3}$$

Notice that, since

$$\Phi_1(\theta) = \mathcal{T}\Phi_0(\theta) = g(\omega^{-1}(\theta)) f(\Phi_0(\omega^{-1}(\theta))) \le M f(R) = R = \Phi_0(\theta),$$

and $\mathcal{T}$ preserves the order of the graphs, the sequence Φ_n is decreasing. The limit point function of the decreasing sequence of non-negative continuous functions Φ_n is a non-negative upper-semicontinuous function Φ.

Notice also that taking limit in (4.3), we obtain $\Phi(\theta) = \mathcal{T}\Phi(\theta)$, where we use that f is continuous. As a result, Γ_Φ is an invariant graph.

Since Φ is upper-semicontinuous, for all $\varepsilon > 0$ the set $\{\theta \in \Theta \ / \ \Phi(\theta) < \varepsilon\}$ is open. Then,

$$Z_\Phi = \bigcap_{m>0} \left\{ \theta \in \Theta \ / \ \Phi(\theta) < \frac{1}{m} \right\}$$

is an intersection of a decreasing sequence of open sets, so it is a G_δ set. □

Remark 4.3. *Notice that if $\theta_0 \in Z_g$, then $\Phi(\omega(\theta_0)) = g(\theta_0) f(\Phi(\theta_0)) = 0$, that is $\omega(\theta_0) \in Z_\Phi$. As a result, for all $n \ge 1$, $\omega^n(\theta_0) \in Z_\Phi$. As a result, if the orbit of θ_0 is dense in Θ, the set Z_Φ is a G_δ-dense set (a residual set).*

In particular, if ω is minimal (all its orbits are dense in Θ) and g has zeros, the set of zeros of Φ is a residual set.

With the proof of Proposition 4.2 we have proved (a),(c) of Theorem 1.1. We have also proved that the set of zeros of Φ is "small" from a topological point of view (that it is a G_δ set). We will prove now that it is also "small" from a measure theoretic point of view, that the set of zeros of Φ is null. In fact, we will prove much more, that the function Φ is log-integrable.

Proposition 4.4. *The invariant function* Φ *is log-integrable, that is*

$$\int_\Theta \log \Phi(\theta)\ d\mu > -\infty.$$

In particular, Φ *is positive a.e., that is* $\mu(Z_\Phi) = 0$.

Proof. Let us define the constant

$$\alpha = \sup_{0<t\leq R} q(t) < 1, \tag{4.4}$$

and the function $h :]0,+\infty[\to]0,+\infty[$ by $h(x) = \dfrac{f(x)}{x^\alpha}$. Notice the h is decreasing in $]0,R]$, because

$$h'(x) = \frac{f(x)}{x^{\alpha+1}}(q(x) - \alpha) \leq 0$$

if $0 < x \leq R$.

We also define

$$\gamma = \int_\Theta \log g(\theta)\ d\mu \in]-\infty,+\infty[,$$

and, for all n

$$\varphi_n = \int_\Theta \log \Phi_n(\theta)\ d\mu \in [-\infty, \log R],$$

where the sequence Φ_n is given in Proposition 4.2. In particular, $\varphi_0 = \log R$.

Then, using that $f(x) = x^\alpha h(x)$,

$$\begin{aligned}
\varphi_n &= \int_\Theta \log\left(g(\omega^{-1}(\theta))\Phi_{n-1}(\omega^{-1}(\theta))^\alpha h(\Phi_{n-1}(\omega^{-1}(\theta)))\right)\ d\mu \\
&= \gamma + \alpha\varphi_{n-1} + \int_\Theta \log h(\Phi_{n-1}(\theta))\ d\mu \\
&\geq \alpha\varphi_{n-1} + \gamma + \log h(R)\ ,
\end{aligned}$$

where in the second equality we use that μ is an invariant measure, and in the inequality we use that $\Phi_{n-1}(\theta) \leq R$ and h is a decreasing function in $]0,R]$. The previous arguments prove that the functions Φ_n are log-integrable.

From the Monotone Convergence Theorem, the limit point of the increasing sequence of non-negative integrable functions $\log R - \log \Phi_n$, which is $\log R - \log \Phi$, is a measurable function and

$$\int_\Theta (\log R - \log \Phi(\theta))\ d\mu = \lim_{n\to\infty} \int_\Theta (\log R - \log \Phi_n(\theta))\ d\mu\ .$$

As a result,

$$\log R \geq \int_\Theta \log \Phi(\theta)\ d\mu = \lim_{n\to\infty} \varphi_n \geq \frac{1}{1-\alpha}(\gamma + \log h(R)). \tag{4.5}$$

Hence, $\log \Phi$ is integrable, and the set of zeros of Φ has zero measure. □

With this proof we prove statement (b) of Theorem 1.1.

The following proposition is an immediate consequence of Proposition 3.3, and states that almost all orbits approach the invariant graph exponentially fast. This is (d) of Theorem 1.1.

Proposition 4.5. *For all $\theta \notin Z_\Phi$, for all $x > 0$, there exist $C_{\theta,x} > 0$ such that*

$$|\Phi(\theta_n) - x_n| \leq C_{\theta,x} \alpha^n |\Phi(\theta) - x|,$$

where $\alpha = \sup_{0<t\leq R} q(t) < 1$.

Proof. From Proposition 3.3, it suffices to take $C_{\theta,x} = \frac{R}{\min(x,\Phi(\theta))}$. □

Remark 4.6. *We emphasize that, if Φ is bounded from below (for instance, Φ is continuous and has no zeros, see Proposition 4.11 below) then the approach of orbits to the graph is uniform. That is, the constant $C_{\theta,x}$ can be made independent from θ in a compact neighborhood of the graph.*

On the other side, if Φ has zeros, then the approach in not uniform, and the constants $C_{\theta,x}$ can explode.

We are going now to relate the Lyapunov exponents of orbits and the invariant graph Φ. The first result is a consequence of Birkhoff Ergodic Theorem.

Proposition 4.7. *For a.e. $\theta \in \Theta$, $\lambda(\theta, \Phi(\theta)) = \lambda_\Phi \leq \log \alpha < 0$.*

Proof. We have

$$\begin{aligned} \lambda(\theta, \psi(\theta)) &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log g(\omega^k(\theta)) + \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log f'(\Phi(\omega^k(\theta))) \\ &= \int_\Theta \log g(\theta)\, d\mu + \int_\Theta \log f'(\Phi(\theta))\, d\mu = \lambda_\Phi, \end{aligned}$$

where the second equality holds for a.e. $\theta \in \Theta$, by Birkhoff Ergodic Theorem. Notice that $\log g : \Theta \to \mathbb{R} \cup \{-\infty\}$, $\log^+ g \in L^1_\mu$ because g is bounded from above (in fact, hypothesis (g1) says that $\log g \in L^1_\mu$), and $\log f'{\circ}\Phi : \Theta \to \mathbb{R} \cup \{+\infty\}$, $\log^- f'{\circ}\Phi \in L^1_\mu$ because $f'{\circ}\Phi$ is bounded from below. The proof follows from Proposition 3.4, taking into account that for $x = \Phi(\theta)$, $\alpha_x = \alpha$. □

Remark 4.8. *Notice also that*

$$\begin{aligned}
\lambda_\Phi &= \int_\Theta \log g(\theta)\ d\mu + \int_\Theta \log f'(\Phi(\theta))\ d\mu \\
&= \int_\Theta \log q(\Phi(\theta))\ d\mu + \int_\Theta \log\left(\frac{g(\theta)f(\Phi(\theta))}{\Phi(\theta)}\right)\ d\mu \\
&= \int_\Theta \log q(\Phi(\theta))\ d\mu + \int_\Theta \left(\log(\Phi(\omega(\theta))) - \log(\Phi(\theta))\right)\ d\mu \\
&= \int_\Theta \log q(\Phi(\theta))\ d\mu \le \log\alpha,
\end{aligned}$$

where in the fourth equality we use that Φ is log-integrable (and μ is ω-invariant), and the last inequality follows from the fact that $\Phi(\theta) \le R$, incidentally proving again the estimate in Proposition 4.7. As a result, the Lyapunov multiplier of the invariant graph, $\Lambda_\Phi = \exp\lambda_\Phi$, is the geometric mean of the elasticity of the function f on the invariant graph:

$$\Lambda_\Phi = \exp\left(\int_\Theta \log q(\Phi(\theta))\ d\mu\right). \tag{4.6}$$

Remark 4.9. *Equality* (4.6) *can also be obtained from Keller's hypotheses. One has to apply a general measure theoretic result (Lemma 2 in* [Kel96]*).*

Up to now, we have bounded from above the upper Lyapunov exponent of most orbits (see Proposition 3.4) and we have computed the Lyapunov exponent for orbits on the invariant graph (see Proposition 4.7). Moreover, in the arguments we have used that $\alpha = \sup_{0<t\le R} q(t) < 1$, but not that q is continuous at 0 with $0 < q(0) = q_0 < 1$. Using this hypothesis (f3), we will prove that for almost all orbits the Lyapunov exponent equals the Lyapunov exponent of the graph.

Proposition 4.10. *For a.e. $\theta \in \Theta$, for all $x > 0$, $\lambda(\theta, x) = \lambda_\Phi$.*

Proof. We recall that

$$\lambda(\theta, x) = \lim_{n\to\infty} \frac{1}{n}\log\left(\frac{x_n}{x_0}\right) + \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\log q(x_k) \tag{4.7}$$

(see (3.1)), if both limits exist. These are the limits we are going to compute below.

Firstly, notice that $x_n = y_n + z_n$, with $y_n = \Phi(\theta_n)$ and $z_n = x_n - \Phi(\theta_n)$. If $\theta \notin Z_\Phi$, $|z_n| \le C\alpha^n$, for a certain constant $C = C_{\theta,x}|x - \Phi(\theta)|$, and $\alpha < 1$ (see Proposition 4.5). On the other side, since Φ is log-integrable (see Proposition 4.4) then it is tempered, i.e. for a.e. $\theta \in \Theta$,

$$\lim_{n\to\infty}\frac{1}{n}\log y_n = \lim_{n\to\infty}\frac{1}{n}\log\Phi(\theta_n) = 0$$

(this follows from Birkhoff Ergodic Theorem). Let us take $\varepsilon > 0$ small enough such that $\alpha < 1 - \varepsilon$, so for n sufficiently big we have $(1-\varepsilon)^n < y_n$. Then,

$$\lim_{n\to\infty} \frac{1}{n} \log(y_n + z_n) = \lim_{n\to\infty} \frac{1}{n} \log\left(1 + \frac{z_n}{y_n}\right),$$

and the r.h.s equals zero because

$$\left|\frac{z_n}{y_n}\right| < C\alpha^n (1-\varepsilon)^{-n} \longrightarrow 0.$$

These arguments prove that

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\frac{x_n}{x_0}\right) = 0. \tag{4.8}$$

Secondly, notice that

$$\lim_{n\to\infty} (\log q(x_k) - \log q(\Phi(\theta_k)) = 0,$$

because $\lim_{n\to\infty} |x_k - \Phi(\theta_k)| = 0$ and, by hypothesis (f3), $\log q$ is uniformly continuous in, say, $[0, 2R]$. As a result,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log q(x_k) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log q(\Phi(\theta_k)) = \int_\Theta \log q(\Phi(\theta))\ d\mu, \tag{4.9}$$

for a.e. $\theta \in \Theta$.

From (4.7), (4.8) and (4.9), we end up with the proof of Proposition 4.10. □

With the proof of Proposition 4.10 we prove statement (e) of Theorem 1.1.

Finally, we will show that, if g has no zeros, then the invariant graph Φ is continuous.

Proposition 4.11. *Assume that $Z_g = \emptyset$. Then, $\Phi : \Theta \to]0, +\infty[$ is continuous (and $Z_\Phi = \emptyset$).*

Proof. Recall that for all $\theta \in \Theta$, $0 < m \le g(\theta) \le M$. From Proposition 2.2, there exist (a unique) $r > 0$ such that $p(r) = m^{-1}$. Hence, for all $\theta \in \Theta$,

$$T(\theta, r) = g(\theta) f(r) \ge m r m^{-1} = r.$$

Using similar arguments to those of the proof of Proposition 4.2, the sequence of continuous functions

$$\Psi_n(\theta) = \pi_x \circ T^n(\omega^{-n}(\theta), r)$$

is increasing, and bounded from above by R. The limit point Ψ is then invariant under T, so $\Psi = \Phi$ a.e.

In fact, for all $\theta \in \Theta$, we have the ordering

$$r \le \Psi_1(\theta) \le \dots \le \Psi_n(\theta) \le \dots \Psi(\theta) \le \Phi(\theta) \le \dots \Phi_n(\theta) \le \dots \le \Phi_1(\theta) \le R.$$

Then, by Proposition 3.3, for all $\theta \in \Theta$,

$$\Phi(\theta) - \Psi_n(\theta) \leq \pi_x \circ T^n(\omega^{-n}(\theta), R) - \pi_x \circ T^n(\omega^{-n}(\theta), r) \leq \frac{R}{r}(R - r)\alpha^n,$$

that goes to zero uniformly when n goes to $+\infty$. Hence, the sequence of continuous functions Ψ_n converges uniformly to Φ, so it is a continuous function. $\square$

Hence, we have already proved statement (z0) of Theorem 1.1.

The proof of statement (z1) is just to quote Remark 4.3 and to note that, if one of the pinched points (zeroes of g) has a dense orbit, then the function Φ has a dense set of zeroes. If Φ were also continuous, then it should be constant zero, which is in contradiction with the fact that the set of zeroes of Φ is null.

With these final arguments we are done with the whole proof of Theorem 1.1.

5. A paradigmatic example

In this section we will particularize the results for the skew-product $T_\alpha(\theta, x) = (\omega(\theta), g(\theta)x^\alpha)$, that is the case $f(x) = x^\alpha$. Notice that in such a case $q(x) = \alpha < 1$. We will see that the bounds of the Lyapunov exponents and of the invariant graph obtained in the previous sections saturate. We can make rather explicit calculations.

For instance, given $(\theta, x) \in X$, its (forward) orbit (θ_n, x_n) is given by

$$x_n = \prod_{k=0}^{n-1} g(\omega^k(\theta))^{\alpha^{n-k-1}} x^{\alpha^n} \tag{5.1}$$

We can also compute its Lyapunov exponent.

Proposition 5.1. *For a.e. $\theta \in \Theta$, for all $x > 0$, $\lambda(\theta, x) = \log \alpha < 0$.*

Proof. From (5.1), we obtain that

$$L_n(\theta, x) = \prod_{k=0}^{n-1} g(\omega^k(\theta))^{\alpha^{n-k-1}} \alpha^n x^{\alpha^n - 1},$$

so, for all $\theta \notin \tilde{Z}_g$, for all $x > 0$,

$$\lambda(\theta, x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \alpha^{n-k-1} \log g(\omega^k(\theta)) + \log \alpha.$$

Moreover,

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\alpha^{n-k-1}\log g(\omega^k(\theta)) &= \lim_{n\to\infty}\frac{\sum_{k=0}^{n-1}\alpha^{-k}\log g(\omega^k(\theta))}{n\alpha^{1-n}}\\ &= \lim_{n\to\infty}\frac{\alpha^{-n}\log g(\omega^n(\theta))}{(n+1)\alpha^{-n}-n\alpha^{1-n}}\\ &= \frac{1}{1-\alpha}\lim_{n\to\infty}\frac{1}{n}\log g(\omega^n(\theta)),\end{aligned}$$

and the last limit is 0 a.e. θ, by the Birkhoff Ergodic Theorem. □

We will denote by $G_\alpha : \Theta \to [0,+\infty[$ the positive a.e. invariant function for the skew-product T_α, obtained in Proposition 4.2 (see Theorem 1.1).

Proposition 5.2. *The function G_α is expanded by*

$$G_\alpha(\theta) = \prod_{k=1}^{\infty} g(\omega^{-k}(\theta))^{\alpha^{k-1}}$$

Moreover,

$$\int_\Theta \log G_\alpha(\theta)\ d\mu = \frac{1}{1-\alpha}\int_\Theta \log g(\theta)\ d\mu.$$

Proof. We have just to perform the construction of Proposition 4.2, with $R_\alpha = M^{\frac{1}{1-\alpha}}$, obtain that the nth iterate is

$$G_n(\theta) = \prod_{k=1}^{n} g(\omega^{-k}(\theta))^{\alpha^{k-1}}(R_\alpha)^{\alpha^n},$$

and realize that $\lim\limits_{n\to\infty}(R_\alpha)^{\alpha^n} = 1$.

The integral formula is straightforward. (See also Proposition 4.4). □

Remark 5.3. *A complementary result is that the geometric means of a positive orbit, a.e. θ , $\hat{x}_n = (x_0x_1\dots x_{n-1})^{\frac{1}{n}}$ tend to a power of the geometric mean of g, $\hat{g} = \exp\int_\Theta \log g(\theta)\ d\mu$:*

$$\lim_{n\to\infty}\hat{x}_n = \hat{g}^{\frac{1}{1-\alpha}}. \tag{5.2}$$

For the proof, just notice that we can also write

$$\begin{aligned}\lambda(\theta,x) &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\log g(\omega^k(\theta)) + \log\alpha + \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}(\alpha-1)\log x_k\\ &= \int_\Theta \log g(\theta)\ d\mu + \log\alpha - (1-\alpha)\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\log x_k,\end{aligned}$$

and, since $\lambda(\theta, x) = \log \alpha$, *we obtain*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log x_k = \frac{1}{1-\alpha} \int_\Theta \log g(\theta)\ d\mu,$$

which is equivalent to (5.2).

Finally, we will see that the invariant graph G_α of T_α provides a lower bound for the invariant graph Φ of T.

Proposition 5.4. *Let* Φ *be the positive a.e. invariant graph of the bundle map* T *satisfying the hypothesis of Theorem 1.1. Then, for all* $\theta \in \Theta$,

$$\Phi(\theta) \geq \frac{R}{R_\alpha} G_\alpha(\theta),$$

where $p(R) = M^{-1}$, $\alpha = \sup_{0<t\leq R} q(t)$ *and* $R_\alpha = M^{\frac{1}{1-\alpha}}$.

Proof. Let us denote $\Psi(\theta) = AG_\alpha(\theta)$, with $A = \frac{R}{R_\alpha}$. Notice that $\Psi(\theta) \leq AR_\alpha = R$. It suffices to prove that $\mathcal{T}\Psi(\theta) \geq \Psi(\theta)$. To do so,

$$\begin{aligned}\mathcal{T}\Psi(\theta) &= g(\omega^{-1}(\theta)) f(\Psi(\omega^{-1}(\theta))) = g(\omega^{-1}(\theta)) A^\alpha G_\alpha(\omega^{-1}(\theta))^\alpha h(\Psi(\omega^{-1}(\theta))) \\ &\geq A^\alpha G_\alpha(\theta) h(R) = A^{\alpha-1} h(R) \Psi(\theta),\end{aligned}$$

where in the inequality we use that G_α is invariant for T_α, and h is a decreasing function in $[0, R]$. Finally, notice that

$$A^{\alpha-1} h(R) = R^{\alpha-1} h(R) R_\alpha^{1-\alpha} = p(R) M = 1,$$

and we are done with the proof of Proposition 5.4. □

Remark 5.5. *We are now to give an alternative proof of Proposition 4.4. Notice that, from Proposition 5.4*

$$\int_\Theta \log \Phi(\theta)\ d\mu \geq \log\left(\frac{R}{R_\alpha}\right) + \int_\Theta \log G_\alpha(\theta)\ d\mu = \log\left(\frac{R}{R_\alpha}\right) + \frac{1}{1-\alpha} \int_\Theta \log g(\theta)\ d\mu,$$

incidentally proving the lower bound (4.5), *because*

$$\frac{1}{1-\alpha} \log h(R) = \frac{1}{1-\alpha} \log\left(p(R) R^{1-\alpha}\right) = \frac{1}{1-\alpha} \log\left(\frac{R^{1-\alpha}}{M}\right) = \log\left(\frac{R}{R_\alpha}\right).$$

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